Systems Research Institute
Polish Academy of Sciences

Meizhi Qian

Topological derivative method in control and design of distributed parameter systems on metric graphs

PhD dissertation

…………………………………….

Supervisor: Jan Sokołowski
Systems Research Institute, Polish Academy of Sciences, Poland

Co-supervisor: Shengfeng Zhu
School of Mathematical Sciences, East China Normal University, China

Warszawa, 2024

Abstract

We consider the networks. The network is defined by a metric graph, $G = \{E, V\}$, and state equations on the set of edges E with some compatibility conditions at the vertices. There are applications of such models in transportation networks as well as for mechanical structures of beams. The state equations for dynamic problems are the wave equations on the graphs. The steady state equations are vectorial ordinary differential equations. The control problems on networks are considered for the system of wave equations. The optimum design problems for networks are considered for the steady state equations. The so-called *Turnpike Property* is shown to hold for the control problems with infinite horizon, which makes it possible to consider control and design bilevel optimization problems on networks. The cost for optimum design is the optimal cost for control.

This dissertation focuses on the development and application of the *Topological Derivative Method* in shape and topology optimization of networks. The singular perturbation of the tree graph is obtained, e.g., by nucleation of a small cycle. The size of the cycle is determined by the shape optimization technique in singularly perturbed domains. The small cycle changes the structure of the control system for the wave equation. That is why the small cycle is the singular perturbation of the control system on networks.

The original contribution of the dissertation includes among others:

1) The proof of Turnpike Property for abstract wave equations and the network. The results are obtained for the optimality system including the state equation, the adjoint state equation, and the sufficient optimality conditions for the wave equation and the steady state model.

2) The constructive form of topological derivative for complex networks including the network of Timoshenko beams with the cost for optimum design defined by the control problems with steady state equations.

3) The analysis of optimality conditions for nonlinear steady state equations in the form of local Pontryagin's maximum principle.

4) Numerical methods for solution of combined control and design problems on networks using Matlab with programs presented in Appendix.

Partial Differential Equations (PDEs) are considered on graphs, primarily of the tree structure. The singular perturbation of the shape is defined by a nucleation of a small cycle with the size $\varepsilon \to 0$. The cycle can be called a hole in the network. The location of the hole in the graph can be determined by the topological derivative method.

Optimal Control Problems (OCPs) are considered in the networks. The state equations defined on the graph are of evolution type, e.g., the wave equation for the networks. The steady state of the network is governed by specific Elliptic Boundary Value Problems (EBVPs).

Optimality conditions for optimal control problems arising in network modeling are derived. For the nonlinear state equations, we confine ourselves to the steady state network models. Therefore, we consider only control systems described by nonlinear ordinary differential equations. First, we derive optimality conditions for the nonlinear problem

for a single beam. These conditions are formulated in terms of the local Pontryagin maximum principle and the matrix Riccati equation. Then the optimality conditions for the control problem for networks posed on an arbitrary planar graph are discussed. Two simple numerical examples for the single-beam problem are considered.

The optimal control problems for linear state equations are introduced. The optimal distributed control problems for the Timoshenko beams are considered on networks. We employ the Steklov-Poincaré operator to convert the beam network model into a linear system of equations and utilize domain decomposition methods to enable topological changes. Specifically, the Timoshenko beams are known for their ability to model more complex mechanical behavior compared to simpler beam models. We present a method for computing such an optimality system and the topological derivative for tracking type of cost functionals, thereby enabling effective control and optimization of Timoshenko beam networks. Numerical examples validate our analytical results.

The optimal boundary control problems for the wave equation are considered on networks. Time interval $(0, T)$ for $T \to +\infty$ is fixed for the dynamic control problems. The so-called turnpike property is shown for the state equation, the adjoint state equation as well as the optimal cost. The shape and topology optimization is performed for the network with the shape functional given by the optimality system of control problem. The set of admissible shapes for the network is compact in finite dimensions, thus the use of turnpike property is possible. The topology optimization is analyzed for an example of nucleation of a small cycle at the internal node of network. The topological derivative of the cost is introduced and evaluated in the framework of domain decomposition technique. Numerical examples are provided.

We refer the reader to Lions' book [53] for optimal control theory, to Lewiński et al. [50] for optimum design methods, to Novotny et al. [63, 62, 64] for the topological derivative method in optimum design.

List of Published Articles

1. M. Gugat, M. Qian, and J. Sokołowski. Topological derivative method for control of wave equation on networks. *The 27th International Conference on Methods and Models in Automation and Robotics (MMAR). IEEE*, 2023. https://doi.org/10.1109/MMAR58394.2023.10242484

2. N. P. Osmolovskii, M. Qian, J. Sokołowski. Network optimality conditions. *Control and Cybernetics*, 52(2): 129–180, 2023. https://doi.org/10.2478/candc-2023-0035

3. M. Gugat, M. Qian, and J. Sokołowski. Network design and control: Shape and topology optimization for the turnpike property for wave equation, *Journal of Geometric Analysis*, 34, 273, 2024. https://doi.org/10.1007/s12220-024-01712-8

In addition, the following papers are published:

4. M. Qian, S. Zhu. A level set method for Laplacian eigenvalue optimization subject to geometric constraints. *Computational Optimization and Applications*, 82(2), 499-524, 2022. https://doi.org/10.1007/s10589-022-00371-1

5. M. Qian, X, Hu., S, Zhu. A phase field method based on multi-level correction for eigenvalue topology optimization. *Computer Methods in Applied Mechanics and Engineering*, 401, 115646, 2022. https://doi.org/10.1016/j.cma.2022.115646

6. X, Hu, M. Qian, S. Zhu. Accelerating a phase field method by linearization for eigenfrequency topology optimization. *Structural and Multidisciplinary Optimization*, 66(12), 242, 2023. https://doi.org/10.1007/s00158-023-03692-9

Contents

Chapter 1

Introduction

We consider optimal control problems and optimum design for networks. Such problems arise for example in gas and water transportation pipelines with compressors. Another important example could be the network of Timoshenko beams to model a bike. We are interested primarily in the dynamic control problems which can be reduced to the static control problems under appropriate conditions.

If we consider beams then we could have:

- Single load case and the elastic energy function;
- Multiply load case.

We use a more sophisticated model, where this cost for optimum design takes into account a set of parameters which are called *controls*. Therefore, the multiply load case is achieved by using the optimal control theory. The cost of design is determined by the optimality system for the control of networks. A specific choice is to define the design cost by the optimum value of control problem. For such a general shape functional we can use shape derivatives and topological derivatives for numerical solutions of design problems. This approach is new and efficient and combines the theory of optimal control of distributed parameters system with the modern technique of shape and topology optimization. We use also the so-called *Turnpike Property* of dynamic control problems with infinite horizons in order to use the steady state optimal problems for the numerical solutions of dynamic control problems.

The literature on control and optimization includes various foundational works and advancements. Lions' seminal contribution [53] is important in fundamental principles within optimal control theory. Furthermore, Leugering and Kogut's book [37] uses an optimality system of optimization, enriching our understanding of complex systems. In shape optimization, Sokołowski and Zolésio's book [88] is devoted to the shape sensitivity analysis for unilateral problems describing such physical phenomena as contact problems in elasticity, elasto-plastic torsion problems, and the obstacle problem. In topological derivatives, the works by Sokołowski and Żochowski [87], Sokołowski and Lewiński [51], and Sokołowski and Leugering [46] contribute significantly to understanding optimization in structural design. Asymptotic analysis in singularly perturbed geometrical domains is needed. Moreover, Novotny's books [63, 62, 64] recall the classical approach to shape optimization problems and extend the analysis to some singular perturbations of the reference domains for elliptic boundary value problems. The topological derivative can be considered as the singular limit of the classical shape derivative. In our dissertation, the optimal control theory is combined with the topological derivative method of shape optimization.

1.1 Motivation

In engineering, the optimum design of mechanical structures under specific loads is a popular task. To give a representative example, we refer e.g., to [84] for shape optimization approach in optimum design of prosthetic feet. Common objective functions encompass compliance, average stress, or structure volume [8], while the range of permissible shapes may be constrained by design limitations.

Typically, the optimization of a structure's design or, more broadly, a domain involves an iterative process marked by incremental enhancements. Although shape optimization methods are effective for various modifications, including topology-altering changes like merging or splitting substructures, they are generally unsuitable for generating new subdomains [33], such as introducing holes in a structure (See Fig. 1.1). The strategic determination of where to create openings in the domain stands as a pivotal aspect of topology design [9, 23].

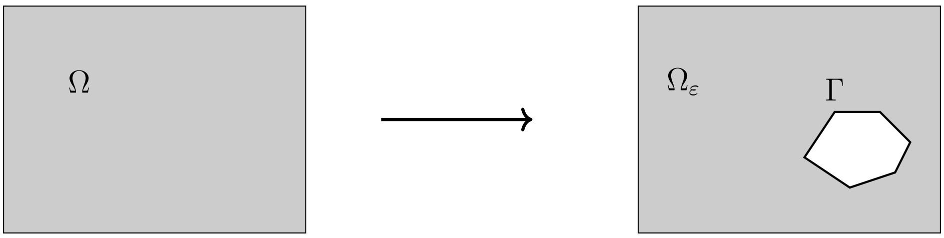


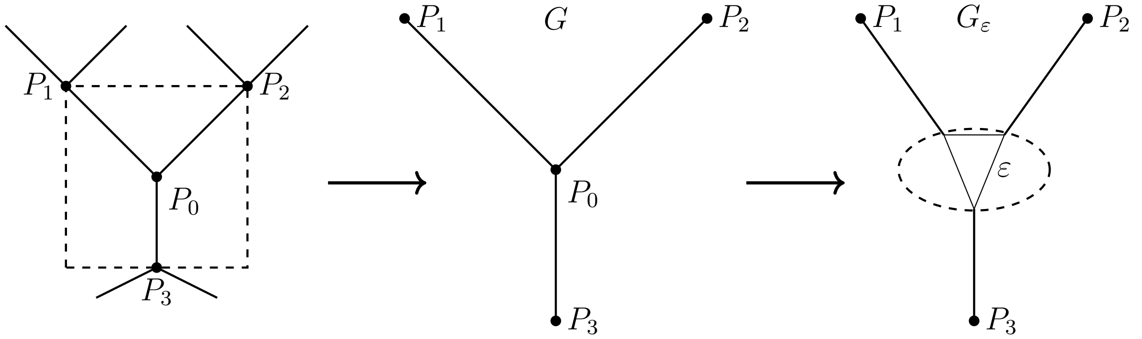
Figure 1.1: Topology change by creation of a hole (continuum).

Contrary to the suggestive nature of the term, the term "holes" within this domain does not necessarily denote emptiness. Depending on the context, these "holes" can signify material inclusions or variations [5, 83], concealed objects within a medium [12], or impediments influencing the flow of gases or liquids [15].

The utilization of topological derivatives extends to diverse scenarios, including optimal compliance problems [52, 86], shape functionals within the realm of linear elasticity [51, 62], as well as applications to semilinear elliptic systems [35], spectral problems [61], heat diffusion [62], and Helmholtz problems [80, 69].

Frequently, optimization methods integrate topological derivatives with level-set techniques to address the limitation of the latter, which prohibits the generation of new subdomains [3, 4, 60, 92]. Overviews of the fields of use for topological gradients are given by [42] and [62].

The utilization of topological derivatives within beam networks revolves around optimizing the objective functional through vertex splitting, thereby inducing changes in the graph's configuration [22, 45, 46]. This process entails initially removing the targeted vertex and segments of its adjoining edges. In contrast to the situation in continuous domains, where the boundaries of a hole formation are typically enclosed, the altered graph lacks such a descriptive feature in relation to the void. Consequently, careful consideration is essential to define the desired structure post-reconnection following this modification (refer to Fig. 1.2). In Fig. 1.2, (a) is a Network, G is a star graph of the network, and G_ε is a graph in which the central node is replaced by a small cycle of size of ε.



(a) Network (b) Star (c) Star with cycle

Figure 1.2: Topology change by creation of a small cycle .

Indeed, the optimum design of arrays of significant trusses and frames, as well as Michell Structures is studied in the monograph by Lewiński et al.[50]. Michell poses the query of safely and economically transmitting given loads to specified support zones without a priori assumptions on bar layout. The request for volume minimization transforms into maximizing virtual work, shifting the problem from trusses to optimal design of continuum bodies. Michell structures, optimal solutions to these problems, have inspired engineers and designers across disciplines, extending into modern fields like mechatronics and biotechnology in recent years. We refer the reader to [93, 49, 70, 1, 48, 47, 85, 7, 91, 16, 71] for modeling and optimum design or/and topology optimization in structural mechanics.

1.2 Main Idea

We consider networks, which are described by partial differential equations (PDEs) on metric graphs. The first problem to solve is the optimal control problem. We consider the dynamic state equation, e.g., the wave equation. We perform shape and topology optimization of a cost functional, i.e., the shape functional. we are interested in problems with the turnpike property, which allows us to simplify numerical methods. In such a case, the dynamic problem could be simply replaced by a static problem, for which we can use shape derivatives and topological derivatives for numerical solutions for the design of networks.

We consider state equations for Timoshenko beams and some other models of elastic structure for networks. We start with the description of shape optimization methods for networks.

The *boundary variations* technique in shape optimization is applied to the minimization of the functional

$$\Omega \mapsto J(\Omega), \tag{1.1}$$

which means the mapping from the geometrical domain to the scalar cost functional. As for applications, we will concentrate on equations of mathematical physics and mechanical engineering. This implies that Ω is representative of a physical body and, as a mathematical object, is the integration domain of a system of ordinary and more importantly PDEs describing the physical process of interest. In such a case, a composed mapping is considered. First

$$\Omega \mapsto y(\Omega) \tag{1.2}$$

is the solution of the state equation in the spatial domain, then the value of the cost function is given by $y(\Omega) \mapsto I(y(\Omega))$. Therefore, the composite mapping is given by

$$\Omega \mapsto y(\Omega) \longmapsto I(y(\Omega)). \tag{1.3}$$

With (1.1) we associate shape gradients and the shape Hessians with respect to changes in Ω. The underlying state equations lead to corresponding material and shape derivatives for (1.2). Finally, with (1.3) we associate a so-called *adjoint state* equation to simplify the expression for the shape gradient of the composed mapping.

For regular perturbations of Ω, we use boundary variation techniques such that we restrict the analysis to the case of a family Ω_τ with $\tau \to 0$ and $\Omega_0 = \Omega$. This approach has its origin in fluid mechanics. As a result, a function $J(\cdot)$, originally defined as a function on domains is represented by a function of one variable in the vicinity of $\tau = 0^+$,

$$\tau \mapsto J\left(\Omega_\tau\right). \tag{1.4}$$

The case of singular domain perturbations is also investigated. By singular perturbations, we mean the introduction and evolution of holes or cracks. If a hole $B_\varepsilon(x)$ is present in Ω_ε, we consider a small parameter $\varepsilon \to 0^+$and the shape functional being parametrized by

$$\varepsilon \mapsto J\left(\Omega_\varepsilon\right).$$

In such a case, an asymptotic analysis of the state equation is performed and the following formula is justified

$$J\left(\Omega_\varepsilon\right) = J(\Omega) + f(\varepsilon)\mathscr{T}_\Omega(x) + o(f(\varepsilon)),$$

with the function

$$x \to \mathscr{T}_\Omega(x)$$

denoted as the *topological derivative* of $J(\Omega)$ at $x \in \Omega$.

Two methods of shape sensitivity analysis can be used for shape and topology optimization. The first method is the so-called *velocity method* originated from the fluid mechanics [88]. The second is the so-called *topological derivative method* which is based on the singular perturbations of the geometrical domains [64]. The first method replaces the mapping defined on the sets $\Omega \to y(x) := y(\Omega; x)$ by the mapping defined on the interval $(-\tau_0, \tau_0) \ni \tau \to y_\tau(x) := y\left(\Omega_\tau; x\right)$. Let us note that the function $x \to y_\tau(x)$ is defined on the set Ω_τ only. The second method uses the approximation of the solution $y\left(x \to \Omega_\varepsilon; x\right)$ in the domain with the small hole $\varepsilon \to \Omega_\varepsilon := \Omega \backslash \overline{\omega}_\varepsilon$ by the solution $x \to y(\Omega; x)$ in the intact domain Ω. The second method will be considered in particular for graphs in this dissertation.

1.3 Optimal Control Problems Governed by PDEs

In optimal control problems, we have a state equation and a tracking cost functional to be minimized. First, we derive the optimality system, which usually contains state equations, adjoint state equations, and optimality conditions. In the case of linear state equations, the associated optimality system admits a unique optimal control. In the case of nonlinear state equations, the solution of the optimality system is not unique. In the case of static problems of ODEs, Pontryagin's Maximum Principle is considered.

For the optimal value of cost, the shape and topological optimization of the network is studied. To this end, the shape and topological derivatives are introduced. We perform the design concerning the length of the edges. The topological derivatives are computed with respect to the nucleation of the small cycles.

1.4 Modeling, Control, and Design of Networks

The network structure is defined by a graph $G = (V, E)$, where V represents the set of nodes, E is the set of edges, and $n_v := \#V$, $n_e := \#E$. In the context of planar graphs, it is crucial to distinguish between one-dimensional "out-of-plane" evolution and two-dimensional "in-plane" evolution. This distinction necessitates the consideration of a function $y(x)$, which can be scalar, two-dimensional, or three-dimensional. In the latter two cases, *local coordinate* systems $\left(e_{i_j}\right)_{i=1,\cdots,n_e}^{j=1,\cdots,d}$, $d \in \{2, 3\}$ are required. For instance, in the $2D$ case, the expression for y on edge E_i takes the form:

$$y_{\mathrm{i}}(x) = y_{1i}(x) e_i + y_{2i}(x) e_i^{\perp},$$

where e_i denotes the local unit vector in the direction of the edge. So y_{1i} describes the longitudinal part of the displacement, while y_{2i} gives the lateral one.

In a network, we distinguish between single and multiple nodes. Multiple nodes inside the graph are called $v_M \in V_M \subset V$, while single nodes are $\partial V = V \backslash V_M$. They differ in their boundary conditions. Dirichlet (clamped) nodes are permitted

$$v_D \in V_D := \{v_j \in \partial V \mid v_j \text{ is jammed }\} \subset V,$$

Neumann (free) nodes

$$v_N \in V_N := \{v_j \in \partial V \mid v_j \text{ is free }\} \subset V,$$

and controlled nodes

$$v_C \in V_C := \{v_j \in \partial V \mid v_j \text{ is controlled }\} \subset V.$$

We have $V = V_D \cup V_N \cup V_C \cup V_M$. Further, we define the index set ϵ_J as the set of indices for all edges adjacent to the node v_J, and d_J as the degree of the node v_J. So it is

$$V_M = \{v \in V \mid d(v) > 1\}.$$

Let L_i denote the length of the edge i and $\epsilon_{i,j}$ the orientation of the outer normal vector on the boundary (the nodes of the edge), i.e.,

$$\epsilon_{i,j} := \begin{cases} -1 & \text{if the } i\text{-th edge starts at node } v_\mathrm{j}, \\ +1 & \text{if the } i\text{-th edge ends at node } v_\mathrm{j}, \\ 0 & \text{otherwise }. \end{cases}$$

Figure 1.3 illustrates a single edge along with its local coordinate system. The coordinate system is centered at the node v_1, with parameters $\epsilon_{i,1} = -1$ and $\epsilon_{i,2} = 1$ applied.

Figure 1.3: The local coordinate system of an edge.

The geometrical set associated with the graph is denoted by Ω. We consider a single edge, a cross of three edges, and a cross with a small cycle of six edges. If the system of PDEs is defined on the graph we can consider the shape sensitivity analysis of the state equation with respect to the domain perturbations of Ω. Thus we have the shape and material derivatives for regular perturbations and the topological derivatives for the singular perturbations. On edges, there are given PDEs, with the solutions y_i on E_i. For a single edge, the model of interest is a beam or wave equation.

1. *Timoshenko Beams model* (TB): A linear system of hyperbolic dynamic or elliptic static equations;

2. *Geometrically Exact Beams model* (GEB): Nonlinear beam;

3. *Intrinsic Geometrically Exact Beams model* (IGEB): A system of semilinear hyperbolic equations (weak formulation is used for approximation in space, then ODEs are solved);

4. Scalar wave equations.

In the vertices, there are two sorts of continuity conditions, the *continuity* for solutions and for fluxes which are the *Kirchhoff* conditions. They are also called *transmission* conditions. We consider the boundary optimal control problems for the second-order hyperbolic problems (wave equations) and distributed optimal control for the TB and static IGEB. The optimal value of the cost for control problems is subject to design with respect to the geometry

$$\Omega \to G = (E, V) \to y \to J(\Omega).$$

We evaluate the shape and the topological derivatives of the shape functional

$$\Omega \to J(\Omega).$$

The singular perturbation of the geometrical domain Ω means e.g. the nucleation of a small cycle of the size $\varepsilon \to 0$.
Remark In section 2.3.2 of [89] we have the Timoshenko beam model derived directly from GEB. We need the steady state models of beams (the models without time derivatives):

- from geometrically exact beams;

- from Timoshenko beams.

This way we could define two optimum design problems for steady state models:

- nonlinear beam or networks of beams by IGEB modeling;

- linear beam or networks of beams by Timoshenko modeling.

The solution of optimization problems will be tested if possible for time-dependent models of beams or networks.

The shape functional may be given by the optimal value of the cost for OCP in the case of a specific state equation. In such a case there are two techniques for solving the optimum design problems. The first strategy is Pontryagin's Maximum Principle for the total optimization problem i.e., control and design together. The second possibility is decomposition into the control problem at the lower level and the design problem at the upper level. In such a case, a unique optimal control is determined by an optimality system. In particular, the solution of an optimality system becomes simple in some cases with quadratic cost and without control and state constraints.

1.5 Principal Notations

Ω	a bounded open domain in $\mathbb{R}$
u	control
y	the state of the system
$SO(3)$	$\left\{\boldsymbol{S} \in \mathbb{R}^{3\times 3} : \det(\boldsymbol{S}) = 1, \boldsymbol{S}^{-1} = \boldsymbol{S}^{\top}\right\}$
$C^k(\Omega)$	space of k-times continuously differentiable functions on $\bar{\Omega}, k$ integer $\geqslant 0$
$\mathscr{D}(\Omega)$	space of infinitely differentiable functions in Ω, with compact support in Ω, endowed with the inductive limit topology of Schwartz
$\mathscr{D}'(\Omega)$	dual space of $\mathscr{D}(\Omega)$, space of distributions on Ω
$L^2(\Omega)$	space (equivalence class) of functions square integrable on Ω
$H^m(\Omega)$	(Sobolev space of order m) space of functions φ such that $\varphi \in L^2(\Omega), \dfrac{\partial \varphi}{\partial x_i} \in L^2(\Omega), \dots, D^{\alpha}\varphi \in L^2(\Omega) \quad \forall \alpha,\ \lvert\alpha\rvert \leqslant m$
$H_0^m(\Omega)$	$\{\varphi \mid \varphi \in H^m(\Omega), D^{\alpha}p = 0 \text{ on } \Gamma, \lvert\alpha\rvert \leqslant m-1\}$
$\mathscr{U}$	a Hilbert space on $\mathbb{R}$
$\mathscr{U}_{\text{ad}}$	(set of admissible controls) a closed, convex subset of $\mathscr{U}$
$\widetilde{u}$	is the skew-symmetric matrix defined by $\widetilde{u} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$ for any $u = (u_1, u_2, u_3)^{\top} \in \mathbb{R}^3$

1.6 Outline and Main Results

We now give a brief description of the contents of the various chapters.

In Chapter 2, we present a list of models under consideration, including the Geometrically Exact Beams model and the Timoshenko Beams model, among others. Then, we study a detailed analysis of the characteristics of these models and their applications.

In Chapter 3, our analysis of the optimality conditions is performed for the nonlinear steady state models. Such an analysis can be useful for the real systems governed by the networks of Nonlinear Partial Differential Equations. The practical examples for our framework include e.g., the Gas and Hydrogen Distribution (GHD) Networks [28], [27], and the Geometrically Exact Beams (GEB) Networks which lead to the Intrinsic Geometrically Exact Beams (IGEB) network models [44], [77]. The GHD Networks are modeled by quasilinear hyperbolic systems. The IGEB Networks are governed by semilinear hyperbolic systems under some assumptions on the transformation of GEB models. The steady state equations for two types of networks are given by ODEs.

In Chapter 4, we present the optimality system for both static and evolution optimization problems. We introduce the network model and provide the exact solution for both the 1D and 3D Timoshenko Beams models. Additionally, we employ the domain decomposition method to facilitate topology changes. Furthermore, we present numerical examples for both models to demonstrate their effectiveness and applicability in practical situations.

In Chapter 5, we consider two optimal control problems. The first problem, termed as *optimal control for evolution wave problem* (OCE), is governed by an evolution equation. In order to define the second problem, denoted as *optimal control for static problem* (OCS), the evolution state equation is reduced to the steady state equation. The optimal controls are given by the appropriate optimality systems, see [53]. In order to justify the approximation of (OCE) by (OCS), we study the turnpike property of the couple (OCE)-(OCS). For recent reviews on the turnpike property in optimal control, see [24] and the references therein. Several forms of the turnpike phenomenon have been studied in detail, for example, the exponential turnpike property (see [90]) and the interval turnpike property, see [21]. Numerical issues and the turnpike phenomenon in optimal shape design with parabolic PDEs have been studied in [41].

Numerical results are presented in Chapter 3,4,5. All programs in Matlab are in the Appendix A. The numerical results confirm the theoretical results of the dissertation. The results of the dissertation are published in *Control and Cybernetics* (2024), a paper is submitted to the Journal of Geometric Analysis and the third paper is in preparation.

Chapter 2

Modeling of Beams

The first part of this dissertation is dedicated to deriving suitable descriptions of thin structures. These thin structures are commonly modeled as one-dimensional continua, i.e., the objects themselves and associated quantities are parameterized by only one spatial variable. All the models here are defined on a single edge.

2.1 Nonlinear Beams

The networks of elastic beams are of primal importance for applications we have in mind. Thus, we describe in detail the nonlinear models of beams that lead to semilinear state equations for static and evolution problems. The optimal steady state can be determined by solving the control problem for the static model. The practical examples for our framework include e.g., the *Geometrically Exact Beams* (GEB) networks which lead to the *Intrinsic Geometrically Exact Beams* (IGEB) network models [44], [77]. The IGEB networks are governed by semilinear hyperbolic systems under some assumptions on the transformation of GEB models. The steady state equations for two kinds of networks are given by ODEs. For convenience, we omit the index of edges.

2.1.1 Geometrically Exact Beams Model

The mathematical framework describing *geometrically exact beams* focuses on the position of the beam's centerline and the orientation of its cross sections with a fixed coordinate system. In the GEB context, the system state is denoted as $(\mathbf{p}, \mathbf{R})$. This state includes the position of the centerline, denoted as $\mathbf{p}(x,t) \in \mathbb{R}^3$, and the orientation of the cross sections, represented by the columns $\{\mathbf{b}^j\}_{j=1}^3$ of the rotation matrix $\mathbf{R}(x,t) \in \mathrm{SO}(3)$. For visual reference, we could refer to Fig. 2.1. The figure illustrates three pivotal states of a deformable beam: the unchanged reference beam; the initial beam characterized by a curvature described as $\Upsilon_c = \mathrm{vec}\left(R^\top \frac{\mathrm{d}}{\mathrm{d}x} R\right)$, where $R = [b^1,\, b^2,\, b^3]$; and the beam at time t, represented by the state variables $\mathbf{p}$ and $\mathbf{R} = [\mathbf{b}^1,\, \mathbf{b}^2,\, \mathbf{b}^3]$, where the operator vec is basically the inverse operation of the operator $\widetilde{(\cdot)}$, that is $\mathrm{vec}(\widetilde{u}) = u$.

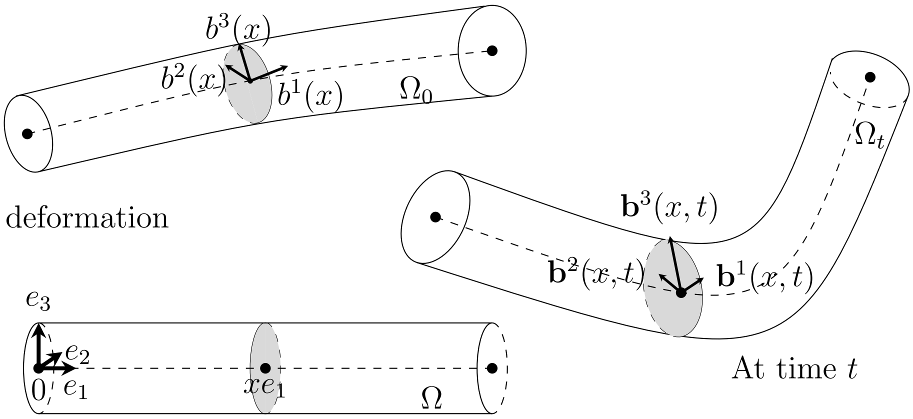


Figure 2.1: The straight reference beam (bottom), the beam before deformation (upper left), and the beam at time t (upper right).

For a beam with a length $L > 0$ positioned within the domain $(0, L) \times (0, T)$, the governing system is defined as follows:

$$\begin{bmatrix} \partial_t & \mathbf{0} \\ (\partial_t \widetilde{\mathbf{p}}) & \partial_t \end{bmatrix} \left[\begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \mathbf{M} y_v \right] = \begin{bmatrix} \partial_x & \mathbf{0} \\ (\partial_x \widetilde{\mathbf{p}}) & \partial_x \end{bmatrix} \left[\begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} z \right] + \begin{bmatrix} \bar{\phi} \\ \bar{\psi} \end{bmatrix}, \tag{2.1}$$

given external forces and moments $\bar{\phi}(x,t), \bar{\psi}(x,t) \in \mathbb{R}^3$, the mass matrix $\mathbf{M}(x) \in \mathbb{S}^6_{++}$(the set of positive definite symmetric matrices), the flexibility (or compliance) matrix $\mathbf{C}(x) \in \mathbb{S}^6_{++}$ and the curvature before deformation $\Upsilon_c(x)$, and where y_v, z depend on $(\mathbf{p}, \mathbf{R})$:

$$y_v = \begin{bmatrix} \mathbf{R}^\top \partial_t \mathbf{p} \\ \operatorname{vec}\left(\mathbf{R}^\top \partial_t \mathbf{R}\right) \end{bmatrix}, \quad s = \begin{bmatrix} \mathbf{R}^\top \partial_x \mathbf{p} - e_1 \\ \operatorname{vec}\left(\mathbf{R}^\top \partial_x \mathbf{R}\right) - \Upsilon_c \end{bmatrix}, \quad z = \mathbf{C}^{-1} s. \tag{2.2}$$

and $y_v(x,t) \in \mathbb{R}^6$ represents linear and angular velocities, and $z(x,t) \in \mathbb{R}^6$ represents internal forces and moment.

2.1.2 Intrinsic Geometrically Exact Beams Model

Consider the *Intrinsic Geometrically Exact Beams* (IGEB) model for a single beam. The governing semilinear system consists of twelve equations. The state variable is denoted as

$$y = \begin{bmatrix} y_v \\ z \end{bmatrix}$$

expressed on a moving basis. That is, GEB model (2.1) and IGEB model (2.3) are related by the nonlinear transformation. We use y_v^f, z^f, y_v^l, and z^l to denote the first and last three components of y_v and z respectively. The notation $\bar{\Phi}(x,t)$ and $\bar{\Psi}(x,t) \in \mathbb{R}^3$ is employed for external forces and moments expressed in the moving basis. Within the domain $(0,L)\times(0,T)$, the governing system of IGEB reads:

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} \partial_t y - \begin{bmatrix} \mathbf{0} & \mathbb{I}_6 \\ \mathbb{I}_6 & \mathbf{0} \end{bmatrix} \partial_x y - \mathcal{A} y = -\mathcal{B}(y_v, z) \begin{bmatrix} \mathbf{M} y_v \\ \mathbf{C} z \end{bmatrix} + \begin{bmatrix} \bar{\Phi} \\ \bar{\Psi} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \tag{2.3}$$

where

$$\mathcal{A} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \widetilde{\Upsilon}_c & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \widetilde{e}_1 & \widetilde{\Upsilon}_c \\ \widetilde{\Upsilon}_c & \widetilde{e}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \widetilde{\Upsilon}_c & \mathbf{0} & \mathbf{0} \end{bmatrix}, \qquad \mathcal{B}(y_v, z) = \begin{bmatrix} \widetilde{y_v^l} & \mathbf{0} & \mathbf{0} & \widetilde{z^f} \\ \widetilde{y_v^f} & \widetilde{y_v^l} & \widetilde{z^f} & \widetilde{z^l} \\ \mathbf{0} & \mathbf{0} & \widetilde{y_v^l} & \widetilde{y_v^f} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \widetilde{y_v^l} \end{bmatrix}, \tag{2.4}$$

and $\mathbb{I}_6$ is the identity matrix with the size 6×6. The system (2.3) is semilinear because of the presence on the right-hand side of the quadratic terms

$$(v,z) \mapsto \mathcal{B}(y_v, z) \begin{bmatrix} \mathbf{M} y_v \\ \mathbf{C} z \end{bmatrix}.$$

We introduce the matrix $\mathbf{E}(x) \in \mathbb{R}^{6\times 6}$, which contains information about curvature and twist at rest, and the matrix $Q^{\mathcal{P}}(x) \in \mathbb{S}^{12}_{++}$, defined by

$$\mathbf{E} = \begin{bmatrix} \widetilde{\Upsilon}_c & \mathbf{0} \\ \widetilde{e}_1 & \widetilde{\Upsilon}_c \end{bmatrix}, \qquad Q^{\mathcal{P}} = \operatorname{diag}(\mathbf{M}, \mathbf{C}).$$

We present in a simple example, that of a single beam clamped at $x=0$ and controlled via velocity free at $x=L$. The IGEB system with boundary conditions reads

$$\begin{cases} \partial_t y + \bar{A}(x)\partial_x y + \bar{B}(x) y = \bar{g}(x,y) & \text{in } (0,L)\times(0,T) \\ v(0,t) = 0 & \text{for } t\in(0,T) \\ z(l,t) = 0 & \text{for } t\in(0,T) \\ y(x,0) = y^0(x) & \text{for } x\in(0,L) \end{cases} \tag{2.5}$$

where the coefficients $\bar{A}, \bar{B}$ and the source $\bar{g}$ depend on $\mathbf{M}, \mathbf{C}$ and $\mathbf{R}$, and $y^0(x)$ is the initial velocity. The governing system is derived by left-multiplying Eq. (2.3) by the inverse of $Q^{\mathcal{P}}$. Specifically, the functions $\bar{A}(x)$ and $\bar{B}(x)$ are defined over the interval $[0,L]$ and map to $\mathbb{R}^{12\times 12}$,

$$\bar{A} = -\left(Q^{\mathcal{P}}\right)^{-1} \begin{bmatrix} \mathbf{0} & \mathbb{I}_6 \\ \mathbb{I}_6 & \mathbf{0} \end{bmatrix}, \quad \bar{B} = \left(Q^{\mathcal{P}}\right)^{-1} \begin{bmatrix} \mathbf{0} & -\mathbf{E} \\ \mathbf{E}^\top & \mathbf{0} \end{bmatrix}. \tag{2.6}$$

The function $\bar{g} : [0,L]\times\mathbb{R}^{12} \to \mathbb{R}^{12}$ is defined by

$$\bar{g}(x,u) = Q^{\mathcal{P}}(x)^{-1} \mathcal{G}(u) Q^{\mathcal{P}}(x) u$$

for all $x \in [0, L]$ and $u = \left(u_1^\top, u_2^\top, u_3^\top, u_4^\top\right)^\top \in \mathbb{R}^{12}$ with each $u_j \in \mathbb{R}^3$, where the map $\overline{\mathcal{G}}$ is defined by

$$\mathcal{G}(u) = -\begin{bmatrix} \widetilde{u}_2 & \mathbf{0} & \mathbf{0} & \widetilde{u}_3 \\ \widetilde{u}_1 & \widetilde{u}_2 & \widetilde{u}_3 & \widetilde{u}_4 \\ \mathbf{0} & \mathbf{0} & \widetilde{u}_2 & \widetilde{u}_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \widetilde{u}_2 \end{bmatrix}.$$

For the static problem, the nonlinear transformation results in $y_v = 0$. Denote $\mathbf{L}(z) := \begin{bmatrix} 0 & \widetilde{z^f} \\ \widetilde{z^f} & \widetilde{z^l} \end{bmatrix}$ and we have a steady state system:

$$\begin{cases} -\partial_x z = \mathbf{E}(x)z - \mathbf{L}(z)\mathbf{C}z + f(x)\boldsymbol{e}_1, & \text{in } (0, L) \\ z(L) = \mathbf{0}. \end{cases} \tag{2.7}$$

where $\partial_x = \frac{\partial}{\partial x}$, $f(x)$ is the distributed control and $\boldsymbol{e}_1 = (1, 0, 0, 0, 0, 0)^\top \in \mathbb{R}^6$.

2.2 Timoshenko Beams

The Timoshenko beam theory allows for cross-sections that are not perpendicular to the center line of the beam, and is more suitable for short or anisotropic beams. It is also well-suited for problems with vibrating beams [34, 55]. More information about the modeling of Timoshenko beams can for example be found in [39, 40].

2.2.1 Notation and Parameters

One can prove the following representation for $\mathbb{E}_i$, the so-called *Rodrigues formula*

$$\begin{aligned} \mathbb{E}_i = \mathbb{E}_i(\boldsymbol{\phi}) &= \cos\phi\mathbb{I} + \sin\phi\widetilde{e}_i + (1 - \cos\phi)e_i e_i^\top \\ &\overset{\boldsymbol{\phi} := \phi e_i}{=} \mathbb{I} + \frac{\sin\phi}{\phi}\widetilde{\boldsymbol{\phi}} + \frac{1 - \cos\phi}{\phi^2}\widetilde{\boldsymbol{\phi}}^2. \end{aligned} \tag{2.8}$$

For dynamic, cross sections' orientation by the columns $\{\mathbf{b}^j\}_{j=1}^3$ of $\mathbb{E}_i(x, t) \in \mathrm{SO}(3)$. The displacements in the x-, y-, and z-directions are denoted by $r_{x,i}$, $r_{y,i}$, and $r_{z,i}$, respectively. These quantities are integrated into the displacement vector for steady state

$$r_i(x) := \mathbb{E}_i(\boldsymbol{\phi})\left(r_{x,i}(x), r_{y,i}(x), r_{z,i}(x)\right)^\top$$

as well as for dynamic model

$$r_i(x, t) := \mathbb{E}_i(\boldsymbol{\phi}(t))\left(r_{x,i}(x, t), r_{y,i}(x, t), r_{z,i}(x, t)\right)^\top.$$

Similarly, the rotations in the x-, y-, and z- directions are represented by $\varphi_{x,i}$, $\varphi_{y,i}$, and $\varphi_{z,i}$, respectively, and combined into the rotation vector for steady state

$$\varphi_i(x) := \mathbb{E}_i(\boldsymbol{\phi})\left(\varphi_{x,i}(x), \varphi_{y,i}(x), \varphi_{z,i}(x)\right)^\top$$

as well as for dynamic problem

$$\varphi_i(x,t) := \mathbb{E}_i(\boldsymbol{\phi}(t)) \left(\varphi_{x,i}(x,t), \varphi_{y,i}(x,t), \varphi_{z,i}(x,t)\right)^\top .$$

External loads, including distributed forces acting along the beam, are assumed to be piecewise continuous and can be summarized for steady state into the function

$$f_i \in \mathcal{PC}\left(0, L_i\right)^3 ,\ f_i(x) := \mathbb{E}_i(\boldsymbol{\phi}) \left(f_{x,i}(x), f_{y,i}(x), f_{z,i}(x)\right)^\top ,$$

as well as for dynamic problems

$$f_i \in L^1 \left(\mathcal{PC}\left(0, L_i\right); (0,T)\right)^3 ,\ f_i(x,t) := \mathbb{E}_i(\boldsymbol{\phi}(t)) \left(f_{x,i}(x,t), f_{y,i}(x,t), f_{z,i}(x,t)\right)^\top .$$

Similarly, distributed moments are combined for steady state into the vector function

$$m_i \in \mathcal{PC}\left(0, L_i\right)^3 ,\ m_i(x) := \mathbb{E}_i(\boldsymbol{\phi}) \left(m_{x,i}(x), m_{y,i}(x), m_{z,i}(x)\right)^\top ,$$

as well as for dynamic model

$$m_i \in L^1 \left(\mathcal{PC}\left(0, L_i\right); (0,T)\right)^3 ,\ m_i(x,t) := \mathbb{E}_i(\boldsymbol{\phi}(t)) \left(m_{x,i}(x,t), m_{y,i}(x,t), m_{z,i}(x,t)\right)^\top .$$

The combined displacement and rotation of a Timoshenko beam can be represented for steady state by six-dimensional vector functions as follows:

$$u_i(x) := \begin{pmatrix} r_i(x) \\ \varphi_i(x) \end{pmatrix} \quad \text{and} \quad q_i(x) := \begin{pmatrix} f_i(x) \\ m_i(x) \end{pmatrix} .$$

For the dynamic model, we use the notation:

$$u_i(x,t) := \begin{pmatrix} r_i(x,t) \\ \varphi_i(x,t) \end{pmatrix} \quad \text{and} \quad q_i(x,t) := \begin{pmatrix} f_i(x,t) \\ m_i(x,t) \end{pmatrix} .$$

These vector functions will be extensively used in the subsequent sections of this study. The notation and parameters employed in the model are presented 2.1. The coefficients k_i^x, k_i^s, c_i^x, c_i^y, and c_i^z are the stiffness and damping coefficients in the i-th edge, which are beam-specific and depend on the material properties and geometry of the beam. The internal forces $F_i^x, F_i^y, F_i^z : [0, L_i] \to \mathbb{R}^3$ of a Timoshenko beam are given by

$$\begin{aligned} F_i^x(x) &:= k_i^x r'_{x,i}(x), \\ F_i^y(x) &:= k_i^s \left(r'_{y,i}(x) - \varphi_{z,i}(x)\right) , \\ F_i^z(x) &:= k_i^s \left(r'_{z,i}(x) + \varphi_{y,i}(x)\right) , \end{aligned}$$

as well as for dynamic problems,

$$\begin{aligned} F_i^x(x,t) &:= k_i^x r'_{x,i}(x,t), \\ F_i^y(x,t) &:= k_i^s \left(r'_{y,i}(x,t) - \varphi_{z,i}(x,t)\right) , \\ F_i^z(x,t) &:= k_i^s \left(r'_{z,i}(x,t) + \varphi_{y,i}(x,t)\right) , \end{aligned}$$

Table 2.1: Physical parameters of the Timoshenko beam model

Parameter	Physical significance	Parameter	Physical significance
κ_i	shear coefficient	$k_i^x := \overline{\mathrm{E}}_i\overline{\mathrm{A}}_i$	axial stiffness
$\overline{\mathrm{G}}_i(\mathrm{N/m}^2)$	shear modulus	$k_i^s := \kappa_i\overline{\mathrm{G}}_i\overline{\mathrm{A}}_i$	shear stiffness
$\overline{\mathrm{A}}_i(\mathrm{m}^2)$	cross-section area	$c_i^x := \overline{\mathrm{G}}_i\overline{\mathrm{I}}_i^t$	torsional stiffness
$\overline{\mathrm{E}}_i(\mathrm{N/m}^2)$	elastic modulus	$c_i^y := \overline{\mathrm{E}}_i\overline{\mathrm{I}}_i^y$	bending stiffnesses
$\overline{\mathrm{I}}_i^t(\mathrm{m}^4)$	torsion constant		(w.r.t. the y-axis)
$\overline{\mathrm{I}}_i^y(\mathrm{m}^4)$	second moments of area (w.r.t. the y-axis)	$c_i^z = \overline{\mathrm{E}}_i\overline{\mathrm{I}}_i^z$	bending stiffnesses (w.r.t. the z-axis)
$\overline{\mathrm{I}}_i^z(\mathrm{m}^4)$	second moments of area (w.r.t. the z-axis)	$\rho_i(\mathrm{kg/m}^3)$	mass per unit length

and the internal moments $M_i^x, M_i^y, M_i^z : [0, L_i] \to \mathbb{R}^3$ by

$$\begin{aligned} M_i^x(x) &:= c_i^x \varphi'_{x,i}(x), \\ M_i^y(x) &:= c_i^y \varphi'_{y,i}(x), \\ M_i^z(x) &:= c_i^z \varphi'_{z,i}(x), \end{aligned}$$

as well as for dynamic problems

$$\begin{aligned} M_i^x(x,t) &:= c_i^x \varphi'_{x,i}(x,t), \\ M_i^y(x,t) &:= c_i^y \varphi'_{y,i}(x,t), \\ M_i^z(x,t) &:= c_i^z \varphi'_{z,i}(x,t). \end{aligned}$$

These functions can also be represented in global coordinates as:

$$\begin{aligned} F_i(x) : &= \mathbb{E}_i(\boldsymbol{\phi}) \left(F_i^x(x), F_i^y(x), F_i^z(x)\right)^\top, \\ M_i(x) : &= \mathbb{E}_i(\boldsymbol{\phi}) \left(M_i^x(x), M_i^y(x), M_i^z(x)\right)^\top. \end{aligned}$$

For dynamic, we have

$$\begin{aligned} F_i(x,t) : &= \mathbb{E}_i(\boldsymbol{\phi}(t)) \left(F_i^x(x,t), F_i^y(x,t), F_i^z(x,t)\right)^\top, \\ M_i(x,t) : &= \mathbb{E}_i(\boldsymbol{\phi}(t)) \left(M_i^x(x,t), M_i^y(x,t), M_i^z(x,t)\right)^\top. \end{aligned}$$

These equations show the complex interrelation between internal forces and moments, providing insightful information about the behavior of Timoshenko beam structures.

2.2.2 Dynamic Modeling of Timoshenko Beams

We use the notation $\partial_{tt} r := \ddot{r}$ and $\partial_{xx} r := r''$. The PDE dynamic equations of the Timoshenko beam model in 3D in local coordinates are described in the following forms

[89],

$$\begin{aligned}
\rho_i \overline{\mathrm{A}}_i \ddot{r}_{x,i}(x,t) &= k_i^x r''_{x,i}(x,t) + f_{x,i}(x,t), && \text{(2.9a)}\\
\rho_i \overline{\mathrm{I}}_i^t \ddot{\varphi}_{x,i}(x,t) &= c_i^x \varphi''_{x,i}(x,t) + m_{x,i}(x,t), && \text{(2.9b)}\\
\rho_i \overline{\mathrm{A}}_i \ddot{r}_{y,i}(x,t) &= k_i^s \left(r''_{y,i}(x,t) - \varphi'_{z,i}(x,t)\right) + f_{y,i}(x,t), && \text{(2.9c)}\\
\rho_i \overline{\mathrm{A}}_i \ddot{r}_{z,i}(x,t) &= k_i^s \left(r''_{z,i}(x,t) + \varphi'_{y,i}(x,t)\right) + f_{z,i}(x,t), && \text{(2.9d)}\\
\rho_i \overline{\mathrm{I}}_i^y \ddot{\varphi}_{y,i}(x,t) &= c_i^y \varphi''_{y,i}(x,t) - k_i^s \left(r'_{z,i}(x,t) + \varphi_{y,i}(x,t)\right) + m_{y,i}(x,t) && \text{(2.9e)}\\
\rho_i \overline{\mathrm{I}}_i^z \ddot{\varphi}_{z,i}(x,t) &= c_i^z \varphi''_{z,i}(x,t) + k_i^s \left(r'_{y,i}(x,t) - \varphi_{z,i}(x,t)\right) + m_{z,i}(x,t). && \text{(2.9f)}
\end{aligned}$$

We also call this the *full model*. This system comprises six equations that govern the displacements and rotations of the beam. The first two equations relate to the displacements along the x-axis and rotations about the x-axis, respectively. The next two equations are concerned with the forces acting along the y and z axes, respectively. And the final two relate to the moments.

When there is no initial shear and either the initial twist is zero or the second moments of area coincide, the simplified diagonal uncoupled constitutive laws are employed. This is expressed as:

$$C_N := \begin{pmatrix} k_i^x & 0 & 0 \\ 0 & k_i^s & 0 \\ 0 & 0 & k_i^s \end{pmatrix}, \quad C_M := \begin{pmatrix} c_i^x & 0 & 0 \\ 0 & c_i^y & 0 \\ 0 & 0 & c_i^z \end{pmatrix}.$$

The material inertia matrix is defined as follows:

$$C_I = \begin{pmatrix} \overline{\mathrm{I}}_i^t & 0 & 0 \\ 0 & \overline{\mathrm{I}}_i^y & 0 \\ 0 & 0 & \overline{\mathrm{I}}_i^z \end{pmatrix}.$$

The Eq. (2.9) in global coordinates can be written as

$$\begin{aligned}
\rho_i \overline{\mathrm{A}}_i \ddot{r}_i(x,t) &= C_N(r''_i(x,t) + \hat{e}_{i_1}\varphi'_i(x,t)) + F_i(x,t),\\
\rho_i C_I \ddot{\varphi}_i &= C_M \varphi_i(x,t) + \hat{e}_{i_1} C_N(r'_i(x,t) + \hat{e}_{i_1}\varphi_i(x,t)) + M_i(x,t).
\end{aligned} \tag{2.10}$$

Beam models are employed to characterize thin deformable structures, implying that, according to common understanding, their characteristic dimension is significantly greater in one spatial direction than in the perpendicular plane. Using the geometrical attributes of beams facilitates the establishment of a constrained motion, leading to the reduction of the full three-dimensional unknowns to spatially one-dimensional quantities based on specific assumptions. For example, we reduce the three-dimensional beam equations to one-dimension in the following case with governing PDE holding for $(x,t) \in [0, L_i] \times [0, T]$ (See [56]):

$$\begin{aligned}
\rho_i \overline{\mathrm{A}}_i \ddot{r}_{z,i}(x,t) &= k_i^s \left(r''_{z,i}(x,t) + \varphi'_{y,i}(x,t)\right) + f_{z,i}(x,t),\\
\rho_i \overline{\mathrm{I}}_i^y \ddot{\varphi}_{y,i}(x,t) &= c_i^y \varphi''_{y,i}(x,t) - k_i^s \left(r'_{z,i}(x,t) + \varphi_{y,i}(x,t)\right) + m_{y,i}(x,t).
\end{aligned} \tag{2.11}$$

There is no transformation from local coordinate to global. We only consider the $r_{z,i}$ and $\varphi_{y,i}$. The Timoshenko beams model of concern is shown in Fig. 2.2 and we call this the *reduced model*. At the origin 0, the deflection and the rotational angle are zero for time t, namely, $r_{z,i}(0,t) = \varphi_{y,i}(0,t) = 0$.

Assumption 1 The axial deformation of the Timoshenko beam is not taken into account in this model.

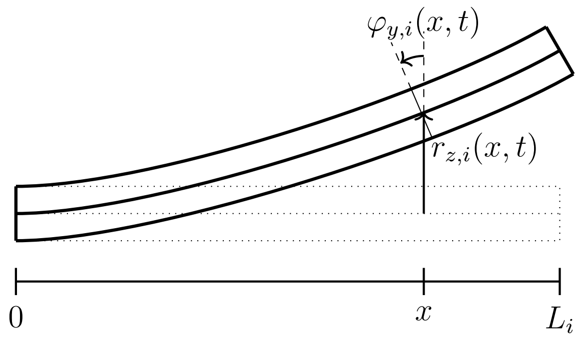


Figure 2.2: Displacement and bending of a single beam.

2.2.3 Static Modeling of Timoshenko Beams

In addition to the dynamic model, we extend our analysis to the static model, thereby providing a comprehensive overall perspective on the behavior of the beam under varying conditions. In 3D, the static Timoshenko beam equation system for a single beam in local coordinate is represented in Eq. (2.12),

$$f_{x,i}(x) + k_i^x r''_{x,i}(x) = 0, \tag{2.12a}$$
$$m_{x,i}(x) + c_i^x \varphi''_{x,i}(x) = 0, \tag{2.12b}$$
$$f_{y,i}(x) + k_i^s \left(r''_{y,i}(x) - \varphi'_{z,i}(x)\right) = 0, \tag{2.12c}$$
$$f_{z,i}(x) + k_i^s \left(r''_{z,i}(x) + \varphi'_{y,i}(x)\right) = 0, \tag{2.12d}$$
$$m_{y,i}(x) + c_i^y \varphi''_{y,i}(x) - k_i^s \left(r'_{z,i}(x) + \varphi_{y,i}(x)\right) = 0, \tag{2.12e}$$
$$m_{z,i}(x) + c_i^z \varphi''_{z,i}(x) + k_i^s \left(r'_{y,i}(x) - \varphi_{z,i}(x)\right) = 0. \tag{2.12f}$$

Similarly, the static system in the global coordinate is given by,

$$\begin{aligned} C_N(r''_i(x,t) + \widetilde{e}_{i_1}\varphi'_i(x,t)) + F_i(x,t) &= 0, \\ C_M\varphi_i(x,t) + \hat{e}_{i_1} C_N(r'_i(x,t) + \widetilde{e}_{i_1}\varphi_i(x,t)) + M_i(x,t) &= 0. \end{aligned} \tag{2.13}$$

The reduced system is described in the following forms:

$$\begin{aligned} f_{z,i}(x) + k_i^s \left(r''_{z,i}(x) + \varphi'_{y,i}(x)\right) &= 0, \\ m_{y,i}(x) + c_i^y \varphi''_{y,i}(x) - k_i^s \left(r'_{z,i}(x) + \varphi_{y,i}(x)\right) &= 0. \end{aligned} \tag{2.14}$$

This static model provides additional insights into the structural response of Timoshenko beams and complements the dynamic analysis by considering the beam's behavior under equilibrium conditions. The optimal control problem is formulated based on this static model, thus highlighting its significance in guiding optimization strategies for the Timoshenko beams systems.

Chapter 3

Necessary and Sufficient Optimality Conditions for the Nonlinear Model

Networks are governed by state equations on metric graphs. Evaluation state equations are of type wave equation, i.e., hyperbolic partial differential equations static state equations are ordinary differential equations. We consider optimal control problems for evolution state equations. In such a case, the so-called *Turnpike Property* occurs, therefore the static state equation can be used for control and design of the network.

In this chapter, the optimization problems are considered for nonlinear steady state ordinary differential equations. The results obtained can be applied to control and design of networks with nonlinear evolution equations. The results are obtained in the framework of the local Pontryagin maximum principle, see [68].

The control and shape problems are considered within the same mathematical framework for necessary and sufficient optimality conditions. For numerical methods, two approaches are possible. The first is approximate then optimize. The second is optimized and then approximate. Numerical results are obtained for the latter approach.

The results are published in [68].

3.1 Preliminaries

3.1.1 Formulation of the First-Order Necessary Optimality Conditions for an Autonomous Problem on the Interval $[0, 1]$

Consider the following autonomous problem of optimal control:

$$\left.\begin{array}{l} J(x,u) = \int_0^1 F(x(t),u(t))\mathrm{d}t \to \min, \\ \dot{x}(t) = f(x(t),u(t)) \ \forall t \in [0,1],\ \kappa(x(0),x(1)) \le 0,\ K(x(0),x(1)) = 0. \end{array}\right\} \tag{3.1}$$

Here $x : [0,1] \to \mathbb{R}^n$ is a continuously differentiable function, $u : [0,1] \to \mathbb{R}^m$ is a continuous function, and $\dot{x} = \mathrm{d}x/\mathrm{d}t$. Hence the problem is considered in the space

$$\mathcal{W} := C^1([0,1], \mathbb{R}^n) \times C([0,1], \mathbb{R}^m).$$

A local minimum in this space is called a *weak local minimum*. We call x the *state variable* and u the *control*. All data $F : \mathbb{R}^{n+m} \to \mathbb{R}$, $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$, $\kappa : \mathbb{R}^{2n} \to \mathbb{R}^k$, $K : \mathbb{R}^{2n} \to \mathbb{R}^s$ are assumed to be continuously differentiable.

We say that $w = (x, u) \in \mathcal{W}$ is an *admissible point* if it satisfies all the constraints of the problem. For brevity we set $\xi = (x(0), x(1))$.

Let us formulate first-order necessary optimality conditions for this problem. We introduce the Hamiltonian (Pontryagin) function and the endpoint Lagrange function:

$$H(x, u, p, \alpha_0) = \langle p, f \rangle(x, u) + \alpha_0 F(x, u), \quad L = \alpha\kappa(\xi) + \beta K(\xi),$$

where p, α, β are row vectors of the same dimensions as the column vectors f, κ, K, respectively, α_0 is a number. By definition, $\langle p, f \rangle = \sum_{i=1}^{n} p_i f_i$, where p_i and f_i are the components of the vectors p and f, respectively.

Denote by $\mathbb{R}^{n\top}$ the space of row vectors of dimension n.

By F_x and F_u we denote the partial derivatives $\partial F/\partial x$ and $\partial F/\partial u$, respectively, considered as row vectors, i.e. $F_x \in \mathbb{R}^{n\top}$, $F_u \in \mathbb{R}^{m\top}$. Similarly, $f_x := \partial f/\partial x$ and $f_u := \partial f/\partial u$, which are matrices of order $n \times n$ and $n \times m$, respectively. Note that $H_x \in \mathbb{R}^{n\top}$, $H_u \in \mathbb{R}^{m\top}$ are row vectors, and $H_p = f \in \mathbb{R}^n$ is a column vector.

We say that at an admissible point $w^0 = (x^0, u^0) \in \mathcal{W}$ the *local minimum principle* (LMP) is satisfied if there exists a continuously differentiable function $p : [0,1] \to \mathbb{R}^{n\top}$, a number α_0, and row vectors $\alpha \in \mathbb{R}^{k\top}$, $\beta \in \mathbb{R}^{s\top}$ such that the following system of optimality conditions holds:

(a) the nonnegativity conditions: $\alpha_0 \geq 0$, $\alpha \geq 0$,

(b) the nontriviality condition: $\alpha_0 + |\alpha| + |\beta| > 0$,

(c) the complementary slackness condition: $\alpha\kappa(\xi^0) = 0$, where $\xi^0 = (x^0(0), x^0(1))$,

(d) the adjoint equation: $-\dot{p}(t) = H_x(w^0(t), p(t), \alpha_0)\ \forall\, t \in [0,1]$,

(e) the transversality conditions: $(-p(0), p(1)) = L_\xi(\xi^0, \alpha, \beta)$,

(f) the stationarity of the Hamiltonian with respect to the control: $H_u(w^0(t), p(t), \alpha_0) = 0\ \forall t \in [0,1]$.

From the equation $\dot{x}^0 = f(w^0)$ and conditions (d) and (f) it follows

(g) the *condition for the Hamiltonian to be constant*: there exists a constant c_H such that $H(w^0(t), p(t), \alpha_0) = c_H\ \forall\, t \in [0,1]$.

Indeed, $\frac{\mathrm{d}}{\mathrm{d}t} H(w^0(t), p(t), \alpha_0) = H_x(w^0(t), p(t), \alpha_0)\dot{x}^0(t) + H_u(w^0(t), p(t), \alpha_0)\dot{u}^0(t)$
$+ \dot{p}(t) H_p(w^0(t), p(t), \alpha_0) = -\dot{p}(t)\dot{x}^0(t) + \dot{p}(t)\dot{x}^0(t) = 0.$ □

The following theorem is well known, see, for example, [2], [19], [59], [58], [73].

Theorem 1. *If w^0 is a weak local minimum in problem (3.1), then it satisfies the LMP.*

The case, when the cost Lagrange multiplier α_0 is not equal to zero (for any quadruple $(\alpha_0, \alpha, \beta, p(\cdot))$ satisfying the LMP conditions), is called *normal*. Let us formulate a condition that guarantees the normal case for the point w^0. Introduce a *set of active indices*

$$I = \{i \in \{1, \ldots, k\} : \kappa_i(\xi^0) = 0\}.$$

We say that the *Mangasyan-Fromowitz constraint qualification* (MFCQ) is satisfied for the point $w^0 = (x^0, u^0) \in \mathcal{W}$ if there exists a pair $(x, u) \in \mathcal{W}$ such that

$$\kappa_i'(\xi^0)\xi < 0 \quad \forall\, i \in I, \quad K'(\xi^0)\xi = 0, \quad \xi = (x(0), x(1)), \quad \dot{x} = f'(w^0)w,$$

where, for example, $f'(w^0)w = f_x(w^0)x + f_u(w^0)u$. In this case, in the LMP conditions, we can set $\alpha_0 = 1$.

3.1.2 Formulation of the Second-Order Sufficient Optimality Conditions for an Autonomous Problem on the Interval $[0, 1]$

Consider again the autonomous problem (3.1). Now we suppose that all data F, f, κ, K are twice continuously differentiable.

Let us formulate sufficient second-order conditions for a weak local minimum at an admissible point $w^0 = (x^0, u^0) \in \mathcal{W}$, satisfying necessary first-order conditions with the adjoint variable p and Lagrange multipliers α_0, α, β. Define the *critical cone* at the point w^0:

$$\begin{aligned} \mathcal{C} := \big\{\delta w = (\delta x, \delta u) \in \mathcal{W} : \quad & \delta\dot{x}(t) = f'(w^0(t))\delta w(t),\ K'(\xi^0)\delta\xi = 0, \\ & \kappa_i'(\xi^0)\delta\xi \le 0,\ i \in I,\ \textstyle\int_0^1 F'(w^0(t))\delta w(t)\mathrm{d}t \le 0\,\big\}, \end{aligned}$$

where $\delta\xi = (\delta x(0), \delta x(1))$. The equation $\delta\dot{x} = f'(w^0)\delta w$ is called the *equation in variations*.

In the normal case, where $\alpha_0 = 1$, the inequality $\int_0^1 F'(w^0(t))\delta w(t)\mathrm{d}t \le 0$ can be excluded from the definition of the critical cone, but then we must add the equalities $\alpha_i\kappa_i'(\xi^0)\delta\xi = 0$, $i \in I$. Thus, in the normal case, we have

$$\begin{aligned} \mathcal{C} := \{\delta w = (\delta x, \delta u) \in \mathcal{W} : \quad & \delta\dot{x}(t) = f'(w^0(t))\delta w(t),\ K'(\xi^0)\delta\xi = 0, \\ & \kappa_i'(\xi^0)\delta\xi \le 0,\ i \in I,\ \alpha_i\kappa_i'(\xi^0)\delta\xi = 0,\ i \in I\}. \end{aligned}$$

This is easy to prove using the LMP conditions. Later, in Section 3.2, where we consider the normal case, we will use this critical cone representation.

Define the *strengthened Legendre condition*: there exists $c_L > 0$ such that for all $t \in [0, 1]$ we have $\langle H_{uu}(w^0(t), p(t), \alpha_0)u, u\rangle \ge c_L|u|^2 \quad \forall\, u \in \mathbb{R}^m$. Here $H_{uu} = \partial^2 H/\partial u^2$ stands for the second partial derivative of H with respect to the control.

Next, define a *quadratic form*:

$$2\Theta(\delta w) = \langle L_{\xi\xi}(\xi^0, \alpha, \beta)\delta\xi, \delta\xi\rangle + \int_0^1 \langle H_{ww}(w^0(t), p(t), \alpha_0)\delta w(t), \delta w(t)\rangle \mathrm{d}t.$$

Note that if $\kappa(\xi)$ and $K(\xi)$ are affine functions, then $L = \alpha\kappa + \beta K$ is also an affine function of ξ, and therefore, $L_{\xi\xi} = 0$. In this case, the endpoint term $\langle L_{\xi\xi}(\xi^0, \alpha, \beta)\delta\xi, \delta\xi\rangle$ vanishes, and Θ reduces to the integral only.

The following theorem holds, see, for example, [67].

Theorem 2. *Assume that for the point* w^0

(a) the strengthened Legendre condition is satisfied,

(b) there exists a constant $c_\Theta > 0$ *such that* $\Theta(\delta w) \geq c_\Theta(|\delta x(0)|^2 + \|\delta u\|_2^2) \quad \forall\, \delta w \in \mathcal{C}$.

Then there are $c > 0$ *and* $\varepsilon > 0$ *such that* $J(w) - J(w^0) \geq c\Big(\|x - x^0\|_\infty^2 + \int_0^1 |u(t) - u^0(t)|^2 \mathrm{d}t\Big)$ *for all admissible* $w = (x, u)$ *such that* $\|w - w^0\|_\infty < \varepsilon$, *and hence* w^0 *is a weak local minimum in the problem.*

Remark 1. *Since* $\Theta(-\delta w) = \Theta(\delta w)$ *for all* $\delta w \in \mathcal{W}$, *condition (b) in this theorem is equivalent to the condition* $\Theta(\delta w) \geq c_\Theta(|\delta x(0)|^2 + \|\delta u\|_2^2)\ \forall\, \delta w \in \Sigma$, *where* $\Sigma = \mathcal{C} \cup (-\mathcal{C})$. *In particular, let* $\mathcal{C} = \{\delta w \in \Gamma,\ l(\delta w) \leq 0\}$, *where* Γ *is a subspace, and* l *is a linear functional. Then, obviously,* $\Sigma = \Gamma$.

3.1.3 Matrix Riccati Equation

Now we consider a sufficient condition for positive definiteness of the quadratic form Θ on the subspace Γ. Assume that Γ has the form:

$$\Gamma = \Big\{\delta w = (\delta x, \delta u) \in \mathcal{W} :\ \delta\dot{x} = f_x(w^0)\delta x + f_u(w^0)\delta u,\ \mathcal{E}\delta\xi = 0\Big\},$$

where $\mathcal{E}$ is a constant matrix, $\delta\xi = (\delta x(0), \delta x(1))$. Let us show that the quadratic form Θ could be transformed into a perfect square if the corresponding Riccati equation has a solution $Q(t)$ defined on $[0, 1]$. Assume that the strengthened Legendre condition is satisfied. Define the Riccati matrix equation along $(x^0(t), u^0(t), p(t))$ by

$$\dot{Q} + Qf_x + f_x^T Q + H_{xx} - (H_{xu} + Qf_u)H_{uu}^{-1}(H_{ux} + f_u^T Q) = 0, \quad t \in [0, 1], \tag{3.2}$$

where $Q = Q(t)$ is a symmetric matrix of order n whose elements belong to C^1, $f_x = f_x(w^0)$, $H_{xx} = H_{xx}(w^0, p, \alpha_0)$, etc., $f_x^\top$ means the transposed matrix f_x.

Theorem 3. *Assume that the strengthened Legendre condition is satisfied and there exists a symmetric solution* Q *(with the entries belonging to* C^1*) of the matrix Riccati equation* *on* $[0,1]$. *Then the quadratic form* Θ *has the following transformation into a perfect square on the subspace* Γ*:*

$$2\Theta(\delta w) = \int_0^1 \langle H_{uu}^{-1}\delta v, \delta v\rangle \mathrm{d}t + \langle M\delta\xi, \delta\xi\rangle \quad \forall\, \delta w \in \Gamma, \tag{3.3}$$

where $\delta v := (H_{ux} + f_u^T Q)\delta x + H_{uu}\delta u$, H_{uu}^{-1} *is the inverse matrix of matrix* H_{uu}, *and*

$$M := \begin{pmatrix} L_{x_0x_0} + Q(0) & L_{x_0x_1} \\ L_{x_1x_0} & L_{x_1x_1} - Q(1) \end{pmatrix}.$$

For the reader's convenience, we give a proof of this theorem. We follow [67] (see also [57]).

Proof. Let $(\delta x, \delta u) \in \Gamma$. Then

$$\begin{aligned} 2\langle Q\delta\dot{x}, \delta x\rangle &= 2\langle Q(f_x\delta x + f_u\delta u), \delta x\rangle \\ &= \langle (Qf_x + f_x^\top Q)\delta x, \delta x\rangle + \langle Qf_u\delta u, \delta x\rangle + \langle f_u^\top Q\delta x, \delta u\rangle. \end{aligned}$$

Consequently,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\langle Q\delta x, \delta x\rangle &= \langle \dot{Q}\delta x, \delta x\rangle + 2\langle Q\delta\dot{x}, \delta x\rangle \\ &= \langle \dot{Q}\delta x, \delta x\rangle + \langle (Qf_x + f_x^\top Q)\delta x, \delta x\rangle + \langle Qf_u\delta u, \delta x\rangle + \langle f_u^\top Q\delta x, \delta u\rangle \\ &= \langle (\dot{Q} + Qf_x + f_x^\top Q)\delta x, \delta x\rangle + \langle Qf_u\delta u, \delta x\rangle + \langle f_u^\top Q\delta x, \delta u\rangle. \end{aligned}$$

Integrating over $[0,1]$, we get

$$\begin{aligned} &\langle Q(1)\delta x(1), \delta x(1)\rangle - \langle Q(0)\delta x(0), \delta x(0)\rangle \\ &= \textstyle\int_0^1 \Big(\langle (\dot{Q} + Qf_x + f_x^\top Q)\delta x, \delta x\rangle + \langle Qf_u\delta u, \delta x\rangle + \langle f_u^\top Q\delta x, \delta u\rangle\Big)\, \mathrm{d}t. \end{aligned}$$

Consequently,

$$\begin{aligned} &\textstyle\int_0^1 \Big(\langle (\dot{Q} + Qf_x + f_x^\top Q)\delta x, \delta x\rangle + \langle Qf_u\delta u, \delta x\rangle + \langle f_u^\top Q\delta x, \delta u\rangle\Big)\, \mathrm{d}t \\ &+\langle Q(0)\delta x(0), \delta x(0)\rangle - \langle Q(1)\delta x(1), \delta x(1)\rangle = 0. \end{aligned}$$

Adding this zero form to the form $2\Theta(\delta w)$, we obtain

$$\begin{aligned} 2\Theta(\delta w) = &\textstyle\int_0^1 \Big(\langle (\dot{Q} + Qf_x + f_x^\top Q + H_{xx})\delta x, \delta x\rangle \\ &+\langle (Qf_u + H_{xu})\delta u, \delta x\rangle + \langle (f_u^\top Q + H_{ux}\delta x, \delta u\rangle + \langle H_{uu}\delta u, \delta u\rangle\Big)\mathrm{d}t \\ &+\langle Q(0)\delta x(0), \delta x(0)\rangle - \langle Q(1)\delta x(1), \delta x(1)\rangle + \langle L_{\xi\xi}\delta\xi, \delta\xi\rangle. \end{aligned}$$

Now let Q satisfy the Riccati equation (3.2). Then

$$\begin{aligned} 2\Theta(\delta w) = &\textstyle\int_0^1 \Big(\langle (H_{xu} + Qf_u)H_{uu}^{-1}(H_{ux} + f_u^\top Q)\delta x, \delta x\rangle \\ &+\langle (Qf_u + H_{xu})\delta u, \delta x\rangle + \langle (f_u^\top Q + H_{ux}\delta x, \delta u\rangle + \langle H_{uu}\delta u, \delta u\rangle\Big)\mathrm{d}t \\ &+\langle Q(0)\delta x(0), \delta x(0)\rangle - \langle Q(1)\delta x(1), \delta x(1)\rangle + \langle L_{\xi\xi}\delta\xi, \delta\xi\rangle. \end{aligned}$$

Since $\langle H_{uu}\delta u, \delta u\rangle = \langle (H_{uu})^{-1}H_{uu}\delta u, H_{uu}\delta u\rangle$ and $\langle Q(0)\delta x(0), \delta x(0)\rangle - \langle Q(1)\delta x(1), \delta x(1)\rangle + \langle L_{\xi\xi}\delta\xi, \delta\xi\rangle = \langle M\delta\xi, \delta\xi\rangle$, we obtain

$$\begin{aligned} 2\Theta(\delta w) \;=\; & \int_0^1 \Big(\langle (H_{xu} + Qf_u)H_{uu}^{-1}(H_{ux} + f_u^\top Q)\delta x, \delta x\rangle \\ & + \langle (Qf_u + H_{xu})\delta u, \delta x\rangle + \langle (f_u^\top Q + H_{ux}\delta x, \delta u\rangle + \langle (H_{uu})^{-1}H_{uu}\delta u, H_{uu}\delta u\rangle\Big)\mathrm{d}t \\ & + \langle M\delta\xi, \delta\xi\rangle. \end{aligned}$$

Further,

$$\begin{aligned} & \langle (H_{xu} + Qf_u)H_{uu}^{-1}(H_{ux} + f_u^\top Q)\delta x, \delta x\rangle + \langle (Qf_u + H_{xu})\delta u, \delta x\rangle \\ & \qquad + \langle (f_u^\top Q + H_{ux}\delta x, \delta u\rangle + \langle (H_{uu})^{-1}H_{uu}\delta u, H_{uu}\delta u\rangle \\ & = \langle (H_{uu})^{-1}((H_{ux} + f_u^\top Q)\delta x + H_{uu}\delta u), ((H_{ux} + f_u^\top Q)\delta x + H_{uu}\delta u)\rangle \\ & = \langle (H_{uu})^{-1}\delta v, \delta v\rangle, \end{aligned}$$

where $\delta v = (H_{ux} + f_u^\top Q)\delta x + H_{uu}\delta u$. Consequently, $2\Theta(\delta w) = \int_0^1 \langle (H_{uu})^{-1}\delta v, \delta v\rangle\,\mathrm{d}t + \langle M\delta\xi, \delta\xi\rangle$. □

Assume that M is non-negative definite. Recall that H_{uu} is positive definite, and then $(H_{uu})^{-1}$ is positive definite too. Hence $\Theta(\delta w) \geq 0$, $\forall\, \delta w = (\delta x, \delta u) \in \Gamma$.

Suppose that $\Theta(\delta w) = 0$ for some $\delta w = (\delta x, \delta u) \in \Gamma$. Then, given (3.3) both non-negative terms $\int_0^1 \langle (H_{uu})^{-1}\delta v, \delta v\rangle$ and $\langle M\delta\xi, \delta\xi\rangle$ are equal zero. Condition

$$\int_0^1 \langle (H_{uu})^{-1}\delta v, \delta v\rangle \mathrm{d}t = 0$$

implies $\delta v = 0$, i.e. $(H_{ux} + f_u^\top Q)\delta x + H_{uu}\delta u = 0$. Hence $\delta u = -(H_{uu})^{-1}(H_{ux} + f_u^\top Q)\delta x$. It follows that δx is a solution to the homogeneous differential equation $\delta\dot{x} = f_x(\hat{w})\delta x - f_u(\hat{w})(H_{uu})^{-1}(H_{ux} + f_u^\top Q)\delta x$. Let us now assume that the conditions $\mathcal{E}\delta\xi = 0$, $\langle M\delta\xi, \delta\xi\rangle = 0$ imply that $\delta x(0) = 0$ or $\delta x(1) = 0$. Then $\delta x = 0$ and hence $\delta u = 0$. Consequently, $\Theta(\delta w) > 0$ for all $\delta w \in \Gamma \setminus \{0\}$. Since Θ is a Legendre form, its positiveness on the subspace Γ implies positive definiteness on Γ. Thus we obtain the following result.

Theorem 4. *Assume that the strengthened Legendre condition is satisfied and there exists a symmetric solution Q (with the entries belonging to C^1) of the Riccati matrix equation on $[0, 1]$ such that*

(a) the matrix M is non-negative definite;

(b) for all $\xi = (x_0, x_1) \in \mathbb{R}^{2n}$ the conditions $\mathcal{E}\xi = 0$, $\langle M\xi, \xi\rangle = 0$ imply that $x_0 = 0$ or $x_1 = 0$.

Then the quadratic form Θ is positive definite on the subspace Γ.

Other designations

Let Γ has the form:

$$\Gamma = \Big\{\delta w = (\delta x, \delta u) : \delta\dot{x}(t) = A(t)\delta x(t) + B(t)\delta u(t),\, \mathcal{E}\delta\xi = 0\Big\},$$

and

$$2\Theta(\delta w) = \langle N\delta\xi, \delta\xi\rangle + \int_0^1 \left(\langle R(t)\delta x(t), \delta x(t)\rangle + 2\langle S(t)\delta u(t), \delta x(t)\rangle + \langle U(t)\delta u(t), \delta u(t)\rangle\right) \mathrm{d}t, \tag{3.4}$$

where $\mathcal{E}$ and N are constant matrices, $A(t)$, $B(t)$, $R(t)$, $S(t)$, $U(t)$ are matrices with continuous entries. Assume that the matrices $R(t)$ and $U(t)$ are symmetric and, moreover, the matrix $U(t)$ is positive definite for all $t \in [0, 1]$, and the constant symmetric matrix N of the order $2n$ has the form

$$N = \begin{pmatrix} N_{00} & N_{01} \\ N_{10} & N_{11} \end{pmatrix},$$

where N_{00}, N_{01}, N_{10}, N_{11} are constant $n \times n$ matrices, N_{00} and N_{11} are symmetric, and $N_{10} = N_{01}^\top$. Previously, we had $A = f_x$, $B = f_u$, $R = H_{xx}$, $S = H_{xu}$, $U = H_{uu}$. We can prove similar results for the new quadratic form and subspace in the same way as before. Now the Riccati equation and the matrix M are:

$$\dot{Q} + QA + A^\top Q + R - (S + QB)U^{-1}(S^\top + B^\top Q) = 0, \tag{3.5}$$

$$M = \begin{pmatrix} N_{00} + Q(0) & N_{01} \\ N_{10} & N_{11} - Q(1) \end{pmatrix}. \tag{3.6}$$

3.2 Statement of Control Problem for Single Beam

Consider the following optimal control problem. Let $z(x)$ be a state variable, $f(x)$ be a control, where $x \in [0, l]$. Here $z = (z_1, \ldots, z_n)^\top \in \mathbb{R}^n$, $f \in \mathbb{R}^1$, $l > 0$. We assume that $z(x)$ is a continuously differentiable function and $f(x)$ is a continuous function. The control system has the form

$$\frac{\mathrm{d}z(x)}{\mathrm{d}x} = \varphi(z(x)) + e_1 f(x), \quad x \in [0, l], \quad K(z(0), z(l)) = 0 \tag{3.7}$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a twice continuously differentiable function, $e_1 = (1, 0, \ldots, 0)^T \in \mathbb{R}^n$, and $K : \mathbb{R}^{2n} \to \mathbb{R}^s$ is an affine function of its arguments $\zeta_0 := z(0)$ and $\zeta_l := z(l)$. Set $\zeta = (\zeta_0, \zeta_l)$. The cost that needs to be minimized is:

$$J = \int_0^l F(x, z(x), f(x))\mathrm{d}x, \tag{3.8}$$

where $F(x, z, f)$ is a twice continuously differentiable function. In this problem, l is not fixed but satisfies the constraint

$$l \in [a, b], \quad \text{where} \quad 0 < a < b. \tag{3.9}$$

An arbitrary admissible process in this problem is defined by the triple $(l, z(\cdot), f(\cdot))$, where $z : [0, l] \to \mathbb{R}^n$, $f : [0, l] \to \mathbb{R}$. We will consider a fixed admissible process

$$(l^0, z^0(\cdot), f^0(\cdot)), \tag{3.10}$$

where z^0 and f^0 are defined on $[0, l^0]$.

Let us represent this problem as a problem on the interval $[0, 1]$. To do this, we use the following change of the independent variable x. Let $t \in [0, 1]$ be a new independent variable. We set

$$\tilde{x}(t) = lt, \quad t \in [0, 1].$$

Then $\tilde{x} : [0, 1] \to [0, l]$. We treat $\tilde{x}(t)$ as a new state variable. We also treat $l = \tilde{l}(t)$ as another state variable, constant on $[0, 1]$. Hence

$$\frac{\mathrm{d}\tilde{l}(t)}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}\tilde{x}(t)}{\mathrm{d}t} = \tilde{l}(t), \quad t \in [0, 1], \quad \tilde{x}(0) = 0.$$

To any admissible process (l, z, f) in the original problem, we associate the process $(\tilde{l}, \tilde{x}, \tilde{z}, \tilde{f})$ in the new problem by the formulas

$$\tilde{l}(t) = l, \quad \tilde{x}(t) = lt, \quad \tilde{z}(t) = z(\tilde{x}(t)) = z(lt), \quad \tilde{f}(t) = f(\tilde{x}(t)) = f(lt) \quad \forall\, t \in [0, 1].$$

This is one-to-one correspondence. In what follows, we will continue to use the tilde for variables in the interval $[0, 1]$.

Thus, we obtain an autonomous problem with a new independent variable $t \in [0, 1]$:

$$\frac{\mathrm{d}\tilde{l}(t)}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}\tilde{x}(t)}{\mathrm{d}t} = \tilde{l}(t), \quad t \in [0, 1], \tag{3.11}$$

$$\frac{\mathrm{d}\tilde{z}(t)}{\mathrm{d}t} = \tilde{l}(t)\big(\varphi(\tilde{z}(t)) + e_1\tilde{f}(t)\big), \quad t \in [0, 1], \tag{3.12}$$

$$\tilde{x}(0) = 0, \quad K(\tilde{z}(0), \tilde{z}(1)) = 0, \tag{3.13}$$

$$-\tilde{l}(0) + a \leq 0. \quad \tilde{l}(0) - b \leq 0, \tag{3.14}$$

$$J = \int_0^1 \tilde{l}(t)F(\tilde{x}(t), \tilde{z}(t), \tilde{f}(t))\mathrm{d}t \to \min. \tag{3.15}$$

We study the local minimum at the point

$$(\tilde{l}^0(\cdot), \tilde{x}^0, \tilde{z}^0(\cdot), \tilde{f}^0(\cdot)), \tag{3.16}$$

such that

$$\tilde{l}^0(t) = l^0, \quad \tilde{x}^0(t) = l^0 t, \quad \tilde{z}^0(t) = z^0(l^0 t), \quad \tilde{f}^0(t) = f^0(l^0 t), \quad t \in [0, 1].$$

This point corresponds to the process (3.10) in the original problem (3.7)-(3.9). Clearly, the minimum at (3.10) in problem (3.7)-(3.9) implies the minimum at (3.16) in problem (3.11)-(3.15) and vice versa.

Local Minimum Principle for Problem with One Beam

Denote by $\tilde{p}^z(t)$ the adjoint variable which corresponds to the equation for $\tilde{z}$ in the new problem. We consider $\tilde{p}^z = (\tilde{p}^z_1, \dots, \tilde{p}^z_n)$ as a row vector. We also introduce one-dimensional adjoint variables $\tilde{p}^x(t)$ and $\tilde{p}^l(t)$. The Hamiltonian and the endpoint Lagrange function are:

$$\begin{aligned}
&\tilde{H}(\tilde{l}, \tilde{x}, \tilde{z}, \tilde{f}, \tilde{p}^l, \tilde{p}^x, \tilde{p}^z, \alpha_0) = \tilde{p}^x \tilde{l} + \tilde{p}^z \tilde{l}\big(\varphi(\tilde{z}) + e_1 \tilde{f}\big) + \alpha_0 \tilde{l} F(\tilde{x}, \tilde{z}, \tilde{f}),\\
&\tilde{L} = \alpha_a(-\tilde{l}(0) + a) + \alpha_b(\tilde{l}(0) - b) + \beta_x \tilde{x}(0) + \beta K(\tilde{z}(0), \tilde{z}(1)).
\end{aligned}$$

Note that $\tilde{L}$ is an affine function of the endpoint values $\tilde{l}(0)$, $\tilde{x}(0)$, $\tilde{z}(0)$, $\tilde{l}(1)$, $\tilde{x}(1)$, $\tilde{z}(1)$ of the states $\tilde{l}$, $\tilde{x}$, and $\tilde{z}$, since K is an affine function by assumption.

Let us write down the first-order necessary optimality conditions at the point (3.16) in problem (3.11)-(3.15). The partial derivatives of $\tilde{H}$ with respect to $\tilde{l}, \tilde{x}, \tilde{z}, \tilde{f}$ have the form

$$\begin{aligned}
&\tilde{H}_{\tilde{l}} = \tilde{p}^x + \tilde{p}^z(\varphi(\tilde{z}) + e_1 \tilde{f}) + \alpha_0 F(\tilde{x}, \tilde{z}, \tilde{f}),\\
&\tilde{H}_{\tilde{x}} = \alpha_0 \tilde{l} F_{\tilde{x}}(\tilde{x}, \tilde{z}, \tilde{f}),\\
&\tilde{H}_{\tilde{z}} = \tilde{p}^z \tilde{l} \varphi'(\tilde{z})^T + \alpha_0 \tilde{l} F_{\tilde{z}}(\tilde{x}, \tilde{z}, \tilde{f}),\\
&\tilde{H}_{\tilde{f}} = \tilde{p}^z \tilde{l} e_1 + \alpha_0 \tilde{l} F_{\tilde{f}}(\tilde{x}, \tilde{z}, \tilde{f}).
\end{aligned}$$

(a) The nonnegativity conditions: $\alpha_0 \geq 0, \quad \alpha_a \geq 0, \quad \alpha_b \geq 0.$

(b) The nontriviality condition: $\alpha_0 + \alpha_a + \alpha_b + |\beta_x| + |\beta| > 0.$

(c) The complementary slackness conditions: $\alpha_a(\tilde{l}^0(0) - a) = 0$, $\alpha_b(\tilde{l}^0(0) - b) = 0$.

(d) The adjoint equations:

$$-\frac{\mathrm{d}\tilde{p}^l(t)}{\mathrm{d}t} = \tilde{p}^x(t) + \tilde{p}^z(t)\big(\varphi(\tilde{z}^0(t)) + e_1 \tilde{f}^0(t)\big) + \alpha_0 F(\tilde{x}^0(t), \tilde{z}^0(t), \tilde{f}^0(t)), \tag{3.17}$$

$$-\frac{\mathrm{d}\tilde{p}^x(t)}{\mathrm{d}t} = \alpha_0 \tilde{l}^0 F_{\tilde{x}}(\tilde{x}^0(t), \tilde{z}^0(t), \tilde{f}^0(t)), \tag{3.18}$$

$$-\frac{\mathrm{d}\tilde{p}^z(t)}{\mathrm{d}t} = \tilde{p}^z(t) \tilde{l}^0 \varphi'(\tilde{z}^0(t)) + \alpha_0 \tilde{l}^0 F_{\tilde{z}}(\tilde{x}^0(t), \tilde{z}^0(t), \tilde{f}^0(t)),\ t \in [0, 1]. \tag{3.19}$$

(e) The transversality conditions:

$$\begin{aligned}
-\tilde{p}^l(0) &= -\alpha_a + \alpha_b, & \tilde{p}^l(1) &= 0,\\
-\tilde{p}^x(0) &= \beta_x, & \tilde{p}^x(1) &= 0,\\
-\tilde{p}^z(0) &= \beta K_{\tilde{\zeta}_0}(\tilde{z}^0(0), \tilde{z}^0(1)), & \tilde{p}^z(1) &= \beta K_{\tilde{\zeta}_1}(\tilde{z}^0(0), \tilde{z}^0(1)),
\end{aligned}$$

where $\tilde{\zeta}_0 = \tilde{z}(0)$, $\tilde{\zeta}_1 = \tilde{z}(1)$.

(f) The condition $\tilde{H}_{\tilde{f}} = 0$: $\tilde{p}^z(t)\tilde{l}^0 e_1 + \alpha_0 \tilde{l}^0 F_{\tilde{f}}(\tilde{x}^0(t), \tilde{z}^0(t), \tilde{f}^0(t)) = 0$. Since $\tilde{l}^0 > 0$ and $\tilde{p}^z(t)e_1 = \tilde{p}^z_1(t)$, we get

$$\tilde{p}^z_1(t) + \alpha_0 \tilde{l}^0 F_{\tilde{f}}(\tilde{x}^0(t), \tilde{z}^0(t), \tilde{f}^0(t)) = 0, \quad t \in [0,1].$$

(g) Finally, the condition $\tilde{H} = \text{const}$ has the form: there exists a constant $\hat{c}_H$ such that

$$\tilde{p}^x(t)\tilde{l}^0 + \tilde{p}^z(t)\tilde{l}^0\big(\varphi(\tilde{z}^0(t)) + e_1\tilde{f}^0(t)\big) + \alpha_0 \tilde{l}^0 F(\tilde{x}^0(t), \tilde{z}^0(t), \tilde{f}^0(t)) = \tilde{c}_H \quad \forall\, t \in [0,1].$$

Denote the left hand side of this equality by $\tilde{H}(t)$. Dividing this equality by $\tilde{l}^0$, we obtain

$$\tilde{p}^x(t) + \tilde{p}^z(t)\big(\varphi(\tilde{z}^0(t)) + e_1\tilde{f}^0(t)\big) + \alpha_0 F(\tilde{x}^0(t), \tilde{z}^0(t), \tilde{f}^0(t)) = \frac{\tilde{c}_H}{\tilde{l}^0} \quad \forall\, t \in [0,1].$$

Integrating equation (3.17) over the interval $[0,1]$ and using the above condition, we get $\tilde{p}^l(0) - \tilde{p}^l(1) = \frac{\tilde{c}_H}{\tilde{l}^0}$. This and the transversality conditions $-\tilde{p}^l(0) = -\alpha_a + \alpha_b$, $\tilde{p}^l(1) = 0$ give

$$\frac{\tilde{c}_H}{\tilde{l}^0} = \alpha_a - \alpha_b.$$

This relation means the following.

(1) If $a < \tilde{l}^0 < b$ then by the complementary slackness conditions (c) we have $\alpha_a = \alpha_b = 0$ and therefore $\tilde{c}_H = 0$.

(2) If $\tilde{l}^0 = a$, then by (c) we have $\alpha_b = 0$ and, therefore, $\tilde{c}_H = \alpha_a \tilde{l}^0 \geq 0$.

(3) If $\tilde{l}^0 = b$, then by (c) we have $\alpha_a = 0$ and, therefore, $\tilde{c}_H = -\alpha_b \tilde{l}^0 \leq 0$.

(4) Moreover, if $\tilde{c}_H > 0$, then $\alpha_a > 0$, and, therefore, by (c) $\tilde{l}^0 = a$; if $\tilde{c}_H < 0$, then $\alpha_b > 0$, and, therefore, by (c) $\tilde{l}^0 = b$.

Note that the transversality condition $\tilde{p}^x(1) = 0$ and adjoint equation (3.18) imply

$$\tilde{p}^x(t) = \alpha_0 \tilde{l}^0 \int_t^1 F_{\tilde{x}}(\tilde{z}^0(\tau), \tilde{f}^0(\tau))\mathrm{d}\tau, \quad t \in [0,1]. \tag{3.20}$$

Thus, we obtain the following result. If (3.16) is a local minimum in problem (3.11)–(3.15), then there exist a number $\alpha_0 \geq 0$, a row vector $\beta \in \mathbb{R}^{s\top}$, and a continuously such that the following system of optimality conditions holds:

$$\begin{aligned}
\frac{\mathrm{d}\tilde{z}}{\mathrm{d}t} &= \tilde{l}^0\big(\varphi(\tilde{z}^0(t)) + e_1\tilde{f}^0(t)\big), \quad t \in [0,1], \quad K(\tilde{z}^0(0), \tilde{z}^0(1)) = 0,\\
-\frac{\mathrm{d}\tilde{p}^z}{\mathrm{d}t} &= \tilde{p}^z(t)\tilde{l}^0\varphi'(\tilde{z}^0(t)) + \alpha_0\tilde{l}^0 F_{\tilde{z}}(\tilde{x}^0(t), \tilde{z}^0(t), \tilde{f}^0(t)), \quad t \in [0,1],\\
-\tilde{p}^z(0) &= \beta K_{\tilde{\zeta}_0}(\tilde{z}^0(0), \tilde{z}^0(1)), \quad \tilde{p}^z(1) = \beta K_{\tilde{\zeta}_1}(\tilde{z}^0(0), \tilde{z}^0(1)),\\
\tilde{p}^z_1(t) &+ \alpha_0\tilde{l}^0 F_{\tilde{f}}(\tilde{x}^0(t), \tilde{z}^0(t), \tilde{f}^0(t)) = 0, \quad t \in [0,1].
\end{aligned}$$

These conditions imply the condition of the constancy of the Hamiltonian: there exists a constant $\tilde{c}_H$ such that

$$\tilde{p}^x(t)\tilde{l}^0 + \tilde{p}^z(t)\tilde{l}^0\big(\varphi(\tilde{z}^0(t)) + e_1\tilde{f}^0(t)\big) + \alpha_0\tilde{l}^0 F(\tilde{x}^0(t), \tilde{z}^0(t), \tilde{f}^0(t)) = \tilde{c}_H \quad \forall\, t \in [0,1],$$

where $\tilde{p}^x(t)$ is defined by (3.20).

Moreover, the following is true. If $a < \tilde{l}^0 < b$, then $\tilde{c}_H = 0$. If $\tilde{l}^0 = a$, then $\tilde{c}_H \geq 0$; if $\tilde{c}_H > 0$, then $\tilde{l}^0 = a$. If $\tilde{l}^0 = b$, then $\tilde{c}_H \leq 0$; if $\tilde{c}_H < 0$, then $\tilde{l}^0 = b$. We now equivalently represent this system on the interval $[0, l^0]$. Introduce a function $p^z : [0, l^0] \to \mathbb{R}^{n\top}$ such that $\tilde{p}^z(t) = p^z(\tilde{x}^0(t)) = p^z(l^0 t)$, that is $p^z(x) = \tilde{p}^z\left(\frac{x}{l^0}\right)$, $x \in [0, l^0]$. Then $\frac{\mathrm{d}\tilde{p}^z}{\mathrm{d}t} = \frac{\mathrm{d}p^z}{\mathrm{d}x} l^0$. Hence the adjoint equation for $\tilde{p}^z$ takes the form

$$-\frac{\mathrm{d}p^z(x)}{\mathrm{d}x} = p^z(x)\varphi'(z^0(x)) + \alpha_0 F_z(x^0(t), z^0(t), f^0(t)), \quad x \in [0, l^0].$$

So, the obtained result has the following formulation on the interval $[0, l^0]$. Below we replace p^z with p, and we also replace $(l^0, z^0(\cdot), f^0(\cdot))$ with $(l, z(\cdot), f(\cdot))$ omitting the superscript zero.

Theorem 5. *If $(l, z(\cdot), f(\cdot))$ is a local minimum in problem (3.7)-(3.9), then there exist a number $\alpha_0 \geq 0$, a row vector $\beta \in \mathbb{R}^{s\top}$, and a continuously differentiable function $p : [0, l] \to \mathbb{R}^{n\top}$ such that the following system of optimality conditions holds:*

$$\begin{aligned}
\frac{\mathrm{d}z(x)}{\mathrm{d}x} &= \varphi(z(x)) + e_1 f(x), \quad x \in [0,l],\ l \in [a,b],\ K(z(0), z(l)) = 0,\\
-\frac{\mathrm{d}p(x)}{\mathrm{d}x} &= p(x)\varphi'(z(x)) + \alpha_0 F_z(x, z(x), f(x)), \quad x \in [0,l],\\
-p(0) &= \beta K_{\zeta_0}(z(0), z(1)),\\
p(l) &= \beta K_{\zeta_1}(z(0), z(l)),\\
p_1(x) &+ \alpha_0 F_f(x, z(x), f(x)) = 0, \quad x \in [0,l].
\end{aligned}$$

These conditions imply the condition of the constancy of the Hamiltonian: there exists a constant c_H such that

$$p^x(x) + p(x)\big(\varphi(z(x)) + e_1 f(x)\big) + \alpha_0 F(x, z(x), f(x)) = c_H \quad \forall\, x \in [0,l],$$

where $p^x(x) = \alpha_0 \int_x^l F_x(y, z(y), f(y))\mathrm{d}y$, $x \in [0,l]$. Moreover, the following is true. If $a < l < b$, then $c_H = 0$. If $l = a$, then $c_H \geq 0$; if $c_H > 0$, then $l = a$. If $l = b$, then $c_H \leq 0$; if $c_H < 0$, then $l = b$.

Since $p^x(l) = 0$ and $c_H = H(l)$, we get

$$c_H = p(l)\big(\varphi(z(l)) + e_1 f(l)\big) + \alpha_0 F(l, z(l), f(l)).$$

This formula does not use the adjoint variable p^x.

In what follows, we consider the case of

$$F(x,z,f) = \frac{1}{2}|z - z^*(x)|^2 + \frac{1}{2}(f - f^*(x))^2, \tag{3.21}$$

where $|z| = \sqrt{\langle z, z\rangle}$ and $z^*(x)$ and $f^*(x)$ are twice continuously differentiable functions defined on $[0,b]$.

Second-Order Sufficient Conditions for Problem with One Beam

For problem (3.11) -(3.15) on $[0,1]$ with the function F defined by formula (3.21), we formulate sufficient second-order conditions for a weak local minimum at the point $\tilde{w}(\cdot) = (\tilde{l}(\cdot), \tilde{x}(\cdot), \tilde{z}(\cdot), \tilde{f}(\cdot))$.

Now suppose that the normal case holds for this point. Therefore, there are a row vector $\beta \in \mathbb{R}^{s\top}$ and a continuously differentiable function $\tilde{p} : [0,1] \to \mathbb{R}^{n\top}$ such that the necessary optimality conditions in Section 3.2 are satisfied with $\alpha_0 = 1$. In problem (3.11)-(3.15) on $[0,1]$, by definition $\tilde{\xi} = (\tilde{l}(0), \tilde{x}(0), \tilde{z}(0); \tilde{l}(1), \tilde{x}(1), \tilde{z}(1))$. Since $\tilde{L}$ is an affine function of $\tilde{\xi}$, we have $\tilde{L}_{\tilde{\xi}\tilde{\xi}} = 0$. Since $\alpha_0 = 1$ we have

$$\tilde{H}(\tilde{l}, \tilde{x}, \tilde{z}, \tilde{f}, \tilde{p}^l, \tilde{p}^x, \tilde{p}^z) = \tilde{p}^x\tilde{l} + \tilde{p}^z\tilde{l}\big(\varphi(\tilde{z}) + e_1\tilde{f}\big) + \tilde{l}F(\tilde{x}, \tilde{z}, \tilde{f}).$$

Recall that $\tilde{H}_{\tilde{f}} = \tilde{p}^z\tilde{l}e_1 + \alpha_0\tilde{l}(\tilde{f} - f^*(\tilde{x}))$. Consequently, $\tilde{H}_{\tilde{f}\tilde{f}} = \tilde{l}$. Since $\tilde{l} = l \geq a > 0$, the strengthened Legendre condition is satisfied.

Let us write down the definition of the critical cone $\tilde{\mathcal{C}}$. Equations in variations for the system

$$\frac{\mathrm{d}\tilde{l}}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}\tilde{x}}{\mathrm{d}t} = \tilde{l}, \quad \frac{\mathrm{d}\tilde{z}}{\mathrm{d}t} = \tilde{l}(\varphi(\tilde{z}(t)) + e_1\tilde{f}(t))$$

at the point $\tilde{w}$ have the form

$$\delta\dot{\tilde{l}} = 0, \quad \delta\dot{\tilde{x}}(t) = \delta\tilde{l}, \quad \delta\dot{\tilde{z}}(t) = \tilde{l}\big(\varphi'(\tilde{z}(t))\delta\tilde{z}(t) + e_1\delta\tilde{f}(t)\big) + \big(\varphi(\tilde{z}(t)) + e_1\tilde{f}(t)\big)\delta\tilde{l}.$$

The endpoint conditions $\tilde{x}(0) = 0$ and $K(\tilde{z}(0), \tilde{z}(1)) = 0$ imply the following conditions in the critical cone

$$\delta\tilde{x}(0) = 0, \quad K'(\tilde{\zeta})\delta\tilde{\zeta} = 0,$$

where $\tilde{\zeta} = (\tilde{z}(0), \tilde{z}(1))$, $\delta\tilde{\zeta} = (\delta\tilde{z}(0), \delta\tilde{z}(1))$.

Further, recall that

$$\frac{\tilde{c}_H}{\tilde{l}} = \alpha_a - \alpha_b.$$

The initial conditions $-\tilde{l}(0) + a \leq 0$ and $\tilde{l}(0) - b \leq 0$ imply:

- if $a < \tilde{l} < b$, i.e., these constraints are not active, then $\tilde{c}_H = 0$, and we have no conditions on $\delta\tilde{l}(0)$,

- if $a = \tilde{l}$ and, therefore, $\tilde{l} < b$, then the following conditions are satisfied $\delta\tilde{l}(0) \geq 0$, $\tilde{c}_H \delta\tilde{l}(0) = 0$,

- if $\tilde{l} = b$ and, therefore, $\tilde{l} > a$, then the following conditions are satisfied $\delta\tilde{l}(0) \leq 0$, $\tilde{c}_H \delta\tilde{l}(0) = 0$.

Consequently,

$$\begin{aligned}\tilde{\mathcal{C}} = \Big\{\delta\tilde{w} = (\delta\tilde{l}, \delta\tilde{x}, \delta\tilde{z}, \delta\tilde{f}) :\ & \delta\dot{\tilde{l}} = 0,\ \delta\dot{\tilde{x}}(t) = \delta\tilde{l},\ \delta\tilde{x}(0) = 0,\ K'(\tilde{\zeta})\delta\tilde{\zeta} = 0,\ c_H\delta\tilde{l}(0) = 0,\\ & \delta\dot{\tilde{z}}(t) = \tilde{l}\big(\varphi'(\hat{z}(t))\delta\tilde{z}(t) + e_1\delta\tilde{f}(t)\big) + \big(\varphi(\tilde{z}(t)) + e_1\tilde{f}(t)\big)\delta\tilde{l},\\ & \tilde{l} = a \implies \delta\tilde{l}(0) \geq 0;\quad \tilde{l} = b \implies \delta\tilde{l}(0) \leq 0\Big\}.\end{aligned}$$

As stated in Remark 1, if Θ is positive definite on $\tilde{\mathcal{C}}$, then it is positive definite on $(-\tilde{\mathcal{C}})$. Only one of the two conditions $\tilde{l} = a$ or $\tilde{l} = b$ could be realized. Therefore, the conditions $\tilde{l} = a \implies \delta\tilde{l}(0) \geq 0$; $\tilde{l} = b \implies \delta\tilde{l}(0) \leq 0$ in the definition of $\tilde{\mathcal{C}}$ can be ommitted. More precisely, we can replace $\tilde{\mathcal{C}}$ with a subspace

$$\begin{aligned}\tilde{\Sigma} = \Big\{\delta\tilde{w} = (\delta\tilde{l}, \delta\tilde{x}, \delta\tilde{z}, \delta\tilde{f}) :\quad & \delta\dot{\tilde{l}} = 0,\ \delta\dot{\tilde{x}}(t) = \delta\tilde{l},\ \delta\tilde{x}(0) = 0,\ K'(\tilde{\zeta})\delta\tilde{\zeta} = 0,\ \tilde{c}_H\delta\tilde{l}(0) = 0,\\ & \delta\dot{\tilde{z}}(t) = \tilde{l}\big(\varphi'(\hat{z}(t))\delta\tilde{z}(t) + e_1\delta\tilde{f}(t)\big) + \big(\varphi(\tilde{z}(t)) + e_1\tilde{f}(t)\big)\delta\tilde{l}\ \Big\}.\end{aligned}$$

Note that if $\tilde{c}_H \neq 0$, then in the definition of $\tilde{\Sigma}$ we have $\delta\tilde{l}(0) = 0$,which gives $\delta\tilde{l} = 0$, and this means that $\delta\tilde{x} = 0$. In this case,

$$\tilde{\Sigma} = \Big\{\delta\tilde{w} = (\delta\tilde{l}, \delta\tilde{x}, \delta\tilde{z}, \delta\tilde{f}) :\ \delta\tilde{l} = 0,\ \delta\tilde{x} = 0,\ \delta\dot{\tilde{z}}(t) = \tilde{l}\varphi'(\hat{z}(t))\delta\tilde{z}(t) + \tilde{l}e_1\delta\tilde{f}(t),\ K'(\tilde{\zeta})\delta\tilde{\zeta} = 0\Big\}.$$

Let us write down the quadratic form $\tilde{\Theta}$. Since $\alpha_0 = 1$,

$$\begin{aligned}\tilde{H}_{\tilde{l}} &= \tilde{p}^x + \tilde{p}^z(\varphi(\tilde{z}) + e_1\tilde{f}) + F(\tilde{x}, \tilde{z}, \tilde{f}),\\ \tilde{H}_{\tilde{x}} &= \tilde{l}F_{\tilde{x}} = -\tilde{l}\big(\tilde{z} - z^*(\tilde{x})\big)^{\top}(z^*)'(\tilde{x}) - \tilde{l}(\tilde{f} - f^*(\tilde{x}))(f^*)'(\tilde{x}),\\ \tilde{H}_{\tilde{z}} &= \tilde{p}^z\tilde{l}\varphi'(\tilde{z}) + \tilde{l}\big(\tilde{z} - z^*(\tilde{x})\big)^{\top},\\ \tilde{H}_{\tilde{f}} &= \tilde{p}^z\tilde{l}e + \tilde{l}F_{\tilde{f}}(\tilde{x}, \tilde{z}, \tilde{f}) = \tilde{p}^z\tilde{l}e_1 + \tilde{l}(\tilde{f} - f^*(\tilde{x})).\end{aligned}$$

Once again we emphasize that we consider z, $\tilde{z}$, z^* as column vectors, and p^z, $\tilde{p}^z$, $\tilde{H}_{\tilde{z}}$ as row vectors. Therefore, $\big(\tilde{z} - z^*(\tilde{x})\big)^{\top}(z^*)'(\tilde{x}) = \sum_i \big(\tilde{z}_i - z_i^*(\tilde{x})\big)(z_i^*)'(\tilde{x})$.

The second-order partial derivatives have the form

$$
\begin{aligned}
&\tilde{H}_{\tilde{l}\tilde{l}} = 0,\\
&\tilde{H}_{\tilde{l}\tilde{x}} = \tilde{H}_{\tilde{x}\tilde{l}} = -\big(\tilde{z} - z^*(\tilde{x})\big)^\top (z^*)'(\tilde{x}) - (\tilde{f} - f^*(\tilde{x}))(f^*)'(\tilde{x}),\\
&\tilde{H}_{\tilde{l}\tilde{z}} = \tilde{H}_{\tilde{z}\tilde{l}}^\top = \tilde{p}^z\varphi'(\tilde{z}) + \big(\tilde{z} - z^*(\tilde{x})\big)^\top,\\
&\tilde{H}_{\tilde{l}\tilde{f}} = \tilde{H}_{\tilde{f}\tilde{l}} = \tilde{p}^z_1 + \tilde{f} - f^*(\tilde{x}),\\
&\tilde{H}_{\tilde{x}\tilde{x}} = \tilde{l}[(z^*)'(\tilde{x})]^\top (z^*)'(\tilde{x}) - \tilde{l}(\tilde{z} - z^*(\tilde{x}))^\top (z^*)''(\tilde{x})\\
&\qquad + \tilde{l}[(f^*)'(\tilde{x})]^2 - \tilde{l}(\tilde{f} - f^*(\tilde{x}))(f^*)''(\tilde{x}),\\
&\tilde{H}_{\tilde{x}\tilde{z}} = \tilde{H}_{\tilde{z}\tilde{x}}^\top = -\tilde{l}[(z^*)'(\tilde{x})]^\top,\\
&\tilde{H}_{\tilde{x}\tilde{f}} = \tilde{H}_{\tilde{f}\tilde{x}} = -\tilde{l}(f^*)'(\tilde{x}),\\
&\tilde{H}_{\tilde{z}\tilde{z}} = \tilde{p}^z\tilde{l}\varphi'(\tilde{z})^\top + \tilde{l}I_n,\\
&\tilde{H}_{\tilde{z}\tilde{f}} = \tilde{H}_{\tilde{f}\tilde{z}}^\top = 0,\\
&\tilde{H}_{\tilde{f}\tilde{f}} = \tilde{l}.
\end{aligned}
\tag{3.22}
$$

Here I_n is the identity matrix of size n and

$$\big(\tilde{z} - z^*(\tilde{x})\big)^\top (z^*)''(\tilde{x}) = \sum_i \big(\tilde{z}_i - z_i^*(\tilde{x})\big)(z_i^*)''(\tilde{x}).$$

Denoting $\tilde{w} = (\tilde{l}, \tilde{x}, \tilde{z}, \tilde{f})$, we get

$$
\begin{aligned}
&\langle \tilde{H}_{ww}(\tilde{l}, \tilde{x}, \tilde{z}, \tilde{f}, \tilde{p}^l, \tilde{p}^x, \tilde{p}^z)\delta w, \delta w\rangle = \tilde{H}_{\tilde{l}\tilde{l}}(\delta\tilde{l})^2 + \tilde{H}_{\tilde{x}\tilde{x}}(\delta\tilde{x})^2 + \langle \tilde{H}_{\tilde{z}\tilde{z}}\delta\tilde{z}, \delta\tilde{z}\rangle + \tilde{H}_{\tilde{f}\tilde{f}}(\delta\tilde{f})^2\\
&+2\tilde{H}_{\tilde{l}\tilde{x}}\delta\tilde{x}\cdot\delta\tilde{l} + 2\tilde{H}_{\tilde{l}\tilde{z}}\delta\tilde{z}\cdot\delta\tilde{l} + 2\tilde{H}_{\tilde{l}\tilde{f}}\delta\tilde{f}\cdot\delta\tilde{l} + 2\tilde{H}_{\tilde{x}\tilde{z}}\delta\tilde{z}\cdot\delta\tilde{x} + 2\tilde{H}_{\tilde{x}\tilde{f}}\delta\tilde{f}\cdot\delta\tilde{x} + 2\tilde{H}_{\tilde{f}\tilde{z}}\delta\tilde{z}\cdot\delta\tilde{f}.
\end{aligned}
$$

Using the above formulas, we obtain

$$
\begin{aligned}
&\big\langle \tilde{H}_{\tilde{w}\tilde{w}}(\tilde{l}(t), \tilde{x}(t), \tilde{z}(t), \tilde{f}(t), \tilde{p}^l(t), \tilde{p}^x(t), \tilde{p}^z(t))\delta\tilde{w}(t), \delta\tilde{w}(t)\big\rangle\\
&= \tilde{l}\Big([(z^*)'(\tilde{x}(t))]^\top (z^*)'(\tilde{x}(t)) - (\tilde{z}(t) - z^*(\tilde{x}(t)))^\top (z^*)''(\tilde{x}(t))\\
&\quad + [(f^*)'(\tilde{x}(t))]^2 - (\tilde{f}(t) - f^*(\tilde{x}(t)))(f^*)''(\tilde{x}(t))\Big)(\delta\tilde{x}(t))^2\\
&\quad + \tilde{l}\big\langle\big(\tilde{p}^z(t)\varphi''(\tilde{z}(t)) + I_n\big)\delta\tilde{z}(t), \delta\tilde{z}(t)\big\rangle + \tilde{l}\big(\delta\tilde{f}(t)\big)^2\\
&\quad - 2\Big(\big(\tilde{z}(t) - z^*(\tilde{x}(t))\big)(z^*)'(\tilde{x}(t)) + \big(\tilde{f}(t) - f^*(\tilde{x}(t))\big)(f^*)'(\tilde{x}(t))\Big)\delta\tilde{x}(t)\cdot\delta\tilde{l}\\
&\quad + 2\big(\tilde{p}^z(t)\varphi'(\tilde{z}(t)) + \big(\tilde{z}(t) - z^*(\tilde{x}(t))\big)^\top\big)\delta\tilde{z}(t)\cdot\delta\tilde{l}\\
&\quad + 2\big(\tilde{p}^z_1(t) + \tilde{f}(t) - f^*(\tilde{x}(t))\big)\delta\tilde{f}(t)\cdot\delta\tilde{l}\\
&\quad - 2\tilde{l}\cdot[(z^*)'(\tilde{x}(t))]^\top\delta\tilde{z}(t)\cdot\delta\tilde{x}(t) - 2\tilde{l}\cdot(f^*)'(\tilde{x}(t))\delta\tilde{f}(t)\cdot\delta\tilde{x}(t).
\end{aligned}
$$

Recall that here $\tilde{l} = l = \text{const} > 0$. Since $\tilde{L}_{\tilde{\xi}\tilde{\xi}} = 0$, the quadratic form $\tilde{\Theta}$ is:

$$\tilde{\Theta}(\delta\tilde{w}) = \int_0^1 \langle \tilde{H}_{\tilde{w}\tilde{w}}(\tilde{l}, \tilde{x}, \tilde{z}, \tilde{f}, \tilde{p}^l, \tilde{p}^x, \tilde{p}^z)\delta\tilde{w}, \delta\tilde{w}\rangle \mathrm{d}t.$$

Thus, we obtain the following result: *if there exists a constant* $\tilde{c}_\Theta > 0$ *such that*

$$\tilde{\Theta}(\delta\tilde{w}) \geq \tilde{c}_\Theta((\delta\tilde{l})^2 + |\delta\tilde{z}(0)|^2 + \|\delta\tilde{f}\|_2^2) \quad \forall\, \delta\tilde{w} \in \tilde{\Sigma},$$

then the quadruple $(\tilde{l}(\cdot), \tilde{x}(\cdot), \tilde{z}(\cdot), \tilde{f}(\cdot))$ *is a weak local minimum in problem (3.11)-(3.15) on* $[0,1]$.

Now let us rewrite the obtained sufficient second-order condition in terms of the independent variable $x \in [0,l]$. Let $\delta\tilde{w} = (\delta\tilde{l}, \delta\tilde{x}, \delta\tilde{z}, \delta\tilde{f}) \in \tilde{\Sigma}$. Introduce $\delta z(x)$ such that $\delta z(\tilde{x}(t)) = \delta z(lt) = \delta\tilde{z}(t)$, that is $\delta z(x) = \delta\tilde{z}\Big(\frac{x}{l}\Big)$. Then $\delta\dot{\tilde{z}}(t) = (\delta z)'(\tilde{x}(t))l$, where $(\delta z)'(x) = \frac{\mathrm{d}z(x)}{\mathrm{d}x}$. Define δl such that $\delta\tilde{l} = l\delta l$, that is $\delta l = \dfrac{\delta\tilde{l}}{l}$. Similarly, we define $\delta f(x) = \delta\tilde{f}\left(\frac{x}{l}\right)$, $\delta x(x) = \delta\tilde{x}\left(\frac{x}{l}\right)$. Then the equation $\delta\dot{\tilde{z}}(t) = \tilde{l}\big(\varphi'(\hat{z}(t))\delta\tilde{z}(t) + e_1\delta\tilde{f}(t)\big) + \big(\varphi(\tilde{z}(t)) + e_1\tilde{f}(t)\big)\delta\tilde{l}$ takes the form

$$(\delta z)'(x) = \varphi'(z(x))\delta z(x) + e_1\delta f(x) + \big(\varphi(z(x)) + e_1 f(x)\big)\delta l$$

and the subspace $\tilde{\Sigma}$ in the new variables reads as follows

$$\Sigma = \Big\{\delta w = (\delta l, \delta x, \delta z, \delta f): \; (\delta l)' = 0,\; (\delta x)'(x) = \delta l,\; \delta x(0) = 0,\; K'(\zeta)\delta\zeta = 0,\; c_H\delta l(0) = 0,$$
$$(\delta z)'(x) = \varphi'(z(x))\delta z(x) + e_1\delta f(x) + \big(\varphi(z(x)) + e_1 f(x)\big)\delta l\Big\},$$

where $\zeta = (z(0), z(l))$, $\delta\zeta = (\delta z(0), \delta z(l))$. Recall that δx and δl are one-dimensional, $\delta l = \mathrm{const}$ and $\delta x = x\delta l$. Therefore,

$$\Sigma = \Big\{\delta w = (\delta l, \delta x, \delta z, \delta f): \; (\delta l)' = 0,\; \delta x(x) = x \cdot \delta l,\; K'(\zeta)\delta\zeta = 0,\; c_H\delta l(0) = 0,$$
$$(\delta z)'(x) = \varphi'(z(x))\delta z(x) + e_1\delta f(x) + \big(\varphi(z(x)) + e_1 f(x)\big)\delta l\Big\}.$$

Let us rewrite the quadratic form $\tilde{\Theta}$ in the new variables. Recall that $\tilde{l} = l$. If $x = \tilde{x}(t) = lt$, then $\mathrm{d}x = l\mathrm{d}t$ and $\tilde{z}(t) = z(x)$, $\tilde{f}(t) = f(x)$, $\delta\tilde{l} = l\delta l$, $\delta z(x) = \delta\tilde{z}(t)$, $\delta x(x) = \delta\tilde{x}(t)$,

$\delta f(x) = \delta \tilde{f}(t)$. Therefore, we have

$$
\begin{aligned}
&\tilde{l}\Big([(z^*)'(\tilde{x}(t))]^T (z^*)'(\tilde{x}(t)) - (\tilde{z}(t) - z^*(\tilde{x}(t)))^\top (z^*)''(\tilde{x}(t)) \\
&\qquad\qquad + [(f^*)'(\tilde{x}(t))]^2 - (\tilde{f}(t) - f^*(\tilde{x}(t)))(f^*)''(\tilde{x}(t))\Big)(\delta\tilde{x}(t))^2 \mathrm{d}t \\
&= \Big([(z^*)'(x)]^\top (z^*)'(x) - (z(x) - z^*(x))^\top (z^*)''(x) \\
&\qquad\qquad + [(f^*)'(x)]^2 - (f(x) - f^*(x))(f^*)''(x)\Big)(\delta x(x))^2 \mathrm{d}x, \\
&\big\langle\big(\tilde{p}^z(t)\tilde{l}\varphi''(\tilde{z}(t)) + \tilde{l}I_n\big)\delta\tilde{z}(t), \delta\tilde{z}(t)\big\rangle \mathrm{d}t = \big\langle\big(p^z(x)\varphi''(z(x)) + I_n\big)\delta z(x), \delta z(x)\big\rangle \mathrm{d}x, \\
&l(\delta\tilde{f}(t))^2 \mathrm{d}t = (\delta f(x))^2 \mathrm{d}x, \\
&-2\Big(\big(\tilde{z}(t) - z^*(\tilde{x}(t))\big)^\top (z^*)'(\tilde{x}(t)) + \big(\tilde{f}(t) - f^*(\tilde{x}(t))(f^*)'(\tilde{x}(t))\Big)\delta\tilde{x}(t)\cdot\delta\tilde{l}\cdot \mathrm{d}t \\
&\qquad\qquad = -2\Big(\big(z(x) - z^*(x)\big)^\top (z^*)'(x) + \big(f(x) - f^*(x)\big)(f^*)'(x)\Big)\delta x(x)\cdot\delta l\cdot \mathrm{d}x, \\
&2\Big(\big(\tilde{p}^z(t)\varphi'(\tilde{z}(t)) + \big(\tilde{z}(t) - z^*(\tilde{x}(t))\big)^\top\Big)\delta\tilde{z}(t)\cdot\delta\tilde{l}\cdot \mathrm{d}t \\
&\qquad\qquad = 2\Big(\big(p^z(x)\varphi'(z(x)) + \big(z(x) - z^*(x)\big)^\top\Big)\delta z(x)\cdot\delta l\cdot \mathrm{d}x, \\
&-2\tilde{l}[(z^*)'(\tilde{x}(t))]^\top \delta\tilde{z}(t)\cdot\delta\tilde{x}(t)\cdot \mathrm{d}t - 2\tilde{l}(f^*)'(\tilde{x}(t))\cdot\delta\tilde{f}(t)\cdot\delta\tilde{x}(t)\cdot \mathrm{d}t \\
&\qquad\qquad = -2[(z^*)'(x)]^\top \delta z(x)\cdot\delta x(x)\cdot \mathrm{d}x - 2(f^*)'(x)\delta f(x)\cdot\delta x(x)\cdot \mathrm{d}x.
\end{aligned}
$$

Consequently,

$$\tilde{\Theta}(\delta\tilde{w}) = \Theta(\delta w),$$

where

$$
\begin{aligned}
\Theta(\delta w) = \int_0^l &\Big\{\big([(z^*)'(x)]^\top (z^*)'(x) - \big(z(x) - z^*(x)\big)^\top (z^*)''(x) \\
&+ [(f^*)'(x)]^2 - (f(x) - f^*(x))(f^*)''(x)\big)(\delta x(x))^2 \\
&+ \big\langle\big(p^z\varphi''(z) + I_n\big)\delta z(x), \delta z(x)\big\rangle + (\delta f(x))^2 \\
&- 2\big(\big(z(x) - z^*(x)\big)^\top (z^*)'(x) + (f(x) - f^*(x))(f^*)'(x)\big)\delta x(x)\cdot\delta l \\
&+ 2\big(p^z(x)\varphi'(z(x)) + \big(z(x) - z^*(x)\big)^\top\big)\delta z(x)\cdot\delta l \\
&+ 2\big(p_1^z(x) + f(x) - f^*(x)\big)\delta f(x)\cdot\delta l \\
&- 2[(z^*)'(x)]^\top \delta z(x)\cdot\delta x(x) - 2(f^*)'(x)\delta f(x)\cdot\delta x(x)\Big\}\mathrm{d}x.
\end{aligned}
$$

Below we replace p^z with p, omitting the superscript z. Since $\delta l \in \mathbb{R}$ is a constant,

$\delta x = x \cdot \delta l$, and δf is one-dimensional, we obtain

$$
\begin{aligned}
\Theta(\delta w) = (\delta l)^2 \int_0^l & \big([(z^*)'(x)]^\top (z^*)'(x) - (z(x) - z^*(x))^\top (z^*)''(x) \\
&+ [(f^*)'(x)]^2 - (f(x) - f^*(x)(f^*)''(x)\big) x^2 \mathrm{d}x \\
&+ \int_0^l \big\langle \big(p(x)\varphi''(z(x)) + I_n\big)\delta z(x), \delta z(x)\big\rangle \mathrm{d}x + \int_0^l (\delta f(x))^2 \mathrm{d}x \\
&- 2(\delta l)^2 \int_0^l \Big(\big(z(x) - z^*(x)\big)^\top (z^*)'(x) + (f(x) - f^*(x))(f^*)'(x)\Big) x \mathrm{d}x \\
&+ 2\delta l \int_0^l \Big(p(x)\varphi'(z(x)) + \big(z(x) - z^*(x)\big)^\top\Big) \delta z(x) \mathrm{d}x \\
&+ 2\delta l \int_0^l \big(p_1(x) + f(x) - f^*(x)\big)\delta f(x) \mathrm{d}x \\
&- 2\delta l \int_0^l \big((z^*)'(x)\big)^\top \delta z(x) x \mathrm{d}x \\
&- 2\delta l \int_0^l (f^*)'(x)\delta f(x) x \mathrm{d}x.
\end{aligned}
\tag{3.23}
$$

This quadratic form is independent of δx. We can exclude δx from the definition of Σ as well. Therefore, the quadratic form Θ is considered on a subspace, which we still denote by Σ (we also keep the notation δw for the shorter collection $(\delta l, \delta z, \delta f)$):

$$
\begin{aligned}
\Sigma := \Big\{ \delta w = (\delta l, \delta z, \delta f) \quad : \quad & (\delta l)' = 0,\ c_H \delta l = 0,\ K'(\zeta)\delta\zeta = 0, \\
& (\delta z)'(x) = \varphi'(z(x))\delta z(x) + e_1 \delta f(x) + \big(\varphi(z(x)) + e_1 f(x)\big)\delta l \Big\}.
\end{aligned}
$$

Thus, we obtain the following result:

Theorem 6. *Let an admissible triple $(l, z(\cdot), f(\cdot))$ satisfy the first order necessary optimality conditions of Theorem 5 in problem (3.7)-(3.9) with the corresponding multipliers $\alpha_0 = 1, \beta, p(\cdot)$. Suppose there exists a constant $c_\Theta > 0$ such that*

$$
\Theta(\delta w) \geq c_\Theta \Big((\delta l)^2 + |\delta z(0)|^2 + \|\delta f\|_2^2 \Big) \quad \forall\, \delta w \in \Sigma.
$$

Then the triple $(l, z(\cdot), f(\cdot))$ is a weak local minimum in problem (3.7)-(3.9).

Matrix Riccati equation for One Beam: Case $c_H \neq 0$

In this case, as we know, the condition $c_H > 0$ implies $l = a$, and the condition $c_H < 0$ implies $l = b$. Then in the definition of Σ, we have $\delta l = 0$, so that we can put

$$
\Sigma := \Big\{ \delta w = (\delta z, \delta f) : (\delta z)'(x) = \varphi'(z(x))\delta z(x) + e_1 \delta f(x),\ K'(\zeta)\delta\zeta = 0 \Big\}.
$$

Since $\delta l = 0$, the quadratic form reduces to

$$\Theta(\delta w) = \int_0^l \langle (p(x)\varphi''(z(x)) + I_n)\, \delta z(x), \delta z(x)\rangle \, \mathrm{d}x + \int_0^l (\delta f(x))^2 \mathrm{d}x.$$

We study the question of the positive definiteness of Θ on Σ in terms of the solution of the matrix Riccati equation. Obviously, the strengthened Legendre condition is satisfied.

Comparing the differential equation in the definition of Σ with the equation $(\delta z)' = A\delta z + B\delta f$ (see end of section 3.1.3), we obtain

$$A = \varphi'((z(x)), \quad B = e_1 = (1, 0, \dots, 0)^\top.$$

Comparing Θ with (3.4), we get

$$R = p(x)\varphi''(z(x)) + I_n, \quad S = 0, \quad U = 1.$$

Consequently,

$$(S + QB)U^{-1}(S^\top + B^\top Q) = Qe_1e_1^\top Q = \begin{pmatrix} Q_{11} \\ \dots \\ Q_{1n} \end{pmatrix} \begin{pmatrix} Q_{11} & \dots & Q_{1n} \end{pmatrix}$$

$$= \begin{pmatrix} Q_{11}Q_{11} & \dots & Q_{11}Q_{1n} \\ \dots & \dots & \dots \\ Q_{1n}Q_{11} & \dots & Q_{1n}Q_{1n} \end{pmatrix} = ||Q_{1i}Q_{1j}||_{i,j=1}^n.$$

Thus, the Riccati equation (3.5) reduces to the following

$$\frac{\mathrm{d}}{\mathrm{d}x}Q + QA + A^\top Q + R - Qe_1e_1^\top Q = 0, \quad x \in [0, l]. \tag{3.24}$$

where

$$\begin{aligned} A &= \varphi'((z(x)), & R &= p(x)\varphi''(z(x)) + I_n, \\ e_1 &= (1, 0. \dots, 0)^\top, & Qe_1e_1^T Q &= ||Q_{1i}Q_{1j}||_{i,j=1}^n. \end{aligned}$$

The matrix M has the form

$$M = \begin{pmatrix} Q(0) & 0 \\ 0 & -Q(l) \end{pmatrix}.$$

To this Riccati equation, one can add the initial condition

$$Q(0) = I_n,$$

where I_n is the identity matrix of order n.

Similarly to Theorem 4 the following theorem holds.

Theorem 7. *Assume that the strengthened Legendre condition is satisfied, $c_H \neq 0$, and there exists a symmetric solution Q (with the entries belonging to C^1) of the Riccati matrix equation (3.24) on $[0, l]$ such that*

(a) the matrix M is non-negative definite;

(b) for all $\zeta = (\zeta_0, \zeta_1) \in \mathbb{R}^{2n}$ the conditions $K'(\zeta)\zeta = 0$, $\langle M\zeta, \zeta \rangle = 0$ imply that $\zeta_0 = 0$ or $\zeta_1 = 0$.

Then the quadratic form Θ is positive definite on the subspace Σ.

Matrix Riccati equation for one beam: case $c_H = 0$

In this more complicated case, we have

$$\Sigma := \Big\{ \delta w = (\delta l, \delta z, \delta f) : \quad (\delta l)' = 0, \quad K'(\zeta)\delta\zeta = 0, \\ (\delta z)'(x) = \varphi'(z(x))\delta z(x) + e_1 \delta f(x) + \big(\varphi(z(x)) + e_1 f(x)\big)\delta l \Big\}.$$

Consider again the sufficient condition for the positive definiteness of the quadratic form Θ on the subspace Σ. Now Σ is defined by a linear system of differential equations

$$\begin{cases} (\delta l)' = 0, \\ (\delta z)' = (\varphi((z(x)) + e_1 f(x))\delta l + \varphi'((z(x))\delta z(x) + e_1 \delta f(x). \end{cases}$$

In the sequel, we denote

$$X = \begin{pmatrix} l \\ z \end{pmatrix} = \begin{pmatrix} l \\ z_1 \\ \dots \\ z_n \end{pmatrix} \in \mathbb{R}^{n+1}, \quad \delta X = \begin{pmatrix} \delta l \\ \delta z \end{pmatrix} = \begin{pmatrix} \delta l \\ \delta z_1 \\ \dots \\ \delta z_n \end{pmatrix} \in \mathbb{R}^{n+1},$$

$$w = \begin{pmatrix} X \\ f \end{pmatrix} = \begin{pmatrix} l \\ z \\ f \end{pmatrix} \in \mathbb{R}^{n+2}, \quad \delta w = \begin{pmatrix} \delta X \\ \delta f \end{pmatrix} = \begin{pmatrix} \delta l \\ \delta z \\ \delta f \end{pmatrix} \in \mathbb{R}^{n+2}.$$

Let us represent the above system in matrix form $(\delta X)' = A\delta X + B\delta f$, where A is a $(n+1) \times (n+1)$ matrix, B is a $(n+1) \times 1$ matrix such that

$$A = \begin{pmatrix} 0 & 0_n^\top \\ \varphi(z(x)) + e_1 f(x) & \varphi'((z(x)) \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ e_1 \end{pmatrix}, \quad 0_n^\top = (0, \dots, 0) \in \mathbb{R}^{n\top}.$$

It is convenient to present

$$A := \begin{pmatrix} 0 & 0_n^\top \\ A_{zl} & A_{zz} \end{pmatrix}, \quad \text{where} \quad A_{zl} = \varphi(z(x)) + e_1 f(x), \quad A_{zz} = \varphi'((z(x)).$$

Compare quadratic form (3.23) with the standard form (see (3.4)):

$$\Theta(\delta w) = \int_0^l \Big(\langle R\delta X, \delta X\rangle + 2(\delta X)^\top S\delta f + U(\delta f)^2\Big)\mathrm{d}t,$$

where R is the symmetric $(n+1)\times(n+1)$ matrix, $S \in \mathbb{R}^{n+1}$ is the column vector, U is the number. Let us find the matrix R. Denote

$$R = \begin{pmatrix} R_{ll} & R_{lz} \\ R_{zl} & R_{zz} \end{pmatrix},$$

where R_{ll}, R_{lz}, $R_{zl} = R_{lz}^\top$, $R_{zz} = R_{zz}^\top$ are matrices of orders 1×1, $1\times n$, $n\times 1$, $n\times n$, respectively. Then,

$$\langle R\delta X, \delta X\rangle = R_{ll}(\delta l)^2 + 2R_{lz}\delta z\delta l + \langle R_{zz}\delta z, \delta z\rangle.$$

Using (3.23), we obtain

$$\begin{aligned}\langle R\delta X, \delta X\rangle = & \Big[\Big([(z^*)'(x)]^\top (z^*)'(x) - (z(x) - z^*(x))^\top (z^*)''(x) \\ & +[(f^*)'(x)]^2 - (f(x) - f^*(x))(f^*)''(x)\Big)x^2 \\ & -2\left(\big(z(x) - z^*(x)\big)^\top (z^*)'(x) + (f(x) - f^*(x))(f^*)'(x)\right) x\Big] \cdot (\delta l)^2 \\ & + \langle (p(x)\varphi''(z(x)) + I_n)\,\delta z(x), \delta z(x)\rangle \\ & +2\left[\big(p(x)\varphi'(z(x)) + \big(z(x) - z^*(x)\big)^\top - x\big((z^*)'(x)\big)^\top\right] \delta z(x)\cdot\delta l.\end{aligned}$$

Consequently,

$$\begin{aligned}R_{ll} &= \Big([(z^*)'(x)]^\top (z^*)'(x) - (z(x) - z^*(x))^\top (z^*)''(x) \\ &\quad +[(f^*)'(x)]^2 - (f(x) - f^*(x))(f^*)''(x)\Big)x^2 \\ &\quad -2\Big(\big(z(x) - z^*(x)\big)^\top (z^*)'(x) \\ &\quad +(f(x) - f^*(x))(f^*)'(x)\Big)x, \qquad (3.25)\\ R_{zz} &= p(x)\varphi''(z(x)) + I_n, \qquad (3.26)\\ R_{lz} = R_{zl}^\top &= \big(p(x)\varphi'(z(x)) + \big(z(x) - z^*(x)\big)^\top - x\big((z^*)'(x)\big)^\top. \qquad (3.27)\end{aligned}$$

Further, $U = 1$, and finally, S has the form

$$S = \begin{pmatrix} S_l \\ 0_n \end{pmatrix} \in \mathbb{R}^{n+1},$$

where $S_l = p_1(x) + f(x) - f^*(x) - x(f^*)'(x)$. Recall that the Riccati equation has the form

$$\frac{\mathrm{d}}{\mathrm{d}x}Q + QA + A^TQ + R - (S + QB)U^{-1}(S^T + B^TQ) = 0, \quad x \in [0, l],$$

where

$$Q(x) = \begin{pmatrix} Q_{ll} & Q_{lz} \\ Q_{zl} & Q_{zz} \end{pmatrix}(x)$$

where

$$Q_{ll}(x) \in \mathbb{R}, \quad Q_{zl}(x) = \begin{pmatrix} Q_{z_1 l} \\ \dots \\ Q_{z_n l} \end{pmatrix}(x) \in \mathbb{R}^n,$$

$$Q_{lz}(x) = Q_{zl}^\top(x) = \begin{pmatrix} Q_{lz_1} & \dots & Q_{lz_n} \end{pmatrix}(x) \in \mathbb{R}^{n\top},$$

and

$$Q_{zz}(x) = \begin{pmatrix} Q_{z_1 z_1} & \dots & Q_{z_1 z_n} \\ \dots & \dots & \dots \\ Q_{z_n z_1} & \dots & Q_{z_n z_n} \end{pmatrix}(x)$$

is $n \times n$ symmetric matrix. Since $U = 1$, we have

$$(S + QB)U^{-1}(S^\top + B^\top Q) = (S + QB)(S + QB)^\top.$$

Further,

$$QB = \begin{pmatrix} Q_{ll} & Q_{lz} \\ Q_{zl} & Q_{zz} \end{pmatrix} \begin{pmatrix} 0 \\ e_1 \end{pmatrix} = \begin{pmatrix} Q_{lz}e_1 \\ Q_{zz}e_1 \end{pmatrix}.$$

Hence

$$S + QB = \begin{pmatrix} S_l \\ 0 \end{pmatrix} + \begin{pmatrix} Q_{lz}e_1 \\ Q_{zz}e_1 \end{pmatrix} = \begin{pmatrix} Q_{lz}e_1 + S_l \\ Q_{zz}e_1 \end{pmatrix}.$$

Consequently,

$$\begin{aligned}(S + QB)(S + QB)^\top &= \begin{pmatrix} Q_{lz}e_1 + S_l \\ Q_{zz}e_1 \end{pmatrix} (Q_{lz}e_1 + S_l, e_1^\top Q_{zz}) \\ &= \begin{pmatrix} (Q_{lz}e_1 + S_l)^2 (Q_{lz}e_1 + S_l)e_1^\top Q_{zz} & \\ Q_{zz}e_1(Q_{lz}e_1 + S_l) & (Q_{zz}e_1)(e_1^\top Q_{zz}) \end{pmatrix}.\end{aligned}$$

Moreover,

$$QA = \begin{pmatrix} Q_{ll} & Q_{lz} \\ Q_{zl} & Q_{zz} \end{pmatrix} \begin{pmatrix} 0 & 0_n^\top \\ A_{zl} & A_{zz} \end{pmatrix} = \begin{pmatrix} Q_{lz}A_{zl} & Q_{lz}A_{zz} \\ Q_{zz}A_{zl} & Q_{zz}A_{zz} \end{pmatrix}, \ A^\top Q = \begin{pmatrix} Q_{lz}A_{zl} & A_{zl}^\top Q_{zz} \\ A_{zz}^\top Q_{zl} & A_{zz}^\top Q_{zz} \end{pmatrix}.$$

Here $Q_{lz}A_{zl}$, $Q_{lz}A_{zz}$, $Q_{zz}A_{zl}$, $Q_{zz}A_{zz}$ are matrices of order 1×1, $1 \times n$, $n \times 1$, $n \times n$, respectively. Consequently,

$$QA + A^\top Q = \begin{pmatrix} 2Q_{lz}A_{zl} & Q_{lz}A_{zz} + A_{zl}^\top Q_{zz} \\ Q_{zz}A_{zl} + A_{zz}^\top Q_{zl} & Q_{zz}A_{zz} + A_{zz}^\top Q_{zz} \end{pmatrix}.$$

Thus, according to (3.5), we obtain the matrix Riccati equation in the form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} \begin{pmatrix} Q_{ll} & Q_{lz} \\ Q_{zl} & Q_{zz} \end{pmatrix} &+ \begin{pmatrix} 2Q_{lz}A_{zl} & Q_{lz}A_{zz} + A_{zl}^\top Q_{zz} \\ Q_{zz}A_{zl} + A_{zz}^\top Q_{zl} & Q_{zz}A_{zz} + A_{zz}^\top Q_{zz} \end{pmatrix} + \begin{pmatrix} R_{ll} & R_{lz} \\ R_{zl} & R_{zz} \end{pmatrix} \\ &- \begin{pmatrix} (Q_{lz}e_1 + S_l)^2 & (Q_{lz}e_1 + S_l)e^{\top_1}Q_{zz} \\ (Q_{lz}e_1 + S_l)Q_{zz}e_1 & (Q_{zz}e)(e_1^\top Q_{zz}) \end{pmatrix} = 0, \quad x \in [0, l],\end{aligned} \tag{3.28}$$

where the blocks of the matrix R are determined by formulas (3.25)-(3.27). Further, the matrix M has the form

$$M = \begin{pmatrix} Q(0) & 0 \\ 0 & -Q(l) \end{pmatrix}.$$

We set

$$\delta\xi = \begin{pmatrix} \delta X(0) \\ \delta X(l) \end{pmatrix}.$$

Then

$$\langle M\delta\xi, \delta\xi\rangle = \langle Q(0)\delta X(0), \delta X(0)\rangle - \langle Q(l)\delta X(l), \delta X(l)\rangle,$$

where

$$\delta X(0) = \begin{pmatrix} \delta l \\ \delta z(0) \end{pmatrix}, \quad \delta X(l) = \begin{pmatrix} \delta l \\ \delta z(l) \end{pmatrix}, \quad (\delta l)' = 0, \text{ i.e.,} \quad \delta l = \text{const}.$$

The condition $\mathcal{E}\delta\xi = 0$ in the definition of Σ means $K_{z_0}\delta z(0) + K_{z_l}\delta z(l) = 0$, $(\delta l)' = 0$. Consequently,

$$\delta X_0 := \delta X(0) = \begin{pmatrix} \delta l \\ \delta z(0) \end{pmatrix}, \quad \delta X_l := \delta X(l) = \begin{pmatrix} \delta l \\ \delta z(l) \end{pmatrix}.$$

The following theorem holds.

Theorem 8. *Assume that the strengthened Legendre condition is satisfied, $c_H = 0$, and there exists a symmetric solution Q (with the entries belonging to C^1) of the Riccati matrix equation (3.28) on $[0, l]$ such that*

(a) the matrix M is nonnegative definite;

(b) for all pairs of vectors in $\mathbb{R}^{n+1}$

$$\delta X_0 = \begin{pmatrix} \delta l \\ \delta z_0 \end{pmatrix}, \quad \delta X_l = \begin{pmatrix} \delta l \\ \delta z_l \end{pmatrix}$$

the conditions $K_{z_0}\delta z_0 + K_{z_l}\delta z_l = 0$, $\delta l \in \mathbb{R}$, $\langle Q(0)\delta X_0, \delta X_0\rangle - \langle Q(l)\delta X_l, \delta X_l\rangle = 0$ imply that $\delta X_0 = 0$ or $\delta X_l = 0$. Then the quadratic form Θ is positive definite on the subspace Σ.

3.3 Numerical Examples

In the following examples, we focus on a single edge to show the application of our methods. In the first example, we construct a semilinear differential equation, specifying the desired state and control parameters. Subsequently, we derive the optimal state and control configurations through the optimality system.

For the second example, we consider the static IGEB model, with parameters detailed in [44]. First, we semi-discretize in space the IGEB model by means of the finite element method and using $\mathbb{P}_2$ (quadratic) elements. For the nonlinear ODE, we use the Newton–Raphson method.

3.3.1 Example 1

Consider a steady state scenario involving a single edge, governed by a semilinear differential equation. The control system has the form

$$z'(x) = \varphi(z(x)) + f(x), \quad x \in [0, l], \quad z(l) = 0,$$

where z is one dimensional and l is not fixed. The beam's behavior is described by the function

$$\varphi(z) = z - z^2.$$

Set

$$l^* = 1, \quad z^*(x) = -2 + x + x^2, \quad f^*(x) = 7 - 3x - 4x^2 + 2x^3 + x^4.$$

It is easy to check that the triple $(l^*, z^*, f^*(x))$ defines an admissible process of a given control system.

The cost functional is expressed as:

$$J = \frac{1}{2} \int_0^l \big((z(x) - z^*(x))^2 + (f(x) - f^*(x))^2\big) \mathrm{d}x + \frac{1}{2}(l - l^*)^2 \to \min .$$

The parameter l is constrained $l \in [\frac{1}{2}, \frac{3}{2}]$.

Obviously, $(l^*, z^*, f^*(x))$ is the solution to this problem. But assume that this solution is unknown and let us write down the necessary optimality conditions of Theorem 5.

Since $\int_0^l (x - l^*)\mathrm{d}x = \frac{1}{2}(l - l^*)^2 - \frac{1}{2}(l^*)^2$, we can consider the equivalent problem of minimizing the functional

$$J = \int_0^l F(x, z(x), f(x))\mathrm{d}x$$

with $F(x, z, f) = \frac{1}{2}\big((z - z^*(x))^2 + (f - f^*(x))^2\big) + x - l^*$.

Let the triple $(l, z(\cdot), f(\cdot))$ be a solution to this problem. Then, according to Theorem 5 there are numbers $\alpha_0 \geq 0$, β, and a continuously differentiable function $p : [0, l] \to \mathbb{R}$ such that

$$\begin{aligned} z'(x) &= \varphi(z(x)) + f(x), \quad z(l) = 0 \\ -p'(x) &= p(x)\varphi'(z(x)) + \alpha_0 F_z(x, z(x), f(x)), \quad x \in [0, l] \\ & p(0) = 0, \quad p(l) = \beta, \\ & p(x) + \alpha_0 F_f(x, z(x), f(x)) = 0, \quad x \in [0, l]. \end{aligned}$$

If $\alpha_0 = 0$, then $p(x) = 0$ and $\beta = 0$. Therefore, $\alpha_0 > 0$, and we put $\alpha_0 = 1$. Hence, taking into account that $\varphi(z) = z - z^2$, $\varphi'(z) = 1 - 2z$, $F_z(x, z, f) = (z - z^*(x))$, and $F_f(x, z, f) = f - f^*(x)$, we get a system

$$\left.\begin{aligned} z'(x) &= z(x) - z^2(x) + f(x), \quad z(l) = 0, \\ -p'(x) &= p(x)(1 - 2z(x)) + z - z^*(x), \quad p(0) = 0 \\ & p(x) + f(x) - f^*(x) = 0. \end{aligned}\right\} \tag{3.29}$$

Theorem 5 gives one more necessary optimality condition for determining $(l, z(\cdot), f(\cdot))$. Recall that we are considering l close to $l^* = 1$, which means $a < l < b$ with $a = 0.5$, $b = 1.5$. As we know, in this case $c_H = 0$. Since $\alpha_0 = 1$, this condition looks like

$$c_H = p^x(x) + p(x)\big(\varphi(z(x)) + f(x)\big) + F(x, z(x), f(x)) = 0 \quad \forall\, x \in [0, l],$$

where $p^x(x) = \alpha_0 \int_x^l F_x(y, z(y), f(y))\mathrm{d}y$, $x \in [0, l]$. Considering that $p^x(l) = 0$, $z(l) = 0$, and $-p(l) = f(l) - f^*(l)$, we get

$$\begin{aligned}
0 &= p(l)\big(\varphi(z(l)) + f(l)\big) + F(l, z(l), f(l)) \\
&= p(l)\big(\varphi(0) + f(l)\big) + F(l, 0, f(l)) \\
&= p(l)f(l) + \frac{1}{2}(z^*(l))^2 + \frac{1}{2}((f(l) - f^*(l))^2 + l - 1 \\
&= p(l)f(l) + \frac{1}{2}(z^*(l))^2 + \frac{1}{2}(p(l))^2 + l - 1,
\end{aligned}$$

that is

$$p(l)f(l) + \frac{1}{2}(z^*(l))^2 + \frac{1}{2}(p(l))^2 + l - 1 = 0. \tag{3.30}$$

Conditions (3.29) and (3.30) constitute a complete system of necessary optimality conditions for determining $(l, z(\cdot), f(\cdot))$. Obviously, the triple $p(x) = 0$, $f(x) = f^*(x)$, $z(x) = z^*(x)$ is a solution to this system.

We will now show numerical results for this problem. We conducted the computation using the finite element method and the Newton method to handle the nonlinear component. Here are the results. Fig. 3.1 illustrates the variation of the cost functional with respect to the length parameter. It is observed that the cost functional attains its minimum value at $l = 1 = l^*$, indicating the optimality of this length. This signifies that the length $l = 1$ is the optimal choice based on the minimization of the cost functional.

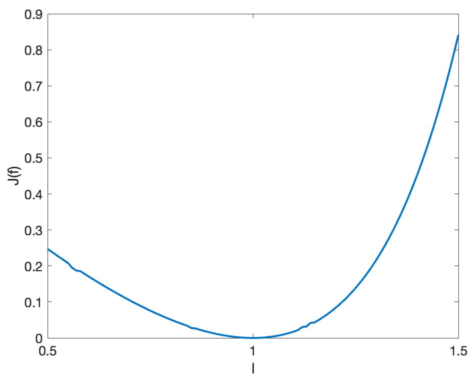


Figure 3.1: Cost with respect to l

In Fig. 3.2, we show the optimal control and state under the optimal length.

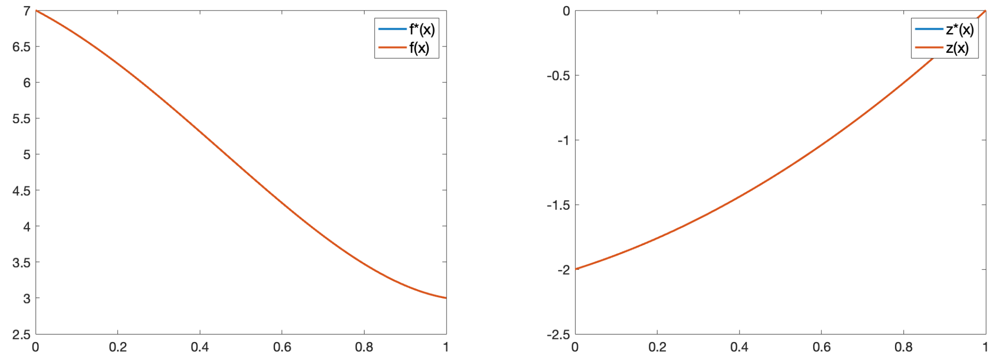


Figure 3.2: Optimal control and optimal state

Then, we computed the L^2 norm error between the numerical solution and the analytical solution to assess the accuracy of the results:

$$f_{err} = ||f(x) - f^*(x)|| = 1.6236e - 12,\ z_{err} = ||z(x) - z^*(x)|| = 4.5426e - 11.$$

3.3.2 Example 2

Consider the steady state for a single beam governed by the semilinear differential equation (Eq. 2.7). We present results from numerical simulations. The cost functional is expressed the same as before:

$$J = \frac{1}{2}\int_0^l \left((z(x) - z^*(x))^2 + (f(x) - f^*(x))^2\right)\mathrm{d}x + \frac{1}{2}(l - l^*)^2 \to \min. \tag{3.31}$$

The flexibility matrix is given by

$$\mathbf{C} = \operatorname{diag}\left(c_1, c_2, c_3, c_4, c_5, c_6\right)^{-1} = \operatorname{diag}\left(10^4, 10^4, 10^4, 500, 500, 500\right)^{-1}.$$

In Eq. (3.7), setting $f^*(x) = -1$, then the steady state values are $z_1^* = -x + 1, z_2^* = 0, \ldots, z_6^* = 0$. For the IGEB model, the function

$$\varphi(r) = -\mathbf{E}(x)r + \mathbf{L}(r)\mathbf{C}r, \tag{3.32}$$

and its derivative is given by

$$\varphi'(r) = -\mathbf{E}(x) + (\mathbf{L}(r)\mathbf{C}r)' := -\mathbf{E}(x) + \bar{G}(r). \tag{3.33}$$

Here,

$$\mathbf{E}(x) = \left(\begin{array}{ccc:c} & \mathbf{0}_{3\times3} & & \mathbf{0}_{3\times3} \\ \hdashline 0 & 0 & 0 & \\ 0 & 0 & -1 & \mathbf{0}_{3\times3} \\ 0 & 1 & 0 & \end{array}\right), \mathbf{L}(z) = \left(\begin{array}{ccc:ccc} & & & 0 & -z_3 & z_2 \\ & \mathbf{0}_{3\times3} & & z_3 & 0 & -z_1 \\ & & & -z_2 & z_1 & 0 \\ \hdashline 0 & -z_3 & z_2 & 0 & -z_6 & z_5 \\ z_3 & 0 & -z_1 & z_6 & 0 & -z_4 \\ -z_2 & z_1 & 0 & -z_5 & z_4 & 0 \end{array}\right).$$

The optimality system of equations can be written as

$$\begin{cases} z'(x) = -\mathbf{E}(x)z(x) + \mathbf{L}(z)\mathbf{C}z(x) + e_1 f(x), & x \in [0,l] \\ -p'(x) = -p(x)\mathbf{E}(x) + p(x)\bar{G}(z) + z(x) - z^*(x), & x \in [0,l] \\ z_i(l) = 0,\ p_i(0) = 0, & i = 1,2,\dots,6 \\ p_1(x) + f(x) - f^* = 0. \end{cases} \tag{3.34}$$

The weak form of this system is given by:

$$\begin{cases} -\int_0^l \left\langle \frac{\mathrm{d}\psi}{\mathrm{d}x}, z \right\rangle dx - z(0)\psi(0) + \int_0^l \langle \mathbf{E}z, \psi \rangle\, dx - \int_0^l \langle \mathbf{CL}(z)z, \psi \rangle\, dx \\ \qquad + \int_0^l e_1 \langle p, \psi \rangle\, dx = \int_0^l e_1 \langle f^*, \psi \rangle\, dx,\ \forall \psi \in V_1 \\ \int_0^l \left\langle \frac{\mathrm{d}\eta}{\mathrm{d}x}, p \right\rangle dx - p(1)\eta(1) + \int_0^l \langle \mathbf{E}p, \eta \rangle\, dx - \int_0^l \left\langle \bar{G}(z)p, \eta \right\rangle dx \\ \qquad - \int_0^l \langle z, \eta \rangle\, dx = -\int_0^l \langle z^*, \eta \rangle\, dx,\ \forall \eta \in V_2, \end{cases} \tag{3.35}$$

where

$$V_1 := \{\psi \in H^1\left(0,l;\mathbb{R}^6\right), \psi(1) = 0\},$$

and

$$V_2 := \{\eta \in H^1\left(0,l;\mathbb{R}^6\right), \eta(0) = 0\}.$$

In numerical discretization, the interval $[0,l]$ is discretized into N_x points $\{x_k\}_{k=1}^{N_x}$, where $x_1 = 0$ and $x_{N_x} = l$. Each subinterval $\omega^e := [x_{2e-1}, x_{2e+1}]$ for $e \in \{1,2,\dots,N_e\}$ constitutes an element. These elements are defined by the points x_{2e-1}, x_{2e}, and x_{2e+1} and have a uniform length $h_e = x_{2e+1} - x_{2e-1}$. It is important to note that $N_x = 2N_e + 1$.

We utilize $\mathbb{P}_2$ (quadratic) elements to define function spaces $V_{1,h}$ and $V_{2,h}$ as described below:

$$\begin{aligned} V_{1,h} : &= \left\{\psi \in C^0\left([0,l];\mathbb{R}^{\mathbb{N}_6}\right) : \psi|_{\omega^e} \in (\mathbb{P}_2)^{\mathbb{N}_6} \text{ for all } e \in \{1,\dots,N_e\}, \psi(1) = 0\right\}, \\ V_{2,h} : &= \left\{\eta \in C^0\left([0,l];\mathbb{R}^{\mathbb{N}_6}\right) : \eta|_{\omega^e} \in (\mathbb{P}_2)^{\mathbb{N}_6} \text{ for all } e \in \{1,\dots,N_e\}, \eta(0) = 0\right\}. \end{aligned}$$

The approximations for $z_i(x)$ and $p_i(x)$ are represented by the following expressions:

$$z_i(x) = \sum_{j=1}^{N_x} Z_{i,j}\psi_j(x), \quad p_i(x) = \sum_{j=1}^{N_x} P_{i,j}\eta_j(x),$$

where $Z_{i,j}$ denotes the value of z_i at the $\mathbb{P}_2$ basis function ψ_j, which value is 1 at node x_j and 0 at other nodes, and similarly for $P_{i,j}$. In the discretized system, we define the following matrices and vectors:

$$A_1 = \int_0^l \boldsymbol{\psi}\boldsymbol{\psi}^\top,\ A_2 = \int_0^l \boldsymbol{\psi}(\boldsymbol{\psi}')^\top,\ A_3[z] = \int_0^l z\boldsymbol{\psi}\boldsymbol{\psi}^\top,$$

$$\bar{A}_1 = \int_0^l \boldsymbol{\eta}\boldsymbol{\eta}^\top,\ \bar{A}_2 = \int_0^l \boldsymbol{\eta}(\boldsymbol{\eta}')^\top,\ \bar{A}_3[z] = \int_0^l z\boldsymbol{\eta}\boldsymbol{\eta}^\top,$$

where $\boldsymbol{\psi} = (\psi_1, \psi_2, \cdots, \psi_{N_x})^\top$ and $\boldsymbol{\eta} = (\eta_1, \eta_2, \cdots, \eta_{N_x})^\top$. The matrix form of system (3.35) can be written as

$$\begin{aligned} -K_{s,1}Z - M_{\mathbf{CL}(z)}Z + \bar{e}_1\bar{M}P &= \bar{e}_1\hat{F}, \\ K_{s,2}P - M_{G(z)}P - \bar{M}Z &= -\hat{Z}, \end{aligned} \tag{3.36}$$

i.e.,

$$\begin{pmatrix} -K_{s,1} & \bar{e}_1\bar{M} \\ -\bar{M} & K_{s,2} \end{pmatrix} \begin{pmatrix} Z \\ P \end{pmatrix} - \begin{pmatrix} M_{\mathbf{CL}(z)} & \\ & M_{G(z)} \end{pmatrix} \begin{pmatrix} Z \\ P \end{pmatrix} = \begin{pmatrix} \bar{e}_1\hat{F} \\ -\hat{Z} \end{pmatrix}, \tag{3.37}$$

where $\bar{e}_1 = \mathrm{diag}(\mathbb{I}_{N_x}, \mathbb{O}_{N_x}, \mathbb{O}_{N_x}, \mathbb{O}_{N_x}, \mathbb{O}_{N_x}, \mathbb{O}_{N_x})$. $\mathbb{O}_{N_x}$ is the matrix of zeros. Furthermore, the vectors Z and P are defined as:

$$\begin{aligned} Z &= (Z_{1,1}, \cdots, Z_{1,N_x}, \cdots, Z_{6,1}, \cdots, Z_{6,N_x})^\top, \\ P &= (P_{1,1}, \cdots, P_{1,N_x}, \cdots, P_{6,1}, \cdots, P_{6,N_x})^\top. \end{aligned}$$

Similarity,

$$\begin{aligned} \hat{Z} &= \left(\hat{Z}_{1,1}, \cdots, \hat{Z}_{1,N_x}, \cdots, \hat{Z}_{6,1}, \cdots, \hat{Z}_{6,N_x}\right)^\top, \\ \hat{F} &= \left(\hat{F}_1, \cdots, \hat{F}_{N_x}, \mathbb{O}_{N_x}, \mathbb{O}_{N_x}, \mathbb{O}_{N_x}, \mathbb{O}_{N_x}, \mathbb{O}_{N_x}\right)^\top, \end{aligned}$$

where $\hat{Z}_{i,j}$ represents the value of z_i^* at the basis function ψ_j and $\hat{F}_j$ represents the value of f^* at the basis function η_j.

The other matrices are defined by:

$$K_{s,1} = \begin{pmatrix} A_2 & & & & & \\ & A_2 & & & & \\ & & A_2 & & & \\ & & & A_2 & & \\ & & A_1 & & A_2 & \\ & -A_1 & & & & A_2 \end{pmatrix},\ K_{s,2} = \begin{pmatrix} \bar{A}_2 & & & & & \\ & \bar{A}_2 & & & & \\ & & \bar{A}_2 & & & \\ & & & \bar{A}_2 & & \\ & & -\bar{A}_1 & & \bar{A}_2 & \\ & \bar{A}_1 & & & & \bar{A}_2 \end{pmatrix},$$

$$M_{G(z)} = \begin{pmatrix} \mathbf{0} & c_6\bar{A}_3(z_6) & -c_5\bar{A}_3(z_5) & \mathbf{0} & -c_5\bar{A}_3(z_3) & c_6\bar{A}_3(z_2) \\ -c_6\bar{A}_3(z_6) & \mathbf{0} & c_4\bar{A}_3(z_4) & c_4\bar{A}_3(z_3) & \mathbf{0} & -c_6\bar{A}_3(z_1) \\ c_5\bar{A}_3(z_5) & -c_4\bar{A}_3(z_4) & 0 & -c_4\bar{A}_3(z_2) & c_5\bar{A}_3(z_1) & \mathbf{0} \\ \mathbf{0} & (c_3-c_2)\bar{A}_3(z_3) & (c_3-c_2)\bar{A}_3(z_2) & \mathbf{0} & (c_6-c_5)\bar{A}_3(z_6) & (c_6-c_5)\bar{A}_3(z_5) \\ (c_1-c_3)\bar{A}_3(z_3) & \mathbf{0} & (c_1-c_3)\bar{A}_3(z_1) & (c_4-c_6)\bar{A}_3(z_6) & \mathbf{0} & (c_4-c_6)\bar{A}_3(z_4) \\ (c_2-c_1)\bar{A}_3(z_2) & (c_2-c_1)\bar{A}_3(z_1) & \mathbf{0} & (c_5-c_4)\bar{A}_3(z_5) & (c_5-c_4)\bar{A}_3(z_4) & \mathbf{0} \end{pmatrix},$$

$$M_{\mathbf{CL}(z)} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -c_5A_3(z_3) & c_6A_3(z_2) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & c_4A_3(z_3) & \mathbf{0} & -c_6A_3(z_1) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -c_4A_3(z_2) & c_5A_3(z_1) & \mathbf{0} \\ \mathbf{0} & -c_2A_3(z_3) & c_3A_3(z_2) & \mathbf{0} & -c_5A_3(z_6) & c_6A_3(z_5) \\ c_1A_3(z_3) & \mathbf{0} & -c_3A_3(z_1) & c_4A_3(z_6) & \mathbf{0} & c_6A_3(z_4) \\ -c_1A_3(z_2) & c_2A_3(z_1) & \mathbf{0} & -c_4A_3(z_5) & c_5A_3(z_4) & \mathbf{0} \end{pmatrix},$$

$$\bar{M} = \mathrm{diag}(A_1, A_1, A_1, A_1, A_1, A_1).$$

Denote

$$A = \begin{pmatrix} -K_{s,1} & \bar{e}_1\bar{M} \\ -\bar{M} & K_{s,2} \end{pmatrix},\ N_L(z) = \begin{pmatrix} M_{\mathbf{CL}(z)} & \\ & M_{G(z)} \end{pmatrix},\ W = \begin{pmatrix} Z \\ P \end{pmatrix},\ \mathbf{F} = \begin{pmatrix} \bar{e}_1\hat{F} \\ -\hat{Z} \end{pmatrix}.$$

So Eq. (3.37) becomes:

$$AW - N_L(z)W = \mathbf{F}, \tag{3.38}$$

where $N_L(z)$ represents the nonlinear component. The iterative process is governed by the equation:

$$AW^{[n+1]} - N_L(z^{[n]})W^{[n+1]} = \mathbf{F}, \tag{3.39}$$

where the superscript $[n]$ denotes the n-th iteration. With an auxiliary function S_n:

$$S_n(\zeta) = A\zeta - N_L(z^{[n]})\zeta - \mathbf{F}, \tag{3.40}$$

equation (3.38) becomes $S_n(W^{[n+1]}) = 0$. To find an approximate solution to $S_n(\zeta) = 0$, we employ the Newton-Raphson method, i.e., find ζ such that $S_n(\zeta) = 0$, by means of the scheme:

$$\zeta_{n+1} = \zeta_n - \left(\mathrm{Jac}\, S_n\left(\zeta_n\right)\right)^{-1} S_n\left(\zeta_n\right), \tag{3.41}$$

where $\mathrm{Jac}\, S_n = A - N_L(z^{[n]})$.

For our problem, the initial data is set to zero. The following Algorithm 1 outlines the steps taken to approximate the solution to Eq. (3.35).

Algorithm 1: Solve the obtained ODE for W

Set $\mathbf{C}$, f^*, z^* ;
Given initial guesses z^0 ;
while convergence **do**
  $\zeta_{n+1} = \zeta_n - \left(\mathrm{Jac}\, S_n\left(\zeta_n\right)\right)^{-1} S_n\left(\zeta_n\right)$;
end
$W = \zeta_{n+1}$

The results of this iterative scheme are visually presented in Fig. 3.3. These figures demonstrate that the optimal state and control closely approach z^* and f^* respectively when $l = l^*$.

Furthermore, Fig. 3.4 illustrates that the cost is convex with respect to the length of the beam with the unique minimizer. The optimal design corresponds to the length $l = l^* = 1$.

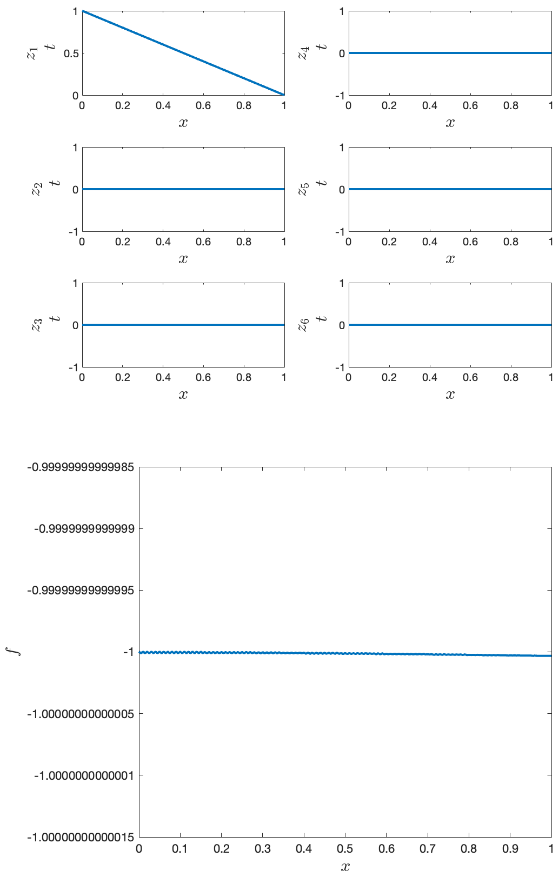


Figure 3.3: Optimal state z(top) and optimal force f(bottom) ($l = l^*$)

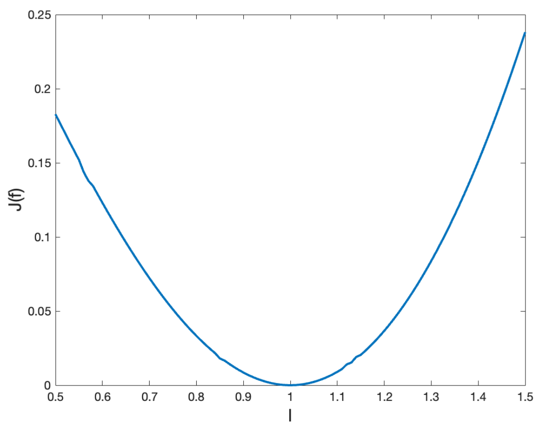


Figure 3.4: Cost with respect to l

Chapter 4

Static Optimization for Timoshenko Beams

We start with the static model for Timoshenko beams. The optimal control problem is solved by using the optimality system. The cost for optimal design of networks is defined as the optimal cost for control problems.

4.1 Introduction

In this chapter, we investigate an optimal control problem with Timoshenko beams as an underlying physical model [66]. Optimal control is - roughly speaking - concerned with minimizing a certain cost functional that includes a partial differential equation as a constraint or as a design-to-state mapping where the state might be part of the arguments of the cost functional [43]. Optimum design problems Timoshenko beams are solved in [38]. Topological derivatives of the energy functional for static problems of Timoshenko beams are derived in [66].

We present new results on topological derivatives for general shape functionals for TB. The results are derived by using the domain decomposition techniques combined with the asymptotic analysis of the Steklov-Pincaré operator. The small parameter of the analysis is the size of the small cycle introduced in the network at some center nodes. In this way, the topology optimization of the network can be performed by nucleation of the small cycles.

Define $\Omega = \overline{\cup_{i\in\mathcal{I}}\Omega_i}$. We are interested in control problems defined on G. The controls are denoted by $v := v(x)$ with $x \in \Omega$, and by $u_T := u_T(x,t)$ with $x \in \Omega$ and $t \in (0,T)$. There are two control problems, the first is the static control problem, and the second is the evolution control problem. The static state equation gives the state $z := z(v;x) = z(x)$ that is determined by the state equation for

$$z \in H \,:\, a(z,\phi) = (L(v),\phi) + (f,\phi)\ \forall\phi \in H\,, \tag{4.1}$$

where $x \mapsto z(v;x)$ lives in the Hilbert space H. In the applications usually H is a Sobolev space. We assume that the bilinear form $a(\cdot,\cdot)$ is symmetric and coercive with a bounded linear operator L from the space of controls to H and $f \in H$.

We need as well the cost functional, the simplest possibility is the quadratic cost with the appropriate choice of norms in Hilbert function spaces. That is, we want to use a simple tracking-type cost functional for the state and velocity of a beam. For the static problem, it is

$$J(v) = \frac{1}{2}\|z - z^d\|^2_{L^2(\Omega)} + \frac{1}{2}\|v - v^d\|^2_{L^2(\Gamma)} \,. \tag{4.2}$$

For the evolution problem, it is

$$J_T(u) = \frac{1}{2}\int_0^T \|y - y^d\|^2_{L^2(\Omega)} + \gamma\,\|\partial_t(y - y^d)\|^2_{L^2(\Omega)} dt + \frac{1}{2}\int_0^T \|u - u^d\|^2_{L^2(\Gamma)} dt \tag{4.3}$$

where $\gamma \in (0, \infty)$. For the sake of simplicity, we consider the control problems without constraints.

The optimality conditions are necessary and sufficient and can be obtained by using the Lagrangian formalism. To this end, we need the adjoint states, which simplify the formulas for the gradients of the cost with respect to controls. We ask two questions now: the first is the shape and topology optimization for the graph with respect to the control problem, this results in the bi-level optimization problem for the graph. The second question is the turnpike property for the two-level optimization problem for the graph.

4.2 Optimality System for the Static Problem

We denote by $(\hat{z}, \hat{v}, \hat{p})$ the unique optimal solution of the static control problem. Let us note that the uniqueness of the optimal control $\hat{u}$ follows by the convexity of the variational problem under consideration with quadratic cost functional and the linear state equation. The optimal solution is given by the optimality system which depends on the *shape, or design* Ω of the network as an infinite dimensional factor or parameter to be selected at the upper level of optimization over the class of admissible shapes. The class of admissible shapes is denoted by S_{ad}, and the continuous variation of the shape is denoted by $\Omega_\tau \in S_{ad}$ with the real parameter $\tau \in (-\delta, \delta)$ for some $\delta > 0$ for shape variations. We use as well the singular perturbations of the shape denoted by $\Omega_\epsilon \in S_{ad}$ for the topology variations with $\epsilon \to 0$.

Remark 2. *In the case of a network, there are at least two possibilities of shape variations. The first is the change of the lengths of edges, it corresponds to the boundary variations in the classical shape optimization. The second, which corresponds to the topology variations in the shape optimization means the presence of a small cycle within the network with the small size of the cycle* $\epsilon \to 0$.

The optimality system for the static control problem is equivalent to the vanishing of the gradient for the cost, hence

$$\min_v \{J(v)\} = J(\hat{v})$$

iff the following optimality system is verified

$$\hat{z} \in H \,:\, a(\hat{z}, \varphi) = (L(\hat{v}), \varphi) + (f, \varphi) \; \forall \varphi \in H \,, \tag{4.4}$$

$$\hat{p} \in H \,:\, a(\hat{p}, \phi) = (z_d - \hat{z}, \phi) \; \forall \phi \in H \,, \tag{4.5}$$

$$(v_d - \hat{v}, v)_U = (L'(\hat{v}) \cdot v, \hat{p}) \; \forall v \in U \,. \tag{4.6}$$

In the linear case we have $L'(\hat{v}) \cdot v = L(v)$. The optimality system is derived using the Lagrangian formalism,

$$\mathcal{L}(v, z, \phi) = \frac{1}{2}\|z - z^d\|^2_{L^2(\Omega)} + \frac{1}{2}\|v - v^d\|^2_{L^2(\Gamma)} + a(z, \phi) - (L(v), \phi) - (f, \phi)$$

Then the adjoint state $p \in H$ is introduced

$$\partial_z \mathcal{L}(v, z, p)(\phi) = (z - z_d, \phi)_\Omega + a(p, \phi) = 0, \;\; \forall \phi \in H$$

and the gradient of the cost is obtained

$$dI(v; \eta) = \partial_v \mathcal{L}(v, z, p)(\eta) = (v - v^d, \eta)_\Gamma - (p, \eta)_\Gamma$$

which leads to the optimality condition

$$v - v^d = p \text{ a.e. on } \Gamma.$$

In the case of distributed control, we have $\Gamma = \Omega$.

Proposition *There exists the unique solution $(\hat{v}, \hat{z}, \hat{p})$ to the optimality system (4.4)-(4.6).* The optimal value of the cost $J(\hat{v}) := J(\hat{v}(\Omega))$ is defined as a shape functional over the set S_{ad}. Therefore, we consider the optimum design of the network

$$\inf_{\Omega \in S_{ad}} J(\hat{v}(\Omega)) \,. \tag{4.7}$$

The analysis of such a variational problem requires:

1. The existence of solutions;
2. The necessary optimality conditions;
3. Finally, numerical methods for solution.

In particular, we perform the shape calculus and determine the shape gradient of the optimal control cost

$$\Omega \longmapsto J(\hat{v}(\Omega))$$

as well as the topological derivative obtained at $\epsilon := 0^+$ for the mapping

$$\epsilon \longmapsto J(\hat{v}(\Omega_\epsilon)).$$

Remark 3. *It is useful for applications to introduce a random right-hand side $f := f(\omega; x)$ to the state equation (4.1).*

4.3 Network Model

In this section, we extend the beam model previously defined on a single edge to a network structure. The model for 3D Timoshenko beams on the network takes the following form:

$$
\begin{aligned}
&\text{Eq.}(2.10), && (0, L_i) \times (0, T), i = 1 : n_e && \text{(4.8a)}\\
&u_d\left(v_D\right) = 0, && v_D \in V_D, d \in \mathcal{E}_D, t \in (0, T) && \text{(4.8b)}\\
&u_i\left(v_J\right) = u_j\left(v_J\right), && \forall i, j \in \mathcal{E}_J, v_J \in V_M, t \in (0, T) && \text{(4.8c)}\\
&\sum_{i \in \mathcal{E}_J} \epsilon_{iJ} F_i\left(v_J\right) = 0, && v_J \in V_M \cup V_N, t \in (0, T) && \text{(4.8d)}\\
&\sum_{i \in \mathcal{E}_J} \epsilon_{iJ} M_i\left(v_J\right) = 0, && v_J \in V_M \cup V_N, t \in (0, T) && \text{(4.8e)}\\
&u_i(x, 0) = u_i^0, && x \in \left(0, l_i\right), i = 1 : n_e && \text{(4.8f)}\\
&\partial_t u_i(x, 0), = u_i^1 && x \in \left(0, l_i\right), i = 1 : n_e && \text{(4.8g)}
\end{aligned}
$$

The first equation describes the evolution of the system, the third, fourth, and fifth ones the continuity and balance of Kirchhoff conditions at inner nodes of the network, the remaining equations constitute the initial and boundary conditions. This system is well-posed in the space setting

$$
\begin{aligned}
&\partial_t u, u^1 \in \prod_{i=1}^{n_e} L^2\left(0, L_i\right);\\
&u, u^0 \in \left\{ u \in \prod_{i=1}^{n_e} L^2\left(0, L_i\right) \mid u \text{ satisfies (4.8b), (4.8c)} \right\}.
\end{aligned}
\tag{4.9}
$$

In conclusion, the model outlined here provides a comprehensive framework for understanding the behavior of 3D Timoshenko beams on networks. By incorporating dynamical equations, boundary conditions, and continuity constraints, we create a robust foundation for further analysis and optimization of network structures.

4.4 Domain Decomposition Technique for Topological Derivatives

The domain decomposition technique was used for derivation of topological derivatives for contact problems in linear elasticity [87].

The domain decomposition technique is applied in the mathematical modeling of distributed parameter systems that are defined on graphs. In this approach, a subdomain is employed to isolate the topology change from the primary section of the original domain (refer to Fig. 4.1). A Dirichlet to Neumann operator operates on the shared boundary

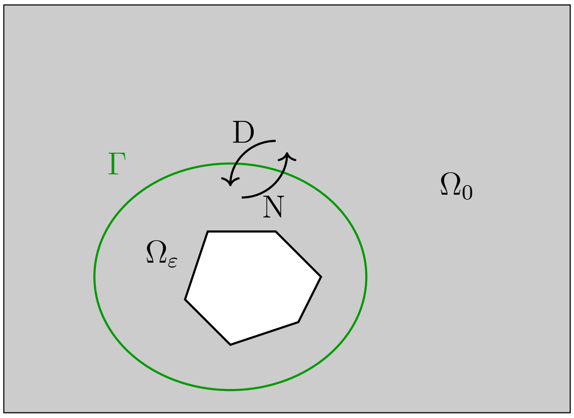


Figure 4.1: Domain Decomposition, concept[87].

of the respective domains [36]. Additionally, the domain decomposition method has been incorporated for optimizing the topology of networks, as illustrated in [46].

The graph G is decomposed into two subgraphs G_0 and G_ε (See Fig. 4.2), where the topology variations of the network are governed by the perturbations of G_ε for $\varepsilon \to 0$, in another word, a small cycle is introduced at some interior node P_0 of the network (Fig. 4.3). We assume that the vertex Q_i is a boundary vertex and the vertex P_i is the interior vertex for the subgraph G_ε which contains a small cycle.

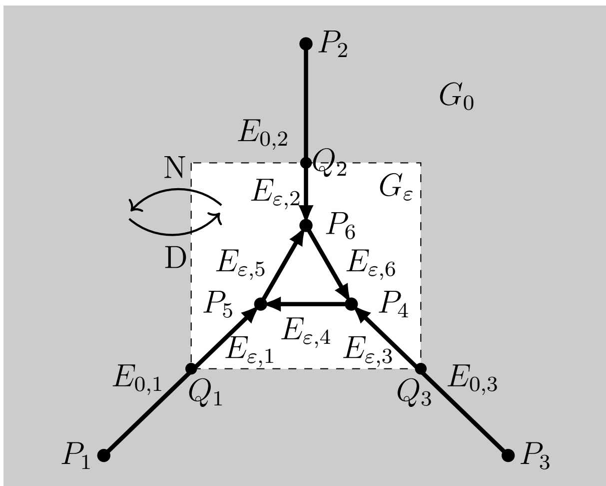


Figure 4.2: Tripod directed network with a cycle for the Timoshenko beam.

Domain decomposition technique allows us to replace the singular domain perturbation of Ω by a regular perturbation of the bilinear form for the elliptic problem considered on Ω_0. The perturbation of bilinear form is given by the *Dirichlet-to-Neumann operator* which is also called the *Steklov-Poincaré operator*. In the case considered the nonlocal operator is represented by a matrix which can be determined explicitly. In this way, the evaluation of topological derivatives for the shape functional becomes simple and does not

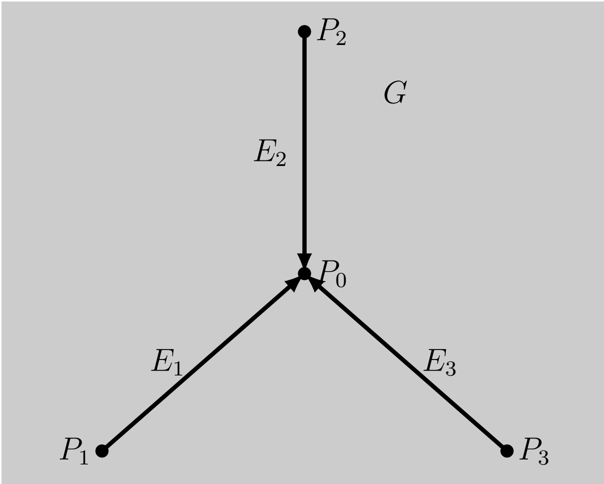


Figure 4.3: The three-star graph.

require the compound asymptotics method. The domain decomposition technique and the Steklov-Poincaré nonlocal boundary operators are used in the topological sensitivity analysis of nonlinear variational problems. Roughly speaking, Steklov-Poincaré operator for G_ε means the Dirichlet-to-Neumann operator for $G\backslash G_\varepsilon$, is a definition by Steklov-Poincaré operator for G_ε. It gives the energy of exterior domain.

4.5 Steklov-Poincaré Operator for a Subgraph G_ε with a cycle

We are going to use the domain decomposition method for a graph $G = G_0 \cup G_\varepsilon$. For example, we have $G_0 = \{E_0, V_0\}$ and $G_\varepsilon = \{E_\varepsilon, V_\varepsilon\}$ in Fig. 4.2. Here, $E_0 = \{P_1Q_1, P_2Q_2, P_3Q_3\}$ and $E_\varepsilon = \{Q_1P_5, Q_2P_6, Q_3P_4, P_4P_5, P_5P_6, P_6P_4\}$. The edges in E_0 are defined by lengths $L_{0,i} := |E_{0,i}| = L_i - \varepsilon_{\max}\,(i = 1, 2, 3)$, where $L_i (i = 1, 2, 3)$ is the length of E_i in Fig. 4.3, while the edges in E_ε are characterized by lengths $|E_{\varepsilon,i}| = \varepsilon_{\max} - \varepsilon\ (i = 1, 2, 3)$, and $|E_{\varepsilon,i}| = \varepsilon\,(i = 4, 5, 6)$. It's crucial to clarify that $\varepsilon_{\max}$ represents the maximum value of ε, serving as a constant offset in E_0. Meanwhile, ε denotes the perturbation magnitude in E_ε, distinguishing the two states. In this way the dependence of the solutions to the state equation with respect to the small parameter $\varepsilon \to 0$ is explicitly given in the weak form of the state equation.

First, we give the weak form for the full model in one beam. We introduce the test functions ψ_i, which are given by $\psi_i = (\phi_{x,i}, \phi_{y,i}, \phi_{z,i}, \eta_{x,i}, \eta_{y,i}, \eta_{z,i})^\top$, where i is the beam index. The bilinear form for a single beam can be obtained by integrating the strong system (2.12) by parts. All functions in the weak form are considered to be in the space

$H^1(0, L_i)$. Thus, the resulting equation for a single beam is given by:

$$a(u_i, \psi_i) = \int_0^{L_i} k^x r'_{x,i}\phi'_{x,i} + \int_0^{L_i} c^x \varphi'_{x,i}\eta'_{x,i} + \int_0^{L_i} k^s \left(r'_{y,i} - \varphi_{z,i}\right)\left(\phi'_{y,i} - \eta_{z,i}\right)$$
$$+ \int_0^{L_i} k^s \left(r'_{z,i} + \varphi_{y,i}\right)\left(\phi'_{z,i} + \eta_{y,i}\right) + \int_0^{L_i} c^z \varphi'_{z,i}\eta'_{z,i} + \int_0^{L_i} c^y \varphi'_{y,i}\eta'_{y,i}.$$

For the distributed loading on the right-hand side, it follows the linear form

$$(q_i, \psi_i) = \int_0^{L_i} f_i^x \phi_{x,i} + \int_0^{L_i} m_i^x \eta_{x,i} + \int_0^{L_i} f_i^y \phi_{y,i} + \int_0^{L_i} m_i^y \eta_{y,i} + \int_0^{L_i} f_i^z \phi_{z,i} + \int_0^{L_i} m_i^z \eta_{z,i}.$$

Hence, the weak form of a beam is given by the equation: find $u_i \in H^1(0, L_i)$, such that

$$a(u_i, \psi_i) = (q_i, \psi_i), \quad \forall \psi_i \in H^1(0, L_i). \tag{4.10}$$

Thus, on the graph $G = \{E, V\}$ the total bilinear form is defined,

$$a(u, \psi) := a(\Omega; u, \psi) = \sum_{i=1}^{i=6} a_i(E_i; u_i, \psi_i).$$

We are able to determine explicitly the solution u_ε of the Dirichlet problem on Ω_ε with the small cycle for $b \in \mathbb{R}^3$, where $u_{\varepsilon,i}(Q_i) = u_b = b_i$ for $i = 1, 2, 3$. The Steklov-Poincaré operator Λ_ε on G_ε is defined by the equality

$$a_\varepsilon(u_\varepsilon, u_\varepsilon) := a(\Omega_\varepsilon; u_\varepsilon, u_\varepsilon) = -b^\top \Lambda_\varepsilon \cdot b.$$

where $a_\varepsilon = (\Omega_\varepsilon; \cdot, \cdot)$ is the symmetry bilinear form on subgraph G_ε.

Proposition 1. *The matrix Λ_ε is only negative semidefinite because the energy vanishes for constant solutions of the Dirichlet problem.*

Let us return to Green's formula for the bilinear form associated with the system (4.10). We have

$$a_i(u_i, \psi_i) = \int_0^{L_i} A_i u_i \cdot \psi_i + \left(N_i u_i \cdot \psi_i\right)\Big|_{x=0}^{x=L_i}, \tag{4.11}$$

where $u_i \mapsto N_i u_i$ is the Neumann operator N_i on E_i.

Proposition 2. *The Neumann operator N_i is given by the following vector*

Proof. We have the expressions

$$\int_0^{L_i} k_i^x r'_{x,i}\phi'_{x,i} = -\int_0^{L_i} k_i^x r''_{x,i}(x)\phi_{x,i} + \left(k_i^x r'_{x,i}\phi_{x,i}\right)\Big|_{x=0}^{x=L_i}$$

$$\int_0^{L_i} k_i^s \left(r'_{y,i} - \varphi_{z,i}\right) \phi'_{y,i} = -\int_0^{L_i} k_i^s \left(r''_{y,i} - \varphi'_{z,i}\right) \phi_{y,i} + \left(k_i^s \left(r'_{y,i} - \varphi_{z,i}\right) \phi_{y,i}\right)\Big|_{x=0}^{x=L_i}$$
$$\int_0^{L_i} k_i^s \left(r'_{z,i} + \varphi_{y,i}\right) \phi'_{z,i} = -\int_0^{L_i} k_i^s \left(r''_{z,i} + \varphi'_{y,i}\right) \phi_{z,i} + \left(k_i^s \left(r'_{z,i} + \varphi_{y,i}\right) \phi_{z,i}\right)\Big|_{x=0}^{x=L_i}$$
$$\int_0^{L_i} c_i^x \varphi'_{x,i} \eta'_{x,i} = -\int_0^{L_i} c_i^x \varphi''_{x,i}(x) \eta_{x,i}(x) + \left(c_i^x \varphi'_{x,i} \eta_{x,i}\right)\Big|_{x=0}^{x=L_i}$$
$$\int_0^{L_i} c_i^z \varphi'_{z,i} \eta'_{z,i} = -\int_0^{L_i} c_i^z \varphi''_{z,i}(x) \eta_{z,i} + \left(c_i^z \varphi'_{z,i} \eta_{z,i}\right)\Big|_{x=0}^{x=L_i}$$
$$\int_0^{L_i} c_i^y \varphi'_{y,i} \eta'_{y,i} = -\int_0^{L_i} c_i^y \varphi''_{y,i}(x) \eta_{y,i} + \left(c_i^y \varphi'_{y,i} \eta_{y,i}\right)\Big|_{x=0}^{x=L_i}$$

Thus we find six components of $N_i u_i$ given by

$$\begin{aligned}(N_i u_i \cdot \psi_i)\big|_{x=0}^{x=L_i} = {} & \left(k_i^x r'_{x,i} \phi_{x,i}\right)\Big|_{x=0}^{x=L_i} + \left(k_i^s \left(r'_{y,i} - \varphi_{z,i}\right) \phi_{y,i}\right)_{x=0}^{x=L_i} + \left(k_i^s \left(r'_{z,i} + \varphi_{y,i}\right) \phi_{z,i}\right)\Big|_{x=0}^{x=L_i} \\ & + \left(c_i^x \varphi'_{x,i} \eta_{x,i}\right)\Big|_{x=0}^{x=L_i} + \left(c_i^y \varphi'_{y,i} \eta_{y,i}\right)\Big|_{x=0}^{x=L_i} + \left(c_i^z \varphi'_{z,i} \eta_{z,i}\right)\Big|_{x=0}^{x=L_i}\end{aligned}$$

This leads to the result on the Steklov-Poincaré operator, we should use the exact solution from Theorem 2.2 in the Ogiermann Dissertation, as well as replace the traces of test functions at $x = 0$ by the Dirichlet conditions. $\square$

Proposition 3. *If we know the exact solution u_b for the Dirichlet problem on the graph G_ε with the polynomials on the edges* $\mathrm{col}(r_x, r_y, r_z, \varphi_x, \varphi_y, \varphi_z)$, *it follows that the associated energy for such a solution takes the form $a(u_b, u_b) = -b^\top \Lambda_\varepsilon \cdot b$, thus the energy functional for the graph G reads*

$$\psi \mapsto a(\Omega; \psi, \psi) = a(\Omega_0; \psi, \psi) - \psi(L - \varepsilon_{\max})^\top \Lambda_\varepsilon \cdot \psi(L - \varepsilon_{\max}), \tag{4.12}$$

where $0 < \varepsilon \leqslant \varepsilon_{\max}$. The restriction of the state equation to Ω_0 is considered under the assumption that the control is supported in Ω_0 and the state is observed in Ω_0. Thus, the state equation becomes: Find $u \in H(\Omega_0)$ such that

$$a(\Omega_0; u, \psi) - u(L - \varepsilon_{\max})^\top \Lambda(\varepsilon) \cdot \psi(L - \varepsilon_{\max}) = (q, \psi)_{\Omega_0}, \quad \forall \psi \in H(\Omega_0). \tag{4.13}$$

In this way, the dependence of the state equation on the small cycle in G is replaced by the dependence of Λ_ε on the cycle.

For the new state equation, the optimality system is defined in the same way as before, on the domain Ω_0. The topological derivative of the optimal cost is defined as the limit of derivatives for $\varepsilon \to 0^+$.

4.6 Explicit Solution to the Static Timoshenko Beams Model

4.6.1 Solution to the 1D Timoshenko Beams Model

Then we derive the solution to the static model. Assume $f_i(x) = m_i(x) = 0$. We will show the explicit solution for both reduced and full models. First, it's the solution for the reduced model.

Proposition 4. *Assume that $f_i^z(x) = 0$ and $m_i^y(x) = 0$. Let us consider the system*

$$
\begin{aligned}
f_i^z(x) + k_i^s\left(r''_{z,i}(x) + \varphi'_{y,i}(x)\right) &:= f_i^z(x) + k_i^s\left(u''_1(x) + u'_2(x)\right) = 0,\\
m_i^y(x) + c_i^y\varphi''_{y,i}(x) - k_i^s\left(r'_{z,i}(x) + \varphi_{y,i}(x)\right) &:= m_i^y(x) + c_i^y u''_2(x) - k_i^s\left(u'_1(x) + u_2(x)\right) = 0,
\end{aligned}
$$

of two linear differential equations for $u(x) = (u_1(x), u_2(x))^\top$ with the Dirichlet boundary conditions $u(0) = (u_1(0), u_2(0))^\top$ and $u(L_i) = (u_1(L_i), u_2(L_i))^\top$. The solution is given by

$$
\begin{aligned}
u_1(x) &= \frac{\lambda_1}{k_i^s}x - \frac{1}{6c_i^y}\lambda_1 x^3 - \frac{1}{2c_i^y}\lambda_2 x^2 - \frac{1}{c_i^y}\lambda_3 x + \lambda_4\,,\\
u_2(x) &= \frac{1}{2c_i^y}\lambda_1 x^2 + \frac{1}{c_i^y}\lambda_2 x + \frac{1}{c_i^y}\lambda_3\,,
\end{aligned}
\tag{4.14}
$$

where $\lambda_i, i = 1, 2, 3, 4,$ are determined in terms of the nonhomogeneous Dirichlet conditions

$$
\begin{aligned}
\lambda_1 &= \frac{12c_i^y}{L_i(12c_i^y k_{s,i}^{-1} + L_i^2)}(u_1(L_i) - u_1(0)) + \frac{6c_i^y}{12c_i^y k_{s,i}^{-1} + L_i^2}(u_2(L_i) + u_2(0))\,,\\
\lambda_2 &= \frac{12c_{i,y}^2}{L_i k(12c_i^y k_{s,i}^{-1} + L_i^2)}(u_2(L_i) - u_2(0))\\
&\quad + \frac{2c_i^y}{12c_i^y k_{s,i}^{-1} + L_i^2}(3u_1(0) - 3u_1(L_i) - 2L_i u_2(0) - L_i u_2(L_i))\,,\\
\lambda_3 &= c_i^y u_2(0)\,,\\
\lambda_4 &= u_1(0)\,.
\end{aligned}
\tag{4.15}
$$

Proof. Set $u(x) = (r_z, \varphi_y)^\top$. The Dirichlet problem for $u(0) = (u_1(0), u_2(0))^\top = (r_z(0), \varphi_y(0))^\top$, $u(L_i) = (u_1(L_i), u_2(L_i))^\top = (r_z(L_i), \varphi_y(L_i))^\top$ are given. The equation becomes

$$
\begin{aligned}
k_i^s\left(r''_{z,i}(x) + \varphi'_{y,i}(x)\right) &= 0,\\
c_i^y\varphi''_{y,i}(x) - k_i^s\left(r'_{z,i}(x) + \varphi_{y,i}(x)\right) &= 0.
\end{aligned}
$$

Hence $k_i^s\left(r'_{z,i}(x) + \varphi_{y,i}(x)\right) = \lambda_1$ thus $c_i^y\varphi''_{y,i}(x) = \lambda_1$, therefore $c_i^y\varphi'_{y,i}(x) = \lambda_1 x + \lambda_2$ and

$c_i^y \varphi_{y,i}(x) = \dfrac{1}{2}\lambda_1 x^2 + \lambda_2 x + \lambda_3$. It leads to

$$\begin{aligned}
\varphi_{y,i}(x) &= \frac{1}{2c_i^y}\lambda_1 x^2 + \frac{1}{c_i^y}\lambda_2 x + \frac{1}{c_i^y}\lambda_3\,,\\
r'_{z,i}(x) &= \frac{\lambda_1}{k_i^s} - \varphi_{y,i}(x) = \frac{\lambda_1}{k_i^s} - \frac{1}{2c_i^y}\lambda_1 x^2 - \frac{1}{c_i^y}\lambda_2 x - \frac{1}{c_i^y}\lambda_3\,,\\
r_{z,i}(x) &= \frac{\lambda_1}{k_i^s}x - \frac{1}{6c_i^y}\lambda_1 x^3 - \frac{1}{2c_i^y}\lambda_2 x^2 - \frac{1}{c_i^y}\lambda_3 x + \lambda_4\,.
\end{aligned}$$

Henceforth, the constants λ_i are determined from the Dirichlet boundary conditions

$$\begin{aligned}
r_{z,i}(0) &= \lambda_4 = u_1(0)\,,\\
\varphi_{y,i}(0) &= \frac{1}{c_i^y}\lambda_3 = u_2(0)\,,\\
r_{z,i}(L_i) &= \frac{\lambda_1}{k_i^s}L_i - \frac{1}{6c_i^y}\lambda_1 L_i^3 - \frac{1}{2c_i^y}\lambda_2 L_i^2 - \frac{1}{c_i^y}\lambda_3 L_i + \lambda_4 = u_1(L_i)\,,\\
\varphi_{y,i}(L_i) &= \frac{1}{2c_i^y}\lambda_1 L_i^2 + \frac{1}{c_i^y}\lambda_2 L_i + \frac{1}{c_i^y}\lambda_3 = u_2(L_i)\,.
\end{aligned}$$

Thus we determine the unique solution Eq. (4.15) of the model in function of Dirichlet conditions. □

Remark 4. *The passage to the limit $L_i \to 0$ in the solution given by Proposition 4 can be performed if the singular terms vanish, i.e., $u_1(L_i) - u_1(0) = 0$ and $u_2(L_i) - u_2(0) = 0$, otherwise boundary layers appear.*

Now, we give the matrix notation for exact solutions of Dirichlet boundary value problems. We define the matrix functions $Q_i, R_i : [0, L_i] \to \mathbb{R}^{2\times 2}$ by

$$Q_i(x) := \begin{pmatrix} a_{11}^{i,0}(x) & a_{12}^{i,0}(x) \\ a_{21}^{i,0}(x) & a_{22}^{i,0}(x) \end{pmatrix}$$

with the coefficient functions

$$\begin{aligned}
a_{11}^{i,0}(x) &:= \mu_i^y\,(L_i - x)\left(L_i^2 + L_i x - 2x^2 + 12k_{s,i}^{-1}c_i^y\right),\\
a_{22}^{i,0}(x) &:= \mu_i^y\,(L_i - x)\left(L_i^2 - 3L_i x + 12k_{s,i}^{-1}c_i^y\right),\\
a_{12}^{i,0}(x) &:= -\mu_i^y x\,(L_i - x)\left(L_i^2 - L_i x + 6k_{s,i}^{-1}c_i^y\right),\\
a_{21}^{i,0}(x) &:= 6\mu_i^y x\,(L_i - x)\,,
\end{aligned}$$

where

$$\mu_i^y := \left(L_i^3 + 12k_{s,i}^{-1}c_i^y L_i\right)^{-1} = \frac{k_i^s}{L_i(k_i^s L_i^2 + 12c_i^y)},$$

and the singular term at $L_i = 0$. Furthermore

$$R_i(x) := \begin{pmatrix} a_{11}^{i,L}(x) & a_{12}^{i,L}(x) \\ a_{21}^{i,L}(x) & a_{22}^{i,L}(x) \end{pmatrix} \tag{4.16}$$

with coefficient functions

$$\begin{aligned}
a_{11}^{i,L}(x) &:= \mu_i^y x \left(3L_i x - 2x^2 + 12k_{s,i}^{-1} c_i^y\right), \\
a_{22}^{i,L}(x) &:= \mu_i^y x \left(-2L_i^2 + 3L_i x + 12k_{s,i}^{-1} c_i^y\right), \\
a_{12}^{i,L}(x) &:= \mu_i^y x \left(L_i - x\right) \left(L_i x + 6k_{s,i}^{-1} c_i^y\right), \\
a_{21}^{i,L}(x) &:= -6\mu_i^y x \left(L_i - x\right).
\end{aligned}$$

Then, the exact solution of the Dirichlet problem for a single beam takes the form

$$u_i(x) = Q_i(x) u_i(0) + R_i(x) u_i(L_i). \tag{4.17}$$

4.6.2 Solution to the 3D Timoshenko Beams Model

For the 3D full model, we refer to the explicit solution presented in [66]. These findings will be unified into a single six-dimensional vector, which will inherently depend on the boundary conditions at the global coordinate level. Here, we define the matrix functions $Q_i, R_i : [0, L_i] \to \mathbb{R}^{6\times 6}$ by

$$Q_i(x) := \begin{pmatrix} \mathbb{E}_i & \mathbb{O}_3 \\ \mathbb{O}_3 & \mathbb{E}_i \end{pmatrix} \begin{pmatrix} a_{11}^{i,0}(x) & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22}^{i,0}(x) & 0 & 0 & 0 & a_{26}^{i,0}(x) \\ 0 & 0 & a_{33}^{i,0}(x) & 0 & a_{35}^{i,0}(x) & 0 \\ 0 & 0 & 0 & a_{44}^{i,0}(x) & 0 & 0 \\ 0 & 0 & a_{53}^{i,0}(x) & 0 & a_{55}^{i,0}(x) & 0 \\ 0 & a_{62}^{i,0}(x) & 0 & 0 & 0 & a_{66}^{i,0}(x) \end{pmatrix} \begin{pmatrix} \mathbb{E}_i^\top & \mathbb{O}_3 \\ \mathbb{O}_3 & \mathbb{E}_i^\top \end{pmatrix} \tag{4.18}$$

with the coefficient functions

$$\begin{aligned}
a_{11}^{i,0}(x) &:= L_i^{-1} \left(L_i - x\right), \\
a_{22}^{i,0}(x) &:= \mu_i^z \left(L_i - x\right) \left(L_i^2 + L_i x - 2x^2 + 12k_{s,i}^{-1} c_i^z\right), \\
a_{33}^{i,0}(x) &:= \mu_i^y \left(L_i - x\right) \left(L_i^2 + L_i x - 2x^2 + 12k_{s,i}^{-1} c_i^y\right), \\
a_{44}^{i,0}(x) &:= L_i^{-1} \left(L_i - x\right), \\
a_{55}^{i,0}(x) &:= \mu_i^y \left(L_i - x\right) \left(L_i^2 - 3L_i x + 12k_{s,i}^{-1} c_i^y\right), \\
a_{66}^{i,0}(x) &:= \mu_i^z \left(L_i - x\right) \left(L_i^2 - 3L_i x + 12k_{s,i}^{-1} c_i^z\right), \\
a_{26}^{i,0}(x) &:= \mu_i^z x \left(L_i - x\right) \left(L_i^2 - L_i x + 6k_{s,i}^{-1} c_i^z\right), \\
a_{35}^{i,0}(x) &:= -\mu_i^y x \left(L_i - x\right) \left(L_i^2 - L_i x + 6k_{s,i}^{-1} c_i^y\right),
\end{aligned}$$

$$
\begin{aligned}
a_{53}^{i,0}(x) &:= 6\mu_i^y x\left(L_i - x\right), \\
a_{62}^{i,0}(x) &:= -6\mu_i^z x\left(L_i - x\right),
\end{aligned}
$$

where

$$
\mu_i^z := \left(L_i^3 + 12k_{s,i}^{-1}c_i^z L_i\right)^{-1} = \frac{k_i^s}{L_i(k_i^s L_i^2 + 12c_i^z)},
$$

as well as

$$
R_i(x) := \begin{pmatrix} \mathbb{E}_i & \mathbb{O}_3 \\ \mathbb{O}_3 & \mathbb{E}_i \end{pmatrix}
\begin{pmatrix}
a_{11}^{i,L}(x) & 0 & 0 & 0 & 0 & 0 \\
0 & a_{22}^{i,L}(x) & 0 & 0 & 0 & a_{26}^{i,L}(x) \\
0 & 0 & a_{33}^{i,L}(x) & 0 & a_{35}^{i,L}(x) & 0 \\
0 & 0 & 0 & a_{44}^{i,L}(x) & 0 & 0 \\
0 & 0 & a_{53}^{i,L}(x) & 0 & a_{55}^{i,L}(x) & 0 \\
0 & a_{62}^{i,L}(x) & 0 & 0 & 0 & a_{66}^{i,L}(x)
\end{pmatrix}
\begin{pmatrix} \mathbb{E}_i^\top & \mathbb{O}_3 \\ \mathbb{O}_3 & \mathbb{E}_i^\top \end{pmatrix}
$$

with the coefficient functions

$$
\begin{aligned}
a_{11}^{i,L}(x) &:= L_i^{-1}x, \\
a_{22}^{i,L}(x) &:= \mu_i^z x\left(3L_i x - 2x^2 + 12k_{s,i}^{-1}c_i^z\right), \\
a_{33}^{i,L}(x) &:= \mu_i^y x\left(3L_i x - 2x^2 + 12k_{s,i}^{-1}c_i^y\right), \\
a_{44}^{i,L}(x) &:= L_i^{-1}x, \\
a_{55}^{i,L}(x) &:= \mu_i^y x\left(-2L_i^2 + 3L_i x + 12k_{s,i}^{-1}c_i^y\right), \\
a_{66}^{i,L}(x) &:= \mu_i^z x\left(-2L_i^2 + 3L_i x + 12k_{s,i}^{-1}c_i^z\right), \\
a_{26}^{i,L}(x) &:= \mu_i^z x\left(L_i - x\right)\left(L_i x + 6k_{s,i}^{-1}c_i^z\right), \\
a_{35}^{i,L}(x) &:= -\mu_i^y x\left(L_i - x\right)\left(L_i x + 6k_{s,i}^{-1}c_i^y\right), \\
a_{53}^{i,L}(x) &:= -6\mu_i^y x\left(L_i - x\right), \\
a_{62}^{i,L}(x) &:= 6\mu_i^z x\left(L_i - x\right).
\end{aligned}
$$

Similarly to the reduced model, the exact solution for the homogeneous full model could also be written as

$$
u_i(x) = Q_i(x)u_i(0) + R_i(x)u_i\left(L_i\right). \tag{4.19}
$$

4.7 Dirichlet-to-Neumann Operator Subgraphs for Networks

We start with the single beam. For the reduced model, we introduce the notation for matrix functions depending on the length L_i of the beam

$$\begin{aligned}
D_i^{\mathrm{I}} &:= \begin{pmatrix} -12c_i^y\mu_i^y & 6c_i^y\mu_i^y L_i \\ 6c_i^y\mu_i^y L_i & -\left(L_i^{-1}+3\mu_i^y L_i^2\right)c_i^y \end{pmatrix},\\
D_i^{\mathrm{II}} &:= \begin{pmatrix} 12c_i^y\mu_i^y & 6c_i^y\mu_i^y L_i \\ -6c_i^y\mu_i^y L_i & \left(L_i^{-1}-3\mu_i^y L_i^2\right)c_i^y \end{pmatrix},\\
D_i^{\mathrm{III}} &:= \begin{pmatrix} -12c_i^y\mu_i^y & 6c_i^y\mu_i^y L_i \\ -6c_i^y\mu_i^y L_i & \left(3\mu_i^y L_i^2-L_i^{-1}\right)c_i^y \end{pmatrix},\\
D_i^{\mathrm{IV}} &:= \begin{pmatrix} 12c_i^y\mu_i^y & 6c_i^y\mu_i^y L_i \\ 6c_i^y\mu_i^y L_i & \left(L_i^{-1}+3\mu_i^y L_i^2\right)c_i^y \end{pmatrix}.
\end{aligned} \tag{4.20}$$

Then, for the full model, we define the matrices

$$\begin{aligned}
D_i^F &:= \mathbb{E}_i \begin{pmatrix} L_i^{-1}k_i^x & 0 & 0 \\ 0 & 12c_i^z\mu_i^z & 0 \\ 0 & 0 & 12c_i^y\mu_i^y \end{pmatrix} \mathbb{E}_i^\top,\\
D_i^+ &:= \mathbb{E}_i \begin{pmatrix} L_i^{-1}c_i^x & 0 & 0 \\ 0 & c_i^y(L_i^{-1}+3\mu_i^y L_i^2) & 0 \\ 0 & 0 & c_i^z(L_i^{-1}+3\mu_i^z L_i^2) \end{pmatrix} \mathbb{E}_i^\top,\\
D_i^- &:= \mathbb{E}_i \begin{pmatrix} L_i^{-1}c_i^x & 0 & 0 \\ 0 & c_i^y(L_i^{-1}+3\mu_i^y L_i^2) & 0 \\ 0 & 0 & c_i^z(L_i^{-1}-3\mu_i^z L_i^2) \end{pmatrix} \mathbb{E}_i^\top,\\
S_i &:= \mathbb{E}_i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 6c_i^y\mu_i^y L_i \\ 0 & -6c_i^z\mu_i^z L_i & 0 \end{pmatrix} \mathbb{E}_i^\top,
\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}
D_i^{\mathrm{I}} &:= -\begin{pmatrix} D_i^F & -S_i^\top \\ -S_i & D_i^+ \end{pmatrix}, D_i^{\mathrm{II}} := \begin{pmatrix} D_i^F & S_i^\top \\ -S_i & D_i^- \end{pmatrix},\\
D_i^{\mathrm{III}} &:= -\begin{pmatrix} D_i^F & -S_i^\top \\ S_i & D_i^- \end{pmatrix}, D_i^{\mathrm{IV}} := \begin{pmatrix} D_i^F & S_i^\top \\ S_i & D_i^+ \end{pmatrix}.
\end{aligned} \tag{4.22}$$

We use the same notation for both models. With this notation, we could obtain the following representation of the *Dirichlet-to-Neumann (DtN)* mapping for the single beam, i.e., the Dirichlet boundary conditions $u_i(0), u_i(L_i)$ to the vectors $F_i(0)$, $M_i(0), F_i(L_i)$ and $M_i(L_i)$.

Theorem 9. *The Dirichlet-to-Neumann operator of a single beam with zero distributed* *loads* $f_i, m_i : [0, L_i] \to \mathbb{R}^d (d = 1, 3)$ *is given by*

$$\begin{aligned} \begin{pmatrix} F_i(0) \\ M_i(0) \end{pmatrix} &= D_i^{\mathrm{I}} u_i(0) + D_i^{\mathrm{II}} u_i(L_i), \\ \begin{pmatrix} F_i(L_i) \\ M_i(L_i) \end{pmatrix} &= D_i^{\mathrm{III}} u_i(0) + D_i^{\mathrm{IV}} u_i(L_i), \end{aligned} \tag{4.23}$$

where $D_i^{\mathrm{I}}, D_i^{\mathrm{II}}, D_i^{\mathrm{III}}, D_i^{\mathrm{IV}}$ *are* (4.20) *and* (4.22) *for 1D and 3D, respectively. And* $f_i(x) = f_i^z(x), m_i(x) = m_i^z(x)$, $F_i(x) = F_i^z(x), M_i(x) = M_i^y(x)$ *for reduced model.*

Then the elastic energy for the beam is evaluated

$$\begin{aligned} \mathcal{E}(u_i(0), u_i(L_i)) &= -\, u_i(0)^\top \begin{pmatrix} F_i(0) \\ M_i(0) \end{pmatrix} - u_i(L_i)^\top \begin{pmatrix} F_i(L_i) \\ M_i(L_i) \end{pmatrix} \\ &= -\, u_i(0)^\top D_i^{\mathrm{I}} u_i(0) - u_i(0)^\top D_i^{\mathrm{II}} u_i(L_i) \\ &\quad - u_i(L_i)^\top D_i^{\mathrm{III}} u_i(0) - u_i(L_i)^\top D_i^{\mathrm{IV}} u_i(L_i). \end{aligned} \tag{4.24}$$

There is the negative semidefinite matrix Λ such that

$$\mathcal{E}(u_i(0), u_i(L_i)) = \begin{pmatrix} u_i(0) \\ u_i(L_i) \end{pmatrix}^\top \Lambda \begin{pmatrix} u_i(0) \\ u_i(L_i) \end{pmatrix}. \tag{4.25}$$

The Steklov-Poincaré operator for the Timoshenko beam maps the Dirichlet boundary conditions $u_i(0), u_i(L_i)$ to the vectors $F_i(0)$, $M_i(0), F_i(L_i)$ and $M_i(L_i)$, which are the Neumann boundary conditions for the remaining part of the network connected to the specific beam.

4.8 Steklov-Poincaré Operator for the Cross with Small Cycle

We obtain the exact solution for the 1D and 3D Timoshenko beam model on the cross with the cycle. In this way, the Steklov-Poincaré operator is identified for the purposes of domain decomposition technique for modeling and design of the network.

This leads to the matrix representation of the Steklov-Poincaré operator for the cross $G = \{E, V\}$ with the small cycle $G_\varepsilon = \{E_\varepsilon, V_\varepsilon\}$ at the center, of the size $\varepsilon \to 0$. Such a system is called a network and its model includes a graph $G = \{V, E\}$ along with the state equations on edges E. Such a model is used e.g., in analysis of real life large scale gas networks with quasilinear hyperbolic equations on the edges. At the interior nodes of the graph there are prescribed transmission conditions, the continuity of solutions as well as the continuity of fluxes. There is no external force at G_ε. So the transmission conditions at the joint verticals on G_ε becomes

$$\begin{pmatrix} F_1(\varepsilon_{\max}-\varepsilon) \\ M_1(\varepsilon_{\max}-\varepsilon) \end{pmatrix} + \begin{pmatrix} F_4(\varepsilon) \\ M_4(\varepsilon) \end{pmatrix} - \begin{pmatrix} F_5(0) \\ M_5(0) \end{pmatrix} = 0, \quad u_1(\varepsilon_{\max}-\varepsilon) = u_4(\varepsilon) = u_5(0), \tag{4.26a}$$

$$\begin{pmatrix} F_2(\varepsilon_{\max}-\varepsilon) \\ M_2(\varepsilon_{\max}-\varepsilon) \end{pmatrix} + \begin{pmatrix} F_5(\varepsilon) \\ M_5(\varepsilon) \end{pmatrix} - \begin{pmatrix} F_6(0) \\ M_6(0) \end{pmatrix} = 0, \quad u_2(\varepsilon_{\max}-\varepsilon) = u_5(\varepsilon) = u_6(0), \tag{4.26b}$$

$$\begin{pmatrix} F_3(\varepsilon_{\max}-\varepsilon) \\ M_3(\varepsilon_{\max}-\varepsilon) \end{pmatrix} + \begin{pmatrix} F_6(\varepsilon) \\ M_6(\varepsilon) \end{pmatrix} - \begin{pmatrix} F_4(0) \\ M_4(0) \end{pmatrix} = 0, \quad u_3(\varepsilon_{\max}-\varepsilon) = u_6(\varepsilon) = u_4(0). \tag{4.26c}$$

Here, $F_i, M_i, u_i\,(i = 1, 2, 3)$ is the defined on the edge $E_{\varepsilon,i}$. Using the formula Eq. (4.23), where the length of edge is $\varepsilon_{\max} - \varepsilon$, the first balanced equation in (4.26a) can be written as

$$D_1^{\mathrm{III}} u_1(0) + D_1^{\mathrm{IV}} u_1(\varepsilon_{\max}-\varepsilon) + D_4^{\mathrm{III}} u_4(0) + D_4^{\mathrm{IV}} u_4(\varepsilon) - D_5^{\mathrm{I}} u_5(0) - D_5^{\mathrm{II}} u_5(\varepsilon) = 0.$$

By the continuity, we have

$$\begin{aligned} D_1^{\mathrm{III}} u_1(0) + D_1^{\mathrm{IV}} u_1(\varepsilon_{\max}-\varepsilon) + D_4^{\mathrm{III}} u_3(\varepsilon_{\max}-\varepsilon) \\ + D_4^{\mathrm{IV}} u_1(\varepsilon_{\max}-\varepsilon) - D_5^{\mathrm{I}} u_1(\varepsilon_{\max}-\varepsilon) - D_5^{\mathrm{II}} u_2(\varepsilon_{\max}-\varepsilon) = 0, \end{aligned}$$

namely,

$$(D_1^{\mathrm{IV}} + D_4^{\mathrm{IV}} - D_5^{\mathrm{I}}) u_1(\varepsilon_{\max}-\varepsilon) - D_5^{\mathrm{II}} u_2(\varepsilon_{\max}-\varepsilon) + D_4^{\mathrm{III}} u_3(\varepsilon_{\max}-\varepsilon) = -D_1^{\mathrm{III}} u_1(0). \tag{4.27}$$

Similarly, from Eq. (4.26b) and Eq. (4.26c), we get the following equations,

$$D_5^{\mathrm{III}} u_1(\varepsilon_{\max}-\varepsilon) + (D_2^{\mathrm{IV}} + D_5^{\mathrm{IV}} - D_6^{\mathrm{I}}) u_2(\varepsilon_{\max}-\varepsilon) - D_6^{\mathrm{II}} u_3(\varepsilon_{\max}-\varepsilon) = -D_2^{\mathrm{III}} u_2(0), \tag{4.28}$$

$$-D_4^{\mathrm{II}} u_1(\varepsilon_{\max}-\varepsilon) + D_6^{\mathrm{III}} u_2(\varepsilon_{\max}-\varepsilon) + (D_3^{\mathrm{IV}} + D_6^{\mathrm{IV}} - D_4^{\mathrm{I}}) u_3(\varepsilon_{\max}-\varepsilon) = -D_3^{\mathrm{III}} u_3(0). \tag{4.29}$$

Combine Eq. (4.27), Eq. (4.28) and Eq. (4.29), we have

$$\begin{aligned} \begin{pmatrix} D_1^{\mathrm{IV}} + D_4^{\mathrm{IV}} - D_5^{\mathrm{I}} & -D_5^{\mathrm{II}} & D_4^{\mathrm{III}} \\ D_5^{\mathrm{III}} & D_2^{\mathrm{IV}} + D_5^{\mathrm{IV}} - D_6^{\mathrm{I}} & -D_6^{\mathrm{II}} \\ -D_4^{\mathrm{II}} & D_6^{\mathrm{III}} & D_3^{\mathrm{IV}} + D_6^{\mathrm{IV}} - D_4^{\mathrm{I}} \end{pmatrix} \begin{pmatrix} u_1(\varepsilon_{\max}-\varepsilon) \\ u_2(\varepsilon_{\max}-\varepsilon) \\ u_3(\varepsilon_{\max}-\varepsilon) \end{pmatrix} \\ = - \begin{pmatrix} D_1^{\mathrm{III}} & 0 & 0 \\ 0 & D_2^{\mathrm{III}} & 0 \\ 0 & 0 & D_3^{\mathrm{III}} \end{pmatrix} \begin{pmatrix} u_1(0) \\ u_2(0) \\ u_3(0) \end{pmatrix}. \end{aligned} \tag{4.30}$$

That is, we could use $u_i(0)$ to present $u_i(\varepsilon_{\max}-\varepsilon)$. The Eq.(4.23) will be

$$\begin{pmatrix} F_{\varepsilon,1}(0) \\ M_{\varepsilon,1}(0) \\ F_{\varepsilon,2}(0) \\ M_{\varepsilon,2}(0) \\ F_{\varepsilon,3}(0) \\ M_{\varepsilon,4}(0) \end{pmatrix} = \Lambda_\varepsilon \begin{pmatrix} u_{\varepsilon,1}(0) \\ u_{\varepsilon,2}(0) \\ u_{\varepsilon,3}(0) \end{pmatrix},$$

where

$$\begin{aligned}
\Lambda_\varepsilon = & \begin{pmatrix} D_1^{\mathrm{I}} & 0 & 0 \\ 0 & D_2^{\mathrm{I}} & 0 \\ 0 & 0 & D_3^{\mathrm{I}} \end{pmatrix} - \begin{pmatrix} D_1^{\mathrm{II}} & 0 & 0 \\ 0 & D_2^{\mathrm{II}} & 0 \\ 0 & 0 & D_3^{\mathrm{II}} \end{pmatrix} \\
& \begin{pmatrix} D_1^{\mathrm{IV}}+D_4^{\mathrm{IV}}-D_5^{\mathrm{I}} & -D_5^{\mathrm{II}} & D_4^{\mathrm{III}} \\ D_5^{\mathrm{III}} & D_2^{\mathrm{IV}}+D_5^{\mathrm{IV}}-D_6^{\mathrm{I}} & -D_6^{\mathrm{II}} \\ -D_4^{\mathrm{II}} & D_6^{\mathrm{III}} & D_3^{\mathrm{IV}}+D_6^{\mathrm{IV}}-D_4^{\mathrm{I}} \end{pmatrix}^{-1} \begin{pmatrix} D_1^{\mathrm{III}} & 0 & 0 \\ 0 & D_2^{\mathrm{III}} & 0 \\ 0 & 0 & D_3^{\mathrm{III}} \end{pmatrix} \\
:= & D_1 - D_2 A^{-1} D_3
\end{aligned} \tag{4.31}$$

is an operator with respect to ε.

In both 1D and 3D model, the structure of the matrix Λ_ε remains uniform. We distinctly utilize the matrix $D_i^j (i=1,2,\cdots,6, j=\mathrm{I},\mathrm{II},\mathrm{III},\mathrm{IV})$ to denote the corresponding cases accordingly. Obviously, in 1D, the operator Λ_ε is a 6×6 matrix while in 3D, it assumes dimensions of 18×18.

Our goal now is to identify such a matrix $\varepsilon \to \Lambda_\varepsilon$, as well as its derivative at ε_0 for the cross with a small cycle of size $\varepsilon \to 0$. This leads us to the topological derivative of shape functionals with respect to nucleation of the cycle. The matrix Λ_ε is a representation of the so-called *Steklov-Poincaré* operator for the network at consideration.

4.9 Topological Derivative of Optimal Control Cost for Static Problem

In this section, a new result of the topological derivative for Timoshenko beams is presented.

We use the standard technique for shape [88] and topology optimization [63]. Namely, the material derivatives are employed in the shape sensitivity analysis in the framework of speed method [88].

Let us construct the prescribed state (z,v) on the graph with a small cycle that $\varepsilon=\varepsilon_0$ first. Control v_i is applied to beams $E_{0,i}$. Then we can obtain the corresponding state z_i from the network modeling. In this way, we evaluate the Topological Derivative of the Optimal Cost for the singular perturbation of the graph by a small cycle. The control problem's cost functional can be represented as follows:

$$J(\Omega) = \frac{1}{2} \sum_{i=1}^{3} \int_0^{L_i - \varepsilon_{\max}} (u_i - z_i)^2 + (q_i - v_i)^2 . \tag{4.32}$$

The weak form on G_0 is

$$a_0(u, \varphi) - u(L - \varepsilon_{\max})^\top \Lambda_\varepsilon \varphi(L - \varepsilon_{\max}) = (q, \varphi) \quad \forall \varphi \in V,$$

where the space V is defined as

$$V = \left\{ \varphi = (\varphi_1, \varphi_2, \varphi_3) \mid \varphi_i \in H^1(0, L_i), \varphi_i(P_i) = 0, i = 1, 2, 3 \right\} . \tag{4.33}$$

The Lagrangian is defined as

$$\begin{aligned} \mathcal{L}(u, q, p) = \frac{1}{2} \sum_{i=1}^{3} \int_0^{L_i - \varepsilon_{\max}} & (u_i - z_i)^2 + (q - v)^2 \\ & + a_0(u, p) - p(L - \varepsilon_{\max})^\top \Lambda_\varepsilon u(L - \varepsilon_{\max}) - (q, p), \end{aligned} \tag{4.34}$$

where p is the adjoint variable, $p(L - \varepsilon_{\max}) = (p_1(L_1 - \varepsilon_{\max}), p_2(L_2 - \varepsilon_{\max}), p_3(L_3 - \varepsilon_{\max}))^\top$ and $u(L - \varepsilon_{\max}) = (u_1(L_1 - \varepsilon_{\max}), u_2(L_2 - \varepsilon_{\max}), u_3(L_3 - \varepsilon_{\max}))^\top$. The optimality system is derived by applying the calculus of variations to the Lagrangian and solving the resulting Euler-Lagrange equations. Specifically, the adjoint state equation is obtained by solving $\dfrac{\partial \mathcal{L}}{\partial u} = 0$, which yields

$$\sum_{i=1}^{3} \int_0^{L_i - \varepsilon_{\max}} (u_i - z_i)\, \eta_i + a_0(\eta, p) - \eta(L - \varepsilon_{\max})^\top \Lambda_\varepsilon p(L - \varepsilon_{\max}) = 0, \tag{4.35}$$

i.e.,

$$(u - z, \eta)_{\Omega_0} + a_0(\eta, p) - \eta(L - \varepsilon_{\max})^\top \Lambda_\varepsilon p(L - \varepsilon_{\max}) = 0, \tag{4.36}$$

where $\eta(L - \varepsilon_{\max})$ defined is similar to $p(L - \varepsilon_{\max})$.

Similarly, solving $\dfrac{\partial \mathcal{L}}{\partial q} = 0$ leads to the optimal control,

$$q = p + v.$$

The optimality system is given by the following coupled equations:

$$\begin{cases} (u, \varphi)_{\Omega_0} + a_0(p, \eta) - p(L - \varepsilon_{\max})^\top \Lambda_\varepsilon \varphi(L - \varepsilon_{\max}) = (z, \varphi)_{\Omega_0} \\ a_0(u, \varphi) - (p, \varphi)_{\Omega_0} - u(L - \varepsilon_{\max})^\top \Lambda_\varepsilon \varphi(L - \varepsilon_{\max}) = (v, \varphi)_{\Omega_0} \end{cases} \tag{4.37}$$

Then we get the topological derivative of cost functional,

$$\frac{dJ}{d\varepsilon} = (u_i - z_i, \dot{u}_i)_{\Omega_0} + (q_i - v_i, \dot{q}_i)_{\Omega_0} . \tag{4.38}$$

The derivative of the system (4.37) is

$$\begin{cases} (\dot{u},\varphi)_{\Omega_0} + a_0(\dot{p},\eta) - \dot{p}(L-\varepsilon_{\max})^\top \Lambda_\varepsilon \varphi(L-\varepsilon_{\max}) - p(L-\varepsilon_{\max})^\top \dot{\Lambda}_\varepsilon \varphi(L-\varepsilon_{\max}) = 0, \\ a_0(\dot{u},\varphi) - (\dot{p},\varphi)_{\Omega_0} - \dot{u}(L-\varepsilon_{\max})^\top \Lambda_\varepsilon \varphi(L-\varepsilon_{\max}) - u(L-\varepsilon_{\max})^\top \dot{\Lambda}_\varepsilon \varphi(L-\varepsilon_{\max}) = 0. \end{cases} \tag{4.39}$$

The second order topological derivatives [75], [76], [17] can be also identified.

Since it's not easy to get the inverse of a 18×18 symbolic matrix, we could use the matrix decomposition trick. For matrix A in (4.31), we use LU decomposition, i.e., $A=\bar{L}\bar{U}$, where the $\bar{L}$ matrix is a lower triangular matrix, with all diagonal elements equal to 1 and all elements above the diagonal equal to 0. The $\bar{U}$ matrix is an upper triangular matrix, with all elements below the diagonal equal to 0. Since $A^{-1}=\bar{U}^{-1}\bar{L}^{-1}$, we have

$$\Lambda_\varepsilon = D_1 - D_2\bar{U}^{-1}\bar{L}^{-1}D_3,$$

therefore,

$$\dot{\Lambda}_\varepsilon = \frac{d\Lambda_\varepsilon}{d\varepsilon} = \frac{dD_1}{d\varepsilon} - \frac{dD_2}{d\varepsilon}\bar{U}^{-1}\bar{L}^{-1}D_3 - D_2\left(\frac{d\bar{U}^{-1}}{d\varepsilon}\bar{L}^{-1} + \bar{U}^{-1}\frac{d\bar{L}^{-1}}{d\varepsilon}\right)D_3 - D_2\bar{U}^{-1}\bar{L}^{-1}\frac{dD_3}{d\varepsilon}.$$

4.10 Numerical Results

We consider a full model and a reduced model in the network structure, obtaining optimal state and shape functionals with respect to the parameter ε.

The following material and geometric parameters apply to the beams considered. The elastic modulus is taken to be $\overline{\mathrm{E}}_i = 2.1\times 10^{11}\mathrm{Nm}^{-2}$. The Poisson ratio ν_i is defined as $\nu_i = \dfrac{\overline{\mathrm{E}}_i}{2\overline{\mathrm{G}}_i} - 1$, and we set $\nu_i = 0.3$, which results in a shear modulus of $\overline{\mathrm{G}}_i = 8.1\times 10^{10}\mathrm{Nm}^{-2}$. These values correspond to the material properties of steel, as stated in [18]. The cross-sectional area is $\overline{\mathrm{A}}_i = 0.01\mathrm{m}^2$. The torsion constant and second moments of area is $\overline{\mathrm{I}}_i^t = 1.41\times 10^{-5}\mathrm{m}^4$ and $\overline{\mathrm{I}}_i^y = \overline{\mathrm{I}}_i^z = 8.33\times 10^{-6}\mathrm{m}^4$, respectively, which are appropriate for a quadratic cross-section of the specified area. The shear coefficient that corresponds to the chosen Poisson ratio and cross-sectional area is given by $\kappa = \dfrac{10\,(1+\nu_i)}{12+11\nu_i} = 0.850$.

See Fig. 4.4 for the exact structure of the network. The vertex set consists of a small cycle with homogeneous Neumann boundary conditions, and three vertices (P_1, P_2, and P_3) with homogeneous Dirichlet conditions. For this particular network, we have $\boldsymbol{\phi}_i = (0,0,\theta_i)^\top$, where the angle θ_i is defined as the angle from the x-coordinate to the edge $E_{\varepsilon,i}$ in the counter-clockwise direction. By (2.8), we have

$$\mathbb{E}_i = \mathbb{E}_i(\boldsymbol{\phi}_i) = \begin{pmatrix} \cos\theta_i & -\sin\theta_i & 0 \\ \sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.40}$$

Specifically, $\theta_1 = \frac{1}{6}\pi, \theta_2 = \frac{1}{2}\pi, \theta_3 = \frac{5}{6}\pi, \theta_4 = \pi, \theta_5 = \frac{1}{3}\pi, \theta_6 = -\frac{1}{3}\pi$. Set $L_i = 2$ $(i = 1,2,3)$, $\varepsilon_0 = 0.5$ and $\varepsilon_{\max} = 1$. For v, there is a given distributed load in $E_{0,2}$, i.e.,

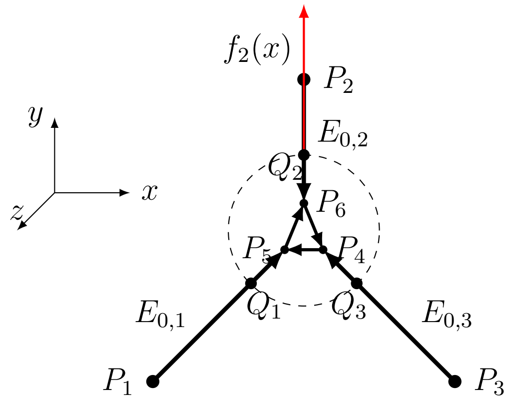


Figure 4.4: Timoshenko beams with cycle.

$v_2 = -10^8 \times (0, 2, 0, 0, 0, 0)^\top$. Then, we could get the exact expression of z by (4.19). Fig. 4.5 shows the z of the full model in local coordinates.

Now we will introduce the Hermite Finite Element Method. It is a high-order finite element method that can accurately solve the Timoshenko beam system. It employs Hermite polynomials as basis functions, which are polynomial functions with the property of having matching function values and derivatives at specified points.

On the unit interval $[0, 1]$, there exist four Hermite Shape Functions with cubic order, denoted by h_{00}, h_{01}, h_{10}, and h_{11} in Eq. (4.41). These functions are referred to as hat functions on the reference unit, and they are graphically displayed in Fig. 4.6.

$$h_{00} = 2x^3 - 3x^2 + 1, h_{01} = -2x^3 + 3x^2, h_{10} = x^3 - 2x^2 + x, h_{11} = x^3 - x^2. \tag{4.41}$$

Let $0 = x_0 < x_1 < \cdots < x_{N_i} = L_{0,i}$ be a local mesh on one edge interval $[0, L_{0,i}]$ which consisting of N_i subintervals of length h_i. After mapping from the reference unit to the local interval, the hat function can be given by $H_{1,i}$ and $H_{2,i}$ as follows:

$$H_{1,i} = \begin{cases} -2\left(\dfrac{x - x_{i-1}}{h_i}\right)^3 + 3\left(\dfrac{x - x_{i-1}}{h_i}\right)^2 & \text{if } x \in [x_{i-1}, x_i]\,; \\ 2\left(\dfrac{x - x_i}{h_i}\right)^3 - 3\left(\dfrac{x - x_i}{h_i}\right)^2 + 1 & \text{if } x \in [x_i, x_{i+1}]\,; \\ 0 & \quad \text{otherwise.} \end{cases}$$

$$H_{2,i} = \begin{cases} \left(\dfrac{x - x_{i-1}}{h_i}\right)^3 - \left(\dfrac{x - x_{i-1}}{h_i}\right)^2 & \text{if } x \in [x_{i-1}, x_i]\,; \\ \left(\dfrac{x - x_i}{h_i}\right)^3 - 2\left(\dfrac{x - x_i}{h_i}\right)^2 + \dfrac{x - x_i}{h_i} & \text{if } x \in [x_i, x_{i+1}]\,; \\ 0 & \quad \text{otherwise.} \end{cases}$$

Each hat function is continuous, piecewise cubic. For $H_{1,i}$, it takes a unit value at its own node x_i, while being zero at all other nodes. For $H_{2,i}$, its derivative takes a unit value at its own node x_i, while being zero at all other nodes (See Fig. 4.7).

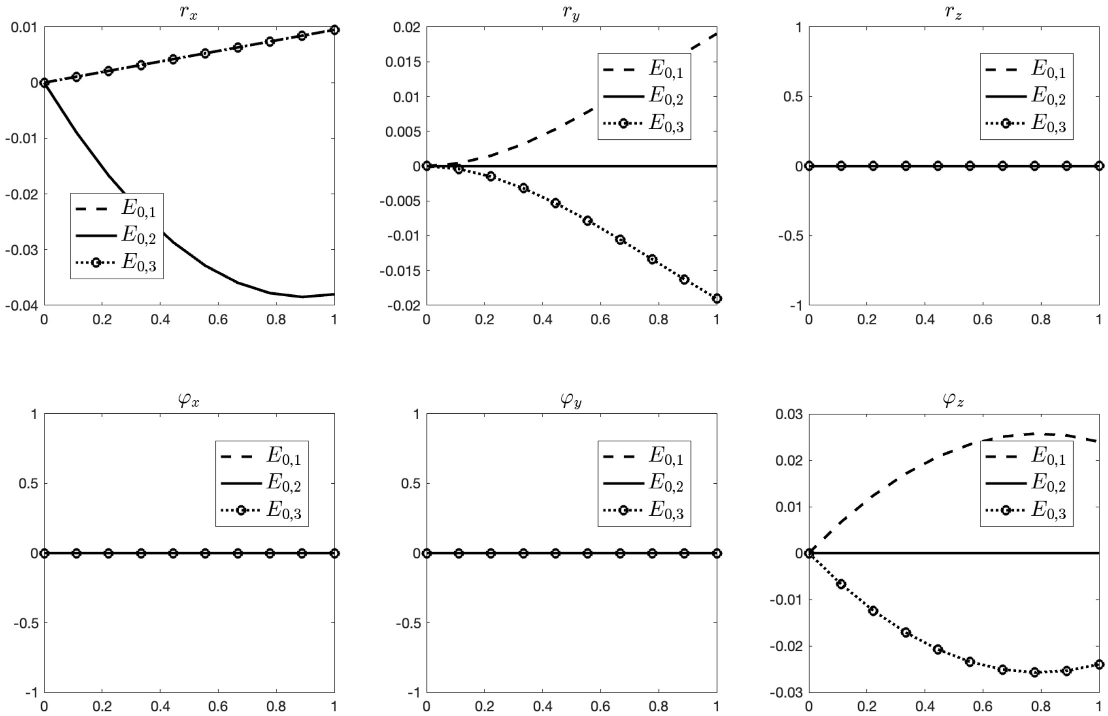


Figure 4.5: Displacements and angles of rotation of the full model in local coordinate.

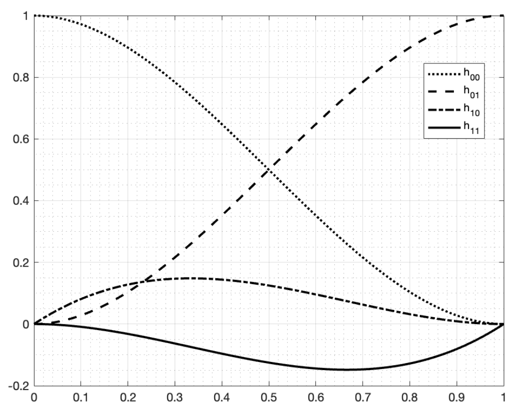


Figure 4.6: The four Hermite basis functions on the unit interval $[0, 1]$.

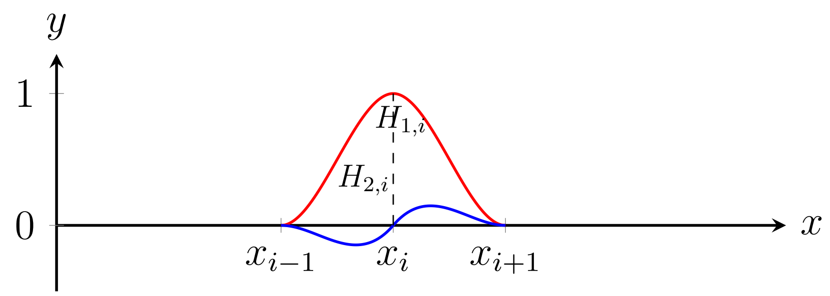


Figure 4.7: Illustration of shape functions (red: $H_{1,i}$, blue: $H_{2,i}$)

The ansatz function $u_h(x)$ is constructed as a linear combination of two sets of basis functions, $H_{1,j}(x)$ and $H_{2,j}(x)$, that are defined over the set of nodes x_j.

$$u(x) \approx u_h(x) = \sum_{j=0}^{N_i} U_j H_{1,j}(x) + \sum_{j=0}^{N_i} \tilde{U}_j H_{2,j}(x).$$

At the j-th node x_j, there are two degrees of freedom U_j and $\tilde{U}_j$, where U_j is the value of $u_h(x)$ at x_j and $\tilde{U}_j$ is the corresponding derivative value. The discrete counterpart of the optimization control in Eq. (4.37) takes the form: find $u_h \in V_h$ and $p_h \in V_h$ such that

$$\begin{cases} (u_h, \varphi)_{\Omega_0} + a_0(p_h, \eta) - p_h(L-1)^\top \Lambda_\varepsilon \varphi(L-1) = (z, \varphi)_{\Omega_0}, \ \forall \varphi \in V_h \\ a_0(u_h, \varphi) - (p_h, \varphi)_{\Omega_0} - u_h(L-1)^\top \Lambda_\varepsilon \varphi(L-1) = (v, \varphi)_{\Omega_0}, \ \forall \varphi \in V_h \end{cases} \tag{4.42}$$

where u_h and p_h are the approximations of the solution and Lagrange multiplier, respectively. The function space V_h is defined as a set of cubic functions that satisfy certain conditions at the nodes and inner vertices.

$$V_h = \{\varphi = (\varphi_1, \cdots, \varphi_6) \mid \varphi_i \text{ is cubic on each}[x_j, x_{j+1}], i = 1, \cdots, 6, \varphi_i(P_i) = 0, i = 1, 2, 3,$$
$$\text{continuity and Kirchhoff conditions at the inner vertices } P_4, P_5, P_6\}.$$

All subscripts h represent discrete forms. Denote by $H_1 = (H_{1,1}, H_{1,2}, \cdots, H_{1,N_i})^\top$, $H_2 = (H_{2,1}, H_{2,2}, \cdots, H_{2,N_i})^\top$ be column vectors. The stiffness matrix K_i and mass matrix M_i could be written as

$$K_i = \begin{pmatrix} \int_0^{L_{0,i}} H_1'(H_1')^\top & \int_0^{L_{0,i}} H_2'(H_1')^\top \\ \int_0^{L_{0,i}} H_1'(H_2')^\top & \int_0^{L_{0,i}} H_2'(H_2')^\top \end{pmatrix}, \ M_i = \begin{pmatrix} \int_0^{L_{0,i}} H_1(H_1)^\top & \int_0^{L_{0,i}} H_2(H_1)^\top \\ \int_0^{L_{0,i}} H_1(H_2)^\top & \int_0^{L_{0,i}} H_2(H_2)^\top \end{pmatrix},$$

We introduce another two matrix K_i^r and K_i^l:

$$K_i^r = \begin{pmatrix} \int_0^{L_{0,i}} H_1(H_1')^\top & \int_0^{L_{0,i}} H_2(H_1')^\top \\ \int_0^{L_{0,i}} H_1(H_2')^\top & \int_0^{L_{0,i}} H_2'(H_2')^\top \end{pmatrix}, \ K_i^l = \begin{pmatrix} \int_0^{L_{0,i}} H_1'(H_1)^\top & \int_0^{L_{0,i}} H_2'(H_1)^\top \\ \int_0^{L_{0,i}} H_1'(H_2)^\top & \int_0^{L_{0,i}} H_2'(H_2)^\top \end{pmatrix}.$$

Then we assemble the matrix

$$A_i = \begin{pmatrix} k^x K_i & 0 & 0 & 0 & 0 & 0 \\ 0 & k^s K_i & 0 & 0 & 0 & -k^s K_i^r \\ 0 & 0 & k^s K_i & 0 & k^s K_i^r & 0 \\ 0 & 0 & 0 & c^x K_i & 0 & 0 \\ 0 & 0 & k^s K_i^l & 0 & k^s M_i + c^y K_i & 0 \\ 0 & -k^s K_i^l & 0 & 0 & 0 & k^s M_i + c^z K_i \end{pmatrix},$$

and

$$B_i = \text{diag}(M_i, M_i, M_i, M_i, M_i, M_i)$$

is a block diagonal matrix. The matrix form of the Eq. (4.42) can be written as

$$\begin{pmatrix} A_1 & -B_1 & & & & \\ B_1 & A_1 & & & & \\ & & \ddots & \ddots & & \\ & & \ddots & \ddots & & \\ & & & & A_6 & -B_6 \\ & & & & B_6 & A_6 \end{pmatrix} \begin{pmatrix} \overline{u}_1 \\ \overline{p}_1 \\ \vdots \\ \vdots \\ \overline{u}_6 \\ \overline{p}_6 \end{pmatrix} = \begin{pmatrix} (\bar{v}_1, H) \\ (Z_1, H) \\ \vdots \\ \vdots \\ (\bar{v}_6, H) \\ (Z_6, H) \end{pmatrix}, \tag{4.43}$$

where $\overline{u}_1 := \begin{pmatrix} (u_1, H_1) \\ (u_1', H_1) \end{pmatrix}$, $(\bar{v}_1, H) := \begin{pmatrix} (v_1, H_1) \\ (v_1', H_1) \end{pmatrix}$ and similar to others. In addition, we impose the coupling continuity and Kirchhoff condition.

Fig. 4.8 shows the state u in $E_{0,1}$, $E_{0,2}$ and $E_{0,3}$ in local coordinate for different ε comparing the desired state z.

Fig. 4.9 depicts the functional shape with respect to ε for the full and reduced models, respectively. It's easy to see that topological derivative is negative as $\varepsilon \to 0^+$. So the cost functional is in fact reduced by the creation of a hole. The minimum value of J is when $\varepsilon = \varepsilon_0$.

We refer e.g., to [82], [81] for numerical methods in more complex geometry.

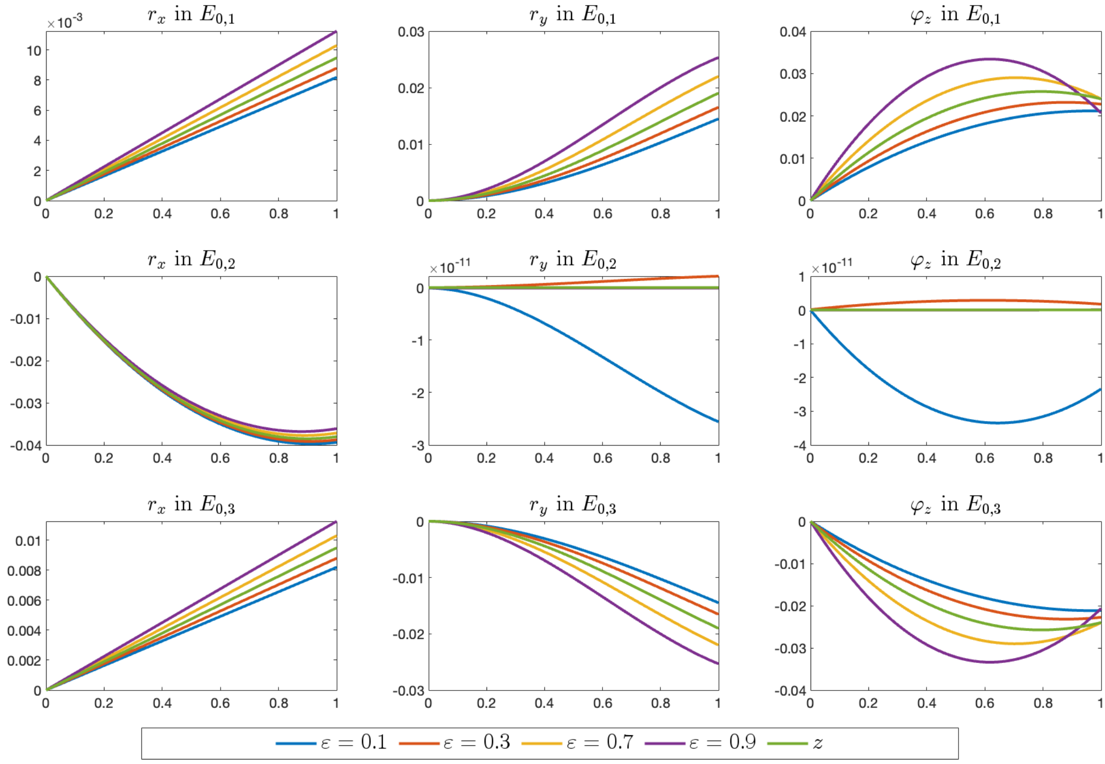


Figure 4.8: The state u for different $\varepsilon = 0.1, 0.3, 0.7, 0.9$ and z of full Timoshenko beam

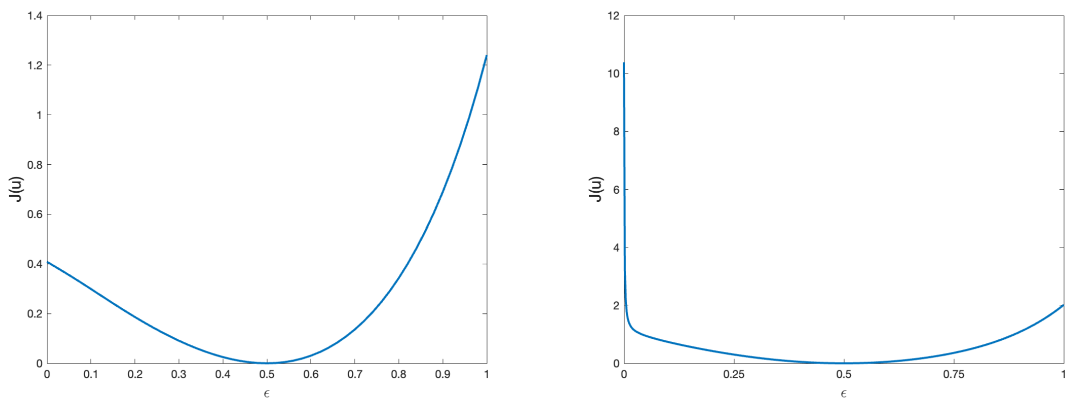


Figure 4.9: The shape functional for $\varepsilon \in (0, 1)$ of the full (left) and reduced (right) Timoshenko beam

Chapter 5

Turnpike Property for Dynamic Optimization and Applications

The Turnpike Property for abstract wave equation is shown in [31]. The state equation is a scalar hyperbolic equation. The cost for control problems contains two components for the position y and the velocity y_t of the wave equation. The constant in the inequality (5.68) is uniform on compact sets of designs of the network. Therefore, the turnpike property holds for an admissible family of the networks. In such a way, the existence of optimum designs assured our control and shape optimization problems of the network by writing the static state equation.

In the proof, it is supposed that $y_t(T) = 0$, for the sake of simplicity. The turnpike property holds for the abstract control problem for the wave equation under the assumption that $y_t(T) = 0$.

5.1 Optimality System for Evolution Problem

We introduce the Lagrangian for the state $y \in L^2(0,T;H^1(\Omega)), y_t \in L^2(0,T;L^2(\Omega))$, $y_t(0) = y^1$ and the adjoint state $p \in L^2(0,T;H^1(\Omega)), p_t \in L^2(0,T;L^2(\Omega))$. Hence $y,p \in C(0,T;L^2(\Omega)$ and the integration by parts

$$\begin{aligned}
&\int_0^T \left(\frac{\partial^2 y}{\partial^2 t}(t), p(t)\right)_{L^2(\Omega)} dt \\
&= -\int_0^T \left(\frac{\partial y}{\partial t}(t), \frac{\partial p}{\partial t}(t)\right)_{L^2(\Omega)} dt + \left(\frac{\partial y}{\partial t}(T), p(T)\right)_{L^2(\Omega)} - \left(\frac{\partial y}{\partial t}(0), p(0)\right)_{L^2(\Omega)} \\
&= -\int_0^T \left(\frac{\partial y}{\partial t}(t), \frac{\partial p}{\partial t}(t)\right)_{L^2(\Omega)} dt + \left(\frac{\partial y}{\partial t}(T), p(T)\right)_{L^2(\Omega)} - \left(y^1, p(0)\right)_{L^2(\Omega)}
\end{aligned}$$

Now, we assume $p(T) = 0$ which leads to the integration by parts formula

$$\int_0^T \left(\frac{\partial^2 y}{\partial^2 t}(t), p(t)\right)_{L^2(\Omega)} dt = -\int_0^T \left(\frac{\partial y}{\partial t}(t), \frac{\partial p}{\partial t}(t)\right)_{L^2(\Omega)} dt - \left(y^1, p(0)\right)_{L^2(\Omega)}$$

Remark 5. *If we add the state constraint $y_t(T) = 0$ then the integration by parts formula is the same but the terminal condition for $p(T)$ becomes undetermined, i.e., we lose homogeneous condition for the terminal adjoint state.*

We derive the optimality system for the optimal control problem with the evolution state equation. To this end, we introduce the Lagrangian with $y \in L^2(0,T;H^1(\Omega)), y_t \in L^2(0,T;\Omega)$, $y(0) = y^0$, and $\varphi \in L^2(0,T;H^1(\Omega)), \varphi_t \in L^2(0,T;\Omega)$, $\varphi(T) = 0$

$$\begin{aligned}\mathcal{L}(u,y,\varphi) &= \frac{1}{2}\int_0^T \|y-y^d\|^2_{L^2(\Omega)}dt + \frac{1}{2}\int_0^T \|u-u^d\|^2_{L^2(\Gamma)}dt \\ &+ \frac{\gamma}{2}\int_0^T \|\partial_t(y-y^d)\|^2_{L^2(\Omega)}dt \\ &- \int_0^T \left(\frac{\partial y}{\partial t}(t), \frac{\partial \varphi}{\partial t}(t)\right)_{L^2(\Omega)} dt - \left(y^1, \varphi(0)\right)_{L^2(\Omega)} + \int_0^T a(y(t),\varphi)dt \\ &- \int_0^T (L(u)(t),\varphi)_{L^2(\Gamma)}dt - \int_0^T (F(t),\varphi)_{L^2(\Omega)}dt.\end{aligned}$$

The adjoint state $p = p(u;x,t)$ is obtained by differentiation of Lagrangian with respect to the state, thus with $Q(T) = (0,T)\times\Omega$

$$(p_{tt},\varphi)_{Q(T)} + \int_0^T a(p,\varphi)\,dt = (y^d-y,\varphi)_{Q(T)} + \gamma\,(\partial_t(y^d-y),\ \varphi_t)_{Q(T)}\ \forall\varphi\in H(Q(T))$$
$$p(T)=0, p_t(T) = \gamma\, y_t(T)\,.$$

The gradient of the cost with respect to the control is obtained by differentiation of the Lagrangian with respect to the control

$$\begin{aligned}\left\langle \frac{dJ}{du}(u),\eta\right\rangle &= \int_0^T (u-u^d,\eta)_{L^2(\Gamma)}dt - \int_0^T (L(\eta)(t),p(t))_{L^2(\Gamma)}dt \\ &= \int_0^T (u-u^d,\eta)_{\Gamma}dt - \int_0^T (L(\eta)(t),p(t))_{\Gamma}dt.\end{aligned}$$

The following lemma contains the necessary optimality conditions for the dynamic problem where $J_T(u)$ as defined in (4.3) is minimized.

Lemma 1. *The optimality system for the optimal control of the evolution control problem is verified for a.e. $t\in(0,T)$:*

$$(\hat{y}_{tt},\varphi)_{Q(T)} + \int_0^T a(\hat{y}(t),\varphi)\,dt = \int_0^T (L(\hat{u})(t),\varphi)_\Gamma\,dt + (F(t),\varphi)_{Q(T)} \tag{5.1}$$
$$\forall\varphi\in H(Q(T))$$
$$\hat{y}(0) = y^0,\ \hat{y}_t(0) = y^1 \tag{5.2}$$

$$(\hat{p}_{tt}, \varphi)_{H(Q(T))} + \int_0^T a(\hat{p}, \varphi)\, dt = (y^d - \hat{y} + \gamma\, \hat{y}_{tt}, \varphi)_{Q(T)} \tag{5.3}$$
$$\forall \varphi \in H(Q(T))$$
$$\hat{p}(T) = 0,\ \hat{p}_t(T) = \gamma\, \hat{y}_t(T) \tag{5.4}$$
$$(\hat{u} - u^d, \eta)_\Gamma - (L(\eta)(t), \hat{p}(t))_\Gamma = 0\ \forall \eta \in L^2(0, T; \Gamma) \tag{5.5}$$

The optimality system (5.1)-(5.5) admits a unique solution $(\hat{u}, \hat{y}, \hat{p})$.

5.2 The difference of the Static and the Dynamic Optimality Systems for Distributed Control

In this section, we study a system that is satisfied by the ordered pair that has the difference between the optimal dynamic state for the time horizon T and the optimal static state as the first component and the difference between the optimal dynamic adjoint state for the time horizon T and the optimal static adjoint state as the second component. The question of long time versus steady state optimal control has already been studied in [74] where the focus is on the turnpike property of the state and the control without the adjoint state. The turnpike phenomenon for optimal boundary control problems with first order hyperbolic systems is considered in [26].

We assume that $\Gamma = \Omega$ and the operators $L(v)$ and $L(u)$ are identity operators in $L^2(\Omega)$ and $F(t) = f$, $u^d(t) = v_d$, $y^d = z_d$ for $t \in [0, T]$ almost everywhere. The optimality system for the static problem reads

$$\hat{z}^\sigma \in H\ :\ a(\hat{z}^\sigma, \varphi) = (\hat{v}^\sigma,\ \varphi)_{L^2(\Omega)} + (f, \varphi)_{L^2(\Omega)}\ \forall \varphi \in H, \tag{5.6}$$
$$\hat{p}^\sigma \in H\ :\ a(\hat{p}^\sigma, \phi) = (z_d - \hat{z}^\sigma, \phi)_{L^2(\Omega)}\ \forall \phi \in H, \tag{5.7}$$
$$\hat{v}^\sigma - v_d = \hat{p}^\sigma\ \text{a.e. in}\, \Omega. \tag{5.8}$$

The optimality system for the evolution problem implies for $t \in [0, T]$ almost everywhere (with $F(t) = f$)

$$(\hat{y}^T_{tt}(t), \varphi)_{L^2(\Omega)} + a(\hat{y}^T(t), \varphi) = (\hat{u}^T(t), \varphi)_{L^2(\Omega)} + (F(t), \varphi)_{L^2(\Omega)}\ \forall \varphi \in H \tag{5.9}$$
$$\hat{y}^T(0) = y^0,\ \ \hat{y}^T_t(0) = y^1 \tag{5.10}$$
$$(\hat{p}^T_{tt}(t), \varphi)_{L^2(\Omega)} + a(\hat{p}^T(t), \varphi) = (y^d - \hat{y}^T(t) + \gamma\, \hat{y}_{tt}, \varphi)_{L^2(\Omega)}\ \forall \varphi \in H \tag{5.11}$$
$$\hat{p}^T(T) = 0,\ \ \hat{p}^T_t(T) = \gamma\, \hat{y}_t(T) \tag{5.12}$$
$$\hat{u}^T(t) - u^d = \hat{p}^T(t)\ \text{a.e. in}\ \Omega \times (0, T). \tag{5.13}$$

Define the differences

$$\omega^T = \hat{y}^T - \hat{z}^\sigma,\ \mu^T = \hat{p}^T - \hat{p}^\sigma, \nu^T = \hat{u}^T - \hat{v}^\sigma.$$

Then for all $\varphi \in H$ we have the initial condition

$$\omega^T(0) = y^0 - \hat{z}^\sigma,\ \ \omega^T_t(0) = y^1, \tag{5.14}$$

the terminal conditions

$$\mu^T(T) = -\hat{p}^\sigma, \; \mu_t^T(T) = \gamma\, \omega_t(T), \tag{5.15}$$

the dynamics

$$(\omega_{tt}^T(t), \varphi)_{L^2(\Omega)} + a(\omega^T(t), \varphi) = (\nu^T(t), \varphi)_{L^2(\Omega)} = (\mu^T(t), \varphi)_{L^2(\Omega)} \tag{5.16}$$

and with the assumption that $y^d = z_d$ and $u^d = v_d$

$$(\mu_{tt}^T(t), \varphi)_{L^2(\Omega)} + a(\mu^T(t), \varphi) = -(\omega^T(t), \varphi)_{L^2(\Omega)} + \gamma\, (\omega_{tt}^T(t), \varphi)_{L^2(\Omega)}. \tag{5.17}$$

Note that for the difference system, the existence of a solution is implied by the construction as the difference between two systems, for which solutions exist.

Note that for the energy

$$E(t) = \frac{(\omega_t^T(t),\, \omega_t^T(t))_{L^2(\Omega)} + a(\omega^T(t),\, \omega^T(t))}{2} + \frac{(\mu_t^T(t),\, \mu_t^T(t))_{L^2(\Omega)} + a(\mu^T(t),\, \mu^T(t))}{2}$$

due to (5.16) and (5.17) for $\gamma = 0$ we have

$$E''(t) \geq (\omega^T(t),\, \omega^T(t))_{L^2(\Omega)} + (\mu^T(t),\, \mu^T(t))_{L^2(\Omega)} \geq 0.$$

Thus E is convex on $[0,\, T]$.

Now we perform a spectral analysis to show the exponential turnpike property.

Assume that there exists a complete orthonormal sequence $(\psi_k)_{k=1}^\infty$ of eigenfunctions with $a(\psi_k,\, \varphi) = \lambda_k(\psi_k,\, \varphi)_{L^2(\Omega)}$ for all $k \in \{0, 1, 2, 3, ..\}$ where

$$\lambda_k \geq \gamma > 0 \tag{5.18}$$

is a real number.

Remark 6. *In the case of optimal design, the bilinear form depends on Ω. In the case of a graph, this means the dependence of the lengths of the edges. In this case, the eigenvalues and eigenfunctions depend on these parameters that we denote by ℓ. Therefore a meaningful analysis has to take into account the sensitivity with respect to ℓ.*

Our assumption on the feasible designs is that the smallest eigenvalues are greater than or equal to the given strictly positive lower-bound $\gamma > 0$ uniformly on the set of admissible designs. In this way, we ensure that the turnpike property is valid for the bilevel optimization problem that we consider in this paper.

It is well known that the smallest eigenvalue that can be characterized as the Rayleigh quotient depends smoothly on the parameters, see [32]. In our analysis, the particular structure of the spectrum is not relevant.

Then we can use the representations

$$\omega^T = \sum_{k=0}^{\infty} a_k(t)\, \psi_k(x), \quad \mu^T = \sum_{k=0}^{\infty} b_k(t)\, \psi_k(x) \tag{5.19}$$

to show that ω^T and μ^T have the turnpike property.

It is clear that the functions a_k and b_k depend on T as a parameter, so a more precise notation would be $a_{k,T}(t)$ and $b_{k,T}(t)$. However, in order to make the text more concise we continue with the shorter notation a_k and b_k.

From (5.16) and (5.17) we obtain $a_k^{(2)} = -\lambda_k\, a_k + b_k$ and $b_k^{(2)} = -\lambda_k\, b_k - a_k + \gamma\, a_k^{(2)}$. Here, (n) means the n-th order derivative. Thus we have the sequence of differential equations

$$
\begin{aligned}
a_k^{(4)} + (2\,\lambda_k - \gamma)\, a_k^{(2)} + (\lambda_k^2 + 1)\, a_k &= 0,\\
b_k^{(4)} + (2\,\lambda_k - \gamma)\, b_k^{(2)} + (\lambda_k^2 + 1)\, b_k &= 0.
\end{aligned} \tag{5.20}
$$

Define the characteristic polynomial

$$p_k(z) = z^4 + (2\,\lambda_k - \gamma)\, z^2 + (\lambda_k^2 + 1).$$

Since p_k is a polynomial in z^2, for the roots $z_k^{[l]}$ of p_k we obtain

$$(z_k^{[l]})^2 = \frac{\gamma}{2} - \lambda_k \pm \frac{i}{2}\sqrt{4 + 4\lambda_k\,\gamma - \gamma^2}$$

for $l \in \{1, 2, 3, 4\}$ which implies $\left|(z_k^{[l]})^2\right| = \sqrt{1 + \lambda_k^2}$. Thus there are two pairs of complex conjugate roots and we have the representation

$$p_k(z) = (z - z_k^{[1]})(z - \overline{z_k^{[1]}})(z + z_k^{[1]})(z + \overline{z_k^{[1]}}).$$

We have $|z_k^{(l)}|^4 = \lambda_k^2 + 1$, $\mathrm{Re}((z^{[l]})_k^2) = \frac{\gamma}{2} - \lambda_k$ and $|\,\mathrm{Im}((z^{[l]})_k^2)| = \frac{\sqrt{4+4\,\lambda_k\,\gamma-\gamma^2}}{2}$. Moreover, we have

$$|\,\mathrm{Re}(z_k^{[l]})| = \sqrt{\frac{\gamma}{4} + \frac{1}{2(\lambda_k + \sqrt{1+\lambda_k^2})}}$$

and

$$|\,\mathrm{Re}(z_k^{[l]})| \geq \frac{\sqrt{\gamma}}{2}. \tag{5.21}$$

The initial condition (5.14) yields the values for $a_k(0)$ and $a_k'(0)$. The terminal condition (5.15) yields the value for $b_k(T)$ and $b_k'(T) = \gamma\, a_k'(T)$. Note that (5.15) implies that the value of $b_k(T)$ is independent of T.

For the sake of conciseness, in the sequel we use the notation $z_k = z_k^{[1]}$.

5.2.1 Representation of the Solution

Since

$$a_{k,T}' = \frac{1}{1 + \gamma\,\lambda_k}\left[-b_{k,T}''' + (\gamma - \lambda_k) b_{k,T}'\right], \tag{5.22}$$

the solution of the optimality system means that for the coefficients $b_{k,T}$ of μ^T as defined in (5.19) we solve a boundary value problem with the ODE of order four (5.20) i.e.

$$\left\{ \begin{array}{rcl} b_{k,T}^{(4)} + (2\,\lambda_k - \gamma)\, b_{k,T}^{(2)} + (\lambda_k^2 + 1)\, b_{k,T} & = & 0 \\ b_{k,T}(T) & = & \beta_k \\ b_{k,T}'(T) & = & \frac{\gamma}{1+\gamma\,\lambda_k}\left[-b_{k,T}'''(T) + (\gamma - \lambda_k)\, b_{k,T}'(T)\right] \\ -b_{k,T}''(0) + (\gamma - \lambda_k)\, b_{k,T}(0) & = & (1+\gamma\,\lambda_k)\, a_{k,T}(0) \\ -b_{k,T}'''(0) + (\gamma - \lambda_k)\, b_{k,T}'(0) & = & (1+\gamma\,\lambda_k)\, a_{k,T}'(0). \end{array} \right. \tag{5.23}$$

where the value of β_k is determined by the terminal condition $\mu^T(T) = -\hat{p}^\sigma$ in (5.15). We represent the solution in the form

$$b_{k,T}(t) = F_{k,T}(t)\,\beta_k + (1+\gamma\,\lambda_k)[G_{k,T}(t)\;a_{k,T}(0) + H_{k,T}(t)\,a_{k,T}'(0)]. \tag{5.24}$$

The following Lemma contains explicit representations of $F_{k,T}$, $G_{k,T}$ and $H_{k,T}$. In the representation, the numbers $d(k,T)$ appear as multipliers, therefore it is important that for T sufficiently large we have $d(k,T) \neq 0$. Since in the study of the turnpike phenomenon we are interested in large time horizons, the assumption that the time horizon T is large is not restrictive for us. We introduce the notation

$$\Xi_k := \gamma - \lambda_k - z_k^2 = \frac{\gamma}{2} \mp \frac{i}{2}\sqrt{4 + 4\,\lambda_k\gamma - \gamma^2}. \tag{5.25}$$

Lemma 2. *Define*

$$q_k = \frac{\gamma^2 - 2\,\gamma\,\lambda_k - 1}{\gamma}.$$

For $k \in \{0, 1, 2, ...\}$ and T sufficiently large define the numbers

$$\begin{aligned} & d(k,T) \\ = \; & -2Re\left(\frac{\Xi_k^2}{z_k^2 - q_k}\right) + 2|\Xi_k|^2 Re\left(\frac{1}{z_k^2 - q_k}|\cosh^2(z_k T)| - \frac{\overline{z_k}}{z_k}\frac{1}{\overline{z_k}^2 - q_k}|\sinh^2(z_k T)|\right). \end{aligned} \tag{5.26}$$

Then we have

$$\begin{aligned} & d(k,T)\,F_{k,T}(t) \\ = \; & 2\,\mathrm{Re}\left(\left[-\frac{\overline{\Xi_k}^2}{\overline{z_k}^2 - q_k} + \frac{|\Xi_k|^2}{z_k^2 - q_k}\cosh(z_k\,T)\cosh(\overline{z_k}T)\right]\cosh(z_k(t-T))\right) \\ + \; & 2\,\mathrm{Re}\left(\frac{|\Xi_k|^2}{z_k^2 - q_k}\cosh(\overline{z_k}T)\sinh(z_k T)\sinh(z_k(t-T))\right) \\ - \; & 2\,\mathrm{Re}\left(\frac{|\Xi_k|^2}{z_k^2 - q_k}\frac{\overline{z_k}}{z_k}\sinh(\overline{z_k}T)\left[\sinh(z_k\,T)\cosh(z_k(t-T)) + \cosh(z_k T)\sinh(z_k(t-T))\right]\right). \end{aligned}$$

Furthermore, it holds

$$\begin{aligned} & d(k,T)\,G_{k,T}(t) \\ = \; & 2\,\mathrm{Re}\left(\frac{\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\cosh(\overline{z_k}T)\left[\cosh(z_k(t-T)) - \cosh(\overline{z_k}(t-T))\right]\right) \\ + \; & 2\,\mathrm{Re}\left(\frac{1}{z_k^2 - q_k}\left[\frac{\overline{z_k}}{z_k}\overline{\Xi_k}\sinh(\overline{z_k}T) - \Xi_k\sinh(z_k T)\right]\sinh(z_k(t-T))\right) \end{aligned} \tag{5.27}$$

and

$$
\begin{aligned}
& d(k,T)\, H_{k,T}(t) \\
= \quad & -2\,\mathrm{Re}\left(\tfrac{\Xi_k}{z_k({z_k}^2-q_k)}\,\sinh(z_k T)\,[\cosh(z_k(t-T))-\cosh(\overline{z_k}(t-T))]\right) \\
& +2\,\mathrm{Re}\left(\tfrac{\overline{\Xi_k}}{z_k({z_k}^2-q_k)}\,\cosh(\overline{z_k}T)\,\sinh(z_k(t-T))-\tfrac{\overline{\Xi_k}}{\overline{z_k}(\overline{z_k}^2-q_k)}\cosh(\overline{z_k}T)\sinh(\overline{z_k}(t-T))\right).
\end{aligned}
$$

Proof: Due to the properties of the hyperbolic functions that can be found for example in [10, 11] we have

$$
d(k,T) = -2\ \mathrm{Re}\left(\frac{\Xi_k^2}{z_k^2-q_k}\right) \tag{5.28}
$$

$$
+|\Xi_k|^2\left(\frac{\mathrm{Re}\,(({z_k}^2-q_k)(|z_k|^2-z_k^2))}{|z_k^2|\,|{z_k}^2-q_k|}\right)\cosh(2\ \mathrm{Re}(z_k)\,T)
$$

$$
+|\Xi_k|^2\left(\frac{\mathrm{Re}\,(({z_k}^2-q_k)(|z_k|^2+z_k^2))}{|z_k^2|\,|{z_k}^2-q_k|}\right)\cosh(2\ \mathrm{Im}(z_k)\,T).
$$

This implies $\lim_{T\to\infty}|d(k,T)|=\infty$, hence for all sufficiently large T we have $d(k,T)\neq 0$. In the remaining part of the proof, we assume that $d(k,T)\neq 0$. Moreover, we see that $d(k,T)$ grows exponentially fast with T with the growth rate

$$
2\mathrm{Re}(z_k) \geq \sqrt{\gamma} \tag{5.29}
$$

due to (5.21).

To be more precise, note that $|\Xi_k|^2 = 1+\gamma\,\lambda_k$ grows with the order λ_k.

At this point, elementary computations show that $F_{k,T}(t)$, $G_{k,T}(t)$ and $H_{k,T}(t)$ satisfy the differential equation and the boundary conditions where $(\beta_k, a_{k,T}(0),\ a'_{k,T}(0))$ have the values $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ respectively. Now we present details of the verification of these basic functions.

Since $b'''_{k,T}(T) = q_k\, b'_{k,T}(T)$, we can make the ansatz

$$
b_{k,T}(t) = A_k^T\cosh(z_k(t-T)) + B_k^T\cosh(\overline{z_k}(t-T)) \tag{5.30}
$$

$$
+C_k^T\left[\frac{1}{z_k\ (z_k^2-q_k)}\sinh(z_k(t-T)) - \frac{1}{\overline{z_k}\ (\overline{z_k}^2-q_k)}\sinh(\overline{z_k}(t-T))\right].
$$

Now we return again to the sloppy notation a_k, b_k. Then

$$
A_k^T + B_k^T = b_k(T).
$$

We have

$$
a_k = \frac{-b_k^{(2)} + (\gamma-\lambda_k)\,b_k}{1+\gamma\,\lambda_k}. \tag{5.31}
$$

Hence we obtain

$$
(1+\gamma\,\lambda_k)\,a_k(t) =
$$

$$A_k^T\left((\gamma-\lambda_k)-z_k^2\right)\cosh(z_k(t-T))+B_k^T\left((\gamma-\lambda_k)-\overline{z_k}^2\right)\cosh(\overline{z_k}(t-T))$$
$$+C_k^T\left[\frac{\gamma-\lambda_k-z_k^2}{z_k\,(z_k{}^2-q_k)}\sinh(z_k(t-T))-\frac{\gamma-\lambda_k-\overline{z_k}^2}{\overline{z_k}\,(\overline{z_k}^2-q_k)}\sinh(\overline{z_k}(t-T))\right].$$

Therefore, we have

$$(1+\gamma\,\lambda_k)\,a_k(0)=$$
$$A_k^T\left((\gamma-\lambda_k)-z_k^2\right)\cosh(z_k\,T)+B_k^T\left((\gamma-\lambda_k)-\overline{z_k}^2\right)\cosh(\overline{z_k}\,T)$$
$$+C_k^T\left[-\frac{\gamma-\lambda_k-z_k^2}{z_k\,(z_k{}^2-q_k)}\sinh(z_k\,T)+\frac{\gamma-\lambda_k-\overline{z_k}^2}{\overline{z_k}\,(\overline{z_k}^2-q_k)}\sinh(\overline{z_k}\,T)\right].$$

Due to (5.22) we have

$$(1+\gamma\,\lambda_k)\,a_k'(t)=-b_k'''(t)+(\gamma-\lambda_k)b_k'(t)$$
$$=A_k^T(-z_k^3+(\gamma-\lambda_k)z_k)\sinh(z_k(t-T))+B_k^T(-\overline{z}_k^3+(\gamma-\lambda_k)\overline{z}_k)\sin(\overline{z_k}(t-T))$$
$$+C_k^T\left[\frac{(-z_k^3+(\gamma-\lambda_k)z_k)}{z_k\,(z_k^2-q_k)}\cosh(z_k(t-T))-\frac{(-\overline{z}_k^3+(\gamma-\lambda_k)\overline{z}_k)}{\overline{z_k}\,(\overline{z_k}^2-q_k)}\cosh(\overline{z_k}(t-T))\right].$$

Hence we have

$$(1+\gamma\,\lambda_k)\,a_k'(0)=-b_k'''(0)+(\gamma-\lambda_k)b_k'(0)$$
$$=A_k^T(z_k^3-(\gamma-\lambda_k)z_k)\sinh(z_kT)+B_k^T(\overline{z}_k^3-(\gamma-\lambda_k)\overline{z}_k)\sin(\overline{z_k}T)$$
$$+C_k^T\left[\frac{(-z_k^3+(\gamma-\lambda_k)z_k)}{z_k\,(z_k^2-q_k)}\cosh(z_k\,T)-\frac{(-\overline{z}_k^3+(\gamma-\lambda_k)\overline{z}_k)}{\overline{z_k}\,(\overline{z_k}^2-q_k)}\cosh(\overline{z_k}T)\right].$$

In order to obtain a unique solution for the vector of coefficients (A_k^T, B_k^T, C_k^T) we investigate the determinant of the corresponding 3×3 matrix $M(T)$. For this purpose, we introduce the notation

$$\begin{aligned}
m_{21}(T) &= \left(\gamma-\lambda_k-z_k^2\right)\cosh(z_kT),\\
m_{22}(T) &= \left(\gamma-\lambda_k-\overline{z_k}^2\right)\cosh(\overline{z_k}T),\\
m_{23}(T) &= -\frac{\gamma-\lambda_k-z_k^2}{z_k\,(z_k{}^2-q_k)}\sinh(z_k\,T)+\frac{\gamma-\lambda_k-\overline{z_k}^2}{\overline{z_k}\,(\overline{z_k}^2-q_k)}\sinh(\overline{z_k}\,T)\\
m_{31}(T) &= -z_k\left(\gamma-\lambda_k-z_k^2\right)\sinh(z_kT)=-m_{21}'(T),\\
m_{32}(T) &= -\overline{z_k}\left(\gamma-\lambda_k-\overline{z_k}^2\right)\sinh(\overline{z_k}T)=-m_{22}'(T),\\
m_{33}(T) &= \frac{((\gamma-\lambda_k)-z_k^2)}{(z_k^2-q_k)}\cosh(z_k\,T)-\frac{((\gamma-\lambda_k)-\overline{z}_k^2)}{(\overline{z_k}^2-q_k)}\cosh(\overline{z_k}T)=-m_{23}'(T).
\end{aligned}$$

Note that we have the equations

$$m_{23}(T)=\frac{1}{z_k^2\,(z_k{}^2-q_k)}m_{31}(T)-\frac{1}{\overline{z_k}^2\,(\overline{z_k}^2-q_k)}m_{32}(T).$$

$$m_{33}(T) = \frac{1}{(z_k{}^2 - q_k)} m_{21}(T) - \frac{1}{(\overline{z_k}^2 - q_k)} m_{22}(T),$$

We have the matrix

$$M(T) = \begin{pmatrix} 1 & 1 & 0 \\ m_{21}(T) & m_{22}(T) & m_{23}(T) \\ m_{31}(T) & m_{32}(T) & m_{33}(T) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ m_{21}(T) & m_{22}(T) & m_{23}(T) \\ -m'_{21}(T) & -m'_{22}(T) & -m'_{23}(T) \end{pmatrix}$$

and the right-hand side

$$r(T) = \begin{pmatrix} b_k(T) \\ (1 + \gamma\,\lambda_k)\, a_k(0) \\ (1 + \gamma\,\lambda_k)\, a'_k(0) \end{pmatrix}$$

that is in fact independent of T, as stated earlier.

5.2.2 Computation of the characteristic determinant and an inverse matrix

For the determinant of $M(T)$ we obtain

$$\begin{aligned} \det M(T) =\,& m_{22}(T)m_{33}(T) - m_{32}(T)m_{23}(T) - m_{21}(T)m_{33}(T) + m_{31}(T)m_{23}(T) \\ =\,& \frac{1}{(z_k{}^2 - q_k)} m_{22}(T)m_{21}(T) - \frac{1}{(\overline{z_k}^2 - q_k)} m_{22}(T)m_{22}(T) \\ & - \frac{1}{z_k^2\,(z_k{}^2 - q_k)} m_{32}(T)m_{31}(T) + \frac{1}{\overline{z_k}^2\,(\overline{z_k}^2 - q_k)} m_{32}(T)m_{32}(T) \\ & - \frac{1}{(z_k{}^2 - q_k)} m_{21}(T)m_{21}(T) + \frac{1}{(\overline{z_k}^2 - q_k)} m_{21}(T)m_{22}(T) \\ & + \frac{1}{z_k^2\,(z_k{}^2 - q_k)} m_{31}(T)m_{31}(T) - \frac{1}{\overline{z_k}^2\,(\overline{z_k}^2 - q_k)} m_{31}(T)m_{32}(T). \end{aligned} \tag{5.32}$$

We have $m_{21}(T)^2 - \frac{1}{z_k^2} m_{31}(T)^2 = \left(\gamma - \lambda_k - z_k^2\right)^2$ and $m_{22}(T)^2 - \frac{1}{\overline{z_k}^2} m_{32}(T)^2 = \left(\gamma - \lambda_k - \overline{z_k}^2\right)^2$. This yields

$$\begin{aligned} \det M(T) =& - \frac{\left(\gamma - \lambda_k - z_k^2\right)^2}{z_k^2 - q_k} - \frac{\left(\gamma - \lambda_k - \overline{z_k}^2\right)^2}{\overline{z_k}^2 - q_k} \\ & + \left(\frac{1}{z_k{}^2 - q_k} + \frac{1}{\overline{z_k}^2 - q_k} \right) m_{22}(T)m_{21}(T) \\ & - \left(\frac{1}{z_k{}^2\,(z_k{}^2 - q_k)} + \frac{1}{\overline{z_k}^2\,(\overline{z_k}^2 - q_k)} \right) m_{32}(T)m_{31}(T). \end{aligned} \tag{5.33}$$

We introduce the notation

$$\Xi_k = \gamma - \lambda_k - z_k^2.$$

Since

$$\cosh(z_k T)\cosh(\overline{z_k}T) = \frac{1}{2}\left[\cosh((z_k + \overline{z_k})T) + \cosh((z_k - \overline{z_k})T)\right] \tag{5.34}$$

and

$$\sinh(z_k T)\sinh(\overline{z_k}T) = \frac{1}{2}\left[\cosh((z_k + \overline{z_k})T) - \cosh((z_k - \overline{z_k})T)\right] \tag{5.35}$$

we obtain the equation

$$\det M(T) = -\frac{\Xi_k^2}{z_k^2 - q_k} - \frac{\overline{\Xi_k}^2}{\overline{z_k}^2 - q_k} + \left(\frac{1}{{z_k}^2 - q_k} + \frac{1}{\overline{z_k}^2 - q_k}\right)|\Xi_k|^2\cosh(z_k T)\cosh(\overline{z_k}T)$$

$$-\left(\frac{1}{{z_k}^2\left({z_k}^2 - q_k\right)} + \frac{1}{\overline{z_k}^2\left(\overline{z_k}^2 - q_k\right)}\right) z_k\overline{z_k}\,|\Xi_k|^2\sinh(z_k T)\sinh(\overline{z_k}T).$$

This yields the representation

$$\det M(T) = -2\,\mathrm{Re}\left(\frac{\Xi_k^2}{z_k^2 - q_k}\right) \tag{5.36}$$

$$+2|\Xi_k|^2\,\mathrm{Re}\left(\frac{1}{z_k^2 - q_k}\right)|\cosh^2(z_k T)| - 2|\Xi_k|^2\,\mathrm{Re}\left(\frac{\overline{z_k}}{z_k}\frac{1}{\left({z_k}^2 - q_k\right)}\right)|\sinh^2(z_k T)|.$$

Due to (5.34) and (5.35) this can also be written in the form

$$\det M(T) = -\frac{\Xi_k^2}{z_k^2 - q_k} - \frac{\overline{\Xi_k}^2}{\overline{z_k}^2 - q_k}$$

$$+\frac{|\Xi_k|^2}{2}\left[\frac{1}{{z_k}^2 - q_k}\left(1 - \frac{\overline{z_k}^2}{|z_k|^2}\right) + \frac{1}{\overline{z_k}^2 - q_k}\left(1 - \frac{{z_k}^2}{|z_k|^2}\right)\right]\cosh((z_k + \overline{z_k})T)$$

$$+\frac{|\Xi_k|^2}{2}\left[\frac{1}{{z_k}^2 - q_k}\left(1 + \frac{\overline{z_k}^2}{|z_k|^2}\right) + \frac{1}{\overline{z_k}^2 - q_k}\left(1 + \frac{{z_k}^2}{|z_k|^2}\right)\right]\cosh((z_k - \overline{z_k})T)$$

We have

$$\frac{1}{{z_k}^2 - q_k}\left(1 - \frac{\overline{z_k}^2}{|z_k|^2}\right) + \frac{1}{\overline{z_k}^2 - q_k}\left(1 - \frac{{z_k}^2}{|z_k|^2}\right)$$

$$= \frac{2\ \mathrm{Re}\left(({z_k}^2 - q_k)(|z_k|^2 - z_k^2)\right)}{|z_k^2|\,|{z_k}^2 - q_k|}.$$

It follows that

$$\mathrm{Re}\left(({z_k}^2 - q_k)(|z_k|^2 - z_k^2)\right) = \mathrm{Re}\left((|z_k|^2 + q_k)z_k^2 - q_k\,|z_k|^2 - z_k^4\right)$$

$$= \mathrm{Re}\left(\left(|z_k|^2 + q_k + 2\,\lambda_k - \gamma\right)z_k^2 + \lambda_k^2 + 1 - q_k\,|z_k|^2\right)$$

$$= \mathrm{Re}\left(\left(|z_k|^2 - \frac{1}{\gamma}\right)z_k^2 + \lambda_k^2 + 1 - q_k\,|z_k|^2\right)$$

$$= \left(\sqrt{\lambda_k^2+1} - \frac{1}{\gamma}\right)\left(\frac{\gamma}{2} - \lambda_k\right) + \lambda_k^2 + 1 + (2\lambda_k + \frac{1}{\gamma} - \gamma)\sqrt{\lambda_k^2+1}$$

$$= \sqrt{\lambda_k^2+1}\left(\lambda_k + \frac{1}{\gamma} - \frac{\gamma}{2}\right) + \frac{1}{\gamma}\lambda_k + \lambda_k^2 + \frac{1}{2} > 0$$

and

$$\frac{1}{{z_k}^2 - q_k}\left(1 + \frac{\overline{z_k}^2}{|z_k|^2}\right) + \frac{1}{\overline{z_k}^2 - q_k}\left(1 + \frac{{z_k}^2}{|z_k|^2}\right)$$
$$= \frac{2 \text{ Re}\left(({z_k}^2 - q_k)(|z_k|^2 + z_k^2)\right)}{|z_k^2|\,|{z_k}^2 - q_k|},$$

Hence we have

$$\begin{aligned}\det M(T) = & -2 \text{ Re}\left(\frac{\Xi_k^2}{z_k^2 - q_k}\right) \\ & + \frac{|\Xi_k|^2}{2}\frac{2 \text{ Re}\left(({z_k}^2 - q_k)(|z_k|^2 - z_k^2)\right)}{|z_k^2|\,|{z_k}^2 - q_k|}\cosh(2 \text{ Re}(z_k)\,T) \\ & + \frac{|\Xi_k|^2}{2}\frac{2 \text{ Re}\left(({z_k}^2 - q_k)(|z_k|^2 + z_k^2)\right)}{|z_k^2|\,|{z_k}^2 - q_k|}\cosh(2 \text{ Im}(z_k)\,T).\end{aligned} \tag{5.37}$$

This implies $\lim_{T\to\infty} |\det M(T)| = \infty$, hence for all sufficiently large T we have $\det M(T) \neq 0$. Moreover, we see that $\det M(T)$ grows exponentially fast with T with the growth rate

$$2\text{Re}(z_k) \geq \sqrt{\gamma} \tag{5.38}$$

due to (5.21). Hence for all sufficiently large T the coefficients $(A_k^T,\, B_k^T,\, C_k^T)$ are uniquely determined.

For the computation of the inverse of $M(T)$ we use the representation

$$\det(M(T)\ [M(T)]^{-1} = \begin{pmatrix}\beta_{11}(T) & \beta_{12}(T) & \beta_{13}(T)\\ \beta_{21}(T) & \beta_{22}(T) & \beta_{23}(T)\\ \beta_{31}(T) & \beta_{32}(T) & \beta_{33}(T)\end{pmatrix}$$

$$= \begin{pmatrix} m_{22}(T)m_{33}(T) - m_{23}(T)m_{32}(T) & -m_{33}(T) & m_{23}(T)\\ m_{23}(T)m_{31}(T) - m_{21}(T)m_{33}(T) & m_{33}(T) & -m_{23}(T)\\ m_{21}(T)\,m_{32}(T) - m_{22}(T)\,m_{31}(T) & m_{31}(T) - m_{32}(T) & m_{22}(T) - m_{21}(T)\end{pmatrix}.$$

The element $\beta_{11}(T)$ in the top-left corner of $\det(M(T))\,M(T)^{-1}$ is given by the minor

$$\beta_{11}(T) = m_{22}(T)\,m_{33}(T) - m_{23}(T)\,m_{32}(T)$$

$$= \overline{\Xi}_k \cosh(\overline{z_k}T)\left\{\frac{\Xi_k}{(z_k^2 - q_k)}\cosh(z_k\,T) - \frac{\overline{\Xi}_k}{(\overline{z_k}^2 - q_k)}\cosh(\overline{z_k}T)\right\}$$

$$+\left\{-\frac{\Xi_k}{z_k\,({z_k}^2 - q_k)}\sinh(z_k\,T) + \frac{\overline{\Xi}_k}{\overline{z_k}\,(\overline{z_k}^2 - q_k)}\sinh(\overline{z_k}\,T)\right\}\left\{\overline{z_k}\,\overline{\Xi}_k\sinh(\overline{z_k}T)\right\}$$

$$= \frac{|\Xi_k|^2}{z_k^2 - q_k} \cosh(z_k T) \cosh(\overline{z_k} T) - \frac{\overline{\Xi_k}^2}{\overline{z_k}^2 - q_k} [\cosh^2(\overline{z_k} T) - \sinh^2(\overline{z_k} T)] - \frac{\overline{z_k}}{z_k} \frac{|\Xi_k|^2}{z_k^2 - q_k} \sinh(z_k T) \sinh(\overline{z_k} T)$$

$$= -\frac{\overline{\Xi_k}^2}{\overline{z_k}^2 - q_k} + \frac{|\Xi_k|^2}{{z_k}^2 - q_k} \left[\cosh(z_k T) \cosh(\overline{z_k} T) - \frac{\overline{z_k}}{z_k} \sinh(z_k T) \sinh(\overline{z_k} T) \right]. \tag{5.39}$$

Due to (5.34) and (5.35) this yields

$$\beta_{11}(T) = -\frac{\overline{\Xi_k}^2}{\overline{z_k}^2 - q_k} + \frac{1}{2} \frac{|\Xi_k|^2}{{z_k}^2 - q_k} \left[\left(1 - \frac{\overline{z_k}}{z_k}\right) \cosh((z_k + \overline{z_k})T) + \left(1 + \frac{\overline{z_k}}{z_k}\right) \cosh((z_k - \overline{z_k})T) \right].$$

The second element $\beta_{12}(T)$ in the top row of $\det(M(T))\, M(T)^{-1}$ is given by

$$\beta_{12}(T) = -m_{33}(T) = -\frac{\Xi_k}{(z_k^2 - q_k)} \cosh(z_k\, T) + \frac{\overline{\Xi_k}}{(\overline{z_k}^2 - q_k)} \cosh(\overline{z_k} T).$$

The third element $\beta_{13}(T)$ in the top row of $\det(M(T))\, M(T)^{-1}$ is given by

$$\beta_{13}(T) = m_{23}(T) = -\frac{\Xi_k}{z_k\, ({z_k}^2 - q_k)} \sinh(z_k\, T) + \frac{\overline{\Xi_k}}{\overline{z_k}\, (\overline{z_k}^2 - q_k)} \sinh(\overline{z_k}\, T).$$

Now we consider the entries in the second row. For the first element in the second row of the matrix $\det(M(T))\, M(T)^{-1}$ we obtain

$$\beta_{21}(T) = -[m_{21}(T) m_{33}(T) - m_{23}(T) m_{31}(T)]$$

$$= -\frac{\Xi_k^2}{(z_k^2 - q_k)} \cosh^2(z_k\, T) + \frac{|\Xi_k|^2}{(\overline{z_k}^2 - q_k)} \cosh(z_k\, T) \cosh(\overline{z_k} T)$$
$$+ \frac{\Xi_k^2}{({z_k}^2 - q_k)} \sinh^2(z_k\, T) - \frac{z_k\, |\Xi_k|^2}{\overline{z_k}\, (\overline{z_k}^2 - q_k)} \sinh(\overline{z_k}\, T) \sinh(z_k T)$$

$$= -\frac{\Xi_k^2}{(z_k^2 - q_k)} + \frac{|\Xi_k|^2}{(\overline{z_k}^2 - q_k)} \cosh(z_k\, T) \cosh(\overline{z_k} T) - \frac{z_k}{\overline{z_k}} \frac{|\Xi_k|^2}{(\overline{z_k}^2 - q_k)} \sinh(\overline{z_k}\, T) \sinh(z_k T). \tag{5.40}$$

Due to (5.34) and (5.35) this yields

$$\beta_{21}(T) = -\frac{\Xi_k^2}{(z_k^2 - q_k)}$$

$$+ \frac{1}{2} \frac{|\Xi_k|^2}{(\overline{z_k}^2 - q_k)} \left[\left(1 - \frac{z_k}{\overline{z_k}}\right) \cosh((z_k + \overline{z_k})T) + \left(1 + \frac{z_k}{\overline{z_k}}\right) \cosh((z_k - \overline{z_k})T) \right].$$

The second element in the second row of the matrix $\det(M(T))\, M(T)^{-1}$ is

$$\beta_{22}(T) = m_{33}(T) = \frac{\Xi_k}{(z_k^2 - q_k)} \cosh(z_k\, T) - \frac{\overline{\Xi_k}}{(\overline{z_k}^2 - q_k)} \cosh(\overline{z_k} T).$$

For the third element in the second row of the matrix $\det(M(T))\, M(T)^{-1}$ we obtain

$$\beta_{23}(T) = -m_{23}(T) = \frac{\Xi_k}{z_k\, ({z_k}^2 - q_k)} \sinh(z_k\, T) - \frac{\overline{\Xi_k}}{\overline{z_k}\, (\overline{z_k}^2 - q_k)} \sinh(\overline{z_k}\, T).$$

Finally, let us look at the elements in the third row.

For the first element in the third row $\beta_{31}(T)$ of the matrix $\det(M(T))\, M(T)^{-1}$ we obtain

$$\beta_{31}(T) = m_{21}(T)\, m_{32}(T) - m_{22}(T)\, m_{31}(T)$$

$$= |\Xi_k|^2 \left[-\overline{z_k} \cosh(z_k\, T)\, \sinh(\overline{z_k} T) + z_k \cosh(\overline{z_k} T)\, \sinh(z_k\, T)\right].$$

For the hyperbolic functions, we have the general identity

$$\cosh(z_k T) \sinh(\overline{z_k} T) = \frac{1}{2} \left[\sinh((z_k + \overline{z_k})T) - \sinh((z_k - \overline{z_k})T)\right] \tag{5.41}$$

and

$$\sinh(z_k T) \cosh(\overline{z_k} T) = \frac{1}{2} \left[\sinh((z_k + \overline{z_k})T) + \sinh((z_k - \overline{z_k})T)\right]. \tag{5.42}$$

Using (5.41) and (5.42) we obtain

$$\beta_{31}(T) = |\Xi_k|^2 \left[-\overline{z_k} \cosh(z_k\, T)\, \sinh(\overline{z_k} T) + z_k \cosh(\overline{z_k} T)\, \sinh(z_k\, T)\right]$$

$$= |\Xi_k|^2 \left[-\overline{z_k} \frac{1}{2} \left[\sinh((z_k + \overline{z_k})T) - \sinh((z_k - \overline{z_k})T)\right] + z_k \frac{1}{2} \left[\sinh((z_k + \overline{z_k})T) + \sinh((z_k - \overline{z_k})T)\right]\right]$$

$$= |\Xi_k|^2 \left[\frac{z_k - \overline{z_k}}{2} \sinh((z_k + \overline{z_k})T) + \frac{z_k + \overline{z_k}}{2} \sinh((z_k - \overline{z_k})T)\right].$$

The second element in the third row $\beta_{32}(T)$ of the matrix $\det(M(T))\, M(T)^{-1}$ is

$$\beta_{32}(T) = -m_{32}(T) + m_{31}(T) = \overline{z_k}\, \overline{\Xi_k} \sinh(\overline{z_k} T) - z_k\, \Xi_k \sinh(z_k T).$$

Finally the element $\beta_{33}(T)$ of the matrix $\det(M(T))\, M(T)^{-1}$ is

$$\beta_{33}(T) = m_{22}(T) - m_{21}(T) = \overline{\Xi_k} \cosh(\overline{z_k} T) - \Xi_k \cosh(z_k T).$$

We summarize the entries of $\det(M(T))\, M(T)^{-1}$ in a table:

$$\beta_{11}(T) \;=\; -\frac{\overline{\Xi_k}^2}{\overline{z_k}^2 - q_k} + \frac{1}{2} \frac{|\Xi_k|^2}{{z_k}^2 - q_k} \left[\left(1 - \frac{\overline{z_k}}{z_k}\right) \cosh((z_k + \overline{z_k})T) + \left(1 + \frac{\overline{z_k}}{z_k}\right) \cosh((z_k - \overline{z_k})T)\right]$$

$$\begin{aligned}
\beta_{12}(T) &= -\frac{\Xi_k}{(z_k^2 - q_k)}\cosh(z_k\, T) + \frac{\overline{\Xi_k}}{(\overline{z_k}^2 - q_k)}\cosh(\overline{z_k} T).\\
\beta_{13}(T) &= -\frac{\Xi_k}{z_k\,({z_k}^2 - q_k)}\sinh(z_k\, T) + \frac{\overline{\Xi_k}}{\overline{z_k}\,(\overline{z_k}^2 - q_k)}\sinh(\overline{z_k}\, T).\\
\beta_{21}(T) &= -\frac{\Xi_k^2}{(z_k^2 - q_k)}\\
&+ \frac{1}{2}\frac{|\Xi_k|^2}{(\overline{z_k}^2 - q_k)}\left[\left(1 - \frac{z_k}{\overline{z_k}}\right)\cosh((z_k + \overline{z_k})T) + \left(1 + \frac{z_k}{\overline{z_k}}\right)\cosh((z_k - \overline{z_k})T)\right].\\
\beta_{22}(T) &= \frac{\Xi_k}{(z_k^2 - q_k)}\cosh(z_k\, T) - \frac{\overline{\Xi_k}}{(\overline{z_k}^2 - q_k)}\cosh(\overline{z_k} T).\\
\beta_{23}(T) &= \frac{\Xi_k}{z_k\,({z_k}^2 - q_k)}\sinh(z_k\, T) - \frac{\overline{\Xi_k}}{\overline{z_k}\,(\overline{z_k}^2 - q_k)}\sinh(\overline{z_k}\, T).\\
\beta_{31}(T) &= |\Xi_k|^2\left[\frac{z_k - \overline{z_k}}{2}\sinh((z_k + \overline{z_k})T) + \frac{z_k + \overline{z_k}}{2}\sinh((z_k - \overline{z_k})T)\right].\\
\beta_{32}(T) &= \overline{z_k}\,\overline{\Xi_k}\ \sinh(\overline{z_k}T) - z_k\,\Xi_k\ \sinh(z_k T).\\
\beta_{33}(T) &= \overline{\Xi_k}\ \cosh(\overline{z_k}T) - \Xi_k\ \cosh(z_k T).
\end{aligned}$$

For

$$b_{k,T}(t) = A_k^T\cosh(z_k(t-T)) + B_k^T\cosh(\overline{z_k}(t-T)) + C_k^T\left[\frac{\sinh(z_k(t-T))}{z_k\ (z_k^2 - q_k)} - \frac{\sinh(\overline{z_k}(t-T))}{\overline{z_k}\,(\overline{z_k}^2 - q_k)}\right]$$

we have

$$V(T) = \begin{pmatrix} A_k^T\\ B_k^T\\ C_k^T\end{pmatrix} = \frac{1}{\det M(T)}\begin{pmatrix}\beta_{11}(T) & \beta_{12}(T) & \beta_{13}(T)\\ \beta_{21}(T) & \beta_{22}(T) & \beta_{23}(T)\\ \beta_{31}(T) & \beta_{32}(T) & \beta_{33}(T)\end{pmatrix}\begin{pmatrix} b_k(T)\\ (1+\gamma\,\lambda_k)a_k(0)\\ (1+\gamma\,\lambda_k)\,a_k'(0)\end{pmatrix}. \tag{5.43}$$

Due to (5.30) we have

$$b_{k,T}(t) = F_{k,T}(t)\,b_k(T) + G_{k,T}(t)\,(1+\gamma\,\lambda_k)\,a_k(0) + H_{k,T}(t)\,(1+\gamma\,\lambda_k)\,a_k'(0), \tag{5.44}$$

where the terms that are multiplied with $b_k(T)$ and come from the first column of $M(T)^{-1}$ are collected in $F_{k,T}(t)$, and analogously for $G_{k,T}(t)$ and $H_{k,T}(t)$.

Thus we have

$$\begin{aligned}
F_{k,T}(t) = &\frac{\beta_{11}(T)}{\det M(T)}\cosh(z_k(t-T)) + \frac{\beta_{21}(T)}{\det M(T)}\cosh(\overline{z_k}(t-T))\\
&+ \frac{\beta_{31}(T)}{\det M(T)}\left[\frac{\sinh(z_k(t-T))}{z_k\ (z_k^2 - q_k)} - \frac{\sinh(\overline{z_k}(t-T))}{\overline{z_k}\,(\overline{z_k}^2 - q_k)}\right],
\end{aligned} \tag{5.45}$$

$$G_{k,T}(t) = \frac{\beta_{12}(T)}{\det M(T)}\cosh(z_k(t-T)) + \frac{\beta_{22}(T)}{\det M(T)}\cosh(\overline{z_k}(t-T)) \tag{5.46}$$

$$+\frac{\beta_{32}(T)}{\det M(T)}\left[\frac{\sinh(z_k(t-T))}{z_k\ (z_k^2-q_k)}-\frac{\sinh(\overline{z_k}(t-T))}{\overline{z_k}\ (\overline{z_k}^2-q_k)}\right],$$

$$\begin{aligned}H_{k,T}(t)&=\frac{\beta_{13}(T)}{\det M(T)}\cosh(z_k(t-T))+\frac{\beta_{23}(T)}{\det M(T)}\cosh(\overline{z_k}(t-T))\\&+\frac{\beta_{33}(T)}{\det M(T)}\left[\frac{\sinh(z_k(t-T))}{z_k\ (z_k^2-q_k)}-\frac{\sinh(\overline{z_k}(t-T))}{\overline{z_k}\ (\overline{z_k}^2-q_k)}\right].\end{aligned}\tag{5.47}$$

Hence (with $\det M(T)$ as in (5.36))

$$\begin{aligned}&\det M(T)F_{k,T}(t)\\&=\left[-\frac{\overline{\Xi_k}^2}{\overline{z_k}^2-q_k}+\frac{1}{2}\frac{|\Xi_k|^2}{z_k{}^2-q_k}\left[\left(1-\frac{\overline{z_k}}{z_k}\right)\cosh((z_k+\overline{z_k})T)\right.\right.\\&\qquad\left.\left.+\left(1+\frac{\overline{z_k}}{z_k}\right)\cosh((z_k-\overline{z_k})T)\right]\right]\cosh(z_k(t-T))\\&+\left[\frac{\Xi_k^2}{(z_k^2-q_k)}+\frac{1}{2}\frac{|\Xi_k|^2}{(\overline{z_k}^2-q_k)}\left[\left(1-\frac{z_k}{\overline{z_k}}\right)\cosh((z_k+\overline{z_k})T)\right.\right.\\&\qquad\left.\left.+\left(1+\frac{z_k}{\overline{z_k}}\right)\cosh((z_k-\overline{z_k})T)\right]\right]\cosh(\overline{z_k}(t-T))\\&+|\Xi_k|^2\left[\frac{z_k-\overline{z_k}}{2}\sinh((z_k+\overline{z_k})T)+\frac{z_k+\overline{z_k}}{2}\sinh((z_k-\overline{z_k})T)\right]\left[\frac{\sinh(z_k(t-T))}{z_k\ (z_k^2-q_k)}-\frac{\sinh(\overline{z_k}(t-T))}{\overline{z_k}\ (\overline{z_k}^2-q_k)}\right]\\&=2\,\mathrm{Re}\left(\left[-\frac{\overline{\Xi_k}^2}{\overline{z_k}^2-q_k}+\frac{1}{2}\frac{|\Xi_k|^2}{z_k{}^2-q_k}\left[\left(1-\frac{\overline{z_k}}{z_k}\right)\cosh((z_k+\overline{z_k})T)\right.\right.\right.\\&\qquad\left.\left.\left.+\left(1+\frac{\overline{z_k}}{z_k}\right)\cosh((z_k-\overline{z_k})T)\right]\right]\cosh(z_k(t-T))\right)\\&+|\Xi_k|^2\left[\frac{z_k-\overline{z_k}}{2}\sinh((z_k+\overline{z_k})T)+\frac{z_k+\overline{z_k}}{2}\sinh((z_k-\overline{z_k})T)\right]\left[\frac{\sinh(z_k(t-T))}{z_k\ (z_k^2-q_k)}-\frac{\sinh(\overline{z_k}(t-T))}{\overline{z_k}\ (\overline{z_k}^2-q_k)}\right].\end{aligned}$$

Whence we have

$$\begin{aligned}&\det M(T)F_{k,T}(t)\\&=2\,\mathrm{Re}\left(\left[-\frac{\overline{\Xi_k}^2}{\overline{z_k}^2-q_k}+\frac{|\Xi_k|^2}{z_k{}^2-q_k}\left[\cosh(z_kT)\cosh(\overline{z_k}T)-\frac{\overline{z_k}}{z_k}\sinh(z_kT)\sinh(\overline{z_k}T)\right]\right]\cosh(z_k(t-T))\right)\\&+|\Xi_k|^2\left[z_k\sinh(z_kT)\cosh(\overline{z_k}T)-\overline{z_k}\cosh(z_kT)\sinh(\overline{z_k}T)\right]\left[\frac{\sinh(z_k(t-T))}{z_k\ (z_k^2-q_k)}-\frac{\sinh(\overline{z_k}(t-T))}{\overline{z_k}\ (\overline{z_k}^2-q_k)}\right]\\&=2\,\mathrm{Re}\left(\left[-\frac{\overline{\Xi_k}^2}{\overline{z_k}^2-q_k}+\frac{|\Xi_k|^2}{z_k{}^2-q_k}\left[\cosh(z_kT)\cosh(\overline{z_k}T)-\frac{\overline{z_k}}{z_k}\sinh(z_kT)\sinh(\overline{z_k}T)\right]\right]\cosh(z_k(t-T))\right)\\&+|\Xi_k|^2 2\,\mathrm{Re}\left(\sinh(z_kT)\cosh(\overline{z_k}T)\frac{\sinh(z_k(t-T))}{(z_k^2-q_k)}-\frac{\overline{z_k}}{z_k}\cosh(z_kT)\sinh(\overline{z_k}T)\left[\frac{\sinh(z_k(t-T))}{(z_k^2-q_k)}\right]\right).\end{aligned}$$

Verification of the Basis Function $F_{k,T}(t)$ We start with the function $F_{k,T}(t)$ from (5.24) that is multiplied with β_k, i.e. we consider the product $d(k,T)\,F_{k,T}(t)$. We will show that $F_{k,T}(t)$ satisfies the boundary conditions

$$\begin{aligned}
F_{k,T}(T) &= 1,\\
F'''_{k,T}(T) &= \frac{\gamma^2 - 2\,\gamma\,\lambda_k - 1}{\gamma}\,F'_{k,T}(T),\\
-F''_{k,T}(0) + (\gamma - \lambda_k)\,F_{k,T}(0) &= 0,\\
-F'''_{k,T}(0) + (\gamma - \lambda_k)\,F'_{k,T}(0) &= 0.
\end{aligned}$$

We have

$$\begin{aligned}
d(k,T)\,F_{k,T}(t) = {} & 2\,\mathrm{Re}\left(-\frac{\overline{\Xi_k}^2}{\overline{z_k}^2 - q_k}\cosh(z_k(t-T))\right)\\
& + 2\,\mathrm{Re}\left(\frac{|\Xi_k|^2}{{z_k}^2 - q_k}\cosh(\overline{z_k}T)\cosh(z_k\,T)\cosh(z_k(t-T))\right)\\
& + 2\,\mathrm{Re}\left(\frac{|\Xi_k|^2}{z_k^2 - q_k}\cosh(\overline{z_k}T)\sinh(z_kT)\sinh(z_k(t-T))\right)\\
& - 2\,\mathrm{Re}\left(\frac{|\Xi_k|^2}{z_k^2 - q_k}\frac{\overline{z_k}}{z_k}\sinh(\overline{z_k}T)\left[\sinh(z_k\,T)\cosh(z_k(t-T)) + \cosh(z_kT)\sinh(z_k(t-T))\right]\right).
\end{aligned}$$

Hence for $t = T$ we have

$$\begin{aligned}
d(k,T)\,F_{k,T}(T) = {} & 2\,\mathrm{Re}\left(-\frac{\overline{\Xi_k}^2}{\overline{z_k}^2 - q_k}\right)\\
& + 2\,\mathrm{Re}\left(\frac{|\Xi_k|^2}{{z_k}^2 - q_k}\cosh(\overline{z_k}T)\cosh(z_k\,T) - \frac{|\Xi_k|^2}{z_k^2 - q_k}\frac{\overline{z_k}}{z_k}\sinh(\overline{z_k}T)\sinh(z_k\,T)\right)\\
= {} & d(k,T).
\end{aligned}$$

Since $d(k,T) \neq 0$, this implies $F_{k,T}(T) = 1$. For the derivative we obtain

$$\begin{aligned}
d(k,T)\,F'_{k,T}(t) = {} & 2\,\mathrm{Re}\left(z_k\left[-\overline{\Xi_k}^2 + |\Xi_k|^2\left[\cosh(z_kT)\cosh(\overline{z_k}T)\right]\right]\sinh(z_k\,(t-T))\right)\\
& + 2\,\mathrm{Re}\left(z_k\frac{|\Xi_k|^2}{{z_k}^2 - q_k}\cosh(\overline{z_k}T)\cosh(z_k\,T)\sinh(z_k(t-T))\right)\\
& + 2\,\mathrm{Re}\left(z_k\frac{|\Xi_k|^2}{z_k^2 - q_k}\cosh(\overline{z_k}T)\sinh(z_kT)\cosh(z_k(t-T))\right)\\
& - 2\,\mathrm{Re}\left(\frac{|\Xi_k|^2}{z_k^2 - q_k}\overline{z_k}\sinh(\overline{z_k}T)\left[\sinh(z_k\,T)\sinh(z_k(t-T)) + \cosh(z_kT)\cosh(z_k(t-T))\right]\right).
\end{aligned}$$

Thus we have

$$d(k,T)\,F'_{k,T}(T) = 2\,\mathrm{Re}\left(z_k\frac{|\Xi_k|^2}{z_k^2 - q_k}\cosh(\overline{z_k}T)\sinh(z_kT)\right)$$

$$- 2\,\mathrm{Re}\left(\frac{|\Xi_k|^2}{z_k^2 - q_k}\overline{z_k}\sinh(\overline{z_k}T)\cosh(z_kT)\right).$$

Therefore we have $F'_{k,T}(T) = 0$. For $t = 0$ we have

$$\begin{aligned}
d(k,T)\,F_{k,T}(0) &= 2\,\mathrm{Re}\left(-\frac{\overline{\Xi_k}^2}{\overline{z_k}^2 - q_k}\cosh(z_kT)\right)\\
&+ 2\,\mathrm{Re}\left(\frac{|\Xi_k|^2}{z_k{}^2 - q_k}\cosh(\overline{z_k}T)\cosh(z_k\,T)\cosh(z_kT)\right)\\
&- 2\,\mathrm{Re}\left(\frac{|\Xi_k|^2}{z_k^2 - q_k}\cosh(\overline{z_k}T)\sinh(z_kT)\sinh(z_kT)\right)\\
&- 2\,\mathrm{Re}\left(\frac{|\Xi_k|^2}{z_k^2 - q_k}\frac{\overline{z_k}}{z_k}\sinh(\overline{z_k}T)\left[\sinh(z_k\,T)\cosh(z_kT) - \cosh(z_kT)\sinh(z_kT)\right]\right)\\
&= 2\,\mathrm{Re}\left(-\frac{\overline{\Xi_k}^2}{\overline{z_k}^2 - q_k}\cosh(z_kT) + \frac{|\Xi_k|^2}{z_k{}^2 - q_k}\cosh(\overline{z_k}T)\right).
\end{aligned}$$

For the second derivative, we have

$$\begin{aligned}
&d(k,T)\,F''_{k,T}(t)\\
&= 2\,\mathrm{Re}\left(-z_k^2\frac{\overline{\Xi_k}^2}{\overline{z_k}^2 - q_k}\cosh(z_k(t-T))\right)\\
&+ 2\,\mathrm{Re}\left(z_k^2\frac{|\Xi_k|^2}{z_k{}^2 - q_k}\cosh(\overline{z_k}T)\cosh(z_k\,T)\cosh(z_k(t-T))\right)\\
&+ 2\,\mathrm{Re}\left(z_k^2\frac{|\Xi_k|^2}{z_k^2 - q_k}\cosh(\overline{z_k}T)\sinh(z_kT)\sinh(z_k(t-T))\right)\\
&- 2\,\mathrm{Re}\left(\frac{|\Xi_k|^2}{z_k^2 - q_k}|z_k|^2\sinh(\overline{z_k}T)\left[\sinh(z_k\,T)\cosh(z_k(t-T)) + \cosh(z_kT)\sinh(z_k(t-T))\right]\right).
\end{aligned}$$

For $t = 0$ this yields

$$\begin{aligned}
d(k,T)\,F''_{k,T}(0) &= 2\,\mathrm{Re}\left(-z_k^2\frac{\overline{\Xi_k}^2}{\overline{z_k}^2 - q_k}\cosh(z_kT)\right)\\
&+ 2\,\mathrm{Re}\left(z_k^2\frac{|\Xi_k|^2}{z_k{}^2 - q_k}\cosh(\overline{z_k}T)\cosh(z_k\,T)\cosh(z_kT)\right)\\
&- 2\,\mathrm{Re}\left(z_k^2\frac{|\Xi_k|^2}{z_k^2 - q_k}\cosh(\overline{z_k}T)\sinh(z_kT)\sinh(z_kT))\right)\\
&- 2\,\mathrm{Re}\left(\frac{|\Xi_k|^2}{z_k^2 - q_k}|z_k|^2\sinh(\overline{z_k}T)\left[\sinh(z_k\,T)\cosh(z_kT) - \cosh(z_kT)\sinh(z_kT)\right]\right)\\
&= 2\,\mathrm{Re}\left(z_k^2\left[-\frac{\overline{\Xi_k}^2}{\overline{z_k}^2 - q_k}\cosh(z_kT)\frac{|\Xi_k|^2}{z_k{}^2 - q_k}\cosh(\overline{z_k}T)\right)\right].
\end{aligned}$$

Hence we have

$$d(k,T)\ \left[-F''_{k,T}(0) + (\gamma - \lambda_k)\, F_{k,T}(0)\right] = 0.$$

Thus we have $-F''_{k,T}(0) + (\gamma - \lambda_k)\, F_{k,T}(0) = 0$.

To proceed, let us observe that

$$\begin{aligned}
d(k,T)\, F'_{k,T}(0) &= 2\,\mathrm{Re}\left(z_k \frac{\overline{\Xi_k}^2}{\overline{z_k}^2 - q_k} \sinh(z_k T)\right) \\
&- 2\,\mathrm{Re}\left(z_k \frac{|\Xi_k|^2}{{z_k}^2 - q_k} \cosh(\overline{z_k} T) \cosh(z_k\, T) \sinh(z_k T)\right) \\
&+ 2\,\mathrm{Re}\left(z_k \frac{|\Xi_k|^2}{z_k^2 - q_k} \cosh(\overline{z_k} T) \sinh(z_k T) \cosh(z_k T)\right) \\
&- 2\,\mathrm{Re}\left(\frac{|\Xi_k|^2}{z_k^2 - q_k} \overline{z_k} \sinh(\overline{z_k} T) \left[-\sinh(z_k\, T) \sinh(z_k T) + \cosh(z_k T) \cosh(z_k T)\right]\right) \\
&= 2\,\mathrm{Re}\left(z_k \frac{\overline{\Xi_k}^2}{\overline{z_k}^2 - q_k} \sinh(z_k T)\right) - 2\,\mathrm{Re}\left(\overline{z_k} \frac{|\Xi_k|^2}{z_k^2 - q_k} \sinh(\overline{z_k} T)\right).
\end{aligned}$$

For the third derivative, we have

$$\begin{aligned}
& d(k,T)\, F'''_{k,T}(t) \\
=&\ 2\ \mathrm{Re}\left(-z_k^3 \frac{\overline{\Xi_k}^2}{\overline{z_k}^2 - q_k} \sinh(z_k(t-T))\right) \\
&+ 2\ \mathrm{Re}\left(z_k^3 \frac{|\Xi_k|^2}{{z_k}^2 - q_k} \cosh(\overline{z_k} T) \cosh(z_k\, T) \sinh(z_k(t-T))\right) \\
&+ 2\ \mathrm{Re}\left(z_k^3 \frac{|\Xi_k|^2}{z_k^2 - q_k} \cosh(\overline{z_k} T) \sinh(z_k T) \cosh(z_k(t-T))\right) \\
&- 2\ \mathrm{Re}\left(\frac{|\Xi_k|^2}{z_k^2 - q_k} |z_k|^2 z_k \sinh(\overline{z_k} T) \left[\sinh(z_k\, T) \sinh(z_k(t-T)) + \cosh(z_k T) \cosh(z_k(t-T))\right]\right).
\end{aligned} \tag{5.48}$$

This yields

$$
\begin{aligned}
&d(k,T)\, F'''_{k,T}(0)\\
&= \ 2\ \mathrm{Re}\left(z_k^3 \frac{\overline{\Xi_k}^2}{\overline{z_k}^2 - q_k} \sinh(z_k T)\right)\\
&+ 2\ \mathrm{Re}\left(-z_k^3 \frac{|\Xi_k|^2}{\overline{z_k}^2 - q_k} \cosh(\overline{z_k}T)\cosh(z_k\, T)\sinh(z_k T)\right)\\
&+ 2\ \mathrm{Re}\left(z_k^3 \frac{|\Xi_k|^2}{z_k^2 - q_k} \cosh(\overline{z_k}T)\sinh(z_k T)\cosh(z_k T)\right)\\
&- 2\ \mathrm{Re}\left(\frac{|\Xi_k|^2}{z_k^2 - q_k} |z_k|^2 z_k \sinh(\overline{z_k}T)\left[-\sinh(z_k\, T)\sinh(z_k T) + \cosh(z_k T)\cosh(z_k T)\right]\right)\\
&= \ 2\ \mathrm{Re}\left(z_k^3 \frac{\overline{\Xi_k}^2}{\overline{z_k}^2 - q_k} \sinh(z_k T)\right) - 2\ \mathrm{Re}\left(\frac{|\Xi_k|^2}{z_k^2 - q_k} z_k^2 \overline{z_k} \sinh(\overline{z_k}T)\right).
\end{aligned}
\tag{5.49}
$$

Hence we have

$$
\begin{aligned}
d(k,T)\ &\left[-F'''_{k,T}(0) + (\gamma - \lambda_k)\, F'_{k,T}(0)\right]\\
&= 2\,\mathrm{Re}\left(-z_k^3 \frac{\overline{\Xi_k}^2}{\overline{z_k}^2 - q_k} \sinh(z_k T)\right) + 2\,\mathrm{Re}\left(\frac{|\Xi_k|^2}{z_k^2 - q_k} z_k^2 \overline{z_k} \sinh(\overline{z_k}T)\right)\\
&+ (\gamma - \lambda_k)\left[2Re\left(z_k \frac{\overline{\Xi_k}^2}{\overline{z_k}^2 - q_k} \sinh(z_k T)\right) - 2\,\mathrm{Re}\left(\overline{z_k} \frac{|\Xi_k|^2}{z_k^2 - q_k} \sinh(\overline{z_k}T)\right)\right]\\
&= 2\,\mathrm{Re}\left(z_k\, \Xi_k \frac{\overline{\Xi_k}^2}{\overline{z_k}^2 - q_k} \sinh(z_k T)\right) - 2\,\mathrm{Re}\left(\frac{|\Xi_k|^2}{z_k^2 - q_k} \Xi_k \overline{z_k} \sinh(\overline{z_k}T)\right) = 0.
\end{aligned}
$$

We have

$$
\begin{aligned}
d(k,T)\, F'''_{k,T}(T) = 2\,\mathrm{Re}&\left(z_k^3 \frac{|\Xi_k|^2}{z_k^2 - q_k} \cosh(\overline{z_k}T)\sinh(z_k T)\right)\\
- 2\,\mathrm{Re}&\left(\frac{|\Xi_k|^2}{z_k^2 - q_k} |z_k|^2 z_k \sinh(\overline{z_k}T)\cosh(z_k T)\right).
\end{aligned}
$$

Whence we have

$$
\begin{aligned}
d(k,T)\left[F'''_{k,T}(T) - q_k\, F'_{k,T}(T)\right] = \ 2\ \mathrm{Re}&\left((z_k^2 - q_k) z_k \frac{|\Xi_k|^2}{z_k^2 - q_k} \cosh(\overline{z_k}T)\sinh(z_k T)\right)\\
- 2\ \mathrm{Re}&\left((z_k^2 - q_k)\overline{z_k} \frac{|\Xi_k|^2}{z_k^2 - q_k} \sinh(\overline{z_k}T)\cosh(z_k T)\right) = 0.
\end{aligned}
\tag{5.50}
$$

Thus we have shown that $F_{k,T}$ satisfies the required boundary conditions.

Verification of the Basis Function $G_{k,T}(t)$. We continue with $G_{k,T}(t)$ from (5.24) that is multiplied with $(1+\gamma\,\lambda_k)\,a_{k,T}(0)$, i.e.

$$\begin{aligned} d(k,T)\,G_{k,T}(t) &= 2\,\mathrm{Re}\left(\frac{\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\cosh(\overline{z_k}T)\cosh(z_k(t-T))\right) \\ &\quad - 2\,\mathrm{Re}\left(\frac{\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\cosh(\overline{z_k}T)\cosh(\overline{z_k}(t-T))\right) \\ &\quad + 2\,\mathrm{Re}\left(\frac{\overline{z_k}\;\overline{\Xi_k}}{z_k({z_k}^2 - q_k)}\sinh(\overline{z_k}T)\sinh(z_k(t-T))\right) \\ &\quad - 2\,\mathrm{Re}\left(\frac{\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\sinh(\overline{z_k}T)\sinh(\overline{z_k}(t-T))\right). \end{aligned}$$

Due to the definition of z_k, the function $G_{k,T}(t)$ satisfies the ODE (5.20). Moreover, for $t = T$ we have

$$d(k,T)\,G_{k,T}(T) = 2\,\mathrm{Re}\left(\frac{\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\cosh(\overline{z_k}T)\right) - 2\,\mathrm{Re}\left(\frac{\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\cosh(\overline{z_k}T)\right) = 0.$$

In addition, for the derivative, we obtain

$$\begin{aligned} d(k,T)\,G'_{k,T}(t) &= 2\,\mathrm{Re}\left(\frac{z_k\;\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\cosh(\overline{z_k}T)\sinh(z_k(t-T))\right) \\ &\quad - 2\,\mathrm{Re}\left(\frac{\overline{z_k}\;\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\cosh(\overline{z_k}T)\sinh(\overline{z_k}(t-T))\right) \\ &\quad + 2\,\mathrm{Re}\left(\frac{\overline{z_k}\;\overline{\Xi_k}}{{z_k}^2 - q_k}\sinh(\overline{z_k}T)\cosh(z_k(t-T))\right) \\ &\quad - 2\,\mathrm{Re}\left(\frac{\overline{z_k}\;\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\sinh(\overline{z_k}T)\cosh(\overline{z_k}(t-T))\right). \end{aligned}$$

Hence for $t = T$, we have the derivative

$$d(k,T)\,G'_{k,T}(T) = 2\,\mathrm{Re}\left(\frac{\overline{z_k}\;\overline{\Xi_k}}{{z_k}^2 - q_k}\sinh(\overline{z_k}T)\right) - 2\,\mathrm{Re}\left(\frac{\overline{z_k}\;\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\sinh(\overline{z_k}T)\right).$$

For the second derivative, we obtain

$$\begin{aligned} d(k,T)\,G''_{k,T}(t) &= 2\,\mathrm{Re}\left(\frac{{z_k}^2\;\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\cosh(\overline{z_k}T)\cosh(z_k(t-T))\right) \\ &\quad - 2\,\mathrm{Re}\left(\frac{\overline{z_k}^2\;\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\cosh(\overline{z_k}T)\cosh(\overline{z_k}(t-T))\right) \\ &\quad + 2\,\mathrm{Re}\left(\frac{z_k\overline{z_k}\;\overline{\Xi_k}}{{z_k}^2 - q_k}\sinh(\overline{z_k}T)\sinh(z_k(t-T))\right) \end{aligned}$$

$$- 2\,\mathrm{Re}\left(\frac{\overline{z_k}^2\,\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\sinh(\overline{z_k}T)\sinh(\overline{z_k}(t-T))\right).$$

For $t=0$, equation (5.27) implies

$$\begin{aligned} d(k,T)\,G''_{k,T}(0) &= 2\,\mathrm{Re}\left(\frac{{z_k}^2\,\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\cosh(\overline{z_k}T)\cosh(z_kT)\right) - 2\,\mathrm{Re}\left(\frac{\overline{z_k}^2\,\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\cosh^2(\overline{z_k}T))\right)\\ &- 2\,\mathrm{Re}\left(\frac{z_k\overline{z_k}\,\overline{\Xi_k}}{{z_k}^2 - q_k}\sinh(\overline{z_k}T)\sinh(z_kT)\right) + 2\,\mathrm{Re}\left(\frac{\overline{z_k}^2\,\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\sinh^2(\overline{z_k}T)\right). \end{aligned}$$

For $t=0$, equation (5.27) implies

$$\begin{aligned} d(k,T)\,G_{k,T}(0) &= 2\,\mathrm{Re}\left(\frac{\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\cosh(\overline{z_k}T)\cosh(z_kT)\right) - 2\,\mathrm{Re}\left(\frac{\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\cosh^2(\overline{z_k}T)\right)\\ &- 2\,\mathrm{Re}\left(\frac{\overline{z_k}\,\overline{\Xi_k}}{z_k({z_k}^2 - q_k)}\sinh(\overline{z_k}T)\sinh(z_kT)\right) + 2\,\mathrm{Re}\left(\frac{\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\sinh^2(\overline{z_k}T)\right). \end{aligned}$$

Hence in view of (5.25) we obtain

$$\begin{aligned} &d(k,T)\,\left[-G''_{k,T}(0) + (\gamma-\lambda_k)\,G_{k,T}(0)\right]\\ &\quad= -2\,\mathrm{Re}\left(\frac{{z_k}^2\,\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\cosh(\overline{z_k}T)\cosh(z_kT)\right) + 2\,\mathrm{Re}\left(\frac{\overline{z_k}^2\,\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\cosh^2(\overline{z_k}T))\right)\\ &\quad+ 2\,\mathrm{Re}\left(\frac{z_k\overline{z_k}\,\overline{\Xi_k}}{{z_k}^2 - q_k}\sinh(\overline{z_k}T)\sinh(z_kT)\right) - 2\,\mathrm{Re}\left(\frac{\overline{z_k}^2\,\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\sinh^2(\overline{z_k}T)\right)\\ &\quad+ (\Xi_k + z_k^2) d(k,T)\,G_{k,T}(0)\\ &\quad= -2\,\mathrm{Re}\left(\frac{\Xi_k^2}{z_k^2 - q_k}\right) + 2|\Xi_k|^2\,\mathrm{Re}\left(\frac{1}{z_k^2 - q_k}|\cosh^2(z_kT)| - \frac{\overline{z_k}}{z_k}\frac{1}{{z_k}^2 - q_k}|\sinh^2(z_kT)|\right)\\ &\quad= d(k,T) \end{aligned}$$

due to the definition (5.26). Since $d(k,T)\neq 0$, this yields

$$-G''_{k,T}(0) + (\gamma-\lambda_k)\,G_{k,T}(0) = 1.$$

For $t=0$, the derivative satisfies the equation

$$\begin{aligned} d(k,T)\,G'_{k,T}(0) &= -2\,\mathrm{Re}\left(\frac{z_k\,\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\cosh(\overline{z_k}T)\sinh(z_kT)\right) + 2\,\mathrm{Re}\left(\frac{\overline{z_k}\,\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\cosh(\overline{z_k}T)\sinh(\overline{z_k}T)\right)\\ &+ 2\,\mathrm{Re}\left(\frac{\overline{z_k}\,\overline{\Xi_k}}{{z_k}^2 - q_k}\sinh(\overline{z_k}T)\cosh(z_kT)\right) - 2\,\mathrm{Re}\left(\frac{\overline{z_k}\,\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\sinh(\overline{z_k}T)\cosh(\overline{z_k}T)\right)\\ &= -2\,\mathrm{Re}\left(\frac{z_k\,\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\cosh(\overline{z_k}T)\sinh(z_kT)\right) + 2\,\mathrm{Re}\left(\frac{\overline{z_k}\,\overline{\Xi_k}}{{z_k}^2 - q_k}\sinh(\overline{z_k}T)\cosh(z_kT)\right). \end{aligned}$$

For the third derivative, we obtain

$$
\begin{aligned}
d(k,T)\, G'''_{k,T}(t) = {} & 2\,\mathrm{Re}\left(\frac{{z_k}^3\,\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\cosh(\overline{z_k}T)\sinh(z_k(t-T))\right)\\
& - 2\,\mathrm{Re}\left(\frac{\overline{z_k}^3\,\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\cosh(\overline{z_k}T)\sinh(\overline{z_k}(t-T))\right)\\
& + 2\,\mathrm{Re}\left(\frac{{z_k}^2\overline{z_k}\,\overline{\Xi_k}}{{z_k}^2 - q_k}\sinh(\overline{z_k}T)\cosh(z_k(t-T))\right)\\
& - 2\,\mathrm{Re}\left(\frac{\overline{z_k}^3\,\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\sinh(\overline{z_k}T)\cosh(\overline{z_k}(t-T))\right).
\end{aligned}
$$

Hence for $t = 0$ we have

$$
\begin{aligned}
d(k,T)\, G'''_{k,T}(0) = {} & -2\,\mathrm{Re}\left(\frac{{z_k}^3\,\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\cosh(\overline{z_k}T)\sinh(z_kT)\right) + 2\,\mathrm{Re}\left(\frac{\overline{z_k}^3\,\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\cosh(\overline{z_k}T)\sinh(\overline{z_k}T)\right)\\
& + 2\,\mathrm{Re}\left(\frac{{z_k}^2\overline{z_k}\,\overline{\Xi_k}}{{z_k}^2 - q_k}\sinh(\overline{z_k}T)\cosh(z_kT)\right) - 2\,\mathrm{Re}\left(\frac{\overline{z_k}^3\,\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\sinh(\overline{z_k}T)\cosh(\overline{z_k}T)\right)\\
= {} & -2\,\mathrm{Re}\left(\frac{{z_k}^3\,\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\cosh(\overline{z_k}T)\sinh(z_kT)\right) + 2\,\mathrm{Re}\left(\frac{{z_k}^2\overline{z_k}\,\overline{\Xi_k}}{{z_k}^2 - q_k}\sinh(\overline{z_k}T)\cosh(z_kT)\right).
\end{aligned}
$$

In order to verify the fourth boundary condition in (5.23), in view of (5.25) we obtain the equation

$$
\begin{aligned}
& d(k,T)\left[-G'''_{k,T}(0) + (\gamma - \lambda_k)\, G'_{k,T}(0)\right]\\
& = 2\,\mathrm{Re}\left(\frac{{z_k}^3\,\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\cosh(\overline{z_k}T)\sinh(z_kT)\right) - 2\,\mathrm{Re}\left(\frac{{z_k}^2\overline{z_k}\,\overline{\Xi_k}}{{z_k}^2 - q_k}\sinh(\overline{z_k}T)\cosh(z_kT)\right)\\
& + (\Xi_k + z_k^2)\left(-2\,\mathrm{Re}\left(\frac{z_k\,\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\cosh(\overline{z_k}T)\sinh(z_kT)\right)\right)\\
& + (\Xi_k + z_k^2) 2\,\mathrm{Re}\left(\frac{\overline{z_k}\,\overline{\Xi_k}}{{z_k}^2 - q_k}\sinh(\overline{z_k}T)\cosh(z_kT)\right)\\
& = 0.
\end{aligned}
$$

We have

$$
d(k,T)\, G'''_{k,T}(T) = 2\,\mathrm{Re}\left(\frac{{z_k}^2\overline{z_k}\,\overline{\Xi_k}}{{z_k}^2 - q_k}\sinh(\overline{z_k}T)\right) - 2\,\mathrm{Re}\left(\frac{\overline{z_k}^3\,\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\sinh(\overline{z_k}T)\right).
$$

Hence we have

$$
\begin{aligned}
&d(k,T)\left[G'''_{k,T}(T) - q_k G'_{k,T}(T)\right] \\
&= 2\mathrm{Re}\left(z_k^2\overline{z_k}\frac{\overline{\Xi_k}}{z_k^2-q_k}\sinh(\overline{z_k}T)\right) - 2\mathrm{Re}\left(\overline{z_k}^3\frac{\overline{\Xi_k}}{\overline{z_k}^2-q_k}\sinh\left(\overline{z_k}T\right)\right) \\
&- 2\mathrm{Re}\left(q_k\overline{z_k}\frac{\overline{\Xi_k}}{z_k^2-q_k}\sinh\left(\overline{z_k}T\right)\right) + 2\mathrm{Re}\left(q_k\overline{z_k}\frac{\overline{\Xi_k}}{\overline{z_k}^2-q_k}\sinh\left(\overline{z_k}T\right)\right) \\
&= 2\mathrm{Re}\left(\overline{z_k}\overline{\Xi_k}\sinh\left(\overline{z_k}T\right)\right) - 2\mathrm{Re}\left(\overline{z_k}\overline{\Xi_k}\sinh\left(\overline{z_k}T\right)\right) \\
&= 0.
\end{aligned}
$$

Thus we have shown that

$$
\begin{aligned}
G_{k,T}(T) &= 0, \\
G'''_{k,T}(T) &= \frac{\gamma^2 - 2\,\gamma\,\lambda_k - 1}{\gamma}\,G'_{k,T}(T), \\
-G''_{k,T}(0) + (\gamma - \lambda_k)\,G_{k,T}(0) &= 1, \\
-G'''_{k,T}(0) + (\gamma - \lambda_k)\,G'_{k,T}(0) &= 0.
\end{aligned}
$$

Verification of the Basis Function $H_{k,T}(t)$. Now we consider $H_{k,T}(t)$ from (5.24) that is multiplied with $(1+\gamma\,\lambda_k)\,a'_{k,T}(0)$, i.e.

$$
\begin{aligned}
d(k,T)\,H_{k,T}(t) = &-2\,\mathrm{Re}\left(\frac{\Xi_k}{z_k({z_k}^2-q_k)}\sinh(z_kT)\cosh(z_k(t-T))\right) \\
&+ 2\,\mathrm{Re}\left(\frac{\Xi_k}{z_k({z_k}^2-q_k)}\sinh(z_kT)\cosh(\overline{z_k}(t-T))\right) \\
&+ 2\,\mathrm{Re}\left(\frac{\overline{\Xi_k}}{z_k({z_k}^2-q_k)}\cosh(\overline{z_k}T)\sinh(z_k(t-T))\right) \\
&- 2\,\mathrm{Re}\left(\frac{\overline{\Xi_k}}{\overline{z_k}(\overline{z_k}^2-q_k)}\cosh(\overline{z_k}T)\sinh(\overline{z_k}(t-T))\right).
\end{aligned}
$$

Due to the definition of z_k, the function $H_{k,T}(t)$ satisfies the ODE (5.20). Moreover, for $t = T$ we have

$$
d(k,T)\,H_{k,T}(T) = -2\,\mathrm{Re}\left(\frac{\Xi_k}{z_k({z_k}^2-q_k)}\sinh(z_kT)\right) + 2\,\mathrm{Re}\left(\frac{\Xi_k}{z_k({z_k}^2-q_k)}\sinh(z_kT)\right) = 0.
$$

For the derivative we have

$$
\begin{aligned}
d(k,T)\,H'_{k,T}(t) = &-2\,\mathrm{Re}\left(\frac{\Xi_k}{{z_k}^2-q_k}\sinh(z_kT)\sinh(z_k(t-T))\right) \\
&+ 2\,\mathrm{Re}\left(\frac{\overline{z_k}\Xi_k}{z_k({z_k}^2-q_k)}\sinh(z_kT)\sinh(\overline{z_k}(t-T))\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\,\mathrm{Re}\left(\frac{\overline{\Xi_k}}{{z_k}^2 - q_k}\cosh(\overline{z_k}T)\cosh(z_k(t-T))\right)\\
&- 2\,\mathrm{Re}\left(\frac{\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\cosh(\overline{z_k}T)\cosh(\overline{z_k}(t-T))\right).
\end{aligned}
$$

Hence for $t = T$ we obtain

$$
d(k,T)\,H'_{k,T}(T) = 2\,\mathrm{Re}\left(\frac{\overline{\Xi_k}}{{z_k}^2 - q_k}\cosh(\overline{z_k}T)\right) - 2\,\mathrm{Re}\left(\frac{\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\cosh(\overline{z_k}T)\right).
$$

For $t = 0$, we have

$$
\begin{aligned}
&d(k,T)\,H_{k,T}(0)\\
&= -2\,\mathrm{Re}\left(\frac{\Xi_k}{z_k({z_k}^2 - q_k)}\sinh(z_kT)\cosh(z_kT)\right) + 2\,\mathrm{Re}\left(\frac{\Xi_k}{z_k({z_k}^2 - q_k)}\sinh(z_kT)\cosh(\overline{z_k}T)\right)\\
&\quad - 2\,\mathrm{Re}\left(\frac{\overline{\Xi_k}}{z_k({z_k}^2 - q_k)}\cosh(\overline{z_k}T)\sinh(z_kT)\right) + 2\,\mathrm{Re}\left(\frac{\overline{\Xi_k}}{\overline{z_k}(\overline{z_k}^2 - q_k)}\cosh(\overline{z_k}T)\sinh(\overline{z_k}T)\right).
\end{aligned}
$$

For the second derivative, we have

$$
\begin{aligned}
d(k,T)\,H''_{k,T}(t) = &-2\,\mathrm{Re}\left(\frac{z_k\Xi_k}{{z_k}^2 - q_k}\sinh(z_kT)\cosh(z_k(t-T))\right)\\
&+ 2\,\mathrm{Re}\left(\frac{\overline{z_k}^2\Xi_k}{z_k({z_k}^2 - q_k)}\sinh(z_kT)\cosh(\overline{z_k}(t-T))\right)\\
&+ 2\,\mathrm{Re}\left(\frac{z_k\overline{\Xi_k}}{{z_k}^2 - q_k}\cosh(\overline{z_k}T)\sinh(z_k(t-T))\right)\\
&- 2\,\mathrm{Re}\left(\frac{\overline{z_k}\,\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\cosh(\overline{z_k}T)\sinh(\overline{z_k}(t-T))\right).
\end{aligned}
$$

For $t = 0$ this implies

$$
\begin{aligned}
d(k,T)\,H''_{k,T}(0) = &-2\,\mathrm{Re}\left(\frac{z_k\Xi_k}{{z_k}^2 - q_k}\sinh(z_kT)\cosh(z_kT)\right) + 2\,\mathrm{Re}\left(\frac{\overline{z_k}^2\Xi_k}{z_k({z_k}^2 - q_k)}\sinh(z_kT)\cosh(\overline{z_k}T)\right)\\
&- 2\,\mathrm{Re}\left(\frac{z_k\overline{\Xi_k}}{{z_k}^2 - q_k}\cosh(\overline{z_k}T)\sinh(z_kT)\right) + 2\,\mathrm{Re}\left(\frac{\overline{z_k}\,\overline{\Xi_k}}{\overline{z_k}^2 - q_k}\cosh(\overline{z_k}T)\sinh(\overline{z_k}T)\right).
\end{aligned}
$$

Hence in view of (5.25) we obtain

$$
-d(k,T)\,H''_{k,T}(0) + (\gamma - \lambda_k)\,\det(M(T))\,H_{k,T}(0) = 0.
$$

Now we verify that the fourth boundary condition in (5.23) is satisfied. For the third derivative, we have

$$
d(k,T)\,H'''_{k,T}(t) = -2\,\mathrm{Re}\left(\frac{{z_k}^2\Xi_k}{{z_k}^2 - q_k}\sinh(z_kT)\sinh(z_k(t-T))\right)
$$

$$
\begin{aligned}
&+ 2\,\mathrm{Re}\left(\frac{\overline{z_k}^3 \Xi_k}{z_k({z_k}^2 - q_k)} \sinh(z_k T)\sinh(\overline{z_k}(t-T))\right) \\
&+ 2\,\mathrm{Re}\left(\frac{{z_k}^2 \overline{\Xi_k}}{{z_k}^2 - q_k} \cosh(\overline{z_k} T)\cosh(z_k(t-T))\right) \\
&- 2\,\mathrm{Re}\left(\frac{\overline{z_k}^2 \overline{\Xi_k}}{\overline{z_k}^2 - q_k} \cosh(\overline{z_k} T)\cosh(\overline{z_k}(t-T))\right).
\end{aligned}
$$

For $t = 0$ this yields

$$
\begin{aligned}
d(k,T)\, H'''_{k,T}(0) = {}& 2\,\mathrm{Re}\left(\frac{{z_k}^2 \Xi_k}{{z_k}^2 - q_k} \sinh(z_k T)\sinh(z_k T)\right) - 2\,\mathrm{Re}\left(\frac{\overline{z_k}^3 \Xi_k}{z_k({z_k}^2 - q_k)} \sinh(z_k T)\sinh(\overline{z_k} T)\right) \\
&+ 2\,\mathrm{Re}\left(\frac{{z_k}^2 \overline{\Xi_k}}{{z_k}^2 - q_k} \cosh(\overline{z_k} T)\cosh(z_k T)\right) - 2\,\mathrm{Re}\left(\frac{\overline{z_k}^2 \overline{\Xi_k}}{\overline{z_k}^2 - q_k} \cosh(\overline{z_k} T)\cosh(\overline{z_k} T)\right).
\end{aligned}
$$

Using again the definition (5.25) of Ξ_k it follows that

$$
-d(k,T)\, H'''_{k,T}(0) + (\gamma - \lambda_k)\, d(k,T)\, H'_{k,T}(0) = d(k,T)
$$

by the definition (5.26) of $d(k,T)$.

Using again the definition (5.25) of Ξ_k it follows that

$$
-d(k,T)\, H'''_{k,T}(0) + (\gamma - \lambda_k)\, d(k,T)\, H'_{k,T}(0) = d(k,T)
$$

by the definition (5.26) of $d(k,T)$. We have

$$
d(k,T)\, H'''_{k,T}(T) = 2\,\mathrm{Re}\left(\frac{{z_k}^2 \overline{\Xi_k}}{{z_k}^2 - q_k} \cosh(\overline{z_k} T)\right) - 2\,\mathrm{Re}\left(\frac{\overline{z_k}^2 \overline{\Xi_k}}{\overline{z_k}^2 - q_k} \cosh(\overline{z_k} T)\right).
$$

Hence, we have

$$
\begin{aligned}
&d(k,T)\left[H'''_{k,T}(T) - q_k H'_{k,T}(T)\right] \\
&= 2\,\mathrm{Re}\left(\frac{{z_k}^2 \overline{\Xi_k}}{{z_k}^2 - q_k} \cosh(\overline{z_k} T)\right) - 2\,\mathrm{Re}\left(\frac{\overline{z_k}^2 \overline{\Xi_k}}{\overline{z_k}^2 - q_k} \cosh(\overline{z_k} T)\right) \\
&\quad + 2\mathrm{Re}\left(\frac{q_k \overline{\Xi_k}}{z_k^2 - q_k} \cosh\left(\overline{z_k} T\right)\right) - 2\,\mathrm{Re}\left(\frac{q_k \overline{\Xi_k}}{\overline{z_k}^2 - q_k} \cosh\left(\overline{z_k} T\right)\right) \\
&= 0.
\end{aligned}
$$

Thus we have shown that

$$
\begin{aligned}
H_{k,T}(T) &= 0, \\
H'''_{k,T}(T) &= \frac{\gamma^2 - 2\,\gamma\,\lambda_k - 1}{\gamma}\, H'_{k,T}(T),
\end{aligned}
$$

$$\begin{aligned}
-H''_{k,T}(0) + (\gamma - \lambda_k)\, H_{k,T}(0) &= 0, \\
-H'''_{k,T}(0) + (\gamma - \lambda_k)\, H'_{k,T}(0) &= 1.
\end{aligned}$$

The proof of Lemma 2 is complete.

In the next section, it is shown that each of the functions $F_{k,T}$, $G_{k,T}$, $H_{k,T}$ in the representation (5.24) of the solution $b_{k,T}$ that are provided in Lemma 2 satisfies an exponential turnpike inequality on the interval $[0, T]$ in the sense that for $I \in \{F, G, H\}$ for all $t \in [0, T]$ the following inequality holds:

$$|I_{k,T}(t)| \leq C_{@} \left[\exp\left(-\frac{\sqrt{\gamma}}{2} t\right) + \exp\left(-\frac{\sqrt{\gamma}}{2}(T-t)\right) \right].$$

Here $C_{@}$ is a constant that is independent of T and k.

The turnpike inequality that we have obtained is used for the applications in shape optimization as described in the subsequent sections. An important point is that the inequality is independent of the properties of the sequence of eigenvalues λ_k as long as (5.18) holds. Therefore the inequality is valid over a compact set of perturbations of the state equation (see Theorem 10 and also Remark 7).

5.3 The Turnpike Property by the Spectral Method for Trees

In this section, we continue our analysis of the structure of the optimal solutions, in particular for the adjoint states. We have in mind the shape optimization problems for the trees. Therefore, we restrict our analysis to the networks in the form of trees. The case of small cycles for the purposes of topology optimization is considered separately. In the latter case, the spectrum is of a specific structure with the branch supported exclusively on the cycle.

We show that the difference between the static optimal adjoint state and the dynamic optimal adjoint state satisfies an exponential turnpike inequality as explained above. Our method is to show that all three basis functions in the representation of the difference between the static optimal adjoint state and the dynamic optimal adjoint state that we have obtained from the respective optimality systems satisfy such an exponential turnpike inequality. More precisely, we show that the basis function with a non-zero value at the time $t = 0$ (namely $G_{k,T}$ and $H_{k,T}$) decay exponentially fast with t. The basis function with a non-zero value at the time $t = T$ (namely $F_{k,T}$) decay exponentially fast as a function of $T - t$, that is backward in time.

5.3.1 The Turnpike Inequalities for the Basis Functions

Exponential Turnpike Inequality for the Basis Function $F_{k,T}(t)$ In order to verify the turnpike property we use the following representation where the hyperbolic tangent appears:

$$
\begin{aligned}
&d(k,T)\, F_{k,T}(t)\\
&= 2\,\mathrm{Re}\left(-\frac{\overline{\Xi_k}^2}{\overline{z_k}^2 - q_k}\cosh(z_k(T-t))\right)\\
&+ 2\,\mathrm{Re}\left(\frac{|\Xi_k|^2}{z_k^2 - q_k}\cosh(\overline{z_k}T)\cosh(z_kT)\left[1-\tanh(z_kT)\tanh(z_k(T-t))\right]\cosh(z_k(T-t))\right)\\
&+ 2\,\mathrm{Re}\left(\frac{|\Xi_k|^2}{z_k^2 - q_k}\frac{\overline{z_k}}{z_k}\cosh(\overline{z_k}T)\tanh(\overline{z_k}T)\cosh(z_k\,T)\left[\tanh(z_k(T-t))-\tanh(z_k\,T)\right]\cosh(z_k(T-t))\right).
\end{aligned}
$$

For our analysis, we define the auxiliary functions

$$
S^1_{k,T}(t) = \frac{2\,\mathrm{Re}\left(-\frac{\overline{\Xi_k}^2}{\overline{z_k}^2 - q_k}\cosh(z_k(T-t))\right)}{d(k,T)},
$$

$$
S^2_{k,T}(t) = \frac{2\,\mathrm{Re}\left(\frac{|\Xi_k|^2}{z_k^2 - q_k}\cosh(\overline{z_k}T)\cosh(z_kT)\left[1-\tanh(z_kT)\tanh(z_k(T-t))\right]\cosh(z_k(T-t))\right)}{d(k,T)},
$$

and

$$
\begin{aligned}
&S^3_{k,T}(t)\\
&= \frac{2\,\mathrm{Re}\left(\frac{|\Xi_k|^2}{z_k^2 - q_k}\frac{\overline{z_k}}{z_k}\cosh(\overline{z_k}T)\tanh(\overline{z_k}T)\cosh(z_k\,T)\left[\tanh(z_k(T-t))-\tanh(z_k\,T)\right]\cosh(z_k(T-t))\right)}{d(k,T)}.
\end{aligned}
$$

Then we have

$$
F_{k,T}(t) = S^1_{k,T}(t) + S^2_{k,T}(t) + S^3_{k,T}(t). \tag{5.51}
$$

Define $I_- = \frac{\mathrm{Re}\left((z_k^2-q_k)(|z_k|^2-z_k^2)\right)}{|z_k^2|\,|z_k^2-q_k|}$, $I_+ = \frac{\mathrm{Re}\left((z_k^2-q_k)(|z_k|^2+z_k^2)\right)}{|z_k^2|\,|z_k^2-q_k|}$.

The numbers $d(k,T)$ can be represented as

$$
d(k,T) = -2\ \mathrm{Re}\left(\frac{\Xi_k^2}{z_k^2 - q_k}\right) + |\Xi_k|^2 I_- \cosh(2\ \mathrm{Re}(z_k)\,T) + |\Xi_k|^2 I_+ \cosh(2\ \mathrm{Im}(z_k)\,T). \tag{5.52}
$$

Hence we obtain the upper bound

$$
|S^1_{k,T}(t)| \le \frac{\left|2\,\mathrm{Re}\left(\frac{1}{\overline{z_k}^2 - q_k}\cosh(z_k(T-t))\right)\right|}{|I_-|\cosh(2\,\mathrm{Re}(z_k)\,T) - \left|I_+\cosh(2\ \mathrm{Im}(z_k)\,T) - 2\mathrm{Re}\left(\frac{\Xi_k^2}{|\Xi_k|^2(z_k^2-q_k)}\right)\right|}.
$$

This implies the inequality

$$
|S^1_{k,T}(t)| \le M_1 \exp(-|\,\mathrm{Re}(z_k)|(T-t)) \tag{5.53}
$$

with a constant $M_1 \geq 1$ that is independent of k and T. We have

$$1 - \tanh(z_k\,T)\tanh(z_k\,(T-t)) = \frac{2[\exp(2\,z_k\,T) + \exp(2\,z_k\,(T-t))]}{(\exp(2\,z_k\,T)+1)\,(\exp(2\,z_k\,(T-t))+1)}$$
$$= \frac{2}{(1+\exp(-2\,z_k\,T))\,(\exp(2\,z_k\,(T-t))+1)} + \frac{2}{(1+\exp(-2\,z_k\,(T-t)))\,(\exp(2\,z_k\,T)+1)}.$$

Hence for $T \to \infty$ this term converges to zero exponentially fast. More precisely, due to (5.21) for all $k \in \{0, 1, 2, ..\}$ for T sufficiently large we have the inequality

$$|1 - \tanh(z_k\,T)\tanh(z_k\,(T-t))| \leq 16\exp(-2\,|\operatorname{Re}(z_k)|\,(T-t)). \tag{5.54}$$

Thus we have

$$|S^2_{k,T}(t)| \leq \frac{32|\operatorname{Re}\left(\frac{1}{z_k^2-q_k}|\cosh(z_kT)|^2\,\cosh(z_k\,(T-t))|\right)\exp(-2\,|\operatorname{Re}(z_k)|\,(T-t))|}{|I_-|\cosh(2\operatorname{Re}(z_k)\,T) - \left|I_+\cosh(2\,\operatorname{Im}(z_k)\,T) - 2\operatorname{Re}\left(\frac{\Xi_k^2}{|\Xi_k|^2(z_k^2-q_k)}\right)\right|}.$$

This implies the inequality

$$|S^2_{k,T}(t)| \leq M_2\exp(-|\operatorname{Re}(z_k)|(T-t)) \tag{5.55}$$

with a constant $M_2 \geq 1$ that is independent of k and T.

In addition, we have

$$\tanh(z_k\,(T-t)) - \tanh(z_kT) = \frac{2[\exp(2\,z_k\,T) - \exp(2\,z_k\,(T-t))]}{(\exp(2\,z_k\,T)+1)\,(\exp(2\,z_k\,(T-t))+1)}$$

so as above for T sufficiently large, we obtain the inequality

$$|\tanh(z_k\,(T-t)) - \tanh(z_kT)| \leq 16\exp(-2\,|\operatorname{Re}(z_k)|\,(T-t)). \tag{5.56}$$

This yields the bound

$$|S^3_{k,T}(t)| \leq$$
$$\frac{32|\operatorname{Re}\left(\frac{1}{z_k^2-q_k}\frac{\overline{z_k}}{z_k}\cosh(z_kT)\cosh(\overline{z_k}T)\cosh(z_k\,(T-t))\tanh(\overline{z_k}T)\right)|\exp(-2\,|\operatorname{Re}(z_k)|\,(T-t))}{|I_-|\cosh(2\,\operatorname{Re}(z_k)\,T) - \left|I_+\cosh(2\,\operatorname{Im}(z_k)\,T) - 2\operatorname{Re}\left(\frac{\Xi_k^2}{|\Xi_k|^2(z_k^2-q_k)}\right)\right|}.$$

Thus we obtain the inequality

$$|S^3_{k,T}(t)| \leq M_3\exp(-|\operatorname{Re}(z_k)|(T-t))$$

with a constant $M_3 \geq 1$ that is independent of k and T. Together with (5.53) and (5.55) due to (5.51) and (5.21) this implies that for $F_{k,T}(t)$ we have

$$|F_{k,T}(t)| \leq (M_1 + M_2 + M_3)\,\exp\left(-\frac{\sqrt{\gamma}}{2}(T-t)\right). \tag{5.57}$$

Exponential Turnpike Inequality for the Basis Function $G_{k,T}(t)$ Now we derive the turnpike inequality for $G_{k,T}(t)$. Again we use a representation with the hyperbolic tangent. We have

$$\begin{aligned} & d(k,T)\, G_{k,T}(t) \\ = \; & -2\,\mathrm{Re}\left(\frac{\Xi_k}{{z_k}^2-q_k}\cosh(z_k T)\,[1+\tanh(z_k T)\tanh(z_k(t-T))]\cosh(z_k(t-T))\right) \\ & +2\,\mathrm{Re}\left(\frac{\overline{\Xi_k}}{{\overline{z_k}}^2-q_k}\cosh(\overline{z_k}T)\left[1-\frac{{\overline{z_k}}^2-q_k}{{z_k}^2-q_k}\frac{\overline{z_k}}{z_k}\tanh(\overline{z_k}T)\tanh(z_k(t-T))\right]\cosh(z_k(t-T))\right), \end{aligned} \tag{5.58}$$

and

$$\lim_{T\to\infty}\left[1-\frac{{\overline{z_k}}^2-q_k}{{z_k}^2-q_k}\frac{\overline{z_k}}{z_k}\tanh(\overline{z_k}T)\tanh(z_k(t-T))\right]=1+\frac{{\overline{z_k}}^2-q_k}{{z_k}^2-q_k}\frac{\overline{z_k}}{z_k}.$$

Similarly, we have

$$\lim_{T\to\infty}[1+\tanh(z_k T)\,\tanh(z_k(t-T))]=0$$

and for the absolute value we have the bound (5.54).

For our analysis, we define the auxiliary functions

$$\tilde{S}^1_{k,T}(t)=\frac{-2\,\mathrm{Re}\left(\frac{\Xi_k}{{z_k}^2-q_k}\cosh(z_k T)\,[1+\tanh(z_k T)\tanh(z_k(t-T))]\cosh(z_k(t-T))\right)}{d(k,T)},$$

$$\tilde{S}^2_{k,T}(t)=\frac{2\,\mathrm{Re}\left(\frac{\overline{\Xi_k}}{{\overline{z_k}}^2-q_k}\cosh(\overline{z_k}T)\left[1-\frac{{\overline{z_k}}^2-q_k}{{z_k}^2-q_k}\frac{\overline{z_k}}{z_k}\tanh(\overline{z_k}T)\tanh(z_k(t-T))\right]\cosh(z_k(t-T))\right)}{d(k,T)}.$$

Then we have

$$G_{k,T}(t)=\tilde{S}^1_{k,T}(t)+\tilde{S}^2_{k,T}(t). \tag{5.59}$$

Due to (5.54) we obtain the inequality

$$|\tilde{S}^1_{k,T}(t)|\le\frac{32\left|\mathrm{Re}\left(\frac{\Xi_k}{|\Xi_k^2|(z_k^2-q_k)}\cosh(z_k T)\,\cosh(z_k(t-T))\right)\right|\exp(-2\,|\,\mathrm{Re}(z_k)|\,(T-t))}{|I_-|\cosh(2\,\mathrm{Re}(z_k)\,T)-\left|I_+\cosh(2\,\mathrm{Im}(z_k)\,T)-2\mathrm{Re}\left(\frac{\Xi_k^2}{|\Xi_k|^2(z_k^2-q_k)}\right)\right|}.$$

This implies the inequality

$$|\tilde{S}^1_{k,T}(t)|\le\frac{1}{|\Xi_k|}\,\tilde{M}_1\exp(-|\,\mathrm{Re}(z_k)|(T-t)) \tag{5.60}$$

with a constant $\tilde{M}_1>0$ that is independent of k and T. Moreover, we have

$$|\tilde{S}^2_{k,T}(t)|\le\frac{32\left|\mathrm{Re}\left(\frac{\overline{\Xi_k}}{|\Xi_k^2|(z_k^2-q_k)}\cosh(\overline{z_k}\,T)\cosh(z_k(t-T))\right)\right|}{|I_-|\cosh(2\,\mathrm{Re}(z_k)\,T)-\left|I_+\cosh(2\,\mathrm{Im}(z_k)\,T)-2\mathrm{Re}\left(\frac{\Xi_k^2}{|\Xi_k|^2(z_k^2-q_k)}\right)\right|}.$$

This implies the inequality

$$|\tilde{S}^2_{k,T}(t)| \le \frac{1}{|\Xi_k|}\, \tilde{M}_2 \left[\exp(-|\operatorname{Re}(z_k)|\, t) + \exp(-|\operatorname{Re}(z_k)|(T-t))\right] \tag{5.61}$$

with a constant $\tilde{M}_2 > 0$ that is independent of k and T.

With (5.60) and (5.61) due to (5.59) and (5.21) this implies that for $G_{k,T}(t)$ we have

$$|G_{k,T}(t)| \le \frac{1}{|\Xi_k|}\, (\tilde{M}_1 + \tilde{M}_2) \left[\exp(-\frac{\sqrt{\gamma}}{2}\, t) + \exp(-\frac{\sqrt{\gamma}}{2}(T-t))\right]. \tag{5.62}$$

Exponential Turnpike Inequality for the Basis Function $H_{k,T}(t)$ Finally, we show the exponential turnpike inequality for the third basis function $H_{k,T}$. We have

$$\begin{aligned}
& d(k,T)\, H_{k,T}(t) \\
= \;& 2\operatorname{Re}\left(\tfrac{\overline{\Xi_k}}{\overline{z_k}(\overline{z_k}^2 - q_k)} \sinh(\overline{z_k}T)\cosh(z_k(t-T)) + \tfrac{\overline{\Xi_k}}{z_k(z_k{}^2-q_k)} \cosh(\overline{z_k}T)\, \sinh(z_k(t-T))\right) \\
- \;& 2\operatorname{Re}\left(\tfrac{\Xi_k}{z_k(z_k{}^2-q_k)} \sinh(z_kT)\cosh(z_k(t-T)) + \tfrac{\Xi_k}{z_k(z_k{}^2-q_k)} \cosh(z_kT)\sinh(z_k(t-T))\right).
\end{aligned}$$

For our analysis, we define the auxiliary functions

$$\hat{S}^1_{k,T}(t) = \frac{2\operatorname{Re}\left(\overline{\Xi_k}\cosh(\overline{z_k}T)\left[\tfrac{1}{\overline{z_k}(\overline{z_k}^2-q_k)}\tanh(\overline{z_k}T) + \tfrac{1}{z_k(z_k{}^2-q_k)}\tanh(z_k(t-T))\right]\cosh(z_k(t-T))\right)}{d(k,\, T)}$$

and

$$\hat{S}^2_{k,T}(t) = \frac{2\operatorname{Re}\left(\tfrac{\Xi_k}{z_k(z_k{}^2-q_k)}\cosh(z_kT)\left[\tanh(z_kT) + \tanh(z_k(t-T))\right]\cosh(z_k(t-T))\right)}{d(k,\, T)}.$$

Then we have

$$H_{k,T}(t) = \hat{S}^1_{k,T}(t) - \hat{S}^2_{k,T}(t). \tag{5.63}$$

For T sufficiently large (uniformly with respect to k due to (5.21)) we have the bound

$$\left|\frac{1}{\overline{z_k}(\overline{z_k}^2 - q_k)}\tanh(\overline{z_k}T) + \frac{1}{z_k(z_k{}^2-q_k)}\tanh(z_k(t-T))\right| \le \frac{4}{|z_k(z_k{}^2-q_k)|}.$$

This yields

$$|\hat{S}^1_{k,T}(t)| \le \frac{\frac{1}{|\Xi_k|}\,\frac{8}{|z_k(z_k{}^2-q_k)|}\,|\cosh(\overline{z_k}T)\cosh(z_k(t-T))|}{|I_-|\cosh(2\operatorname{Re}(z_k)\,T) - \left|I_+\cosh(2\,\operatorname{Im}(z_k)\,T) - 2\operatorname{Re}\left(\frac{\Xi_k^2}{|\Xi_k|^2(z_k^2-q_k)}\right)\right|}.$$

This implies the inequality

$$|\hat{S}^1_{k,T}(t)| \le \frac{1}{|\Xi_k|^2}\, \hat{M}_1 \left[\exp(-|\operatorname{Re}(z_k)|\, t) + \exp(-|\operatorname{Re}(z_k)|(T-t))\right] \tag{5.64}$$

with a constant $\hat{M}_1 > 0$ that is independent of k and T.

For T sufficiently large (uniformly with respect to k due to (5.21)) we have the bound $|\tanh(z_k T) + \tanh(z_k(t-T))| \leq 4$. This yields the bound

$$|\hat{S}^2_{k,T}(t)| \leq \frac{8\frac{1}{|\Xi_k|}\left|\frac{1}{z_k({z_k}^2 - q_k)}\cosh(z_k T)\,\cosh(z_k(t-T))\right|}{|I_-|\cosh(2\,\mathrm{Re}(z_k)\,T) - \left|I_+\cosh(2\,\mathrm{Im}(z_k)\,T) - 2\mathrm{Re}\left(\frac{\Xi_k^2}{|\Xi_k|^2(z_k^2-q_k)}\right)\right|}.$$

Hence we obtain the inequality

$$|\hat{S}^2_{k,T}(t)| \leq \frac{1}{|\Xi_k|^2}\,\hat{M}_2\left[\exp(-|\,\mathrm{Re}(z_k)|\,t) + \exp(-|\,\mathrm{Re}(z_k)|(T-t))\right] \tag{5.65}$$

with a constant $\hat{M}_2 > 0$ that is independent of k and T.

With (5.64) and (5.65) due to (5.63) and (5.21) this implies that for $H_{k,T}(t)$ we have

$$|H_{k,T}(t)| \leq \frac{1}{|\Xi_k|^2}\,(\hat{M}_1 + \hat{M}_2)\left[\exp(-\frac{\sqrt{\gamma}}{2}\,t) + \exp(-\frac{\sqrt{\gamma}}{2}(T-t))\right]. \tag{5.66}$$

5.3.2 Theorem on Turnpike Property Between the Dynamic and Static Optimality Systems

We have unique solutions for the optimality systems governed by static and dynamic state equations. Now, we compare the elements of optimality systems and obtain the turnpike inequalities for the Optimality Systems.

Indeed, the turnpike inequalities that we have derived lead to the useful for applications Turnpike Theorem which means that :

The difference ν^T of the optimal dynamic control and the optimal static control and the corresponding differences ω^T for the state and μ^T the adjoint state admits an exponential turnpike property.

To this end, we need some additional regularity assumptions.

Theorem 10. *Assume that (5.18) holds and that the initial state satisfies the regularity condition*

$$\sum_{k=0}^{\infty} \lambda_k\,|a_k(0)|^2 + |a_k'(0)|^2 < \infty, \tag{5.67}$$

that is the initial state belongs to the energy space of the elliptic problem defined by the bilinear form $a(\cdot,\cdot)$. If $\Omega = \Gamma$ then there exists a constant $\tilde{D} = \tilde{D}(y_0,\, y_1,\, p^\sigma)$ that is independent of T and t such that for all $t \in [0,\, T]$

$$\|\omega^T(t)\|^2_{L^2(\Omega)} + \|\nu^T(t)\|^2_{L^2(\Omega)} + \|\mu^T(t)\|^2_{L^2(\Omega)} \leq \tilde{D}\left[\mathrm{e}^{-\sqrt{\gamma}\,\mathrm{t}} + \mathrm{e}^{-\sqrt{\gamma}(\mathrm{T}-\mathrm{t})}\right]. \tag{5.68}$$

Moreover, the constant $\tilde{D}$ depends on Ω only as a function of the energy norm for the initial state that is determined by Ω as in (5.67).

Remark 7. *In the case of boundary control problems on networks, the Turnpike Property for optimality systems can be shown. The difference between the boundary control and the distributed control is the appearance of a linear operator in the optimality conditions. In the optimality conditions of the boundary control problem, a linear operator maps the adjoint state into the optimal control.*

For the optimal cost the exponential turnpike inequality (5.68) implies the so-called integral turnpike property (see e.g. [26])

$$\sup_{T>0} \int_0^T \|\omega^T(t)\|^2_{L^2(\Omega)} + \|\nu^T(t)\|^2_{L^2(\Omega)} + \|\mu^T(t)\|^2_{L^2(\Omega)}\, dt < \infty.$$

This implies in turn

$$\lim_{T\to\infty} \frac{1}{T} J_T(\hat{u}^T) = I(\hat{v}), \tag{5.69}$$

see for example [29].

In shape sensitivity analysis [88] we avoid, if possible, the dependence of the shape gradient of the cost with respect to given data including e.g., the given initial conditions. Thus, we define the initial conditions, say $y_0(x)$ and $y_1(x)$, $x \in \Omega_\tau$, for variable domains $\tau \to \Omega_\tau$ used for the derivation of shape gradient of the cost. If Ω_τ is e.g., the cross with variable length of the edges, we select the elements y_0, y_1 as the restrictions to Ω_τ of functions $Y_0(x), Y_1(x)$ defined on the cross with the edges of maximal lengths. The shape derivatives of the restrictions are zero. Therefore, there is no contribution from the initial conditions to the shape gradient of the cost.

Remark 8. *We use the material derivative method [88] for the purposes of shape sensitivity analysis for networks. The general rule for the data of initial-boundary value problems is to select the given functions by the restriction to actual domain of some functions defined everywhere. The shape gradients of such initial data are simply zero and the material derivatives are given by the gradients, thus some regularity is required. In this way the shape gradient of the cost is independent of the initial data. In our case, this selection can be used for the initial conditions of the displacement and the velocity. The material derivatives of such initial conditions $Y = Y(x)$ take the form $Y'(x)\mathcal{V}(0,x)$, where $\mathcal{V}(\tau,x)$ is the velocity field of the material derivative method. In other words, the initial conditions are selected in such a way that there is no contribution of the initial conditions to the shape gradient of the cost function.*

The following corollary states a shape-turnpike result. It is a relation between the optimal values of a dynamic optimal shape problem for large time horizons and the optimal values for the static optimal shape problem. For the proof, we suggest to proceed by contradiction.

Assumption 1. *Let us consider a tree $G = \{E, V\}$ with the set of edges $E_i = [0, L_i]$, $i = 1, \ldots, N$, and denote by $\ell = \operatorname{col}(L_1, \ldots, L_N)$. The set of admissible trees $\Omega(\ell) \in \mathcal{A}$*

is defined by the conditions $M_i^{min} \leq L_i \leq M_i^{max}$, *where* $0 < M_0 \leq M_i^{min} < M_i^{max} < M_1 < \infty$. *The set of admissible trees is convex and compact, therefore, for the minimizing sequence* ℓ_n *of optimization problem, there is a subsequence, still denoted by the same symbol such that we have* $\ell_n \to \ell_\infty$ *in* $\mathbb{R}^N$ *and in addition* $\Omega(\ell_\infty) \in \mathcal{A}$.

Corollary 1. *Let Assumption 1 hold. Let a sequence of shape parameters* $(\ell_n)_n$ *and a bounded sequence of controls* $(\hat{u}_n)_n$ *with* $\hat{u}_n(t) \in L^2(\Omega(\ell_n))$ *for all* $n \in \{1, 2, 3, ...\}$ *be given. Let* $\hat{y}_n$ *denote the generated state and* $\hat{p}_n$ *the corresponding adjoint state. Assume that for all* $n \in \{1, 2, 3, ...\}$ *we have*

$$\|\omega_n^T(t)\|^2_{L^2(\Omega(\ell_n))} + \|\nu_n^T(t)\|^2_{L^2(\Omega(\ell_n))} + \|\mu_n^T(t)\|^2_{L^2(\Omega(\ell_n))} \leq \tilde{D}\left[\mathrm{e}^{-\sqrt{\gamma}\,\mathrm{t}} + \mathrm{e}^{-\sqrt{\gamma}(\mathrm{T}-\mathrm{t})}\right] \tag{5.70}$$

where $\omega_n^T = \hat{y}_n^T - \hat{z}_n^\sigma$, $\mu_n^T = \hat{p}_n^T - \hat{p}_n^\sigma, \nu_n^T = \hat{u}_n^T - \hat{v}_n^\sigma$ *and* $(\hat{v}_n, \hat{z}_n, \hat{p}_n)$ *is optimal for* $\Omega(\ell_n)$

Assume that $\lim_{n\to\infty} \ell_{opt}(T_n) = \hat{\ell}$ *and that* $\hat{u}_n$ *converges weakly to* u_{opt}.

Assume that the optimal shape problem with (OCE) *has a solution* (ℓ_{opt}, u_{opt}). *Then* u_{opt} *is a solution of* (OCE).

Let $\hat{v}_{opt}$ *denote the solution of* (OCS) *for* ℓ_{opt}.

Then we have

$$\|\omega_{opt}^T(t)\|^2_{L^2(\Omega(\ell_{opt}))} + \|\nu_{opt}^T(t)\|^2_{L^2(\Omega(\ell_{opt}))} + \|\mu_{opt}^T(t)\|^2_{L^2(\Omega(\ell_{opt}))} \leq \tilde{D}\left[\mathrm{e}^{-\sqrt{\gamma}\,\mathrm{t}} + \mathrm{e}^{-\sqrt{\gamma}(\mathrm{T}-\mathrm{t})}\right]. \tag{5.71}$$

Assume that for a subsequence we have $\lim_{T_n\to\infty} \ell_{opt}(T_n) = \hat{\ell}$.

Then we have

$$\lim_{n\to\infty} \frac{1}{T_n} J_{T_n}(\hat{u}_{opt}^{T_n}, \ell_{opt}(T_n)) = I(\hat{v}, \hat{\ell}). \tag{5.72}$$

and $\hat{v}$ *is the optimal control for the network defined in* $\Omega(\hat{\ell})$, *the limit shape parameter* $\hat{\ell}$ *is optimal for the static problem* $\ell \to (OCS)(\ell)$, *the optimal shape reads* $\Omega(\hat{\ell})$. *In general, an optimal shape is not unique but it does exist for the compact set of admissible shapes.*

Remark 9. *Under Assumption 1 the set of admissible shapes is convex and compact in* $\mathbb{R}^N$. *Denote by* $\hat{\ell}$ *an optimal shape for the static problem, note that the optimal shape is not unique, and let the admissible sequence of shapes* ℓ_n *be convergent to an optimal shape for* $n \to \infty$,

$$\ell_n \to \hat{\ell}$$

then

$$I(\hat{v}, \hat{\ell}) = \lim_{n\to\infty} \frac{1}{T_n} J_{T_n}(\hat{u}_{opt}^{T_n}, \ell_n) \geq \lim_{n\to\infty} \frac{1}{T_n} J_{T_n}(\hat{u}_{opt}^{T_n}, \ell_{opt}(T_n)) = I(\hat{v}, \tilde{\ell}) \tag{5.73}$$

therefore

$$I(\hat{v}, \hat{\ell}) = I(\hat{v}, \tilde{\ell}).$$

Proof of the corollary.

The constant control with the value $\hat{v}$ can be considered as an element of $L^2(\Omega(\ell_{opt}(T_n)))$. We have the inequality

$$\left| \|\hat{u}^{T_n}_{opt}(\ell_{opt}(T_n)) - u_d\|^2_{L^2(\Omega(\ell_{opt}(T_n)))} - \|\hat{v} - u_d\|^2_{L^2(\Omega(\ell_{opt}(T_n)))} \right|$$

$$\leq \|\hat{u}^{T_n}_{opt}(\ell_{opt}(T_n)) - \hat{v}\|_{L^2(\Omega(\ell_{opt}(T_n)))} \left(\|\hat{u}^{T_n}_{opt}(\ell_{opt}(T_n)) - u_d\|_{L^2(\Omega(\ell_{opt}(T_n)))} + \|\hat{v} - u_d\|_{L^2(\Omega(\ell_{opt}(T_n)))} \right).$$

Integration from 0 to T_n and division by T_n yields

$$\left| \frac{\int_0^{T_n} \|\hat{u}^{T_n}_{opt}(\ell_{opt}(T_n)) - u_d\|^2_{L^2(\Omega(\ell_{opt}(T_n)))}\, dt}{T_n} - \|\hat{v} - u_d\|^2_{L^2(\Omega(\ell_{opt}(T_n)))} \right|$$
$$\leq \frac{\int_0^{T_n} \|\hat{u}^{T_n}_{opt}(\ell_{opt}(T_n)) - \hat{v}\|_{L^2(\Omega(\ell_{opt}(T_n)))} \left(\|\hat{u}^{T_n}_{opt}(\ell_{opt}(T_n)) - u_d\|_{L^2(\Omega(\ell_{opt}(T_n)))} + \|\hat{v} - u_d\|_{L^2(\Omega(\ell_{opt}(T_n)))} \right) dt}{T_n}$$
$$\leq \frac{\int_0^{T_n} \|\hat{u}^{T_n}_{opt}(\ell_{opt}(T_n)) - \hat{v}\|^2_{L^2(\Omega(\ell_{opt}(T_n)))}\, dt}{T_n}$$
$$+ \frac{2\|\hat{v} - u_d\|_{L^2(\Omega(\ell_{opt}(T_n)))} \int_0^{T_n} \|\hat{u}^{T_n}_{opt}(\ell_{opt}(T_n)) - \hat{v}\|_{L^2(\Omega(\ell_{opt}(T_n)))}\, dt}{T_n}.$$

For the state we obtain a similar inequality, namely

$$\left| \frac{\int_0^{T_n} \|\hat{y}^{T_n}_{opt}(\ell_{opt}(T_n)) - y_d\|^2_{L^2(\Omega(\ell_{opt}(T_n)))}\, dt}{T_n} - \|\hat{z} - y_d\|^2_{L^2(\Omega(\ell_{opt}(T_n)))} \right|$$
$$\leq \frac{\int_0^{T_n} \|\hat{y}^{T_n}_{opt}(\ell_{opt}(T_n)) - \hat{z}\|^2_{L^2(\Omega(\ell_{opt}(T_n)))}\, dt}{T_n}$$
$$+ \frac{2\|\hat{z} - y_d\|_{L^2(\Omega(\ell_{opt}(T_n)))} \int_0^{T_n} \|\hat{y}^{T_n}_{opt}(\ell_{opt}(T_n)) - \hat{z}\|_{L^2(\Omega(\ell_{opt}(T_n)))}\, dt}{T_n}.$$

For the optimal cost the exponential turnpike inequality (5.68) implies the integral turnpike property (see e.g. [26])

$$\sup_{T>0} \sup_{\ell} \int_0^T \|\omega^T(t)\|^2_{L^2(\Omega)} + \|\nu^T(t)\|^2_{L^2(\Omega)} + \|\mu^T(t)\|^2_{L^2(\Omega)}\, dt < \infty.$$

Moreover, we have

$$\sup_{T>0} \sup_{\ell} \int_0^T \|\omega^T(t)\|_{L^2(\Omega)} + \|\nu^T(t)\|_{L^2(\Omega)} + \|\mu^T(t)\|_{L^2(\Omega)}\, dt < \infty.$$

Thus adding up the inequalities for the control and the state and taking the limit for $T_n \to \infty$ yields (5.72).

Remark 10. *For tree-shaped graphs often the system is exactly controllable in some finite time $t_{\min}$. Recently it was shown that if there is also control action at the interior nodes of the graph, exact controllability is also possible for a graph with cycles, see [6]. In this case, we can choose for all $n \in \{1, 2, 3, ...\}$ a control function $u_{init}^{(n)}(t) \in L^2(0,\, t_{\min}; L^2(\Omega(\ell_{opt}(T_n))))$ that steers the system to the constant state $y(t_{\min},\, \cdot) = \hat{y}$, $y_t(t_{\min},\, \cdot) = 0$ and satisfies*

$$\max_n \|u_{init}^{(n)}\|_{L^2(0,\, t_{\min}; L^2(\Omega(\ell_{opt}(T_n))))} < \infty. \tag{5.74}$$

Such a control can be determined using the classical method of moments as described for example in [78].

We define the control $\tilde{v}^{(n)}(t) = \begin{cases} u_{init}^{(n)}(t),\ t \in (0,\, t_{\min}), \\ \hat{v},\ t \geq t_{\min}. \end{cases}$

Then $\tilde{v}^{(n)}$ is feasible for $(OCE)(T_n,\, \Omega(\ell_{opt}(T_n)))$ and thus we have

$$J_{T_n}(\hat{u}_{opt}^{T_n}, \ell_{opt}(T_n)) \leq J_{T_n}(\tilde{v}^{(n)}, \ell_{opt}(T_n)). \tag{5.75}$$

Moreover, (5.72) implies that for all $\varepsilon > 0$ if $T_n > 0$ is sufficiently large we have

$$\lim_{T_n \to \infty} \frac{1}{T_n} J_{T_n}(\tilde{v}^{(n)}, \ell_{opt}(T_n)) \leq \frac{1}{T_n} J_{T_n}(\hat{u}_{opt}^{T_n}, \ell_{opt}(T_n)) + \varepsilon. \tag{5.76}$$

This can be seen as follows. Since $\lim_{n\to\infty} T_n = \infty$ due to (5.74) the contribution of the integral on the time interval $(0,\, t_{\min})$ vanishes in the limit, that is we have

$$\lim_{n\to\infty} \frac{\int_0^{t_{\min}} \|y(\tilde{v}^{(n)}) - y^d\|_{L^2(\Omega)}^2 + \gamma\, \|\partial_t(y(\tilde{v}^{(n)}) - y^d)\|_{L^2(\Omega)}^2 + \|\tilde{v}^{(n)} - u^d\|_{L^2(\Gamma)}^2\, dt}{T_n} = 0$$

and since $\lim_{T_n\to\infty} \ell_{opt}(T_n) = \hat{\ell}$ we have

$$\lim_{n\to\infty} \frac{J_{T_n}(\tilde{v}^{(n)}, \ell_{opt}(T_n))}{T_n} = \lim_{n\to\infty} \frac{T_n - t_{\min}}{T_n} I(\hat{v},\, \ell_{opt}(T_n)) = I(\hat{v},\, \hat{\ell}).$$

Assuming the exact controllability of the system in the finite time $t_{\min}$ allows to choose for all $n \in \{1, 2, 3, ...\}$, a control function $u_{init}^{(n)}(t) \in L^2(0,\, t_{\min}; L^2(\Omega(\ell_{opt}(T_n))))$ that steers the system to the constant state $y(t_{\min},\, \cdot) = \hat{y}$, $y_t(t_{\min},\, \cdot) = 0$ and satisfies

$$\max_n \|u_{init}^{(n)}\|_{L^2(0,\, t_{\min}; L^2(\Omega(\ell_{opt}(T_n))))} < \infty. \tag{5.77}$$

Such a control can be determined using the classical method of moments as described for example in [78].

We define the control $\tilde{v}^{(n)}(t) = \begin{cases} u_{init}^{(n)}(t),\ t \in (0,\, t_{\min}], \\ \hat{v},\ t \geq t_{\min}. \end{cases}$

For all $n \in \{1, 2, 3, ...\}$ this implies the inequality

$$\frac{1}{T_n} J_{T_n}(\hat{u}_{opt}^{T_n}, \ell_{opt}(T_n)) \leq \frac{1}{T_n} J_{T_n}(\tilde{v}^{(n)}, \ell_{opt}(T_n)).$$

This yields (5.75).

Assume that the optimal shape problem with (OCE) has a solution (ℓ_{opt}, u_{opt}). Then u_{opt} is a solution of (OCE). Let $\hat{v}_{opt}$ denote the solution of (OCS) for ℓ_{opt}. Then Theorem 10 implies that (5.71) holds.

Proof of Theorem 10. Define $M_F = M_1 + M_2 + M_3$, $M_G = \tilde{M}_1 + \tilde{M}_2$, $M_H = \hat{M}_1 + \hat{M}_2$. Due to (5.24), (5.57), (5.62) and (5.66) for all $k \in \{0, 1, 2, 3, ...\}$ we have

$$|b_{k,T}(t)| \leq M_F e^{-\frac{\sqrt{\gamma}}{2}(T-t)} \, |b_{k,T}(T)| \tag{5.78}$$
$$+[\sqrt{1+\gamma\,\lambda_k}\, M_G\, |a_k(0)| + M_H\, |a_k'(0)|] \left[\exp(-\frac{\sqrt{\gamma}}{2}\, t) + \exp(-\frac{\sqrt{\gamma}}{2}(T-t))\right].$$

For the square, this yields the bound

$$\begin{aligned} |b_{k,T}(t)|^2 \leq{}& 3M_F^2 e^{-\sqrt{\gamma}(T-t)} \, |b_{k,T}(T)|^2 \\ &+ \left[6\,(1+\gamma\lambda_k)\, M_G^2\, |a_k(0)|^2 + 6M_H^2\, |a_k'(0)|^2\right] [\exp(-\sqrt{\gamma}t) + \exp(-\sqrt{\gamma}(T-t))]. \end{aligned}$$

Due to Parseval's equation this implies for all $t \in [0,\, T]$

$$\begin{aligned} \|\mu^T(t)\|_{L^2(\Omega)}^2 &= \sum_{k=0}^{\infty} |b_k(t)|^2 \\ &\leq \sum_{k=0}^{\infty} 3\, M_F^2 e^{-\sqrt{\gamma}(T-t)} \, |b_k(T)|^2 \\ &\quad + [6(1+\gamma\,\lambda_k)\, M_G^2\, |a_k(0)|^2 + 6M_H^2\, |a_k'(0)|^2]\, [\exp(-\sqrt{\gamma}\, t) + \exp(-\sqrt{\gamma}(T-t))]\,. \end{aligned}$$

Since the initial state $(y^0,\, y^1)$ satisfies

$$\sum_{k=0}^{\infty} \lambda_k\, |a_k(0)|^2 + |a_k'(0)|^2 < \infty,$$

this yields an exponential turnpike property for the adjoint state. To be precise, we have for all $t \in [0,\, T]$

$$\|\mu^T(t)\|_{L^2(\Omega)}^2 \leq \tilde{C}(y_0,\, y_1,\, p^\sigma)\, [\exp(-\sqrt{\gamma}\, t) + \exp(-\sqrt{\gamma}(T-t))] \tag{5.79}$$

with a real number $\tilde{C}(y_0,\, y_1,\, p^\sigma)$ that is independent of T and k.

For the optimal controls, for $\Gamma = \Omega$ we have $\nu^T = \mu^T$, hence we have a similar inequality as (5.79) for $\|\nu^T(t)\|_{L^2(\Omega)}^2$.

Note that in proof of (5.57), (5.62) and (5.66) in the estimates, we did not take advantage of the real part that appears in the representations. In fact, we have always used upper bound for the modulus of the complex number whose real part appears in the expressions. In the representation of the second order derivatives, the only change is that the factor z_k^2 appears in the complex variable representations compared to the original expression. Therefore our proof also yields the inequality

$$|b''_{k,T}(t)|^2 \le 3|z_k|^2 M_F^2 e^{-\sqrt{\gamma}(T-t)}\, |b_{k,T}(T)|^2 \tag{5.80}$$

$$+|z_k|^2[6(1+\gamma\,\lambda_k)\, M_G^2\, |a_k(0)|^2 + 6M_H^2\, |a'_k(0)|^2]\, [\exp(-\sqrt{\gamma}\, t) + \exp(-\sqrt{\gamma}(T-t))]\,.$$

We have

$$a_k = \frac{-b''_k + (\gamma - \lambda_k)\, b_k}{1+\gamma\,\lambda_k}. \tag{5.81}$$

Define

$$M_\alpha := \sup_k \frac{|z_k|^2}{1+\gamma\,\lambda_k} = \sup_k \frac{\sqrt{1+\lambda_k^2}}{1+\gamma\,\lambda_k} < \infty,\;\; M_\beta := \sup_k \frac{\lambda_k - \gamma}{1+\gamma\,\lambda_k} < \infty$$

and

$$M_\kappa := M_F\, |b_{k,T}(T)| + \sqrt{1+\gamma\,\lambda_k}\, M_G\, |a_k(0)| + M_H\, |a'_k(0)|.$$

Due to (5.81) we also have the following exponential inequality for the coefficients in the expansion of the optimal state:

$$|a_k(t)| \le (M_\alpha + M_\beta)\, M_\kappa \left[\mathrm{e}^{-\frac{\sqrt{\gamma}}{2}\, t} + \mathrm{e}^{-\frac{\sqrt{\gamma}}{2}(T-t)} \right].$$

This yields the turnpike inequality for the state ω^T:

$$\|\omega^T(t)\|^2_{L^2(\Omega)} \le \tilde{C}(y_0,\, y_1,\, p^\sigma)\, [\exp(-\sqrt{\gamma}\, t) + \exp(-\sqrt{\gamma}(T-t))] \tag{5.82}$$

with a real number $\tilde{C}(y_0,\, y_1,\, p^\sigma)$ that is independent of T.

Note that in proof of (5.57), (5.62) and (5.66) in the estimates, we did not take advantage of the structure of the spectrum. In fact, the constants in the turnpike inequalities (5.57) and $M_G = \tilde{M}_1 + \tilde{M}_2$ in (5.62) and $M_H = \hat{M}_1 + \hat{M}_2$ in (5.66) are independent of k.

Remark 11. *From the mathematical point of view, it is of interest to analyze the nucleation of a cycle at the internal node of network. This means that the internal node is replaced by a small cycle, see Figures 5.11, 5.12. The question which internal node is selected can be solved in the steady state case, the topological derivative of the cost is introduced to this end. The domain decomposition method is applied to derive the topological derivative's form. The associated Steklov-Poincaré operator is represented by a matrix* $\varepsilon \to \Lambda(\varepsilon)$ *which is semidefinite positive and differentiable at* $\varepsilon = 0^+$*, the derivative is denoted* $\Lambda'(0)$ *for the sake of simplicity. In the case of wave equation such a result is not known, however, the numerical experiments show that dependence is regular. Therefore,* *we assume that for a given initial conditions the solution of wave equation enjoys the properties of the steady state boundary value problem for the nucleation of the small cycle. Let us note that for $\varepsilon > 0$ the wave equation is well defined and the regularity of initial conditions required for the turnpike property is already given. Here we assume that the limit of the cost for $\varepsilon \to 0^+$ is well defined and the value of the cost is continuous $\varepsilon = 0$. This assumption does not imply the differentiability of the cost at $\varepsilon \to 0^+$.*

Assumption 2. *Let us consider the wave equation on network and let $\mathcal{J}(\Omega)$ be the cost for given initial conditions y_0, y_1 and given time horizon $T > 0$. At the internal node P_0 of the network a cycle of size $\varepsilon > 0$ is introduced, which leads to the cost $\mathcal{J}(\Omega_\varepsilon)$. The topological variation of the network is admissible provided we have*

$$\lim_{\varepsilon \to 0+} \mathcal{J}(\Omega_\varepsilon) = \mathcal{J}(\Omega).$$

5.4 Control Problem for a Single Edge

Let real numbers $L > 0$, $T > 0$, $c > 0$ and $\gamma > 0$ be given. Let $y_0 \in H^1(0, L)$ with $y_0(0) = 0$ and $y_1 \in L^2(0, L)$, $z \in H^1(0, L)$ with $\zeta = z'(L)$ be given. Consider the problem

$$\min \int_0^T \int_0^L |y(t, x) - z(x)|^2 + \gamma \, |y_t(t, x)|^2 \, dx + |u(t) - \zeta|^2 \, dt$$

subject to

$$\begin{cases} y(0, x) = y_0(x), \; x \in (0, L), \\ y_t(0, x) = y_1(x), \; x \in (0, L), \\ y_{tt}(t, x) = c^2 y_{xx}(t, x), \; (t, x) \in (0, T) \times (0, L), \\ y(t, 0) = 0, \; t \in (0, T), \\ y_x(t, L) = u(t), \; t \in (0, T). \end{cases} \tag{5.83}$$

The solution to the initial boundary value problem for a control $u \in L^2(0, T)$ is stated in [25], Theorem 2.3, p. 17 in the form

$$y(t, x) = \sum_{n=0}^{\infty} \alpha_n(t) \, \varphi_n(x) \tag{5.84}$$

with the eigenfunctions $\varphi_n(x) = \frac{\sqrt{2}}{\sqrt{L}} \sin\left(\left(\frac{\pi}{2} + n\pi\right) \frac{x}{L}\right)$, $n \in \{0, 1, 2, ...\}$. For the eigenvalues we have

$$\lambda_n = \frac{1}{L^2} \left(\frac{\pi}{2} + n\pi\right)^2$$

and the minimal eigenvalue is $\lambda_0 = \frac{\pi^2}{4} \, \frac{1}{L^2}$. For $n \in \{0, 1, 2, ...\}$, let

$$\alpha_n^0 = \int_0^L y_0(x) \, \varphi_n(x) \, dx, \; \alpha_n^1 = \int_0^L y_1(x) \, \varphi_n(x) \, dx. \tag{5.85}$$

We have

$$\alpha_n(t) = \alpha_n^0 \cos\left((\frac{\pi}{2}+n\pi)\frac{t}{t_0}\right) + \alpha_n^1 \frac{t_0}{\frac{\pi}{2}+n\pi}\sin\left((\frac{\pi}{2}+n\pi)\frac{t}{t_0}\right) \tag{5.86}$$
$$+(-1)^n\, c^2\, \frac{\sqrt{2}}{\sqrt{L}} \frac{t_0}{\frac{\pi}{2}+n\pi}\int_0^t u(s)\,\sin\left((\frac{\pi}{2}+n\pi)\frac{t-s}{t_0}\right)\,ds,$$

where $t_0 = \frac{L}{c}$. Since Parseval's identity states that almost everywhere on $[0,T]$, we have

$$\int_0^L y(t,x)^2\,dx = \sum_{n=0}^{\infty} |\alpha_n(t)|^2 \quad \text{and} \quad \int_0^L y_t(t,x)^2\,dx = \sum_{n=0}^{\infty} |\alpha_n'(t)|^2.$$

We can represent the objective functional in the form

$$\int_0^T \int_0^L |y(t,x)-z(x)|^2 + \gamma\,|y_t(t,x)|^2\,dx + |u(t)-\zeta|^2\,dt$$
$$= \int_0^T \sum_{n=0}^{\infty} |\alpha_n(t)|^2 + \gamma|\alpha_n'(t)|^2 + |u(t)-\zeta|^2\,dt + \int_0^T \int_0^L z^2(x) + \sum_{n=0}^{\infty} 2\alpha_n(t)\varphi_n(x)z(x)\,dxdt.$$

If the control space $L^2(0,T)$ is replaced by a finite dimensional space of piecewise constant control functions, this yields a finite dimensional quadratic optimization problem. Since in this case, the necessary optimality conditions are a finite dimensional system of linear equations, this can be used to obtain numerical approximations of the optimal control.

5.4.1 Numerical Solutions

In this section, we discuss the numerical solutions for three examples. We consider the following problem data:

$$c := 1;\ \gamma := 0.1;\ T := 1, 10, 100;\ L := 1;$$

We choose $y_1(x) := 0 \quad (x \in (0,L))$.

1. $y_0(x) := x,\ z = 0, \zeta = 0$ (Example 1);
2. $y_0(x) := \pi^{-1}\sin(\pi x),\ z = 0, \zeta = 0$ (Example 2);
3. $y_0(x) := \pi^{-1}\sin(\pi x),\ z = x, \zeta = 1$ (Example 3);

The coefficients α_n^0 and α_n^1 in (5.85) are obtained as the Fourier coefficients of the chosen functions $y_0(x), y_1(x)$.

In order to solve the optimal boundary control problem numerically, a finite-dimensional approximation is used on two sides of (5.83) simultaneously:

First, the series expansion in the objective functional has to be cut after N terms which leads us to the consideration of the problem $(\mathbf{OPT})(\gamma,\,N)$. Second, we compute approximations for the optimal controls $u \in L^2(0,\, T)$ in the space of piecewise constant functions. Let a grid $0 = t_0 < t_1 < t_2 < ... < t_M = T$ be given. For $i \in \{1, \dots, M\}$ let

$$v_j(t) := \begin{cases} 1 & \text{if } t \in [t_{j-1}, t_j), \\ 0 & \text{elsewhere,} \end{cases}$$

and, define the finite dimensional space $X_M(T)$ by

$$X_M(T) := \operatorname{span}\{v_j(\cdot) : j = 1, \dots, M\}.$$

For any $u \in X_M(T)$ we use the representation

$$u(t) = \sum_{j=1}^{M} u(t_{j-1}) v_j(t), \qquad t \in [0, T),$$

where $v_j(t)$ stands for the characteristic function $\chi_{[t_{j-1}, t_j)}(t)$ of the interval $[t_{j-1}, t_j)$ and the approximation of control is defined by the vector $U = \operatorname{col}(U_1, \dots, U_M) \in \mathbb{R}^M$. Hence, we are finally led to solve the problem

$$\begin{aligned} (\mathbf{D_{opt}})(\gamma,\, N,\, M) \ \min_{u \in X_M(T)} \ & \sum_{j=1}^{M} (t_j - t_{j-1}) \left(u(t_{j-1})^2 + 2u(t_{j-1})\zeta + \zeta^2 \right) \\ & + \sum_{j=1}^{M} \sum_{n=0}^{N} \int_{t_{j-1}}^{t_j} |\alpha_n(t)|^2 + \gamma |\alpha_n'(t)|^2 \, dt \\ & + \sum_{j=1}^{M} \sum_{n=0}^{N} \int_{t_{j-1}}^{t_j} \int_0^L 2\alpha_n(t) \varphi_n(x) z(x) + z^2(x) \, dx dt \end{aligned} \tag{5.87}$$

with $\alpha_j(t)$ as defined in (5.86). Problem

$$(\mathbf{D_{opt}})(\gamma,\, N,\, M)$$

can be equivalently formulated as a quadratic programming problem in $\mathbb{R}^M$

$$\min_{U \in \mathbb{R}^M} \ U^\top Q\, U + q^\top U + W$$

where the matrix $Q(N)_{M \times M}$, depending on the fixed number N, and the vector $q \in \mathbb{R}^M$ are to be assembled for fixed N from the cost as stated in (5.87). The assemblage is described below. We use the notation $U_j = u(t_{j-1})$, ($j \in \{1, \dots, M\}$) and take into account a constant term W that is independent of U. We have

$$U^\top Q U + q^\top U + W = \sum_{j=1}^{M} (t_j - t_{j-1}) \left(u(t_{j-1})^2 + 2u(t_{j-1})\zeta + \zeta^2 \right)$$

$$+\sum_{j=1}^{M}\sum_{n=0}^{N}\int_{t_{j-1}}^{t_j}|\alpha_n(t)|^2+\gamma|\alpha_n'(t)|^2\,dt$$
$$+\sum_{j=1}^{M}\sum_{n=0}^{N}\int_{t_{j-1}}^{t_j}\int_0^L 2\alpha_n(t)\varphi_n(x)z(x)+z^2(x)\,dxdt.$$

For the convergence of the approximation, it is important to increase both N and M simultaneously. Otherwise, if only $M\to\infty$ convergence to the optimal control in general does not occur (due to a possible spillover effect). For $t\in(t_{j-1},t_j)$ we have

$$\begin{aligned}\alpha_n(t)&=\alpha_n^0\cos\Big((\frac{\pi}{2}+n\pi)t\Big)+(-1)^n\frac{\sqrt{2}}{\frac{\pi}{2}+n\pi}\int_0^t u(s)\,\sin\Big((\frac{\pi}{2}+n\pi)(t-s)\Big)\,ds\\&=\alpha_n^0\cos\Big((\frac{\pi}{2}+n\pi)t\Big)+(-1)^n\frac{\sqrt{2}}{\frac{\pi}{2}+n\pi}\sum_{k=1}^{j}U_k\int_0^t v_k(s)\,\sin\Big((\frac{\pi}{2}+n\pi)(t-s)\Big)\,ds\\&=\alpha_n^0\cos\Big((\frac{\pi}{2}+n\pi)t\Big)+(-1)^n\frac{\sqrt{2}}{\frac{\pi}{2}+n\pi}V_j^\top U,\end{aligned}$$

where $V_j=\mathrm{col}(V_{j,1},V_{j,2},\cdots,V_{j,j},0,\cdots,0)$,

$$V_{j,k}=\begin{cases}\int_{t_{k-1}}^{t_k}\sin\big((\frac{\pi}{2}+n\pi)(t-s)\big)\,ds & \text{if } k<j,\\ \int_{t_{k-1}}^{t}\sin\big((\frac{\pi}{2}+n\pi)(t-s)\big)\,ds & \text{if } k=j,\\ 0 & \text{if } k>j.\end{cases}$$

and

$$\begin{aligned}|\alpha_n(t)|^2&=(\alpha_n^0)^2\cos^2\Big((\frac{\pi}{2}+n\pi)t\Big)+\frac{2}{(\frac{\pi}{2}+n\pi)^2}U^\top V_jV_j^\top U\\&\quad+(-1)^n\frac{2\sqrt{2}\alpha_n^0}{\frac{\pi}{2}+n\pi}\cos\Big((\frac{\pi}{2}+n\pi)t\Big)V_j^\top U.\end{aligned}\tag{5.88}$$

This implies

$$\begin{aligned}\sum_{j=1}^{M}\sum_{n=0}^{N}\int_{t_{j-1}}^{t_j}|\alpha_n(t)|^2&=\sum_{j=1}^{M}\sum_{n=0}^{N}(\alpha_n^0)^2\int_{t_{j-1}}^{t_j}\cos^2\Big((\frac{\pi}{2}+n\pi)t\Big)\,dt\\&+\sum_{j=1}^{M}\sum_{n=0}^{N}\frac{2}{(\frac{\pi}{2}+n\pi)^2}U^\top\int_{t_{j-1}}^{t_j}V_jV_j^\top dt\,U\\&+\sum_{j=1}^{M}\sum_{n=0}^{N}(-1)^n\frac{2\sqrt{2}\alpha_n^0}{\frac{\pi}{2}+n\pi}\int_{t_{j-1}}^{t_j}\cos\Big((\frac{\pi}{2}+n\pi)t\Big)V_j^\top dt\,U\\&=W_1+U^\top Q_1U+q_1^\top U.\end{aligned}\tag{5.89}$$

Moreover, for the derivatives $\alpha_n'(t)$, we have for $t \in (t_{j-1}, t_j)$

$$\begin{aligned}\alpha_n'(t) &= -(\frac{\pi}{2}+n\pi)\alpha_n^0 \sin\Big((\frac{\pi}{2}+n\pi)t\Big) + (-1)^n\sqrt{2}\sum_{k=1}^{j} U_k \int_0^t v_k(s)\,\cos\Big((\frac{\pi}{2}+n\pi)(t-s)\Big)\,ds \\ &= -(\frac{\pi}{2}+n\pi)\alpha_n^0 \sin\Big((\frac{\pi}{2}+n\pi)t\Big) + (-1)^n\sqrt{2}\tilde{V}_j^\top U,\end{aligned} \tag{5.90}$$

where $\tilde{V}_j = \mathrm{col}(\tilde{V}_{j,1}, \tilde{V}_{j,2}, \cdots, \tilde{V}_{j,j}, 0, \cdots, 0)$,

$$\tilde{V}_{j,k} = \begin{cases} \int_{t_{k-1}}^{t_k} \cos\big((\frac{\pi}{2}+n\pi)(t-s)\big)\,ds & (k<j), \\ \int_{t_{k-1}}^{t} \cos\big((\frac{\pi}{2}+n\pi)(t-s)\big)\,ds & (k=j), \\ 0 & (k>j). \end{cases}$$

This yields

$$\begin{aligned}|\alpha_n'(t)|^2 &= (\alpha_n^0)^2(\frac{\pi}{2}+n\pi)^2 \sin^2\Big((\frac{\pi}{2}+n\pi)t\Big) + 2U^\top \tilde{V}_j\tilde{V}_j^\top U \\ &\quad + (-1)^{n+1}2\sqrt{2}\alpha_n^0(\frac{\pi}{2}+n\pi)\sin\Big((\frac{\pi}{2}+n\pi)t\Big)\,\tilde{V}_j^\top U.\end{aligned}$$

Hence we obtain

$$\begin{aligned}\gamma\sum_{j=1}^{M}\sum_{n=0}^{N}\int_{t_{j-1}}^{t_j}|\alpha_n'(t)|^2 &= \gamma\sum_{j=1}^{M}\sum_{n=0}^{N}(\alpha_n^0)^2(\frac{\pi}{2}+n\pi)^2\int_{t_{j-1}}^{t_j}\sin^2\Big((\frac{\pi}{2}+n\pi)t\Big)\,dt \\ &\quad + \gamma\sum_{j=1}^{M}\sum_{n=0}^{N}2U^\top\int_{t_{j-1}}^{t_j}\tilde{V}_j\tilde{V}_j^\top dt\,U \\ &\quad + \gamma\sum_{j=1}^{M}\sum_{n=0}^{N}(-1)^{n+1}2\sqrt{2}\alpha_n^0(\frac{\pi}{2}+n\pi)\int_{t_{j-1}}^{t_j}\sin\Big((\frac{\pi}{2}+n\pi)t\Big)\,\tilde{V}_j^\top dt\,U \\ &= \gamma(W_2 + U^\top Q_2 U + q_2^\top U).\end{aligned}$$

And we have

$$\begin{aligned}\sum_{j=1}^{M}\sum_{n=0}^{N}\int_{t_{j-1}}^{t_j}\alpha_n(t)\varphi_n(x)z(x) &= \sum_{j=1}^{M}\sum_{n=0}^{N}\int_{t_{j-1}}^{t_j}\alpha_n^0\cos\Big((\frac{\pi}{2}+n\pi)t\Big)\,dt\int_0^L z(x)\varphi(x)dx \\ &\quad + \sum_{j=1}^{M}\sum_{n=0}^{N}(-1)^n\frac{\sqrt{2}}{\frac{\pi}{2}+n\pi}\int_{t_{j-1}}^{t_j}V_j^\top dt\,U \\ &= W_3 + q_3^\top U.\end{aligned} \tag{5.91}$$

For the objective function, this implies

$$\begin{aligned}
&\sum_{j=1}^{M}(t_j - t_{j-1})\left(u(t_{j-1})^2 + 2u(t_{j-1})\zeta + \zeta^2\right) := U^\top Q_3 U + 2q_\tau^\top U + W_\tau,\\
&\sum_{j=1}^{M}\sum_{n=0}^{N}\int_{t_{j-1}}^{t_j} |\alpha_n(t)|^2 + \gamma|\alpha_n'(t)|^2\,dt := U + q_1^\top U + W_1 + \gamma(U^\top Q_2\,U + q_2^\top U + W_2),\\
&\sum_{j=1}^{M}\sum_{n=0}^{N}\int_{t_{j-1}}^{t_j}\int_0^L 2\alpha_n(t)\varphi_n(x)z(x) + z^2(x)\,dxdt = 2q_3^\top U + W_3.
\end{aligned}$$

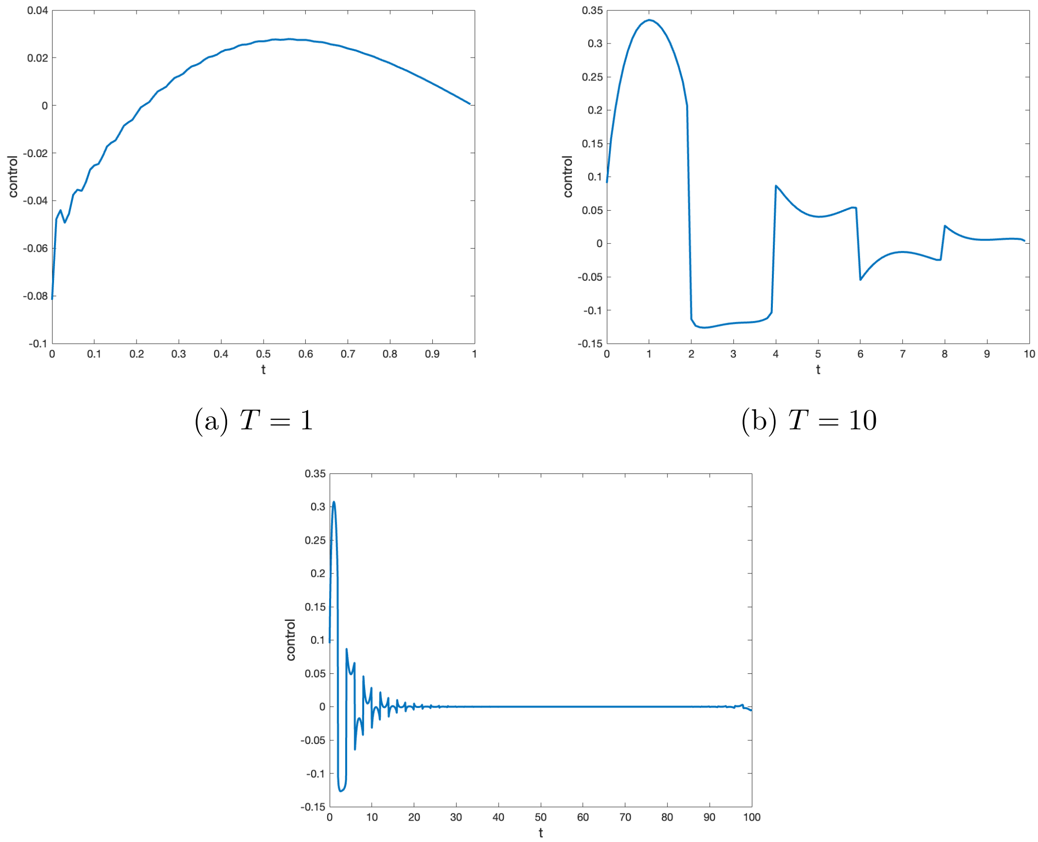
Thus

$$U^\top QU + q^\top U + W = U^\top(Q_1 + \gamma Q_2 + Q_3)U + (q_1 + \gamma q_2 + q_3 + q_\tau)^\top U + W_1 + \gamma W_2 + W_3 + W_\tau.$$

Here we use the notation

$$\begin{aligned}
Q_1 &= \sum_{j=1}^{M}\sum_{n=0}^{N}\frac{2}{(\frac{\pi}{2}+n\pi)^2}\int_{t_{j-1}}^{t_j} V_j V_j^\top\,dt,\\
Q_2 &= \sum_{j=1}^{M}\sum_{n=0}^{N} 2\int_{t_{j-1}}^{t_j}\tilde{V}_j\tilde{V}_j^\top\,dt,\\
Q_3 &= \operatorname{diag}(t_1 - t_0, \cdots, t_k - t_{k-1}, \cdots, t_M - t_{M-1}),\\
q_1 &= \sum_{j=1}^{M}\sum_{n=0}^{N}(-1)^n\frac{2\sqrt{2}\alpha_n^0}{\frac{\pi}{2}+n\pi}\int_{t_{j-1}}^{t_j}\cos\left((\frac{\pi}{2}+n\pi)t\right)V_j\,dt,\\
q_2 &= \sum_{j=1}^{M}\sum_{n=0}^{N}(-1)^{n+1}2\sqrt{2}\alpha_n^0(\frac{\pi}{2}+n\pi)\int_{t_{j-1}}^{t_j}\sin\left((\frac{\pi}{2}+n\pi)t\right)\tilde{V}_j\,dt,\\
q_3 &= \sum_{j=1}^{M}\sum_{n=0}^{N}(-1)^n\frac{\sqrt{2}}{\frac{\pi}{2}+n\pi}\int_{t_{j-1}}^{t_j} V_j^\top\,dt,\\
q_\tau &= (t_1 - t_0, t_2 - t_1, \cdots, t_M - t_{M-1})^\top,\\
W_1 &= \sum_{j=1}^{M}\sum_{n=0}^{N}(\alpha_n^0)^2\int_{t_{j-1}}^{t_j}\cos^2\left((\frac{\pi}{2}+n\pi)t\right)dt,\\
W_2 &= \sum_{j=1}^{M}\sum_{n=0}^{N}(\alpha_n^0)^2\int_{t_{j-1}}^{t_j}\sin^2\left((\frac{\pi}{2}+n\pi)t\right)dt,\\
W_3 &= \sum_{j=1}^{M}\sum_{n=0}^{N}\int_{t_{j-1}}^{t_j}\alpha_n^0\cos\left((\frac{\pi}{2}+n\pi)t\right)dt\int_0^L z(x)\varphi(x)dx.
\end{aligned}$$

We employ Matlab for the computational analysis of all examples. Fig. 5.1 and Fig. 5.2 illustrate the optimal control and state, respectively, for varying values of T in Example 1. Notably, as T increases significantly, the control variable u converges to ζ. Moreover, with the increment in T, there is a discernible trend towards stabilization in the norms of both u and y, as evidenced in Fig. 5.3. Fig. 5.4-5.6 present the results obtained from Example 2, whereas Fig. 5.7 to 5.9 depicts the outcomes of Example 3. Additionally, Fig. 5.10 presents the quotient of the optimal value of $(\mathbf{D_{opt}})(\gamma, N, M)$, which adheres to the turnpike property, thereby providing valuable insights into the behavior of the system concerning varying T values.

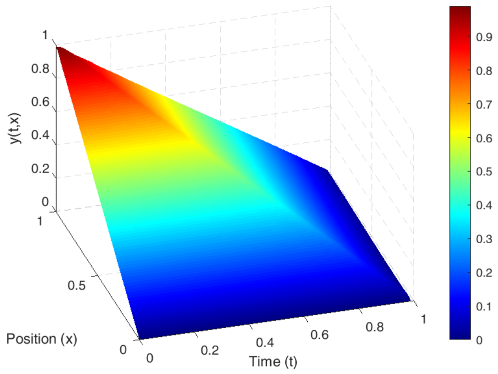


(a) $T = 1$

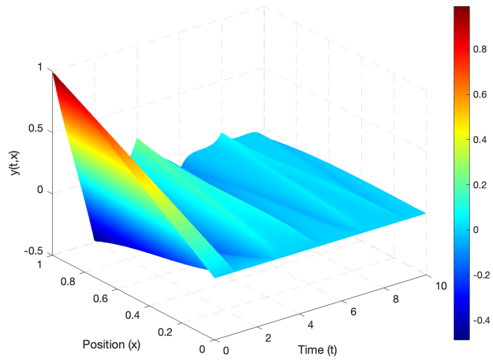
(b) $T = 10$

(c) $T = 100$

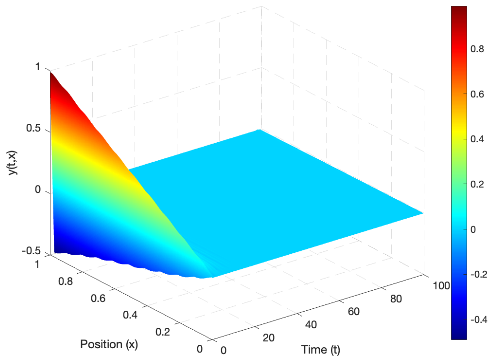
Figure 5.1: Optimal control for different values of T in Example 1



(a) $T = 1$

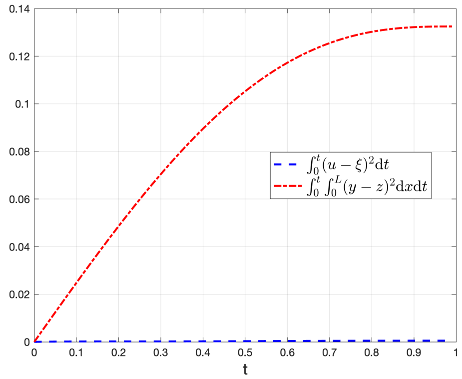


(b) $T = 10$

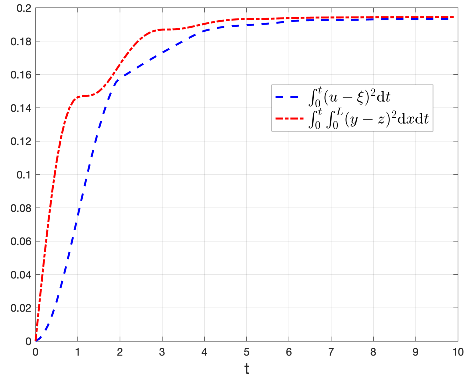


(c) $T = 100$

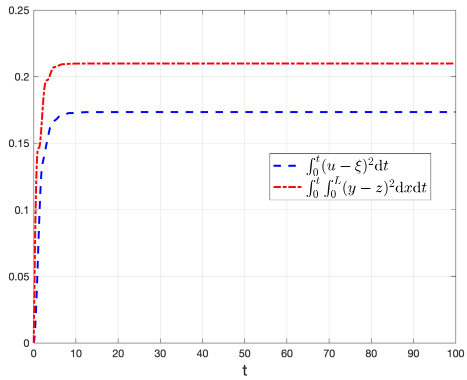
Figure 5.2: Optimal state for different values of T in Example 1



(a) $T = 1$

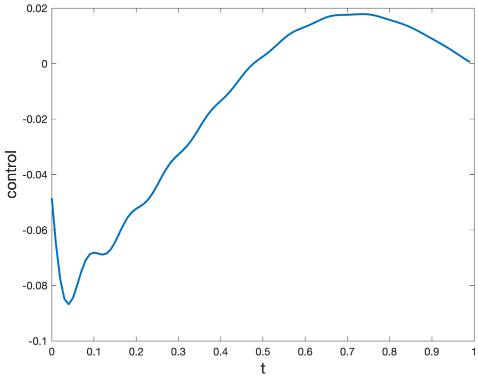


(b) $T = 10$

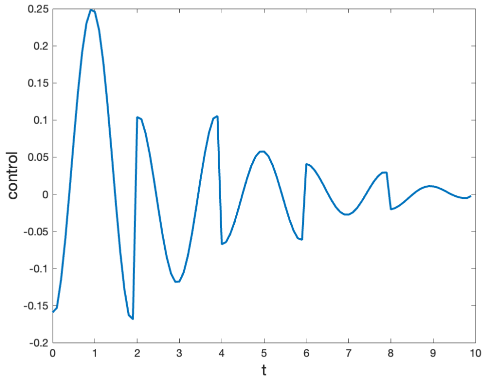


(c) $T = 100$

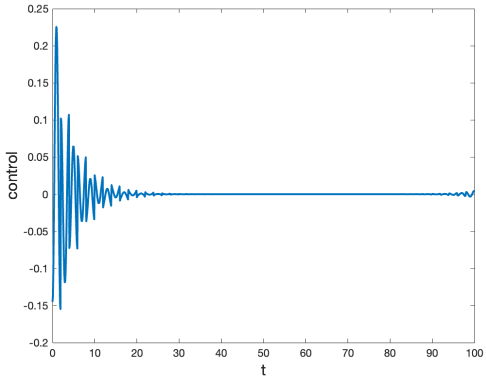
Figure 5.3: Optimal $\int_0^t (u-\xi)^2 \mathrm{d}t$ and $\int_0^t \int_0^L (y-z)^2 \mathrm{d}x\mathrm{d}t$ for different values of T in Example 1 $(\xi = 0,\ z = 0, y_0 = x)$



(a) $T = 1$

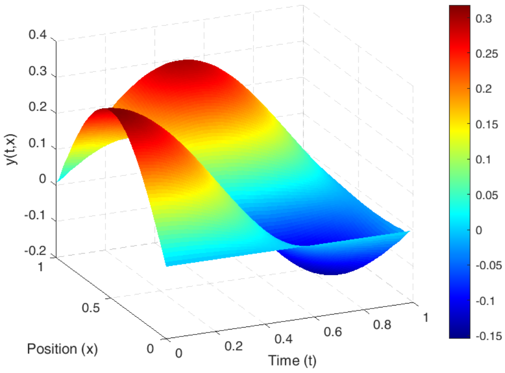


(b) $T = 10$

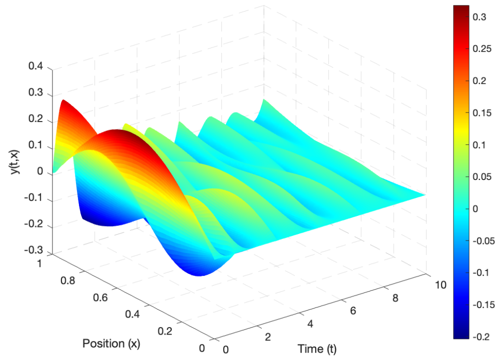


(c) $T = 100$

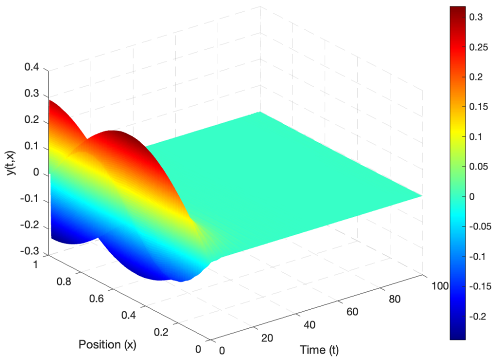
Figure 5.4: Optimal control for different values of T in Example 2



(a) $T = 1$

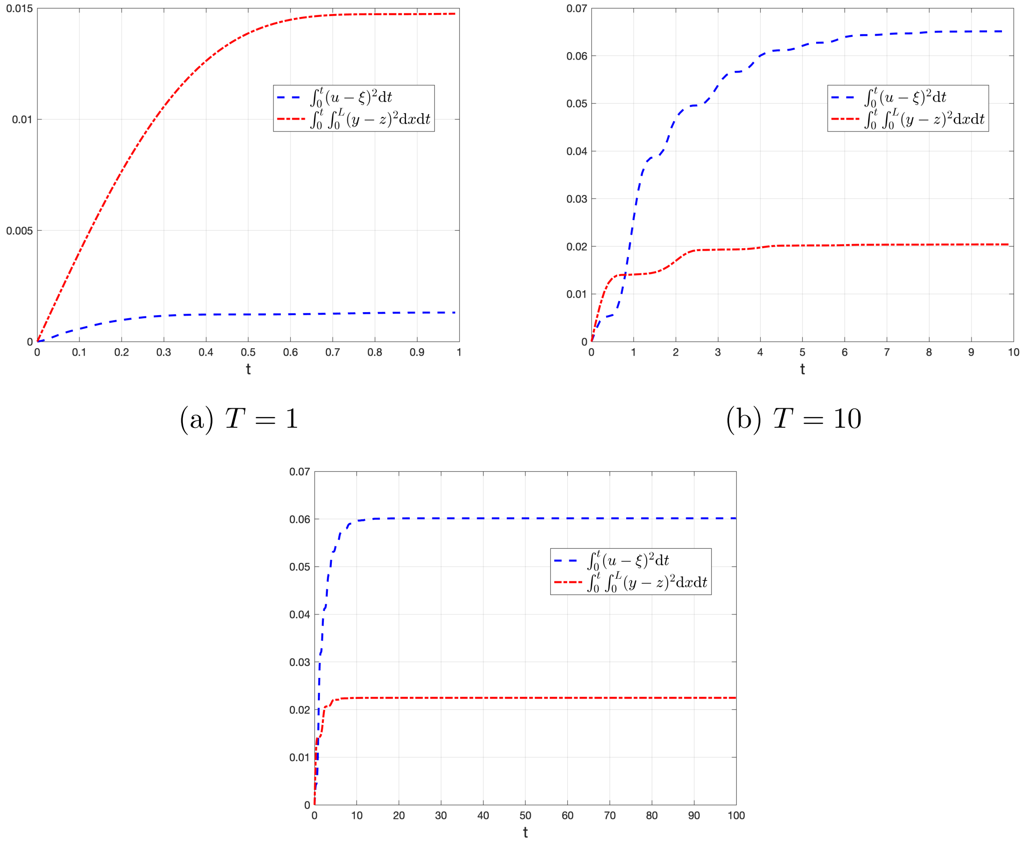


(b) $T = 10$



(c) $T = 100$

Figure 5.5: Optimal state for different values of T in Example 2

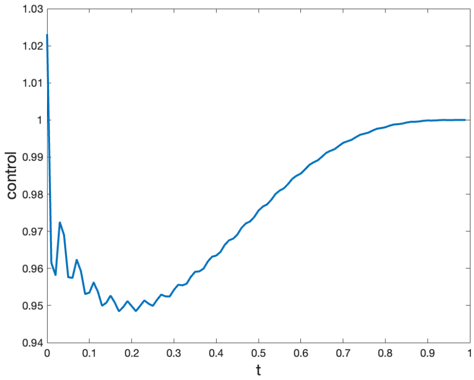


(a) $T = 1$

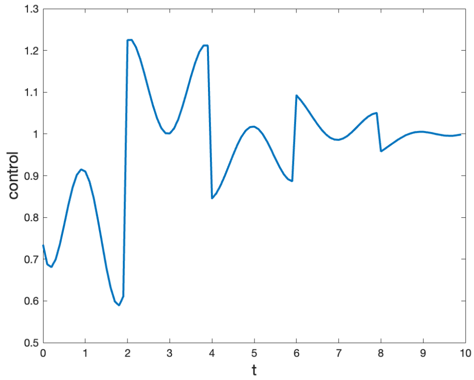
(b) $T = 10$

(c) $T = 100$

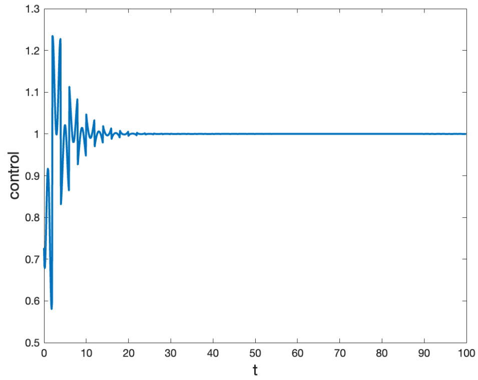
Figure 5.6: Optimal $\int_0^t (u-\xi)^2 \mathrm{d}t$ and $\int_0^t \int_0^L (y-z)^2 \mathrm{d}x\mathrm{d}t$ for different values of T in Example 2 ($\xi = 0,\ z = 0, y_0 = \pi^{-1}\sin(\pi x)$)



(a) $T = 1$

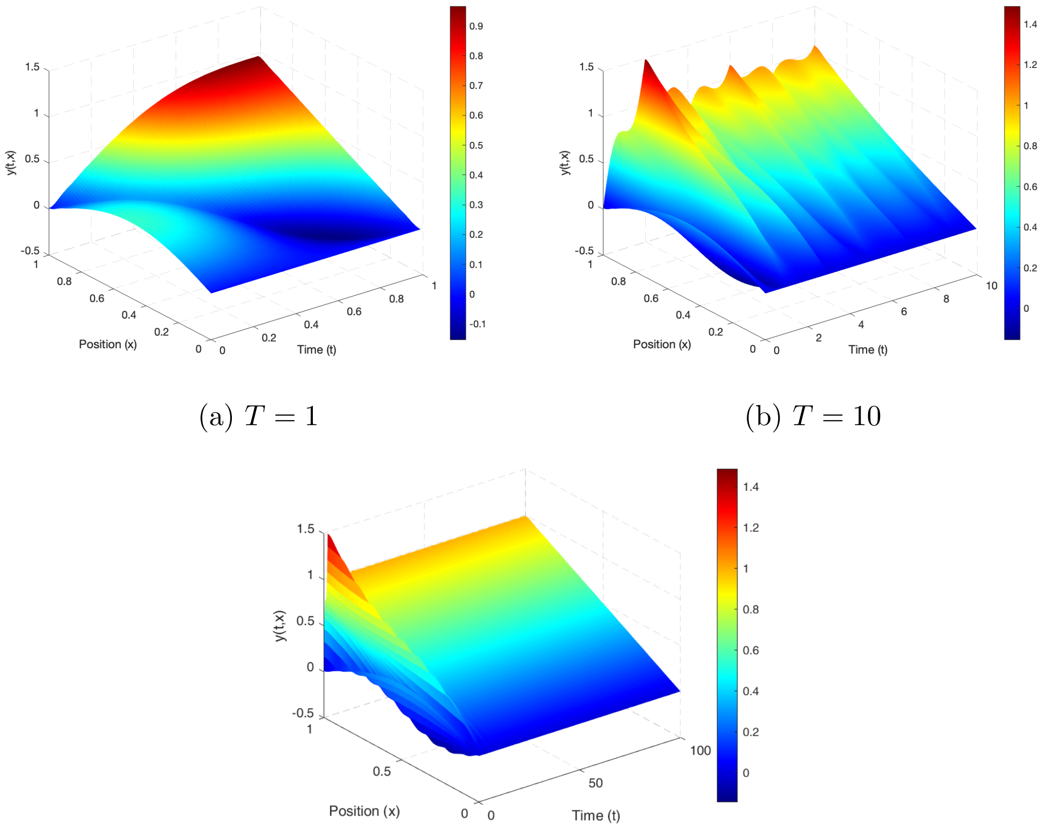


(b) $T = 10$



(c) $T = 100$

Figure 5.7: Optimal control for different values of T in Example 3

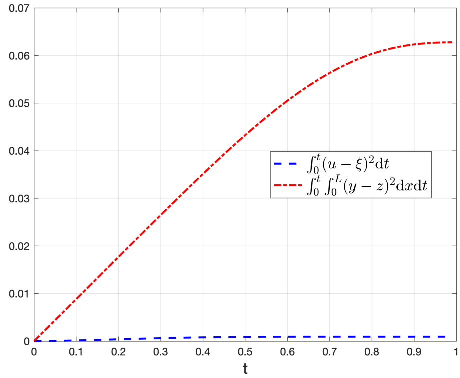


(a) $T = 1$

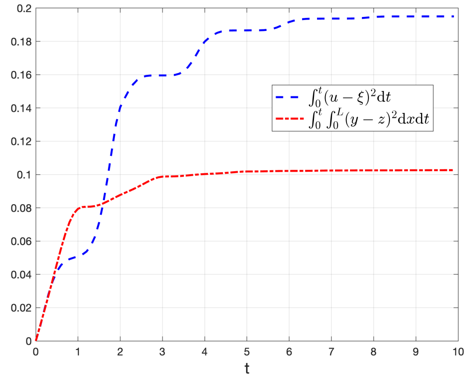
(b) $T = 10$

(c) $T = 100$

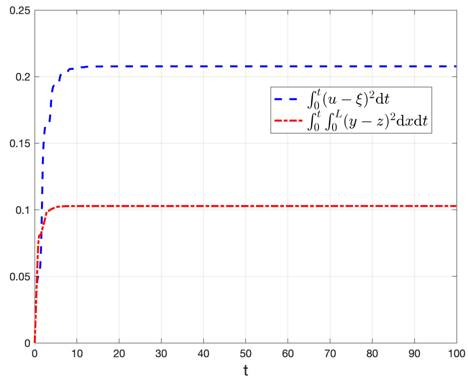
Figure 5.8: Optimal state for different values of T in Example 3



(a) $T = 1$

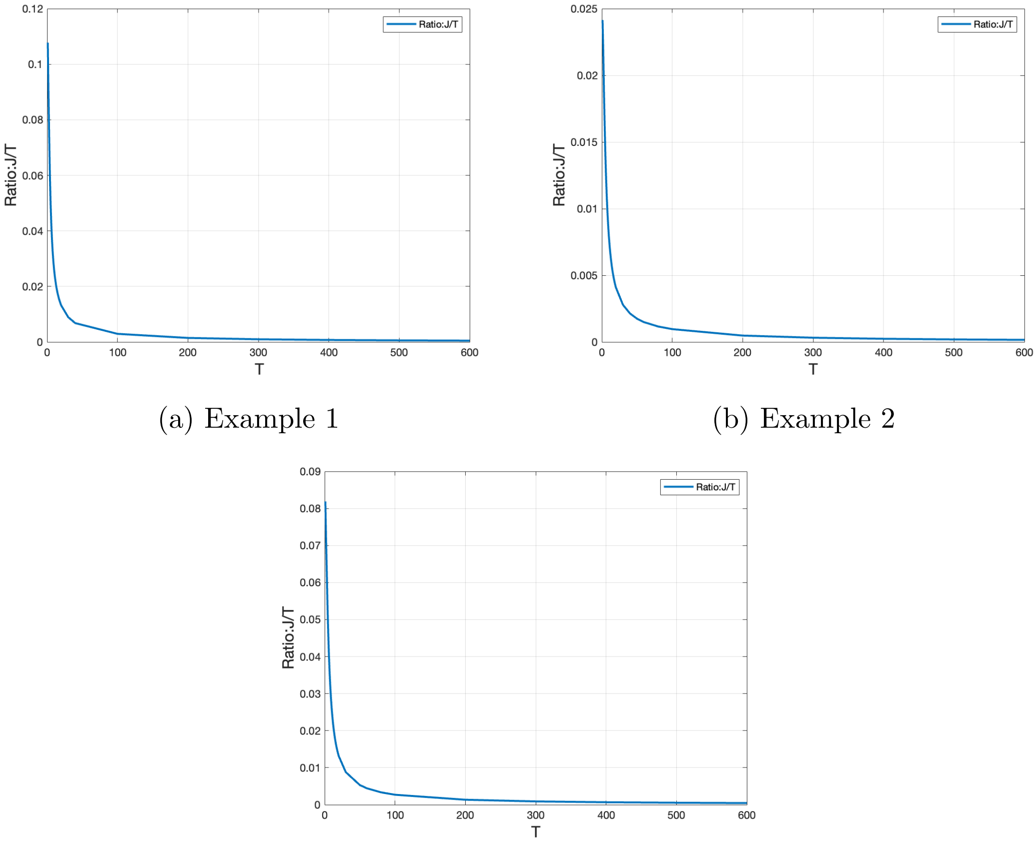


(b) $T = 10$



(c) $T = 100$

Figure 5.9: Optimal $\int_0^t (u-\xi)^2 \mathrm{d}t$ and $\int_0^t \int_0^L (y-z)^2 \mathrm{d}x\mathrm{d}t$ for different values of T in Example 3 ($\xi = 1,\ z = x, y_0 = \pi^{-1}\sin(\pi x)$)



(a) Example 1

(b) Example 2

(c) Example 3

Figure 5.10: Convergence for different examples

5.5 Topological Derivatives for Network Optimal Control Problems

The topology of network for the purposes of optimal control problems is selected in the framework of the topological derivative method for static problems, we refer to [30] for an elementary example. It turns out, that that the topological derivatives for a class of cost functions can be determined by using the domain decomposition technique for the state equation [87]. We describe in detail the topological derivative method and present numerical results for examples.

Let us consider the network static problem. We define the topological derivatives for optimal cost of network control problems with respect to nucleation of a small cycle. We present also numerical examples. The simplest example of a network is the three-star graph with one central vertex P_0 and three boundary vertices P_1, P_2, P_3, thus $V = \{P_0, P_1, P_2, P_3\}$ (See Fig. 5.11). There are three edges $E = \{E_1, E_2, E_3\}$.

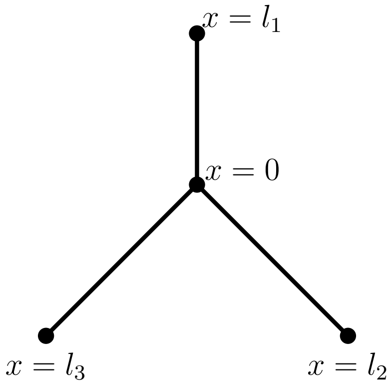


Figure 5.11: The three-star graph

For the steady state problem, singular domain perturbations of the shape are considered. The topological derivatives of the shape functional are defined. The shape and topology optimization is performed. The network is singularly perturbed by a small cycle of the size $\varepsilon \to 0$ (See Fig. 5.12). In such a case the domain decomposition technique is used and the Steklov-Poincaré operator is introduced. The topological derivative technique is employed in order to decide if a small cycle is useful for the topology optimization of the network.

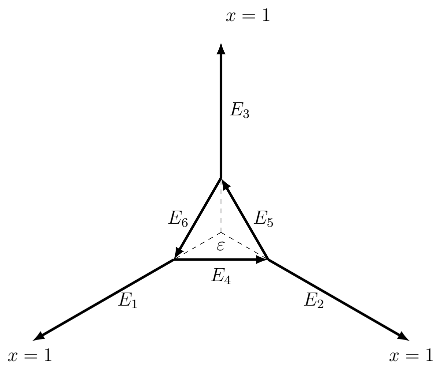


Figure 5.12: Nucleation of a cycle of size ε in three-star graph

We introduce multiple perturbations of network represented in Figure 5.13. The Steklov-Poincaré operator Λ_ε replaces the subgraph G_ε in the state equation of the network. In this way, the topological derivative of the cost for optimal control problem on the perturbed network is obtained for the nucleation of multiple cycles in the three-star graph.

5.5.1 Shape and Topology Optimization on Networks

We recall briefly the shape and topological derivatives of a given cost for the network. We restrict ourselves to static problems however the shape and topology optimization can

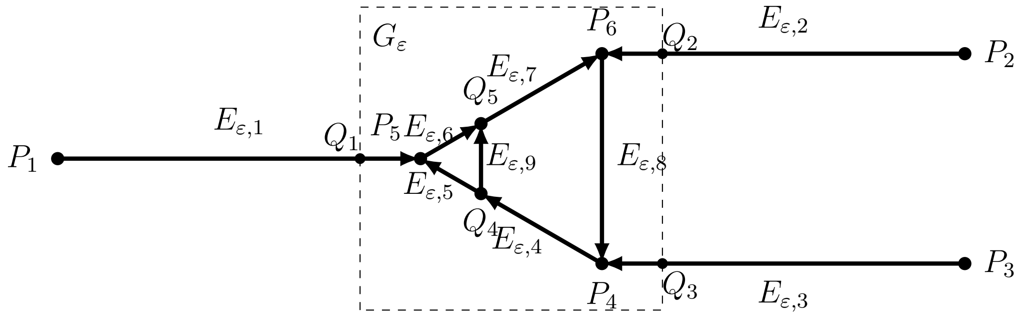


Figure 5.13: Multiple perturbations of the three-star graph for domain decomposition technique.

also be performed for dynamic optimal control problems on networks. For the optimal control problems with the *turnpike Property* the analysis of static problem is useful for the solution of dynamic problem. In particular, the topology of the network is designed using the static problem.

The shape Ω of the network for fixed topology is governed by the finite dimensional vector ℓ, which contains the lengths of edges, $\ell = \operatorname{col}(L_1, \ldots, L_N)$ where $N = \#E = \{E_i | i \in \mathcal{I}\}$. Therefore, $\Omega := \Omega(\ell)$, and the cost $\ell \mapsto \mathcal{I}(\ell) := \mathcal{J}(\Omega(\ell))$ is defined by the optimal cost of control problem $\mathcal{J}(\Omega) := J(\hat{u}(\Omega))$ for evolution problem. For the steady state problem the optimal cost of control problem is denoted by $\ell \mapsto J(\hat{v}(\Omega(\ell)))$, where $\hat{v}(\Omega(\ell))$ is steady state optimal control in the domain defined by the shape $\Omega(\ell)$. We consider the Neumann control in the numerical examples presented for the wave equation.

5.5.2 Examples of Topological Derivatives for Networks

Two examples are presented of singular network perturbation by nucleation of a small cycle. In the first example, the topological derivative is evaluated for the optimal control problem with the static state equation, and then for the dynamic state equation, the optimal size of the cycle is determined. In the second example, the topological derivative is evaluated for multiple singular perturbations of the three-star network.

Example 4 (for one cycle): In this example the optimal control u is computed for the geometry depicted in Figure 5.14. The variables z and ζ satisfy

$$\begin{cases} -z_i'' = 0, x \in [0, L_i],\ i = 1, \cdots, 6, \\ z_1'(0) = \zeta, z_2(L_2) = z_3(L_3) = 0, \\ \text{Continuity and Kirchhoff Condition.} \end{cases} \tag{5.92}$$

Set $\zeta = 1$, $L_1 = L_2 = L_3 = 2$, $\varepsilon_0 = 0.5$, $\varepsilon_{\max} = 1$, and $0 \leq \varepsilon \leq \varepsilon_{\max}$. Here, $G_\varepsilon = \{E_\varepsilon, V_\varepsilon\}$ contains a small cycle. $E_\varepsilon = \{E_{\varepsilon,1}, E_{\varepsilon,2}, \cdots, E_{\varepsilon,6}\}$, $V_\varepsilon = \{Q_1, Q_2, Q_3, P_4, P_5, P_6\}$, $|E_{\varepsilon,1}| = |E_{\varepsilon,2}| = |E_{\varepsilon,3}| = \varepsilon_{\max} - \varepsilon = 1 - \varepsilon$, $|E_{\varepsilon,4}| = |E_{\varepsilon,5}| = |E_{\varepsilon,6}| = \varepsilon$. The cost functional

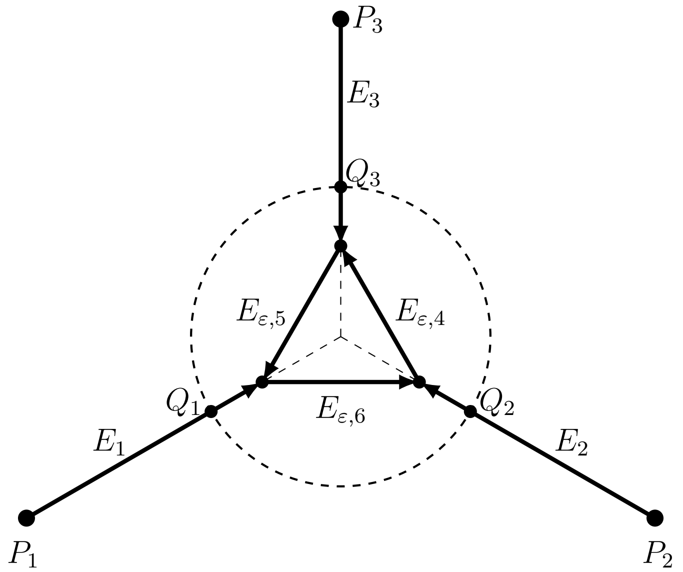


Figure 5.14: Domain decomposition for tripod directed network with an elementary small cycle.

under consideration is defined as:

$$J(u) = \frac{1}{2}\sum_{i=1}^{3}\int_0^{L_i-\varepsilon_{\max}} (y_i - z_i)^2 + \frac{1}{2}|u-\zeta|^2.$$

By the Lagrange method, the optimality system is given by

$$\begin{cases} \sum\limits_{i=1}^{3}\int_0^{L_i-\varepsilon_{\max}} y_i\phi_i + a(\Omega^0; p, \phi) - \phi(L_i-\varepsilon_{\max})^\top \Lambda_\varepsilon p(L_i-\varepsilon_{\max}) = \sum\limits_{i=1}^{3}\int_0^{L_i-\varepsilon_{\max}} z_i\phi_i, \\ a(\Omega^0; y, \phi) - p_1(0)\phi_1(0) - y(L_i-\varepsilon_{\max})^\top \Lambda_\varepsilon \phi(L_i-\varepsilon_{\max}) = -\zeta\phi_1(0), \end{cases}$$

where

$$\phi \in H = \left\{\phi_i, \phi_i' \in L^2(0, L_i),\ \phi_2(0) = \phi_3(0) = 0,\ \text{continuity at interior vertices.}\right\}$$

and

$$\Lambda_\varepsilon = \frac{1}{2\varepsilon - 3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}. \tag{5.93}$$

Fig. 5.15 shows the shape functional with respect to ε. The derivative of J with respect to ε is consistently negative as ε approaches 0. This is the information that allows for the topology variations by nucleation of a small cycle at an interior vertex of the graph. Another observation emerges: as ε approaches ε_0, the cost functional (J) converges to 0, indicating an optimal length for the introduced cycle. This implies that at $\varepsilon = \varepsilon_0$,

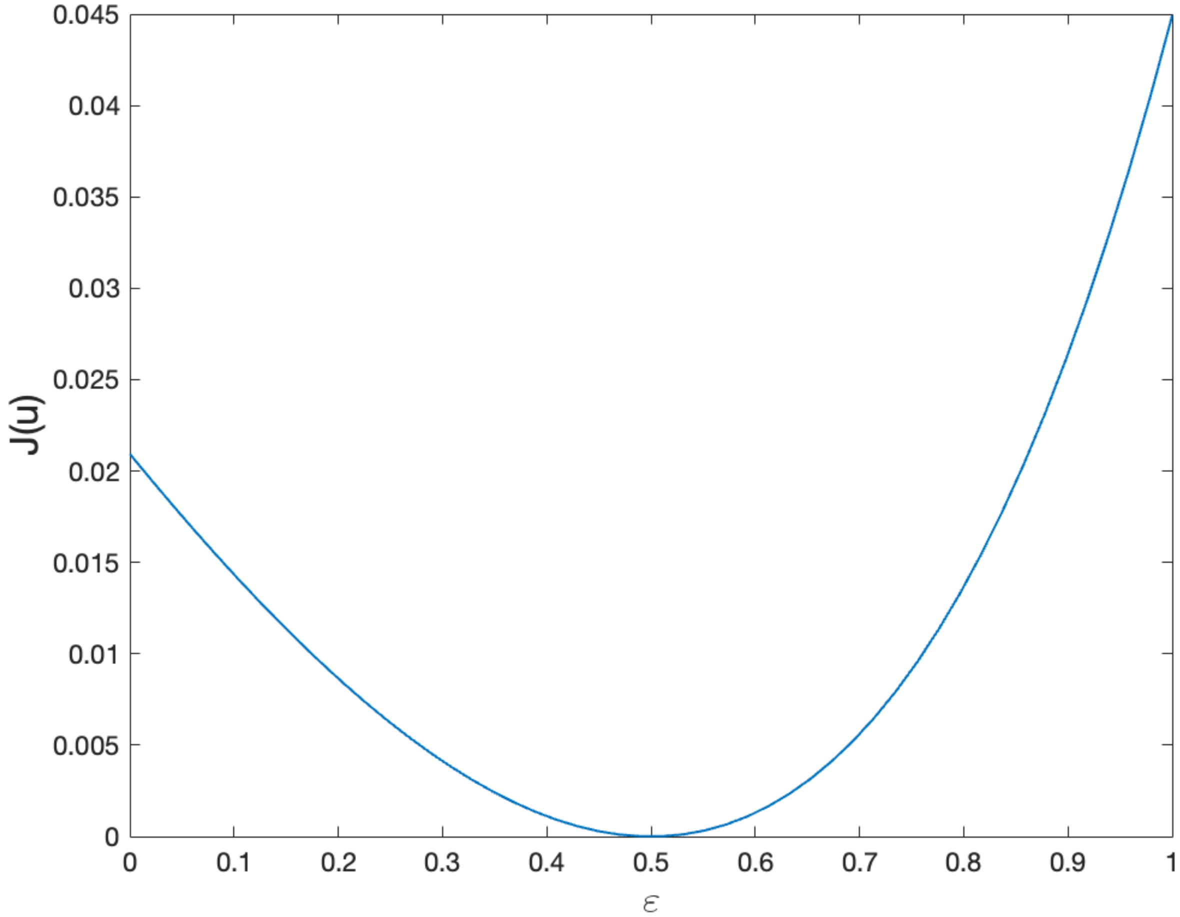


Figure 5.15: The shape functional for $\varepsilon \in [0, 1]$ in Example 4

the network experiences an optimal configuration, emphasizing the critical nature of this parameter in shaping the network.

Example 5 (for multiple cycles): In this example, the geometry is depicted in Figure 5.13. Here, $G_\varepsilon = \{E_\varepsilon, V_\varepsilon\}$ contains a small cycle. $E_\varepsilon = \{E_{\varepsilon,1}, E_{\varepsilon,2}, \cdots, E_{\varepsilon,9}\}$, $V_\varepsilon = \{Q_1, Q_2, \cdots, Q_5, P_4, P_5, P_6\}$, $|Q_1P_5| = |Q_2P_6| = |Q_3P_4| = \varepsilon_{\max} - \varepsilon = 1 - \varepsilon$, $|E_{\varepsilon,5}| = |E_{\varepsilon,6}| = |E_{\varepsilon,9}| = \varepsilon^2$, $|E_{\varepsilon,4}| = |E_{\varepsilon,7}| = \varepsilon - \varepsilon^2$, $|E_{\varepsilon,8}| = \varepsilon$, and $|P_iQ_i| = L_i - \varepsilon_{\max}$ $(i = 1, 2, 3)$. The parameters, cost functional, and optimality system remain consistent with the Example for one cycle (static), with the sole distinction being the form of Λ_ε. The Steklov-Poincaré operator for the small, double cycle is:

$$\Lambda_\varepsilon = -\frac{1}{2\varepsilon - 3}\begin{pmatrix} -2 & 1 & 1 \\ 1 & \dfrac{-5\varepsilon^2 + 20\varepsilon - 18}{2\varepsilon^2 - 10\varepsilon + 9} & \dfrac{3\varepsilon^2 - 10\varepsilon + 9}{2\varepsilon^2 - 10\varepsilon + 9} \\ 1 & \dfrac{3\varepsilon^2 - 10\varepsilon + 9}{2\varepsilon^2 - 10\varepsilon + 9} & \dfrac{-5\varepsilon^2 + 20\varepsilon - 18}{2\varepsilon^2 - 10\varepsilon + 9} \end{pmatrix}.$$

For numerical results, refer to Fig. 5.16. The topological derivative at $\varepsilon = 0^+$ is negative. And the optimal size of the cycle is $\varepsilon = \varepsilon_0 = 0.5$.

All in all, we exploit the properties of shape and topology optimization problems in one space dimension. The optimal control problems are considered in static and dynamic cases. For the tree network, we show the Turnpike Property for the wave equation and consider the geometric shape optimization. The new case is the tree with a small cycle. In such a case the topology of network is determined by using the topological derivatives obtained in static case. Numerically, the case of the cycle also does not pose a problem. This is new: Also here shape optimization works!

This also holds for a graph with multiple cycles as long as in the shape optimization no topology change occurs, that is no cycle vanishes. The Turnpike property also holds if a cycle disappears. We have shown this under the assumption that the initial state is not supported on the cycle. We expect that this condition is not sharp.

The convergence of gradient flow for shape optimization is also relevant to networks. The modeling of shapes of animals is another possibility for the spectral methods. We refer to [72], and [14] for the related results. Further research will concern the corresponding shape optimization problems in the general three dimensional case.

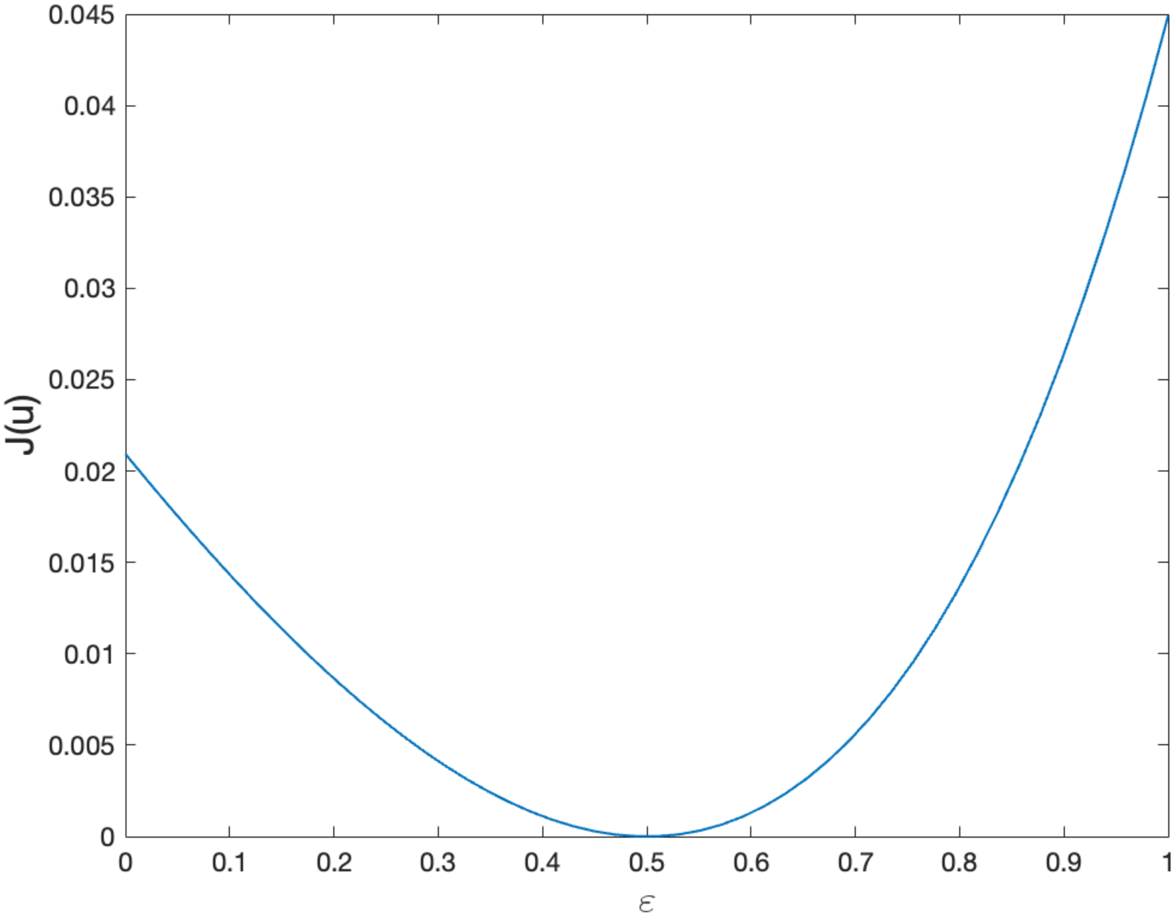


Figure 5.16: The shape functional for the size of cycle $\varepsilon \in [0, 1]$ in Example 5.

Chapter 6

Conclusions and Further Research

The complete results, considering Turnpike Property, on control and design of network with scalar wave equation are presented. The topological derivatives are derived for shape functional on networks. The necessary and sufficient optimality conditions are obtained for control and designed networks with nonlinear ODEs. In the linear case, the topological derivatives are used for singular geometric perturbations of network in order to perform the topology optimization. Numerical examples show that this strategy is efficient in improving the cost of optimal control problems.

Further research and open problems:

- Turnpike Property for linear state equations of Timoshenko beams of networks
- Turnpike Property for nonlinear state equations of geometrically exact beams
- Topological derivatives for network of geometrically exact beams in static setting
- Shape and topology optimization of control problems for networks of Timoshenko beams
- Shape optimum design of networks with geometrically exact beams for optimal control problems
- Shape and control of nonlinear state equations in fluid mechanics on networks.

Bibliography

[1] W. Achtziger. *Topology Optimization of Discrete Structures*, pages 57–100. Springer Vienna, Vienna, 1997.

[2] V. M. Alekseev, V. M. Tikhomirov, and S. V. Fomin. Optimal'noe up-ravlenie [optimal control], Nauka, Moscow [in Russian], 1979.

[3] G. Allaire, F. De Gournay, F. Jouve, and A.-M. Toader. Structural optimization using topological and shape sensitivity via a level set method. *Control Cybern.*, 34(1):59–80, 2005.

[4] G. Allaire and F. Jouve. Coupling the level set method and the topological gradient in structural optimization. In *IUTAM Symposium on Topological Design Optimization of Structures, Machines and Materials*, pages 3–12. Springer, 2006.

[5] H. Ammari, H. Kang, G. Nakamura, and K. Tanuma. Complete asymptotic expansions of solutions of the system of elastostatics in the presence of an inclusion of small diameter and detection of an inclusion. *J. Elasticity*, 67(2):97–129, 2002.

[6] S. Avdonin, J. Edward, and G. Leugering. Controllability for the wave equation on graph with cycle and delta-prime vertex conditions. *Evol. Equ. Control Theory*, pages 1542–1558, 2023.

[7] A. Bazoune, Y.A. Khulief, and N.G. Stephen. Shape functions of three-dimensional timoshenko beam element. *J. Sound Vib.*, 259(2):473–480, 2003.

[8] M. P. Bendsøe and O. Sigmund. *Topology Optimization: Theory, Methods and Applications.* Springer, 2003.

[9] M. Bonnet and G. Delgado. The topological derivative in anisotropic elasticity. *Quart. J. Mech. Appl. Math.*, 66:557–586, 2013.

[10] I. N. Bronstein, K. A. Semendjajew, G. Musiol, and H. Mühlig. *Taschenbuch der Mathematik. Übers. aus dem Russischen. Mit 1 CD-ROM.* Harri Deutsch, Frankfurt am Main, 5., überarb. und erweiterte Aufl. der Neubearbeitung edition, 2001.

[11] I. N. Bronstein, K. A. Semendjajew, G. Musiol, and H. Mühlig. *Handbook of Mathematics. Transl. from the German.* Springer, Berlin, 4th ed. edition, 2004.

[12] A. Carpio and M. L. Rapún. Hybrid topological derivative and gradient-based methods for electrical impedance tomography. *Inverse Problems*, 28(9), 2012. Article 095010, 22 pages.

[13] E. Carrera, G. Giunta, and M. Petrolo. *The Euler–Bernoulli and Timoshenko Theories*, chapter 2, pages 9–22. 1 edition, 2011.

[14] G. Chen, J. Huang, C. Wei, et al. Animal shapes, modal analysis, and visualization of motion(I): Horse and camel. *J. Geom. Anal.*, 33(10):Paper No. 328, 60, 2023.

[15] J. Chetboun and G. Allaire. Flow control of curved air ducts using topological derivatives. 2009.

[16] J. Chróścielewski, J. Makowski, and W. Pietraszkiewicz. *Statyka i dynamika powłok wielopłatowych. Nieliniowa teoria i metoda elementów skończonych.* IPPT PAN, Warsaw, 2004.

[17] J. R. de Faria, A. A. Novotny, R. A. Feijóo, E. Taroco, and C. Padra. Second order topological sensitivity analysis. *Internat. J. Solids Structures*, 44(14-15):4958–4977, 2007.

[18] H. Dubbel. *DUBBEL: Taschenbuch Für den Maschinenbau.* Springer, Verlag, 2013.

[19] A. Y. Dubovitskii and A. A. Milyutin. Extremum problems in the presence of restrictions. *Zh. Vychisl. Mat. Mat. Fiz*, 5(3):395–453, 1965.

[20] Z. Emirsjlow and S. Townley. From PDEs with boundary control to the abstract state equation with an unbounded input operator: a tutorial. *Eur. J. Control*, 6(1):27–53, 2000.

[21] T. Faulwasser, L. Grüne, J.-P. Humaloja, and M. Schaller. The interval turnpike property for adjoints. *Pure Appl. Funct. Anal.*, 7(4):1187–1207, 2022.

[22] J. Friederich, G. Leugering, and P. Steinmann. Adaptive refinement based on asymptotic expansions of finite element solutions for node insertion in 1d. *GAMM-Mitteilungen*, 35(2):175–190, 2012.

[23] S. Garreau, P. Guillaume, and M. Masmoudi. The topological asymptotic for pde systems: the elasticity case. *SIAM J. Control Optim.*, 39(6):1756–1778, 2001.

[24] L. Grüne. Dissipativity and optimal control: examining the turnpike phenomenon. *IEEE Control Systems*, 42(2):74–87, 2022.

[25] M. Gugat. *Optimal Boundary Control and Boundary Stabilization of Hyperbolic Systems.* Birkhäuser, Basel, Switzerland, 2015.

[26] M. Gugat and F. M. Hante. On the turnpike phenomenon for optimal boundary control problems with hyperbolic systems. *SIAM J. Control Optim.*, 57(1):264–289, 2019.

[27] M. Gugat and M. Herty. Existence of classical solutions and feedback stabilization for the flow in gas networks. *ESAIM: Control Optim. Calc. Var.*, 17(1):28–51, 2011.

[28] M. Gugat and M. Herty. Modeling, control, and numerics of gas networks. In *Numerical control. Part A*, volume 23 of *Handb. Numer. Anal.*, pages 59–86. 2022.

[29] M. Gugat and M. Lazar. Turnpike properties for partially uncontrollable systems. *Automatica*, 149:110844, 2023.

[30] M. Gugat, M. Qian, and J. Sokołowski. Topological derivative method for control of wave equation on networks. *2023 27th Int. Conf. on Methods and Models in Automation and Robotics (MMAR), Miedzyzdroje, Poland*, pages 320–325, 2023.

[31] Martin Gugat, Meizhi Qian, and Jan Sokolowski. Network Design and Control: Shape and Topology Optimization for the Turnpike Property for the Wave Equation. *J. Geom. Anal.*, 34(9):Paper No. 273, 2024.

[32] A. Henrot. *Extremum Problems for Eigenvalues of Elliptic Operators.* Springer, Basel, 2006.

[33] M. Hintermüller. A combined shape-newton and topology optimization technique in real-time image segmentation. In *Real-Time PDE-Constrained Optimization*, pages 253–275. SIAM, 2007.

[34] J. R. Hutchinson. Shear coefficients for Timoshenko beam theory. *J. Appl. Mech.*, 68(1):87–92, 2001.

[35] M. Iguernane, S. A. Nazarov, J.-R. Roche, J. Sokołowski, and K. Szulc. Topological derivatives for semilinear elliptic equations. *Int. J. Appl. Math. Comput. Sci.*, 19(2):191–205, 2009.

[36] A. M. Khludnev, A. A. Novotny, J. Sokołowski, and A. Żochowski. Shape and topology sensitivity analysis for cracks in elastic bodies on boundaries of rigid inclusions. *J. Mech. Phys. Solids*, 57(10):1718–1732, 2009.

[37] P. I. Kogut and G. R. Leugering. *Optimal Control Problems for Partial Differential Equations on Reticulated Domains: Approximation and Asymptotic Analysis.* Springer, 2011.

[38] T. Kufner, G. Leugering, A. Martin, et al. Towards a lifecycle oriented design of infrastructure by mathematical optimization. *Optim. Eng.*, 20:215–249, 2019.

[39] J. E. Lagnese, G. Leugering, and E. J. P. G. Schmidt. Modelling of dynamic networks of thin thermoelastic beams. *Math. Methods Appl. Sci.*, 16(5):327–358, 1993.

[40] J. E. Lagnese, G. Leugering, and E. J. P. G. Schmidt. *Modeling, Analysis and Control of Dynamic Elastic Multi-link Structures.* Systems & Control: Foundations & Applications. 1994.

[41] G. Lance, E. Trélat, and E. Zuazua. Numerical issues and turnpike phenomenon in optimal shape design. *Optim. Control Appl. Methods*, 29:343, 2022.

[42] S. Larnier, J. Fehrenbach, and M. Masmoudi. The topological gradient method: From optimal design to image processing. *Milan J. Math.*, 80:411–441, 2012.

[43] G. Leugering. Dynamic domain decomposition of optimal control problems for networks of strings and timoshenko beams. *SIAM J. Control Optim.*, 37(6):1649–1675, 1999.

[44] G. Leugering, C. Rodriguez, and Y. Wang. Nodal profile control for networks of geometrically exact beams. *J. Math. Pures Appl. (9)*, 155:111–139, 2021.

[45] G. Leugering and J. Sokołowski. Topological derivatives for networks of elastic strings. *Z. Angew. Math. Mech.*, 91(12):926–943, 2011.

[46] G. Leugering, J. Sokołowski, et al. Topological sensitivity analysis for elliptic problems on graphs. *Control Cybern.*, 37(4):971–997, 2008.

[47] T. Lewiński. Two versions of woźniak's continuum model of hexagonal-type grid plates. *J. Theoret. Appl. Mech.*, 22(3-4):389–405, 1984.

[48] T. Lewiński. On algebraic equations of elastic trusses, frames and grillages. *J. Theoret. Appl. Mech.*, 39(2):307–322, 2001.

[49] T. Lewiński and S. Czarnecki. On incorporating warping effects due to transverse shear and torsion into the theories of straight elastic bars. *Acta Mech.*, 232(1):247–282, 2021.

[50] T. Lewiński, T. Sokół, and C. Graczykowski. *Michell Structures.* Springer, 2018.

[51] T. Lewiński and J. Sokołowski. Topological derivative for nucleation of non-circular voids. The Neumann problem. *Differential geometric methods in the control of partial differential equations (Boulder, CO, 1999), Contemp. Math*, 268:341–361, 2000.

[52] T. Lewiński and J. Sokołowski. Energy change due to the appearance of cavities in elastic solids. *Internat. J. Solids Structures*, 40(7):1765–1803, 2003.

[53] J. L. Lions. *Contrôle Optimal de Systèmes Gouvernés par des Équations aux Dérivées Partielles.* Dunod, 1968.

[54] M. Lipnicka and A. Nowakowski. On dual dynamic programming in shape optimization of coupled models. *Struct. Multidiscip. Optim.*, 59(1):153–164, 2019.

[55] M Loudini, D Boukhetala, M Tadjine, and MA Boumehdi. Application of Timoshenko beam theory for deriving motion equations of a lightweight elastic link robot manipulator. *ICGST-ARAS Journal*, 5:11–18, 2006.

[56] J. Ma, Z. Wei, H. Wen, and D. Jin. Boundary control of a timoshenko beam with prescribed performance. *Acta Mechanica*, 231:3219–3234, 2020.

[57] H. Maurer and S. Pickenhain. Second-order sufficient conditions for control problems with mixed control-state constraints. *J. Optim. Theory Appl.*, 86:649–667, 1995.

[58] A. A. Milyutin, A. V. Dmitruk, and N. P. Osmolovskii. Maximum principle in optimal control (princip maksimuma v optimal'nom upravlenii, in Russian). *Lomonosov Moscow State University, Faculty of Mathematics and Mechanics, Moscow*, 2004.

[59] A. A. Milyutin and N. P. Osmolovskii. *Calculus of Variations and Optimal Control.* American Mathematical Society, 1998.

[60] A. Myśliński. Level set method for shape and topology optimization of contact problems. In *System Modeling and Optimization*, volume 312 of *IFIP Adv. Inf. Commun. Technol.*, pages 397–410. Springer, Berlin, 2009.

[61] S. A. Nazarov and J. Sokołowski. Spectral problems in the shape optimization. Singular boundary perturbations. *Asymptot. Anal.*, 56(3-4):159–204, 2008.

[62] A. A. Novotny and J. Sokołowski. *Topological Derivatives in Shape Optimization.* Springer, 2013.

[63] A. A. Novotny and J. Sokołowski. *An Introduction to the Topological Derivative Method.* SpringerBriefs in Mathematics. Springer, Cham, 2020.

[64] A. A. Novotny, J. Sokołowski, and A. Żochowski. *Applications of the Topological Derivative Method.* Studies in Systems, Decision and Control. Springer Nature Switzerland, 2019.

[65] A. Nowakowski and J. Sokołowski. On dual dynamic programming in shape control. *Commun. Pure Appl. Anal.*, 11(6):2473–2485, 2012.

[66] E. Ogiermann. *Topological Sensitivity Analysis for Networks of Timoshenko Beams.* Phd thesis, Friedrich-Alexander-Universität Erlangen-Nürnberg, 2015.

[67] N. P. Osmolovskii and H. Maurer. *Applications to Regular and Bang-Bang Control: Second-Order Necessary and Sufficient Optimality Conditions in Calculus of Variations and Optimal Control.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 2012.

[68] N. P. Osmolovskii, M. Qian, and J. Sokołowski. Network optimality conditions. *Control Cybern.*, 52(2):129–180, 2023.

[69] W.-K. Park. On the imaging of thin dielectric inclusions via topological derivative concept. *Prog. Electromagn. Res.*, 110:237–252, 2010.

[70] J. Peł czyński and W. Gilewski. Algebraic formulation for moderately thick elastic frames, beams, trusses, and grillages within Timoshenko theory. *Math. Probl. Eng.*, pages Art. ID 7545473, 9, 2019.

[71] W. Pietraszkiewicz. *Thin Elastic Shells, Lagrangian Geometrically Nonlinear Theory*, pages 2518–2518. Springer Berlin Heidelberg, Berlin, Heidelberg, 2020.

[72] P. I. Plotnikov and J. Sokołowski. Geometric aspects of shape optimization. *J. Geom. Anal.*, 33(7):Paper No. 206, 57, 2023.

[73] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko. Mathematical theory of optimal processes [in Russian], 1961.

[74] A. Porretta and E. Zuazua. Long time versus steady state optimal control. *SIAM J. Control Optim.*, 51(6):4242–4273, 2013.

[75] J. Rocha de Faria and A. A. Novotny. On the second order topological asymptotic expansion. *Struct. Multidiscip. Optim.*, 39(6):547–555, 2009.

[76] J. Rocha de Faria, A. A. Novotny, R. A. Feijóo, and E. Taroco. First- and second-order topological sensitivity analysis for inclusions. *Inverse Probl. Sci. Eng.*, 17(5):665–679, 2009.

[77] C. Rodriguez and G. Leugering. Boundary feedback stabilization for the intrinsic geometrically exact beam model. *SIAM J. Control Optim.*, 58(6):3533–3558, 2020.

[78] S. Rolewicz. On controllability of systems of strings. *Studia Math.*, 36(2):105–110, 1970.

[79] H. L. Royden and P. Fitzpatrick. *Real Analysis*, volume 2. Macmillan, New York, 1968.

[80] B. Samet, S. Amstutz, and M. Masmoudi. The topological asymptotic for the Helmholtz equation. *SIAM J. Control Optim.*, 42(5):1523–1544, 2003.

[81] A. J. B. Santos, A. F. D. Loula, and J. N. C. Guerreiro. Stabilized finite element formation for flexible structures in three-dimensional space. *TEMA Tend. Mat. Apl. Comput.*, 7(2):347–359, 2006.

[82] A. J. B. Santos, A. F. D. Loula, and J. N. C. Guerreiro. Stability and convergence of mixed methods for elastic rods of arbitrary geometry. *Comput. Methods Appl. Mech. Engrg.*, 198(15-16):1283–1297, 2009.

[83] M. Schneider and H. Andrä. The topological gradient in anisotropic elasticity with an eye towards lightweight design. *Math. Methods Appl. Sci.*, 2013.

[84] M. K. Shepherd, D. Gunz, T. Clites, C. Lecomte, and E. J. Rouse. Designing custom mechanics in running-specific prosthetic feet via shape optimization. *IEEE Trans. Biomed. Eng.*, 70(2):747–755, 2022.

[85] A. Shooshtari and R. Khajavi. An efficient procedure to find shape functions and stiffness matrices of nonprismatic euler–bernoulli and timoshenko beam elements. *Eur. J. Mech. A Solids*, 29(5):826–836, 2010.

[86] J. Sokołowski and A. Żochowski. Topological derivatives for elliptic problems. *Inverse problems*, 15(1):123–134, 1999.

[87] J. Sokołowski and A. Żochowski. Modelling of topological derivatives for contact problems. *Numer. Math.*, 102:145–179, 2005.

[88] J. Sokołowski and J. P. Zolésio. *Introduction to Shape Optimization - Shape Sensitivity Analysis.* Springer-Verlag, Berlin, 1992.

[89] C. Strohmeyer. *Networks of Nonlinear Thin Structures-Theory and Applications.* FAU University Press, 2018.

[90] E. Trélat, C. Zhang, and E. Zuazua. Steady-state and periodic exponential turnpike property for optimal control problems in Hilbert spaces. *SIAM J. Control Optim.*, 56(2):1222–1252, 2018.

[91] S. Tudjono, A. Han, D.K. Nguyen, S. Kiryu, and B.S. Gan. Exact shape functions for timoshenko beam element. *IOSR-JCE*, 19(03):12–20, 2017.

[92] X. Wang, M. Yulin, and M. Y. Wang. Incorporating topological derivatives into level set methods for structural topology optimization. *Opt. Shape Des. Model.*, pages 145–157, 2004.

[93] K. Wiśniewski. *Finite rotation shells.* Lecture Notes on Numerical Methods in Engineering and Sciences. International Center for Numerical Methods in Engineering (CIMNE), Barcelona; Springer, Heidelberg, 2010. Basic equations and finite elements for Reissner kinematics.

Acknowledge

Upon finishing this dissertation, I would like to express my great gratitude to all those who have offered me sincere assistance in the past one and a half years.

First and foremost, I sincerely thank the support of the Chinese government and the China Scholarship Council (CSC). Their funding and scholarships enabled me to successfully complete my study abroad program, providing me with invaluable learning and research opportunities. The support from the government and CSC allowed me to receive high-quality education overseas and fully devote myself to academic research. Their support paved the way for me to pursue my academic dreams with greater focus, and I will always remember and strive to repay the care and support from the country and society.

Furthermore, I am deeply grateful to my supervisor Prof. Jan Sokołowski and Prof. Shengfeng Zhu who have given me insightful suggestions and constant encouragement both in my study and in my life. In the many moments of giving up in the process of writing, their unwavering support and guidance have been invaluable, motivating me to persevere and reach new heights in my academic pursuits.

Prof. Sokołowski is an expert in the field of shape topology optimization, and I am deeply honored to be his doctoral student and receive his careful guidance. He is extremely patient. When I encounter confusion, he always patiently answers and even repeatedly explains. His guidance was extremely meticulous, recording all the ideas and deductions in detail in a notebook and handing it to me, demonstrating a rigorous attitude. Although the professor's schedule is very busy, he arranges time every week to guide me. Even if he is not in Poland, we will conduct academic discussions through online platforms.

Prof. Sokołowski not only guides me on weekdays but also values my academic exchange. Whenever he gives academic presentations in other parts of Poland, he invites me to attend together. He also arranged for me to report my research results at a seminar at the research institute and present them at the international IEEE conference. Being invited to give a presentation at the International Conference on Methods and Models in Automation and Robotics (MMAR) was a very meaningful experience for me, as it was my first academic presentation at the conference. In short, thanks to the professor's professional guidance, my research level has been improved and I have had the opportunity to complete this doctoral thesis.

Prof. Sokołowski is very friendly and approachable. He not only took me to visit this city but also shared with me the history of the Old Town. We visited the Royal Castle together and deeply felt that it had been destroyed and plundered by the armies of Sweden, Brandenburg, Germany, and Russia. We also enjoyed the Bernardo Bellootto 300 Birthday Commemorative Exhibition, commemorating this great artist. And we also attended a concert at the Pałac na Wyspie. On Christmas Eve in 2022, the professor invited me to his flat to celebrate, which allowed me to have a richer experience of European Christmas Eve traditions. On weekdays, the professor often invites me to share coffee and desserts and talk freely about other topics. He is full of love for life and passionate about scenery, music, and history, and his extensive knowledge is admirable.

I would like to express my sincere gratitude to my Chinese supervisor, Professor Zhu Shengfeng, for his unwavering support and encouragement, which enabled me to realize my wish to study abroad. His efforts in connecting me with the eminent scholar Jan Sokolowski in the field of shape optimization provided me with the invaluable opportunity to receive guidance and elevate my academic prowess. The mentorship and care provided by Prof. Zhu are treasures on my academic journey, and I will cherish this valuable experience.

Upon arriving in Poland, Prof. Zhu not only provided academic guidance but also extended caring support in daily life. His invaluable experience and unwavering encouragement helped me navigate challenges and smoothly adapt to the new culture and lifestyle. As both a scholarly guide and personal role model, Prof. Zhu consistently engaged in insightful discussions, fostering my intellectual growth and problem-solving skills. His trust and encouragement propelled me to pursue academic excellence, while his generous support, including financial assistance for my doctoral studies in Poland, alleviated any concerns.

I'm profoundly grateful for Prof. Zhu's selfless dedication and steadfast belief in my potential. His mentorship continues to inspire me, and I'm committed to honoring his guidance by striving for continued success. Additionally, the memorable time when Prof. Zhu visited Warsaw, Poland from January 15th, 2024, to February 5th, 2024, further strengthened our bond and provided invaluable insights and inspiration for my research work. His visit was marked by deep academic exchanges and discussions, during which he offered invaluable advice that has shaped the direction of my future research endeavors. This cherished experience has not only enriched my knowledge but also served as a significant milestone in my academic journey.

Thanks to Prof. Jan W. Owsiński, who is the deputy director of the institute. Before I came, he helped me a lot, including delivering kinds of documents, renting the flat, picking me up at the airport and so on. He also cares about me very much in the institute.

Thanks to my colleagues in the IBS PAN. Thanks to Giovanni Bruccola and The Hung Tran. We visited Centrum Nauki Kopernik in the spring, went to the campfire party in the summer, watched Disney's 100th Anniversary show in the Autumn, and had a hotpot in the winter. Especially during the summer, we had a BBQ at Giovanni's flat. It was a good memory. Wishing that we all successfully obtain our degrees and become true mathematicians.

Thanks to Katarzyna Szulc, who was very concerned about my life when I just arrived in Warsaw. She is a kind lady. Thanks to Krzysztof Rutkowski, who organized the Chinese New Year for us. He is a nice man. Thanks to Bartłomiej Solarz-Niesłuchowski, who helped me a lot when my laptop was broken and helped me apply for the resident card. He is a humorous man.

Thanks to Qiang Huo, who is a joint Ph.D. student at the Mathematic Institute (IM PAN). We have known each other while preparing for visas. We exchanged information and prepared to go abroad. Our first meeting was near the Christmas tree in the old town of Warsaw. Sometimes we strolled together, cooked dishes, and witnessed his culinary skills improving. We were in many restaurants and explored food here. We celebrated together during Chinese New Year, Lantern Festival, Mid-Autumn Festival, and birthdays, each occasion filled with warmth and joy. We also frequently studied together, and his passion for learning deeply inspired me. The happy moments in Warsaw always seemed fleeting. Qiang is soon returning to Beijing. I wish him to graduate as expected. I hope that we will cross paths again.

Thanks to Mengyu Li, who is also a joint Ph.D. student but in Roma. It was a good memory to travel with you to Rome, Pisa, Florence, Venice, Warsaw, Prague, and Vienna. We have seen many beautiful scenery and museums and took good photos. During my first year in Poland, she always chatted with me, especially at night. The nights in Poland are so long, but fortunately with your company.

Thanks to Yaqin Lyu, a friend that I knew in Poland. Her kindness shone through when she prepared hot ginger cola and took a one-hour bus to deliver it to me when I had a high temperature. I experienced a profound sense of depression and constant fear and loneliness after that illness, and your visits were a source of great comfort. I'm glad you prepared a graduation gift for me. I am sincerely thankful for your friendship and encouragement.

Thanks to Gege Pan, one of my best friends in Warsaw. On the first night that I arrived in Warsaw, she sought me out at my place of residence, accompanied me to the mall to assist with getting a SIM card, and more. We play badminton, go to the gym, take walks in the park, and make dumplings together. These experiences have fostered a deep friendship between us. I am deeply touched by her kindness and assistance.

Thanks to Yixin Tan, my friend in China, her arrival in Poland has added vitality to my life. We share the same office, motivating each other to study and creating a conducive learning atmosphere. Her earnest dedication to learning and her diligent spirit of inquiry deeply influence me, inspiring me to strive harder in my studies and personal growth. I wish her all the best in her life and studies in Warsaw. I believe that someone to whom her are important will bring her closer to her aim.

Thanks to my Polish friends Łukas and Kaja, who are not only my former landlords but also my friends. Upon my arrival in Poland, during those initial days, their invaluable assistance proved instrumental in acquainting me with the local surroundings and imparting culinary skills. Thanks for their help, which enabled me to survive in a foreign country. I will remember the time when we skated and tasted wine together. Our excursion to Miedzyzdroje in August, a quaint town by the Baltic Sea, stands out as a highlight, as we explored its charm and crossed the German border to stroll through a neighboring town. The end of the year saw us celebrating Christmas Day and ushering in the New Year together, marked by joyous moments of fireworks and relishing traditional Polish cuisine. These cherished times will forever hold a special place in my heart.

Last my thanks would go to my beloved parents for this loving consideration and great confidence in me. They support me and love me even when I least deserve to be loved. In addition, many thanks to my friends who have kindly provided unwavering support and companionship during the past two years.

Appendix A

Program

The topological derivative method can be used for the numerical solutions of optimum design problems for networks. The examples with Matlab codes show the constructive way to proceed.

A.1 Nonlinear Beams

This section provides the MATLAB implementation for the nonlinear beam example discussed in Example 2 in section 3.3.2. The objective is to optimize control problems for a nonlinear beam system, using the Newton method to handle the nonlinear terms efficiently and achieve optimal control.

```
%% This is a contruct example for nonlinear beam
% +------------------------------------------+
% | phi(r) = -E(x)r + L(r)Cr; |
% | z1_hat = -x+1; |
% | zi_hat = 0; i=2,3,4,5,6 |
% | f1_hat = -1; |
% | fi_hat = 0; i=2,3,4,5,6 |
% | |
% | z'(x) = phi(z) + e_1(f_hat - p), |
% | -p'(x) = p(x)phi'(z(x)) + z(x) -z_hat, |
% | z(l) = 0, p(0) = 0; |
% | P2 element |
% +------------------------------------------+

function J = optControl(L)
if nargin == 0
   L = 1; % L \in [1/2,3/2]
end

% Define optimization parameters
Lopt = 1; % Optimal beam length
Lmax = 3/2; % Maximum allowable beam length
```

```
hmax = 0.01;
N0l = ceil((L)/hmax);
Nx = 2*N0l + 1;

% Calculate number of nodes for optimal and maximum beam lengths
Nopt = Lopt/hmax;
Nmax = Lmax/hmax;

% Generate node coordinates for different beam lengths
x0l = linspace(0,L,N0l+1);
xopt = linspace(0,Lopt,Nopt+1)';
xmax = linspace(0,Lmax,Nmax+1)';

% Compute stiffness matrices and mass matrix for finite element analysis
[Pb,A1,A2] = graph1D_matirx_P2(N0l,x0l);
[Pbmax,A1max,~] = graph1D_matirx_P2(Nmax,xmax);

[Ks1,Ks2] = deal(cell(6,6));

for i = 1 : 6
   for j = 1 : 6
      Ks1{i,j} = zeros(size(A1));
      Ks2{i,j} = zeros(size(A1));
   end
end

for i = 1 : 6
   Ks1{i,i} = A2;
   Ks2{i,i} = A2;
end

Ks1{5,3} = A1;
Ks1{6,2} = -A1;

Ks2{5,3} = -A1;
Ks2{6,2} = A1;

for i = 1 : 6
   Ks1{i,i}(1,1) = Ks1{i,i}(1,1) + 1;
   Ks2{i,i}(end,end) = Ks2{i,i}(end,end) - 1;
end

M = deal(cell(6,6));
for i = 1 : 6
   for j = 1 : 6
      M{i,j} = zeros(size(A1));
   end
end

for i = 1 : 6
   M{i,i} = A1;
```

```
end

% Convert cell arrays to numerical matrices
Ks1_matrix = cell2mat(Ks1);
Ks2_matrix = cell2mat(Ks2);
M_matrix = cell2mat(M);

A = [-Ks1_matrix M_matrix;-M_matrix Ks2_matrix];

%% Dirichlet condition
bdyL = (1:12) * Nx;
bdy0 = 1 + (0:11)*Nx;

%% control
xopt = Pbmax';

fhat = -ones(size(xopt))*1; % Desired force distribution
zhat = -xopt+1;

% Calculate force and state distributions
Fhat = A1max*fhat;
Zhat = A1max*zhat;

Fhat(Nx+1:end) = [];
Zhat(Nx+1:end) = [];

% Construct control force vector F
F = zeros(12*Nx,1);
F(bdy0(1):bdyL(1)) = Fhat;
F(bdy0(7):bdyL(7)) = -Zhat;

C = diag([10^4,10^4,10^4,500,500,500])^(-1);

%% Initialization of the state
% Consistent structural linearisation in flexible-body dynamics with ...
    large rigid-body motion
% Henrik Hesse, Rafael Palacios, 2012
EA = 10^4; GAs = 10^4;
EI = 500; GJ = 500;
rhoA = 1; rhoJ = diag([20, 10, 10]);
massMat = blkdiag(rhoA*eye(3), rhoJ); % the MASS matrix
flexMat = inv(diag([EA, GAs, GAs, GJ, EI, EI])); % the FLEXIBILITY matrix

Gamma0 = [0; 0; 0]; % there is no initial shear
W0hat = [0, -1/sqrt(2), 0; 1/sqrt(2), 0, 1/sqrt(2); 0, -1/sqrt(2), 0];
Upsilon0 = func_vec(W0hat);

z0 = [Gamma0; Upsilon0]; % strains
z0 = flexMat\z0; % corresponding stresses

Y0 = zeros(6*Nx, 1);
```

```
p0 = zeros(6*Nx,1);

wm = [Y0;p0]; % initialize wm

disp('Solving the semilinear system..');
%%% M.A. code %%%
tolzero=1e-12;
reltolX=1e-6;
tolF=1e-15;
%%%%%%%%%%%%%%%%%
iter = 0;

while 1
   %%% Newton method: the scheme reads %%%
   % wm1 = wm - (JacFk(wm))^{-1} Fk(wm) %
   %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
   ML_martix = MCL(wm,Nx,C,L,N01);
   MG_martix = MG(wm,Nx,C,L,N01);
   NL = blkdiag(ML_martix,MG_martix);
   Fk_wm = A*wm - NL*wm - F;
   % boundary condition
   Fk_wm(bdyL(1:6),1) = 0;
   Fk_wm(bdy0(7:12),1) = 0;

   JacFk_wm = A - NL;
   for i = 1 : 6
      JacFk_wm(bdyL(i),:) = 0;
      JacFk_wm(bdyL(i),bdyL(i)) = 1;
      JacFk_wm(bdy0(i+6),:) = 0;
      JacFk_wm(bdy0(i+6),bdy0(i+6)) = 1;
   end

   wm1 = wm - JacFk_wm\Fk_wm;

   %%% M.A. code %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
   rel_err = (wm1 - wm)./wm; %
   nan_or_inf = find( isnan(rel_err) + isinf(rel_err) ... %
      + (abs(wm)<=tolzero) ); %
   rel_err(nan_or_inf) = 0; %
   if (norm(rel_err,inf) <= reltolX) && (norm(Fk_wm,inf) <= tolF)%
      break %
   end %
   %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

   wm = wm1;
   iter = iter + 1;
end

z_exp = wm1;
z_solu = cell(6,1);
for i = 1 : 6
```

```
    z_solu{i,1} = zeros(Nx,1);
    z_solu{i,1} = z_exp(bdy0(i):bdyL(i));
end

p_solu = cell(6,1);
for i = 7 : 12
    p_solu{i-6,1} = zeros(Nx,1);
    p_solu{i-6,1} = z_exp(bdy0(i):bdyL(i));
end

f_solu = cell(6,1);
for i = 1 : 6
    f_solu{i,1} = zeros(Nx,1);
    f_solu{1,1} = fhat(1:Nx)-p_solu{1,1};
end

f_err = trapz(Pb,(f_solu{1,1}-fhat(1:Nx)).^2);
z_err = trapz(Pb,(z_solu{1,1}-zhat(1:Nx)).^2);

J = f_err + z_err + (L-Lopt)^2 ;
J = J/2;
disp(['f_err = ', num2str(f_err)]);
disp(['z_err = ', num2str(z_err)]);
disp(['J = ', num2str(J)]);

end
```

A.2 Timoshenko Beam

In this section, we present the exact solution and the optimal control code for both the 1D and 3D Timoshenko models.

The first file contains the MATLAB code implementing the exact solution for the 1D Timoshenko model in section 4.6.1. This solution enables precise calculation of displacements and stresses within the structure under specified loading and boundary conditions. The OptControl function presents the optimal control problem code shown for the 1D/3D Timoshenko model in section 4.10.

```
% This is to get the exact solution of the 1D model
clear

% parameters
% elastic modulus
% em = 2.1e11;
% % shear modulus
% sm = 8.1*1e10 ;
% % cross-section area
% cs = 0.01 ;
```

```
% % torsion constant
% tor = 1.41*1e-5 ;
% % second moments
% secm = 8.33*1e-6 ;
% % axial stiffness
% kx = cs*em ;
% % shear stiffness
% kappa = 0.850 ;
% ks = kappa*sm*cs ;
% % torsional stiffness
% cx = sm*tor ;
% % bending stiffness
% cy = secm*em ;
% cz = secm*em ;

ks = 688500000;
cy = 1749300;

epsVal = 0.5;

Len = 2;

L = [Len-epsVal,Len-epsVal,Len-epsVal,epsVal,epsVal,epsVal];

syms x;
f = piecewise(x <= Len-1.00000, 2e6, x > Len-1.00000 & x <= L(2), 0);
fz = [0,-f,0,0,0,0];
my = [0,0,0,0,0,0];
rz = [0,0,0,0,0,0];
phiy = [0,0,0,0,0,0];
for i = 1 : 3
   q{i} = [fz(i);my(i)];
   u0{i} = [rz(i);phiy(i)];
end

for i = 1 : 6
   D1E{i} = D1(ks,L(i),cy);
   D2E{i} = D2(ks,L(i),cy);
   D3E{i} = D3(ks,L(i),cy);
   D4E{i} = D4(ks,L(i),cy);
   dE{i} = d(fz(i),my(i),cy,L(i));
   wE{i} = w(fz(i),my(i),ks,cy,L(i));
   RE{i} = R(cy,ks,L(i));
   QE{i} = Q(cy,ks,L(i));
end

A = [D4E{1}+D4E{4}-D1E{5}, -D2E{5}, D3E{4};
    D3E{5}, D4E{2}+D4E{5}-D1E{6}, -D2E{6};
    -D2E{4}, D3E{6}, D4E{3}+D4E{6}-D1E{4}]; %symmertic

b = sym(zeros(6,1));
```

```
b(1:2,1) = - q{1} - D3E{1}*u0{1} ...
          - D4E{1}*subs(wE{1},L(1)) - D4E{4}*subs(wE{4},L(4)) + ... D2E{5}*subs(wE{5},L(5)) ...
          + subs(dE{1},L(1)) + subs(dE{4},L(4));
b(3:4,1) = - q{2} - D3E{2}*u0{2} ...
          - D4E{2}*subs(wE{2},L(2)) - D4E{5}*subs(wE{5},L(5)) + ... D2E{6}*subs(wE{6},L(6)) ...
          + subs(dE{2},L(2)) + subs(dE{5},L(5));
b(5:6,1) = - q{3} - D3E{3}*u0{3} ...
          - D4E{3}*subs(wE{3},L(3)) - D4E{6}*subs(wE{6},L(6)) + ... D2E{4}*subs(wE{4},L(4)) ...
          + subs(dE{3},L(3)) + subs(dE{6},L(6));

solu = A\b;
uL{1} = subs(solu(1:2),L(1));
uL{2} = subs(solu(3:4),L(2));
uL{3} = subs(solu(5:6),L(3));

u0{4} = uL{3};
u0{5} = uL{1};
u0{6} = uL{2};

uL{4} = uL{1};
uL{5} = uL{2};
uL{6} = uL{3};

for i = 1 : 6
    u{i} = QE{i}*u0{i} + RE{i}*(subs(uL{i},L(i))+subs(wE{i},L(i)))-wE{i};
end

save('u_data.mat', 'u',"Len");
save('fz.mat', 'fz');
save('my.mat', 'my');
close all

figure
subplot(2,1,1)
fplot(u{1}(1),[0 L(1)])
hold on
fplot(u{2}(1),[0 L(2)])
hold on
fplot(u{3}(1),[0 L(3)])
legend('u_1','u_2','u_3')
title('r_z')

% figure
subplot(2,1,2)
fplot(u{1}(2),[0 L(1)])
hold on
fplot(u{2}(2),[0 L(2)])
hold on
```

```
fplot(u{3}(2),[0 L(3)])
legend('u_1','u_2','u_3')
title('phi_y')

figure
subplot(2,1,1)
fplot(u{4}(1),[0 L(4)])
hold on
fplot(u{5}(1),[0 L(5)])
hold on
fplot(u{6}(1),[0 L(6)])
legend('u_4','u_5','u_6')
title('r_z')

% figure
subplot(2,1,2)
fplot(u{4}(2),[0 L(4)])
hold on
fplot(u{5}(2),[0 L(5)])
hold on
fplot(u{6}(2),[0 L(6)])
legend('u_4','u_5','u_6')
title('phi_y')

function muy = mu(ks,L,cy)
   muy = ks/(ks*L^3+12*cy*L);
% muz = ks/(ks*L^3+12*cz*L);
end

function Ei1 = E(t)
   Ei1 = [cos(t) sin(t);
         -sin(t) cos(t)];
end

function D1i = D1(ks,L,cy)
   muy = mu(ks,L,cy);
   D1i = zeros(2,2);
   D1i(1,1) = -12*cy*muy;
   D1i(2,2) = -cy*(1/L+3*muy*L^2);
   D1i(2,1) = 6*cy*muy*L;
   D1i(1,2) = 6*cy*muy*L;
end

function D2i = D2(ks,L,cy)
   muy = mu(ks,L,cy);
   D2i = zeros(2,2);
   D2i(1,1) = 12*cy*muy;
   D2i(2,2) = cy*(1/L-3*muy*L^2);
   D2i(2,1) = -6*cy*muy*L;
   D2i(1,2) = 6*cy*muy*L;
end
```

```
function D3i = D3(ks,L,cy)
   muy = mu(ks,L,cy);
   D3i = zeros(2,2);
   D3i(1,1) = -12*cy*muy;
   D3i(2,2) = -cy*(1/L-3*muy*L^2);
   D3i(2,1) = -6*cy*muy*L;
   D3i(1,2) = 6*cy*muy*L;
end

function D4i = D4(ks,L,cy)
   muy = mu(ks,L,cy);
   D4i = zeros(2,2);
   D4i(1,1) = 12*cy*muy;
   D4i(2,2) = cy*(1/L+3*muy*L^2);
   D4i(2,1) = 6*cy*muy*L;
   D4i(1,2) = 6*cy*muy*L;
end

function [fInt,fInt2,fInt3] = Int(f,L)
   syms x;
   fInt = int(f,x,0,x);
   fInt2 = int(fInt,x,0,x);
   fInt3 = int(fInt2,x,0,x);
end

function gyi = gy(fz,my,cy,L)
   [fzInt1,~,~] = Int(fz,L);
   gyi = 1/cy*(my+fzInt1);
end

function wi = w(fz,my,ks,cy,L)
   wi = sym(zeros(2,1));
   gyi = gy(fz,my,cy,L);
   [~,fzInt2,~] = Int(fz,L);
   [~,gyInt2,gyInt3] = Int(gyi,L);
   wi(1) = 1/ks*fzInt2 - gyInt3;
   wi(2) = gyInt2;
end

function di = d(fz,my,cy,L)
   di = sym(zeros(2,1));
   gyi = gy(fz,my,cy,L);
   [fzInt1,~,~] = Int(fz,L);
   [gyInt1,~,~] = Int(gyi,L);
   di(1) = fzInt1;
   di(2) = cy*gyInt1;
end

function Qi = Q(cy,ks,L)
   muy = mu(ks,L,cy);
```

```
    Qi = sym(zeros(2,2));
    syms x
    Qi(1,1) = muy*(L-x)*(L^2+L*x-2*x^2+12/ks*cy);
    Qi(2,2) = muy*(L-x)*(L^2-3*L*x+12/ks*cy);
    Qi(1,2) = -muy*x*(L-x)*(L^2-L*x+6/ks*cy);
    Qi(2,1) = 6*muy*x*(L-x);
end

function Ri = R(cy,ks,L)
    muy = mu(ks,L,cy);
    Ri = sym(zeros(2,2));
    syms x
    Ri(1,1) = muy*x*(3*L*x-2*x^2+12/ks*cy);
    Ri(2,2) = muy*x*(-2*L^2+3*L*x+12/ks*cy);
    Ri(1,2) = muy*x*(L-x)*(L*x+6/ks*cy);
    Ri(2,1) = -6*muy*x*(L-x);
end
```

```
function [J,urz,uphiy,vz,vy] = OptControl(eps)

% Optcontrol: Solves an optimization problem
% of the Reduced Timoshenko beam
% with boundary conditions
% for a given epsilon value.
% Input:
% - eps: Value of epsilon. Default is 0.5.
% Output:
% - J: Cost function value.
% - urz, uphiy, vz, vy: Solution vectors.

% Default value for epsilon if not provided
if nargin == 0
    eps = 0.5;
end

% Load data files
load("u_data.mat");
load("LAMBDA.mat");
load("fz.mat");
load("my.mat");

% material coefficient
ksv = 688500000;
cyv = 1749300;

u0 = zeros(6,1);

%% ===== Setting ==========
eps0 = 0.5;
epsmax = 1;
```

```

% The length of each edge.
L = Len;
hmax = 0.01;

l = L - epsmax;
N0l = ceil(l/hmax);
x0l = linspace(0,l,N0l+1);

% number of basis functions
Nfy = 2*(N0l+1);

%% FEM matrices
[A1,A2,A3,A4] = graph1D_matirx_Hermite(N0l,x0l);

AA = cell(4,4);
for i = 1 : 4
    for j = 1 : 4
        AA{i,j} = deal(zeros(Nfy,Nfy));
    end
end

AA{1,1} = A4;
AA{2,2} = A4;
AA{3,3} = -A4;
AA{4,4} = -A4;
AA{1,3} = ksv*A1;
AA{1,4} = ksv*A2;
AA{2,3} = ksv*A3;
AA{2,4} = ksv*A4+cyv*A1;
AA{3,1} = ksv*A1;
AA{3,2} = ksv*A2;
AA{4,1} = ksv*A3;
AA{4,2} = ksv*A4+cyv*A1;

AA = cell2mat(AA);

A = blkdiag(AA,AA,AA);

Nf = [repmat(Nfy,1,12)];
index = cumsum(Nf);
index = [0 index];
for i = 1 : 3
    % index of [y(0),y(L),y'(0),y'(L)];
    urz_bdy{i} = index(4*i-3)+[1,Nf(2*i-1)/2,Nf(2*i-1)/2+1,Nf(2*i-1)];
    uphiy_bdy{i} = index(4*i-2)+[1,Nf(2*i)/2,Nf(2*i)/2+1,Nf(2*i)];
    % index of [p(0),p(L),p'(0),p'(L)];
    prz_bdy{i} = index(4*i-1)+[1,Nf(2*i-1)/2,Nf(2*i-1)/2+1,Nf(2*i-1)];
    pphiy_bdy{i} = index(4*i)+[1,Nf(2*i)/2,Nf(2*i)/2+1,Nf(2*i)];
end
syms epsVal ks cy
```

```
Lval = subs(Lambda,[epsVal,ks,cy],[eps,ksv,cyv]);

A(urz_bdy{1}(2),prz_bdy{1}(2)) = A(urz_bdy{1}(2),prz_bdy{1}(2)) - ...
    Lval(1,1);
A(urz_bdy{1}(2),pphiy_bdy{1}(2)) = A(urz_bdy{1}(2),pphiy_bdy{1}(2)) - ...
    Lval(2,1);
A(urz_bdy{1}(2),prz_bdy{2}(2)) = A(urz_bdy{1}(2),prz_bdy{2}(2)) - ...
    Lval(3,1);
A(urz_bdy{1}(2),pphiy_bdy{2}(2)) = A(urz_bdy{1}(2),pphiy_bdy{2}(2)) - ...
    Lval(4,1);
A(urz_bdy{1}(2),prz_bdy{3}(2)) = A(urz_bdy{1}(2),prz_bdy{3}(2)) - ...
    Lval(5,1);
A(urz_bdy{1}(2),pphiy_bdy{3}(2)) = A(urz_bdy{1}(2),pphiy_bdy{3}(2)) - ...
    Lval(6,1);

A(uphiy_bdy{1}(2),prz_bdy{1}(2)) = A(uphiy_bdy{1}(2),prz_bdy{1}(2)) - ...
    Lval(1,2);
A(uphiy_bdy{1}(2),pphiy_bdy{1}(2)) = A(uphiy_bdy{1}(2),pphiy_bdy{1}(2)) ...
    - Lval(2,2);
A(uphiy_bdy{1}(2),prz_bdy{2}(2)) = A(uphiy_bdy{1}(2),prz_bdy{2}(2)) - ...
    Lval(3,2);
A(uphiy_bdy{1}(2),pphiy_bdy{2}(2)) = A(uphiy_bdy{1}(2),pphiy_bdy{2}(2)) ...
    - Lval(4,2);
A(uphiy_bdy{1}(2),prz_bdy{3}(2)) = A(uphiy_bdy{1}(2),prz_bdy{3}(2)) - ...
    Lval(5,2);
A(uphiy_bdy{1}(2),pphiy_bdy{3}(2)) = A(uphiy_bdy{1}(2),pphiy_bdy{3}(2)) ...
    - Lval(6,2);

A(urz_bdy{2}(2),prz_bdy{1}(2)) = A(urz_bdy{2}(2),prz_bdy{1}(2)) - ...
    Lval(1,3);
A(urz_bdy{2}(2),pphiy_bdy{1}(2)) = A(urz_bdy{2}(2),pphiy_bdy{1}(2)) - ...
    Lval(2,3);
A(urz_bdy{2}(2),prz_bdy{2}(2)) = A(urz_bdy{2}(2),prz_bdy{2}(2)) - ...
    Lval(3,3);
A(urz_bdy{2}(2),pphiy_bdy{2}(2)) = A(urz_bdy{2}(2),pphiy_bdy{2}(2)) - ...
    Lval(4,3);
A(urz_bdy{2}(2),prz_bdy{3}(2)) = A(urz_bdy{2}(2),prz_bdy{3}(2)) - ...
    Lval(5,3);
A(urz_bdy{2}(2),pphiy_bdy{3}(2)) = A(urz_bdy{2}(2),pphiy_bdy{3}(2)) - ...
    Lval(6,3);

A(uphiy_bdy{2}(2),prz_bdy{1}(2)) = A(uphiy_bdy{2}(2),prz_bdy{1}(2)) - ...
    Lval(1,4);
A(uphiy_bdy{2}(2),pphiy_bdy{1}(2)) = A(uphiy_bdy{2}(2),pphiy_bdy{1}(2)) ...
    - Lval(2,4);
A(uphiy_bdy{2}(2),prz_bdy{2}(2)) = A(uphiy_bdy{2}(2),prz_bdy{2}(2)) - ...
    Lval(3,4);
A(uphiy_bdy{2}(2),pphiy_bdy{2}(2)) = A(uphiy_bdy{2}(2),pphiy_bdy{2}(2)) ...
    - Lval(4,4);
A(uphiy_bdy{2}(2),prz_bdy{3}(2)) = A(uphiy_bdy{2}(2),prz_bdy{3}(2)) - ...
    Lval(5,4);
```

```
A(uphiy_bdy{2}(2),pphiy_bdy{3}(2)) = A(uphiy_bdy{2}(2),pphiy_bdy{3}(2)) ...
    - Lval(6,4);

A(urz_bdy{3}(2),prz_bdy{1}(2)) = A(urz_bdy{3}(2),prz_bdy{1}(2)) - ...
    Lval(1,5);
A(urz_bdy{3}(2),pphiy_bdy{1}(2)) = A(urz_bdy{3}(2),pphiy_bdy{1}(2)) - ...
    Lval(2,5);
A(urz_bdy{3}(2),prz_bdy{2}(2)) = A(urz_bdy{3}(2),prz_bdy{2}(2)) - ...
    Lval(3,5);
A(urz_bdy{3}(2),pphiy_bdy{2}(2)) = A(urz_bdy{3}(2),pphiy_bdy{2}(2)) - ...
    Lval(4,5);
A(urz_bdy{3}(2),prz_bdy{3}(2)) = A(urz_bdy{3}(2),prz_bdy{3}(2)) - ...
    Lval(5,5);
A(urz_bdy{3}(2),pphiy_bdy{3}(2)) = A(urz_bdy{3}(2),pphiy_bdy{3}(2)) - ...
    Lval(6,5);

A(uphiy_bdy{3}(2),prz_bdy{1}(2)) = A(uphiy_bdy{3}(2),prz_bdy{1}(2)) - ...
    Lval(1,6);
A(uphiy_bdy{3}(2),pphiy_bdy{1}(2)) = A(uphiy_bdy{3}(2),pphiy_bdy{1}(2)) ...
    - Lval(2,6);
A(uphiy_bdy{3}(2),prz_bdy{2}(2)) = A(uphiy_bdy{3}(2),prz_bdy{2}(2)) - ...
    Lval(3,6);
A(uphiy_bdy{3}(2),pphiy_bdy{2}(2)) = A(uphiy_bdy{3}(2),pphiy_bdy{2}(2)) ...
    - Lval(4,6);
A(uphiy_bdy{3}(2),prz_bdy{3}(2)) = A(uphiy_bdy{3}(2),prz_bdy{3}(2)) - ...
    Lval(5,6);
A(uphiy_bdy{3}(2),pphiy_bdy{3}(2)) = A(uphiy_bdy{3}(2),pphiy_bdy{3}(2)) ...
    - Lval(6,6);


A(prz_bdy{1}(2),urz_bdy{1}(2)) = A(prz_bdy{1}(2),urz_bdy{1}(2)) - ...
    Lval(1,1);
A(prz_bdy{1}(2),uphiy_bdy{1}(2)) = A(prz_bdy{1}(2),uphiy_bdy{1}(2)) - ...
    Lval(2,1);
A(prz_bdy{1}(2),urz_bdy{2}(2)) = A(prz_bdy{1}(2),urz_bdy{2}(2)) - ...
    Lval(3,1);
A(prz_bdy{1}(2),uphiy_bdy{2}(2)) = A(prz_bdy{1}(2),uphiy_bdy{2}(2)) - ...
    Lval(4,1);
A(prz_bdy{1}(2),urz_bdy{3}(2)) = A(prz_bdy{1}(2),urz_bdy{3}(2)) - ...
    Lval(5,1);
A(prz_bdy{1}(2),uphiy_bdy{3}(2)) = A(prz_bdy{1}(2),uphiy_bdy{3}(2)) - ...
    Lval(6,1);

A(pphiy_bdy{1}(2),urz_bdy{1}(2)) = A(pphiy_bdy{1}(2),urz_bdy{1}(2)) - ...
    Lval(1,2);
A(pphiy_bdy{1}(2),uphiy_bdy{1}(2)) = A(pphiy_bdy{1}(2),uphiy_bdy{1}(2)) ...
    - Lval(2,2);
A(pphiy_bdy{1}(2),urz_bdy{2}(2)) = A(pphiy_bdy{1}(2),urz_bdy{2}(2)) - ...
    Lval(3,2);
A(pphiy_bdy{1}(2),uphiy_bdy{2}(2)) = A(pphiy_bdy{1}(2),uphiy_bdy{2}(2)) ...
    - Lval(4,2);
```

```
A(pphiy_bdy{1}(2),urz_bdy{3}(2)) = A(pphiy_bdy{1}(2),urz_bdy{3}(2)) - ...
    Lval(5,2);
A(pphiy_bdy{1}(2),uphiy_bdy{3}(2)) = A(pphiy_bdy{1}(2),uphiy_bdy{3}(2)) ...
    - Lval(6,2);

A(prz_bdy{2}(2),urz_bdy{1}(2)) = A(prz_bdy{2}(2),urz_bdy{1}(2)) - ...
    Lval(1,3);
A(prz_bdy{2}(2),uphiy_bdy{1}(2)) = A(prz_bdy{2}(2),uphiy_bdy{1}(2)) - ...
    Lval(2,3);
A(prz_bdy{2}(2),urz_bdy{2}(2)) = A(prz_bdy{2}(2),urz_bdy{2}(2)) - ...
    Lval(3,3);
A(prz_bdy{2}(2),uphiy_bdy{2}(2)) = A(prz_bdy{2}(2),uphiy_bdy{2}(2)) - ...
    Lval(4,3);
A(prz_bdy{2}(2),urz_bdy{3}(2)) = A(prz_bdy{2}(2),urz_bdy{3}(2)) - ...
    Lval(5,3);
A(prz_bdy{2}(2),uphiy_bdy{3}(2)) = A(prz_bdy{2}(2),uphiy_bdy{3}(2)) - ...
    Lval(6,3);

A(pphiy_bdy{2}(2),urz_bdy{1}(2)) = A(pphiy_bdy{2}(2),urz_bdy{1}(2)) - ...
    Lval(1,4);
A(pphiy_bdy{2}(2),uphiy_bdy{1}(2)) = A(pphiy_bdy{2}(2),uphiy_bdy{1}(2)) ...
    - Lval(2,4);
A(pphiy_bdy{2}(2),urz_bdy{2}(2)) = A(pphiy_bdy{2}(2),urz_bdy{2}(2)) - ...
    Lval(3,4);
A(pphiy_bdy{2}(2),uphiy_bdy{2}(2)) = A(pphiy_bdy{2}(2),uphiy_bdy{2}(2)) ...
    - Lval(4,4);
A(pphiy_bdy{2}(2),urz_bdy{3}(2)) = A(pphiy_bdy{2}(2),urz_bdy{3}(2)) - ...
    Lval(5,4);
A(pphiy_bdy{2}(2),uphiy_bdy{3}(2)) = A(pphiy_bdy{2}(2),uphiy_bdy{3}(2)) ...
    - Lval(6,4);

A(prz_bdy{3}(2),urz_bdy{1}(2)) = A(prz_bdy{3}(2),urz_bdy{1}(2)) - ...
    Lval(1,5);
A(prz_bdy{3}(2),uphiy_bdy{1}(2)) = A(prz_bdy{3}(2),uphiy_bdy{1}(2)) - ...
    Lval(2,5);
A(prz_bdy{3}(2),urz_bdy{2}(2)) = A(prz_bdy{3}(2),urz_bdy{2}(2)) - ...
    Lval(3,5);
A(prz_bdy{3}(2),uphiy_bdy{2}(2)) = A(prz_bdy{3}(2),uphiy_bdy{2}(2)) - ...
    Lval(4,5);
A(prz_bdy{3}(2),urz_bdy{3}(2)) = A(prz_bdy{3}(2),urz_bdy{3}(2)) - ...
    Lval(5,5);
A(prz_bdy{3}(2),uphiy_bdy{3}(2)) = A(prz_bdy{3}(2),uphiy_bdy{3}(2)) - ...
    Lval(6,5);

A(pphiy_bdy{3}(2),urz_bdy{1}(2)) = A(pphiy_bdy{3}(2),urz_bdy{1}(2)) - ...
    Lval(1,6);
A(pphiy_bdy{3}(2),uphiy_bdy{1}(2)) = A(pphiy_bdy{3}(2),uphiy_bdy{1}(2)) ...
    - Lval(2,6);
A(pphiy_bdy{3}(2),urz_bdy{2}(2)) = A(pphiy_bdy{3}(2),urz_bdy{2}(2)) - ...
    Lval(3,6);
A(pphiy_bdy{3}(2),uphiy_bdy{2}(2)) = A(pphiy_bdy{3}(2),uphiy_bdy{2}(2)) ...
```

```
    - Lval(4,6);
A(pphiy_bdy{3}(2),urz_bdy{3}(2)) = A(pphiy_bdy{3}(2),urz_bdy{3}(2)) - ...
    Lval(5,6);
A(pphiy_bdy{3}(2),uphiy_bdy{3}(2)) = A(pphiy_bdy{3}(2),uphiy_bdy{3}(2)) ...
    - Lval(6,6);

syms x
for i = 1 : 3
   z{i} = u{i};
   zNum{i} = double(subs(z{i},x01));
   dz{i} = diff(z{i},x);
   dzNum{i} = double(subs(dz{i},x01));
   dzNum{i}(:,end) = double(subs(dz{i}, x, 0.999999));
   dzNum{i}(:,1) = double(subs(dz{i}, x, 0.000001));

   dfz{i} = diff(fz(i),x);
   dfzNum{i} = double(subs(dfz{i},x01));
   dfzNum{i}(:,end) = double(subs(dfz{i}, x, 0.999999));
   dfzNum{i}(:,1) = double(subs(dfz{i}, x, 0.000001));

   dmy{i} = diff(my(i),x);
   dmyNum{i} = double(subs(dmy{i},x01));
   dmyNum{i}(:,end) = double(subs(dmy{i}, x, 0.999999));
   dmyNum{i}(:,1) = double(subs(dmy{i}, x, 0.000001));
end

b = zeros(size(A,1),1);

b(1:Nfy) = A4*[zNum{1}(1,:) dzNum{1}(1,:)*hmax]';
b(Nfy+1:2*Nfy) = A4*[zNum{1}(2,:) dzNum{1}(2,:)*hmax]';
b(2*Nfy+1:3*Nfy) = A4*[double(subs(fz(1),x01)) dfzNum{1}*hmax]';
b(3*Nfy+1:4*Nfy) = A4*[double(subs(my(2),x01)) dmyNum{2}*hmax]';

b(4*Nfy+1:5*Nfy) = A4*[zNum{2}(1,:) dzNum{2}(1,:)*hmax]';
b(5*Nfy+1:6*Nfy) = A4*[zNum{2}(2,:) dzNum{2}(2,:)*hmax]';
b(6*Nfy+1:7*Nfy) = A4*[double(subs(fz(2),x01)) dfzNum{2}*hmax]';
b(7*Nfy+1:8*Nfy) = A4*[double(subs(my(2),x01)) dmyNum{2}*hmax]';

b(8*Nfy+1:9*Nfy) = A4*[zNum{3}(1,:) dzNum{3}(1,:)*hmax]';
b(9*Nfy+1:10*Nfy) = A4*[zNum{3}(2,:) dzNum{3}(2,:)*hmax]';
b(10*Nfy+1:11*Nfy) = A4*[double(subs(fz(3),x01)) dfzNum{3}*hmax]';
b(11*Nfy+1:12*Nfy) = A4*[double(subs(my(3),x01)) dmyNum{3}*hmax]';

% bounday condition
for i = 1 : 3
   A(urz_bdy{i}(1),:) = 0;
   A(urz_bdy{i}(1),urz_bdy{i}(1)) = 1;
   b(urz_bdy{i}(1)) = u0(i);

   A(uphiy_bdy{i}(1),:) = 0;
   A(uphiy_bdy{i}(1),uphiy_bdy{i}(1)) = 1;
```

```
    b(uphiy_bdy{i}(1)) = u0(i);

    A(prz_bdy{i}(1),:) = 0;
    A(prz_bdy{i}(1),prz_bdy{i}(1)) = 1;
    b(prz_bdy{i}(1)) = u0(i);

    A(pphiy_bdy{i}(1),:) = 0;
    A(pphiy_bdy{i}(1),pphiy_bdy{i}(1)) = 1;
    b(pphiy_bdy{i}(1)) = u0(i);
end

solu = A\b;

for i = 1 : 3
    urz{i} = solu(urz_bdy{i}(1):urz_bdy{i}(2));
    uphiy{i} = solu(uphiy_bdy{i}(1):uphiy_bdy{i}(2));
    prz{i} = solu(prz_bdy{i}(1):prz_bdy{i}(2));
    pphiy{i} = solu(pphiy_bdy{i}(1):pphiy_bdy{i}(2));
    vz{i} = prz{i} + subs(fz(i),x0l)';
    vy{i} = pphiy{i} + subs(my(i),x0l)';
end

J = 0;
for i = 1 : 3
    tmp = norm(urz{i}-double(subs(z{i}(1),x0l))')^2*hmax;
    J = J + tmp;
    tmp = norm(uphiy{i}-double(subs(z{i}(2),x0l))')^2*hmax;
    J = J + tmp;
    tmp = norm(prz{i})^2*hmax;
    J = J + tmp;
    tmp = norm(pphiy{i})^2*hmax;
    J = J + tmp;
end
J = J/2;

end
```

```
% This is to get the exact solution of the 3D model

load("ks.mat")
load("kx.mat")
load("cx.mat")
load("cy.mat")
load("cz.mat")

% Define lengths
L = [2-epsVal,2-epsVal,2-epsVal,epsVal,epsVal,epsVal];

% Define piecewise distributed force
syms x;
```

```
f = piecewise(x <= 1.00000, 2e10, x > 1.00000 & x <= L(2), 0);

% Define local forces and moments
fx = [0,-f,0,0,0,0];
fy = [0,0,0,0,0,0];
fz = [0,0,0,0,0,0];
mx = [0,0,0,0,0,0];
my = [0,0,0,0,0,0];
mz = [0,0,0,0,0,0];

% Define local displacement and rotation
rx = [0,0,0,0,0,0];
ry = [0,0,0,0,0,0];
rz = [0,0,0,0,0,0];
phix = [0,0,0,0,0,0];
phiy = [0,0,0,0,0,0];
phiz = [0,0,0,0,0,0];

for i = 1 : 6
   q{i} = [fx(i);fy(i);fz(i);mx(i);my(i);mz(i)];
   u0{i} = [rx(i);ry(i);rz(i);phix(i);phiy(i);phiz(i)];
end

% Define rotation angles
theta = [-1/6*pi,pi/2,-5/6*pi,-pi,-pi/3,pi/3];

% Compute matrices based on rotation angles
[DFe,DPe,DMe,Se ] = deal(cell(6,1));
for i = 1 : 6
   DFe{i} = simplify(E(theta(i))*DF(kx,ks,L(i),cy,cz)*E(theta(i)).');
   DPe{i} = simplify(E(theta(i))*DP(ks,L(i),cx,cy,cz)*E(theta(i)).');
   DMe{i} = simplify(E(theta(i))*DM(ks,L(i),cx,cy,cz)*E(theta(i)).');
    Se{i} = simplify(E(theta(i))*S(ks,L(i),cy,cz)*E(theta(i)).');
end

for i = 1 : 6
   D1E{i} = -[DFe{i} -Se{i}.';-Se{i} DPe{i}];
   D2E{i} = [DFe{i} Se{i}.';-Se{i} DMe{i}];
   D3E{i} = -[DFe{i} -Se{i}.'; Se{i} DMe{i}];
   D4E{i} = [DFe{i} Se{i}.'; Se{i} DPe{i}];
end

% Compute forces, moments, and displacements based on rotation angles
for i = 1 : 6
   dE{i} = blkdiag(E(theta(i)),E(theta(i)))
          *d(fx(i),fy(i),fz(i),mx(i),my(i),mz(i),cy,cz,L(i));
   wE{i} = blkdiag(E(theta(i)),E(theta(i)))
          *w(fx(i),fy(i),fz(i),mx(i),my(i),mz(i),kx,ks,cx,cy,cz,L(i));
   RE{i} = blkdiag(E(theta(i)),E(theta(i)))
          *R(cy,cz,ks,L(i))*blkdiag(E(theta(i))',E(theta(i))');
   QE{i} = blkdiag(E(theta(i)),E(theta(i)))
```

```
            *Q(cy,cz,ks,L(i))*blkdiag(E(theta(i))',E(theta(i))');
end

% Compute the global A
A = [D4E{1}+D4E{4}-D1E{5}, -D2E{5}, D3E{4};
    D3E{5}, D4E{2}+D4E{5}-D1E{6}, -D2E{6};
    -D2E{4}, D3E{6}, D4E{3}+D4E{6}-D1E{4}]; %symmertic

b = sym(zeros(18,1));
b(1:6,1) = - q{1} - D3E{1}*u0{1} ...
         - D4E{1}*subs(wE{1},L(1)) - D4E{4}*subs(wE{4},L(4)) + ...
            D2E{5}*subs(wE{5},L(5)) ...
         + subs(dE{1},L(1)) + subs(dE{4},L(4));
b(7:12,1) = - q{2} - D3E{2}*u0{2} ...
         - D4E{2}*subs(wE{2},L(2)) - D4E{5}*subs(wE{5},L(5)) + ...
            D2E{6}*subs(wE{6},L(6)) ...
         + subs(dE{2},L(2)) + subs(dE{5},L(5));
b(13:18,1) = - q{3} - D3E{3}*u0{3} ...
         - D4E{3}*subs(wE{3},L(3)) - D4E{6}*subs(wE{6},L(6)) + ...
            D2E{4}*subs(wE{4},L(4)) ...
         + subs(dE{3},L(3)) + subs(dE{6},L(6));

solu = A\b;

% Extract local displacements
uL{1} = subs(solu(1:6),L(1));
uL{2} = subs(solu(7:12),L(2));
uL{3} = subs(solu(13:18),L(3));

u0{4} = uL{3};
u0{5} = uL{1};
u0{6} = uL{2};

uL{4} = uL{1};
uL{5} = uL{2};
uL{6} = uL{3};

% Here u is Global
for i = 1 : 6
   u{i} = QE{i}*u0{i} + RE{i}*(subs(uL{i},L(i))+subs(wE{i},L(i)))-wE{i};
end

for i = 1 : 6
   ulocal{i} = blkdiag(E(theta(i)),E(theta(i)))'*u{i};
end

% save('uglo.mat', 'u');
% save('uloc.mat', 'ulocal');
% save('q.mat', 'q');
% close all

```

```
function DFi = DF(kx,ks,L,cy,cz)
    [muy,muz] = mu(ks,L,cy,cz);
    DFi = sym(zeros(3,3));
    DFi(1,1) = 1/L*kx;
    DFi(2,2) = 12*cz*muz;
    DFi(3,3) = 12*cy*muy;
end

function DPi = DP(ks,L,cx,cy,cz)
    [muy,muz] = mu(ks,L,cy,cz);
    DPi = sym(zeros(3,3));
    DPi(1,1) = 1/L*cx;
    DPi(2,2) = cy*(1/L+3*muy*L^2);
    DPi(3,3) = cz*(1/L+3*muz*L^2);
end

function DMi = DM(ks,L,cx,cy,cz)
    [muy,muz] = mu(ks,L,cy,cz);
    DMi = sym(zeros(3,3));
    DMi(1,1) = 1/L*cx;
    DMi(2,2) = cy*(1/L-3*muy*L^2);
    DMi(3,3) = cz*(1/L-3*muz*L^2);
end

function Si = S(ks,L,cy,cz)
    [muy,muz] = mu(ks,L,cy,cz);
    Si = sym(zeros(3,3));
    Si(2,3) = 6*cy*muy*L;
    Si(3,2) = -6*cz*muz*L;
end

function [muy,muz] = mu(ks,L,cy,cz)
    muy = ks/(ks*L^3+12*cy*L);
    muz = ks/(ks*L^3+12*cz*L);
end

function Ei1 = E(t)
    Ei1 = [cos(t) sin(t) 0;
         -sin(t) cos(t) 0;
          0 0 1];
end

function [fInt,fInt2,fInt3] = Int(f,L)
    syms x;
    fInt = int(f,x,0,x);
    fInt2 = int(fInt,x,0,x);
    fInt3 = int(fInt2,x,0,x);
end

function [gyi,gzi] = gy(fy,fz,my,mz,cy,cz,L)
    [fzInt1,~,~] = Int(fz,L);
```

```
    [fyInt1,~,~] = Int(fy,L);
    gyi = 1/cy*(my+fzInt1);
    gzi = 1/cz*(mz-fyInt1);
end

function wi = w(fx,fy,fz,mx,my,mz,kx,ks,cx,cy,cz,L)
    wi = sym(zeros(6,1));
    [gyi,gzi] = gy(fy,fz,my,mz,cy,cz,L);

    [~,fzInt2,~] = Int(fz,L);
    [~,fxInt2,~] = Int(fx,L);
    [~,fyInt2,~] = Int(fy,L);

    [~,gyInt2,gyInt3] = Int(gyi,L);
    [~,gzInt2,gzInt3] = Int(gzi,L);

    [~,mxInt2,~] = Int(mx,L);

    wi(3) = 1/ks*fzInt2 - gyInt3;
    wi(5) = gyInt2;

    wi(1) = 1/kx*fxInt2;
    wi(2) = 1/ks*fyInt2 + gzInt3;
    wi(4) = 1/cx*(mxInt2);
    wi(6) = gzInt2;
end

function di = d(fx,fy,fz,mx,my,mz,cy,cz,L)
    di = sym(zeros(2,1));
    [gyi,gzi] = gy(fy,fz,my,mz,cy,cz,L);
    [fxInt1,~,~] = Int(fx,L);
    [fzInt1,~,~] = Int(fz,L);
    [fyInt1,~,~] = Int(fy,L);
    [mxInt1,~,~] = Int(mx,L);

    [gyInt1,~,~] = Int(gyi,L);
    [gzInt1,~,~] = Int(gzi,L);
    di(3) = fzInt1;
    di(5) = cy*gyInt1;

    di(1) = fxInt1;
    di(2) = fyInt1;
    di(4) = mxInt1;
    di(6) = cz*gzInt1;

end

function Qi = Q(cy,cz,ks,L)
    [muy,muz] = mu(ks,L,cy,cz);
    Qi = sym(zeros(6,6));
    syms x
```

```
    Qi(3,3) = muy*(L-x)*(L^2+L*x-2*x^2+12/ks*cy);
    Qi(5,5) = muy*(L-x)*(L^2-3*L*x+12/ks*cy);
    Qi(3,5) = -muy*x*(L-x)*(L^2-L*x+6/ks*cy);
    Qi(5,3) = 6*muy*x*(L-x);

    Qi(1,1) = 1 - L^-1*x;
    Qi(2,2) = muz*(L-x)*(L^2+L*x-2*x^2+12*ks^-1*cz);
    Qi(4,4) = 1 - L^-1*x;
    Qi(6,6) = muz*(L-x)*(L^2-3*L*x+12/ks*cz);
    Qi(2,6) = muz*x*(L-x)*(L^2-L*x+6*ks^-1*cz);
    Qi(6,2) = -6*muz*x*(L-x);

end

function Ri = R(cy,cz,ks,L)
    [muy,muz] = mu(ks,L,cy,cz);
    Ri = sym(zeros(6,6));
    syms x
    Ri(3,3) = muy*x*(3*L*x-2*x^2+12/ks*cy);
    Ri(5,5) = muy*x*(-2*L^2+3*L*x+12/ks*cy);
    Ri(3,5) = muy*x*(L-x)*(L*x+6/ks*cy);
    Ri(5,3) = -6*muy*x*(L-x);

    Ri(1,1) = 1/L*x;
    Ri(2,2) = muz*x*(3*L*x-2*x^2+12/ks*cz);
    Ri(4,4) = 1/L*x;
    Ri(6,6) = muz*x*(-2*L^2+3*L*x+12/ks*cz);
    Ri(2,6) = -muz*x*(L-x)*(L*x+6/ks*cz);
    Ri(6,2) = 6*muz*x*(L-x);
end
```

```
function J = OptControlFull(epsVal)
% Function to calculate the optimal control based on certain epsilon ...
    value

% If epsilon value is not provided, set it to 0.5
if nargin == 0
    epsVal = 0.5;
end

% Load necessary data
load("ks.mat")
load("kx.mat")
load("cx.mat")
load("cy.mat")
load("cz.mat")

load("uloc.mat");
load("q.mat");

```

```
% Calculate operator lambda value
Lval = Val_lambdaFullModel(epsVal);

u0 = zeros(6,1);

%% ======== Setting up parameters ==========
epsmax = 1;

theta = [-1/6*pi,pi/2,-5/6*pi,-pi,-pi/3,pi/3];

% Define length and discretization
L = 2;
hmax = 0.02;

l = L-epsmax;
N0l = ceil(l/hmax);
x0l = linspace(0,L-epsmax,N0l+1);

% number of basis functions
Nfy = 2*(N0l+1);

%% FEM matrices
[A1,A2,A3,A4] = graph1D_matirx_Hermite(N0l,x0l);

AA = cell(12,12);
for i = 1 : 12
   for j = 1 : 12
      AA{i,j} = deal(zeros(Nfy,Nfy));
   end
end

for i = 1 : 6
   AA{i,i} = A4;
end

for i = 7 : 12
   AA{i,i} = -A4;
end

AA{3,3+6} = ks*A1;
AA{3,5+6} = ks*A2;
AA{5,3+6} = ks*A3;
AA{5,5+6} = ks*A4+cy*A1;

AA{1,1+6} = kx*A1;
AA{2,2+6} = ks*A1;
AA{4,4+6} = cx*A1;
AA{6,6+6} = ks*A4+cz*A1;
AA{2,6+6} = -ks*A2;
AA{6,2+6} = -ks*A3;
```

```
AA{3+6,3} = ks*A1;
AA{3+6,5} = ks*A2;
AA{5+6,3} = ks*A3;
AA{5+6,5} = ks*A4+cy*A1;

AA{1+6,1} = kx*A1;
AA{2+6,2} = ks*A1;
AA{4+6,4} = cx*A1;
AA{6+6,6} = ks*A4+cz*A1;
AA{2+6,6} = -ks*A2;
AA{6+6,2} = -ks*A3;

AA = cell2mat(AA);

Aold = blkdiag(AA,AA,AA);
A = Aold;

% Set up boundary indices
Nf = [repmat(Nfy,1,36)];
index = cumsum(Nf);
index = [0 index];
for i = 1 : 3
    % index of [y(0),y(L),y'(0),y'(L)];
    urx_bdy{i} = index(12*i-11)+[1,Nf(2*i-1)/2,Nf(2*i-1)/2+1,Nf(2*i-1)];
    ury_bdy{i} = index(12*i-10)+[1,Nf(2*i-1)/2,Nf(2*i-1)/2+1,Nf(2*i-1)];
    urz_bdy{i} = index(12*i-9)+[1,Nf(2*i-1)/2,Nf(2*i-1)/2+1,Nf(2*i-1)];
    uphix_bdy{i} = index(12*i-8)+[1,Nf(2*i)/2,Nf(2*i)/2+1,Nf(2*i)];
    uphiy_bdy{i} = index(12*i-7)+[1,Nf(2*i)/2,Nf(2*i)/2+1,Nf(2*i)];
    uphiz_bdy{i} = index(12*i-6)+[1,Nf(2*i)/2,Nf(2*i)/2+1,Nf(2*i)];
    % index of [p(0),p(L),p'(0),p'(L)];
    prx_bdy{i} = index(12*i-5)+[1,Nf(2*i-1)/2,Nf(2*i-1)/2+1,Nf(2*i-1)];
    pry_bdy{i} = index(12*i-4)+[1,Nf(2*i-1)/2,Nf(2*i-1)/2+1,Nf(2*i-1)];
    prz_bdy{i} = index(12*i-3)+[1,Nf(2*i-1)/2,Nf(2*i-1)/2+1,Nf(2*i-1)];
    pphix_bdy{i} = index(12*i-2)+[1,Nf(2*i)/2,Nf(2*i)/2+1,Nf(2*i)];
    pphiy_bdy{i} = index(12*i-1)+[1,Nf(2*i)/2,Nf(2*i)/2+1,Nf(2*i)];
    pphiz_bdy{i} = index(12*i)+[1,Nf(2*i)/2,Nf(2*i)/2+1,Nf(2*i)];
end

% Rotate lambda
E3 = blkdiag(E(theta(1)),E(theta(1)), ...
          E(theta(2)),E(theta(2)), ...
          E(theta(3)),E(theta(3)));

Lval = E3'*Lval*E3;

% Update A matrix with lambda values at boundaries
for i = 1 : 3
    A(urx_bdy{i}(2),prx_bdy{1}(2)) = A(urx_bdy{i}(2),prx_bdy{1}(2)) - ...
        Lval(1,1+(i-1)*6);
    A(urx_bdy{i}(2),pry_bdy{1}(2)) = A(urx_bdy{i}(2),pry_bdy{1}(2)) - ...
        Lval(2,1+(i-1)*6);
```

```
    A(urx_bdy{i}(2),prz_bdy{1}(2)) = A(urx_bdy{i}(2),prz_bdy{1}(2)) - ...
        Lval(3,1+(i-1)*6);
    A(urx_bdy{i}(2),pphix_bdy{1}(2)) = A(urx_bdy{i}(2),pphix_bdy{1}(2)) ...
        - Lval(4,1+(i-1)*6);
    A(urx_bdy{i}(2),pphiy_bdy{1}(2)) = A(urx_bdy{i}(2),pphiy_bdy{1}(2)) ...
        - Lval(5,1+(i-1)*6);
    A(urx_bdy{i}(2),pphiz_bdy{1}(2)) = A(urx_bdy{i}(2),pphiz_bdy{1}(2)) ...
        - Lval(6,1+(i-1)*6);
    A(urx_bdy{i}(2),prx_bdy{2}(2)) = A(urx_bdy{i}(2),prx_bdy{2}(2)) - ...
        Lval(7,1+(i-1)*6);
    A(urx_bdy{i}(2),pry_bdy{2}(2)) = A(urx_bdy{i}(2),pry_bdy{2}(2)) - ...
        Lval(8,1+(i-1)*6);
    A(urx_bdy{i}(2),prz_bdy{2}(2)) = A(urx_bdy{i}(2),prz_bdy{2}(2)) - ...
        Lval(9,1+(i-1)*6);
    A(urx_bdy{i}(2),pphix_bdy{2}(2)) = A(urx_bdy{i}(2),pphix_bdy{2}(2)) ...
        - Lval(10,1+(i-1)*6);
    A(urx_bdy{i}(2),pphiy_bdy{2}(2)) = A(urx_bdy{i}(2),pphiy_bdy{2}(2)) ...
        - Lval(11,1+(i-1)*6);
    A(urx_bdy{i}(2),pphiz_bdy{2}(2)) = A(urx_bdy{i}(2),pphiz_bdy{2}(2)) ...
        - Lval(12,1+(i-1)*6);
    A(urx_bdy{i}(2),prx_bdy{3}(2)) = A(urx_bdy{i}(2),prx_bdy{3}(2)) - ...
        Lval(13,1+(i-1)*6);
    A(urx_bdy{i}(2),pry_bdy{3}(2)) = A(urx_bdy{i}(2),pry_bdy{3}(2)) - ...
        Lval(14,1+(i-1)*6);
    A(urx_bdy{i}(2),prz_bdy{3}(2)) = A(urx_bdy{i}(2),prz_bdy{3}(2)) - ...
        Lval(15,1+(i-1)*6);
    A(urx_bdy{i}(2),pphix_bdy{3}(2)) = A(urx_bdy{i}(2),pphix_bdy{3}(2)) ...
        - Lval(16,1+(i-1)*6);
    A(urx_bdy{i}(2),pphiy_bdy{3}(2)) = A(urx_bdy{i}(2),pphiy_bdy{3}(2)) ...
        - Lval(17,1+(i-1)*6);
    A(urx_bdy{i}(2),pphiz_bdy{3}(2)) = A(urx_bdy{i}(2),pphiz_bdy{3}(2)) ...
        - Lval(18,1+(i-1)*6);

    A(ury_bdy{i}(2),prx_bdy{1}(2)) = A(ury_bdy{i}(2),prx_bdy{1}(2)) - ...
        Lval(1,2+(i-1)*6);
    A(ury_bdy{i}(2),pry_bdy{1}(2)) = A(ury_bdy{i}(2),pry_bdy{1}(2)) - ...
        Lval(2,2+(i-1)*6);
    A(ury_bdy{i}(2),prz_bdy{1}(2)) = A(ury_bdy{i}(2),prz_bdy{1}(2)) - ...
        Lval(3,2+(i-1)*6);
    A(ury_bdy{i}(2),pphix_bdy{1}(2)) = A(ury_bdy{i}(2),pphix_bdy{1}(2)) ...
        - Lval(4,2+(i-1)*6);
    A(ury_bdy{i}(2),pphiy_bdy{1}(2)) = A(ury_bdy{i}(2),pphiy_bdy{1}(2)) ...
        - Lval(5,2+(i-1)*6);
    A(ury_bdy{i}(2),pphiz_bdy{1}(2)) = A(ury_bdy{i}(2),pphiz_bdy{1}(2)) ...
        - Lval(6,2+(i-1)*6);
    A(ury_bdy{i}(2),prx_bdy{2}(2)) = A(ury_bdy{i}(2),prx_bdy{2}(2)) - ...
        Lval(7,2+(i-1)*6);
    A(ury_bdy{i}(2),pry_bdy{2}(2)) = A(ury_bdy{i}(2),pry_bdy{2}(2)) - ...
        Lval(8,2+(i-1)*6);
    A(ury_bdy{i}(2),prz_bdy{2}(2)) = A(ury_bdy{i}(2),prz_bdy{2}(2)) - ...
        Lval(9,2+(i-1)*6);
```

```
    A(ury_bdy{i}(2),pphix_bdy{2}(2)) = A(ury_bdy{i}(2),pphix_bdy{2}(2)) ...
        - Lval(10,2+(i-1)*6);
    A(ury_bdy{i}(2),pphiy_bdy{2}(2)) = A(ury_bdy{i}(2),pphiy_bdy{2}(2)) ...
        - Lval(11,2+(i-1)*6);
    A(ury_bdy{i}(2),pphiz_bdy{2}(2)) = A(ury_bdy{i}(2),pphiz_bdy{2}(2)) ...
        - Lval(12,2+(i-1)*6);
    A(ury_bdy{i}(2),prx_bdy{3}(2)) = A(ury_bdy{i}(2),prx_bdy{3}(2)) - ...
        Lval(13,2+(i-1)*6);
    A(ury_bdy{i}(2),pry_bdy{3}(2)) = A(ury_bdy{i}(2),pry_bdy{3}(2)) - ...
        Lval(14,2+(i-1)*6);
    A(ury_bdy{i}(2),prz_bdy{3}(2)) = A(ury_bdy{i}(2),prz_bdy{3}(2)) - ...
        Lval(15,2+(i-1)*6);
    A(ury_bdy{i}(2),pphix_bdy{3}(2)) = A(ury_bdy{i}(2),pphix_bdy{3}(2)) ...
        - Lval(16,2+(i-1)*6);
    A(ury_bdy{i}(2),pphiy_bdy{3}(2)) = A(ury_bdy{i}(2),pphiy_bdy{3}(2)) ...
        - Lval(17,2+(i-1)*6);
    A(ury_bdy{i}(2),pphiz_bdy{3}(2)) = A(ury_bdy{i}(2),pphiz_bdy{3}(2)) ...
        - Lval(18,2+(i-1)*6);

    A(urz_bdy{i}(2),prx_bdy{1}(2)) = A(urz_bdy{i}(2),prx_bdy{1}(2)) - ...
        Lval(1,3+(i-1)*6);
    A(urz_bdy{i}(2),pry_bdy{1}(2)) = A(urz_bdy{i}(2),pry_bdy{1}(2)) - ...
        Lval(2,3+(i-1)*6);
    A(urz_bdy{i}(2),prz_bdy{1}(2)) = A(urz_bdy{i}(2),prz_bdy{1}(2)) - ...
        Lval(3,3+(i-1)*6);
    A(urz_bdy{i}(2),pphix_bdy{1}(2)) = A(urz_bdy{i}(2),pphix_bdy{1}(2)) ...
        - Lval(4,3+(i-1)*6);
    A(urz_bdy{i}(2),pphiy_bdy{1}(2)) = A(urz_bdy{i}(2),pphiy_bdy{1}(2)) ...
        - Lval(5,3+(i-1)*6);
    A(urz_bdy{i}(2),pphiz_bdy{1}(2)) = A(urz_bdy{i}(2),pphiz_bdy{1}(2)) ...
        - Lval(6,3+(i-1)*6);
    A(urz_bdy{i}(2),prx_bdy{2}(2)) = A(urz_bdy{i}(2),prx_bdy{2}(2)) - ...
        Lval(7,3+(i-1)*6);
    A(urz_bdy{i}(2),pry_bdy{2}(2)) = A(urz_bdy{i}(2),pry_bdy{2}(2)) - ...
        Lval(8,3+(i-1)*6);
    A(urz_bdy{i}(2),prz_bdy{2}(2)) = A(urz_bdy{i}(2),prz_bdy{2}(2)) - ...
        Lval(9,3+(i-1)*6);
    A(urz_bdy{i}(2),pphix_bdy{2}(2)) = A(urz_bdy{i}(2),pphix_bdy{2}(2)) ...
        - Lval(10,3+(i-1)*6);
    A(urz_bdy{i}(2),pphiy_bdy{2}(2)) = A(urz_bdy{i}(2),pphiy_bdy{2}(2)) ...
        - Lval(11,3+(i-1)*6);
    A(urz_bdy{i}(2),pphiz_bdy{2}(2)) = A(urz_bdy{i}(2),pphiz_bdy{2}(2)) ...
        - Lval(12,3+(i-1)*6);
    A(urz_bdy{i}(2),prx_bdy{3}(2)) = A(urz_bdy{i}(2),prx_bdy{3}(2)) - ...
        Lval(13,3+(i-1)*6);
    A(urz_bdy{i}(2),pry_bdy{3}(2)) = A(urz_bdy{i}(2),pry_bdy{3}(2)) - ...
        Lval(14,3+(i-1)*6);
    A(urz_bdy{i}(2),prz_bdy{3}(2)) = A(urz_bdy{i}(2),prz_bdy{3}(2)) - ...
        Lval(15,3+(i-1)*6);
    A(urz_bdy{i}(2),pphix_bdy{3}(2)) = A(urz_bdy{i}(2),pphix_bdy{3}(2)) ...
        - Lval(16,3+(i-1)*6);
```

```
    A(urz_bdy{i}(2),pphiy_bdy{3}(2)) = A(urz_bdy{i}(2),pphiy_bdy{3}(2)) ...
        - Lval(17,3+(i-1)*6);
    A(urz_bdy{i}(2),pphiz_bdy{3}(2)) = A(urz_bdy{i}(2),pphiz_bdy{3}(2)) ...
        - Lval(18,3+(i-1)*6);

    A(uphix_bdy{i}(2),prx_bdy{1}(2)) = A(uphix_bdy{i}(2),prx_bdy{1}(2)) ...
        - Lval(1,4+(i-1)*6);
    A(uphix_bdy{i}(2),pry_bdy{1}(2)) = A(uphix_bdy{i}(2),pry_bdy{1}(2)) ...
        - Lval(2,4+(i-1)*6);
    A(uphix_bdy{i}(2),prz_bdy{1}(2)) = A(uphix_bdy{i}(2),prz_bdy{1}(2)) ...
        - Lval(3,4+(i-1)*6);
    A(uphix_bdy{i}(2),pphix_bdy{1}(2)) = ...
        A(uphix_bdy{i}(2),pphix_bdy{1}(2)) - Lval(4,4+(i-1)*6);
    A(uphix_bdy{i}(2),pphiy_bdy{1}(2)) = ...
        A(uphix_bdy{i}(2),pphiy_bdy{1}(2)) - Lval(5,4+(i-1)*6);
    A(uphix_bdy{i}(2),pphiz_bdy{1}(2)) = ...
        A(uphix_bdy{i}(2),pphiz_bdy{1}(2)) - Lval(6,4+(i-1)*6);
    A(uphix_bdy{i}(2),prx_bdy{2}(2)) = A(uphix_bdy{i}(2),prx_bdy{2}(2)) ...
        - Lval(7,4+(i-1)*6);
    A(uphix_bdy{i}(2),pry_bdy{2}(2)) = A(uphix_bdy{i}(2),pry_bdy{2}(2)) ...
        - Lval(8,4+(i-1)*6);
    A(uphix_bdy{i}(2),prz_bdy{2}(2)) = A(uphix_bdy{i}(2),prz_bdy{2}(2)) ...
        - Lval(9,4+(i-1)*6);
    A(uphix_bdy{i}(2),pphix_bdy{2}(2)) = ...
        A(uphix_bdy{i}(2),pphix_bdy{2}(2)) - Lval(10,4+(i-1)*6);
    A(uphix_bdy{i}(2),pphiy_bdy{2}(2)) = ...
        A(uphix_bdy{i}(2),pphiy_bdy{2}(2)) - Lval(11,4+(i-1)*6);
    A(uphix_bdy{i}(2),pphiz_bdy{2}(2)) = ...
        A(uphix_bdy{i}(2),pphiz_bdy{2}(2)) - Lval(12,4+(i-1)*6);
    A(uphix_bdy{i}(2),prx_bdy{3}(2)) = A(uphix_bdy{i}(2),prx_bdy{3}(2)) ...
        - Lval(13,4+(i-1)*6);
    A(uphix_bdy{i}(2),pry_bdy{3}(2)) = A(uphix_bdy{i}(2),pry_bdy{3}(2)) ...
        - Lval(14,4+(i-1)*6);
    A(uphix_bdy{i}(2),prz_bdy{3}(2)) = A(uphix_bdy{i}(2),prz_bdy{3}(2)) ...
        - Lval(15,4+(i-1)*6);
    A(uphix_bdy{i}(2),pphix_bdy{3}(2)) = ...
        A(uphix_bdy{i}(2),pphix_bdy{3}(2)) - Lval(16,4+(i-1)*6);
    A(uphix_bdy{i}(2),pphiy_bdy{3}(2)) = ...
        A(uphix_bdy{i}(2),pphiy_bdy{3}(2)) - Lval(17,4+(i-1)*6);
    A(uphix_bdy{i}(2),pphiz_bdy{3}(2)) = ...
        A(uphix_bdy{i}(2),pphiz_bdy{3}(2)) - Lval(18,4+(i-1)*6);

    A(uphiy_bdy{i}(2),prx_bdy{1}(2)) = A(uphiy_bdy{i}(2),prx_bdy{1}(2)) ...
        - Lval(1,5+(i-1)*6);
    A(uphiy_bdy{i}(2),pry_bdy{1}(2)) = A(uphiy_bdy{i}(2),pry_bdy{1}(2)) ...
        - Lval(2,5+(i-1)*6);
    A(uphiy_bdy{i}(2),prz_bdy{1}(2)) = A(uphiy_bdy{i}(2),prz_bdy{1}(2)) ...
        - Lval(3,5+(i-1)*6);
    A(uphiy_bdy{i}(2),pphix_bdy{1}(2)) = ...
        A(uphiy_bdy{i}(2),pphix_bdy{1}(2)) - Lval(4,5+(i-1)*6);
    A(uphiy_bdy{i}(2),pphiy_bdy{1}(2)) = ...
```

```
        A(uphiy_bdy{i}(2),pphiy_bdy{1}(2)) - Lval(5,5+(i-1)*6);
    A(uphiy_bdy{i}(2),pphiz_bdy{1}(2)) = ...
        A(uphiy_bdy{i}(2),pphiz_bdy{1}(2)) - Lval(6,5+(i-1)*6);
    A(uphiy_bdy{i}(2),prx_bdy{2}(2)) = A(uphiy_bdy{i}(2),prx_bdy{2}(2)) ...
        - Lval(7,5+(i-1)*6);
    A(uphiy_bdy{i}(2),pry_bdy{2}(2)) = A(uphiy_bdy{i}(2),pry_bdy{2}(2)) ...
        - Lval(8,5+(i-1)*6);
    A(uphiy_bdy{i}(2),prz_bdy{2}(2)) = A(uphiy_bdy{i}(2),prz_bdy{2}(2)) ...
        - Lval(9,5+(i-1)*6);
    A(uphiy_bdy{i}(2),pphix_bdy{2}(2)) = ...
        A(uphiy_bdy{i}(2),pphix_bdy{2}(2)) - Lval(10,5+(i-1)*6);
    A(uphiy_bdy{i}(2),pphiy_bdy{2}(2)) = ...
        A(uphiy_bdy{i}(2),pphiy_bdy{2}(2)) - Lval(11,5+(i-1)*6);
    A(uphiy_bdy{i}(2),pphiz_bdy{2}(2)) = ...
        A(uphiy_bdy{i}(2),pphiz_bdy{2}(2)) - Lval(12,5+(i-1)*6);
    A(uphiy_bdy{i}(2),prx_bdy{3}(2)) = A(uphiy_bdy{i}(2),prx_bdy{3}(2)) ...
        - Lval(13,5+(i-1)*6);
    A(uphiy_bdy{i}(2),pry_bdy{3}(2)) = A(uphiy_bdy{i}(2),pry_bdy{3}(2)) ...
        - Lval(14,5+(i-1)*6);
    A(uphiy_bdy{i}(2),prz_bdy{3}(2)) = A(uphiy_bdy{i}(2),prz_bdy{3}(2)) ...
        - Lval(15,5+(i-1)*6);
    A(uphiy_bdy{i}(2),pphix_bdy{3}(2)) = ...
        A(uphiy_bdy{i}(2),pphix_bdy{3}(2)) - Lval(16,5+(i-1)*6);
    A(uphiy_bdy{i}(2),pphiy_bdy{3}(2)) = ...
        A(uphiy_bdy{i}(2),pphiy_bdy{3}(2)) - Lval(17,5+(i-1)*6);
    A(uphiy_bdy{i}(2),pphiz_bdy{3}(2)) = ...
        A(uphiy_bdy{i}(2),pphiz_bdy{3}(2)) - Lval(18,5+(i-1)*6);

    A(uphiz_bdy{i}(2),prx_bdy{1}(2)) = A(uphiz_bdy{i}(2),prx_bdy{1}(2)) ...
        - Lval(1,6+(i-1)*6);
    A(uphiz_bdy{i}(2),pry_bdy{1}(2)) = A(uphiz_bdy{i}(2),pry_bdy{1}(2)) ...
        - Lval(2,6+(i-1)*6);
    A(uphiz_bdy{i}(2),prz_bdy{1}(2)) = A(uphiz_bdy{i}(2),prz_bdy{1}(2)) ...
        - Lval(3,6+(i-1)*6);
    A(uphiz_bdy{i}(2),pphix_bdy{1}(2)) = ...
        A(uphiz_bdy{i}(2),pphix_bdy{1}(2)) - Lval(4,6+(i-1)*6);
    A(uphiz_bdy{i}(2),pphiy_bdy{1}(2)) = ...
        A(uphiz_bdy{i}(2),pphiy_bdy{1}(2)) - Lval(5,6+(i-1)*6);
    A(uphiz_bdy{i}(2),pphiz_bdy{1}(2)) = ...
        A(uphiz_bdy{i}(2),pphiz_bdy{1}(2)) - Lval(6,6+(i-1)*6);
    A(uphiz_bdy{i}(2),prx_bdy{2}(2)) = A(uphiz_bdy{i}(2),prx_bdy{2}(2)) ...
        - Lval(7,6+(i-1)*6);
    A(uphiz_bdy{i}(2),pry_bdy{2}(2)) = A(uphiz_bdy{i}(2),pry_bdy{2}(2)) ...
        - Lval(8,6+(i-1)*6);
    A(uphiz_bdy{i}(2),prz_bdy{2}(2)) = A(uphiz_bdy{i}(2),prz_bdy{2}(2)) ...
        - Lval(9,6+(i-1)*6);
    A(uphiz_bdy{i}(2),pphix_bdy{2}(2)) = ...
        A(uphiz_bdy{i}(2),pphix_bdy{2}(2)) - Lval(10,6+(i-1)*6);
    A(uphiz_bdy{i}(2),pphiy_bdy{2}(2)) = ...
        A(uphiz_bdy{i}(2),pphiy_bdy{2}(2)) - Lval(11,6+(i-1)*6);
    A(uphiz_bdy{i}(2),pphiz_bdy{2}(2)) = ...
```

```
        A(uphiz_bdy{i}(2),pphiz_bdy{2}(2)) - Lval(12,6+(i-1)*6);
    A(uphiz_bdy{i}(2),prx_bdy{3}(2)) = A(uphiz_bdy{i}(2),prx_bdy{3}(2)) ...
        - Lval(13,6+(i-1)*6);
    A(uphiz_bdy{i}(2),pry_bdy{3}(2)) = A(uphiz_bdy{i}(2),pry_bdy{3}(2)) ...
        - Lval(14,6+(i-1)*6);
    A(uphiz_bdy{i}(2),prz_bdy{3}(2)) = A(uphiz_bdy{i}(2),prz_bdy{3}(2)) ...
        - Lval(15,6+(i-1)*6);
    A(uphiz_bdy{i}(2),pphix_bdy{3}(2)) = ...
        A(uphiz_bdy{i}(2),pphix_bdy{3}(2)) - Lval(16,6+(i-1)*6);
    A(uphiz_bdy{i}(2),pphiy_bdy{3}(2)) = ...
        A(uphiz_bdy{i}(2),pphiy_bdy{3}(2)) - Lval(17,6+(i-1)*6);
    A(uphiz_bdy{i}(2),pphiz_bdy{3}(2)) = ...
        A(uphiz_bdy{i}(2),pphiz_bdy{3}(2)) - Lval(18,6+(i-1)*6);
end

for i = 1 : 3
    A(prx_bdy{i}(2),urx_bdy{1}(2)) = A(prx_bdy{i}(2),urx_bdy{1}(2)) - ...
        Lval(1,1+(i-1)*6);
    A(prx_bdy{i}(2),ury_bdy{1}(2)) = A(prx_bdy{i}(2),ury_bdy{1}(2)) - ...
        Lval(2,1+(i-1)*6);
    A(prx_bdy{i}(2),urz_bdy{1}(2)) = A(prx_bdy{i}(2),urz_bdy{1}(2)) - ...
        Lval(3,1+(i-1)*6);
    A(prx_bdy{i}(2),uphix_bdy{1}(2)) = A(prx_bdy{i}(2),uphix_bdy{1}(2)) ...
        - Lval(4,1+(i-1)*6);
    A(prx_bdy{i}(2),uphiy_bdy{1}(2)) = A(prx_bdy{i}(2),uphiy_bdy{1}(2)) ...
        - Lval(5,1+(i-1)*6);
    A(prx_bdy{i}(2),uphiz_bdy{1}(2)) = A(prx_bdy{i}(2),uphiz_bdy{1}(2)) ...
        - Lval(6,1+(i-1)*6);
    A(prx_bdy{i}(2),urx_bdy{2}(2)) = A(prx_bdy{i}(2),urx_bdy{2}(2)) - ...
        Lval(7,1+(i-1)*6);
    A(prx_bdy{i}(2),ury_bdy{2}(2)) = A(prx_bdy{i}(2),ury_bdy{2}(2)) - ...
        Lval(8,1+(i-1)*6);
    A(prx_bdy{i}(2),urz_bdy{2}(2)) = A(prx_bdy{i}(2),urz_bdy{2}(2)) - ...
        Lval(9,1+(i-1)*6);
    A(prx_bdy{i}(2),uphix_bdy{2}(2)) = A(prx_bdy{i}(2),uphix_bdy{2}(2)) ...
        - Lval(10,1+(i-1)*6);
    A(prx_bdy{i}(2),uphiy_bdy{2}(2)) = A(prx_bdy{i}(2),uphiy_bdy{2}(2)) ...
        - Lval(11,1+(i-1)*6);
    A(prx_bdy{i}(2),uphiz_bdy{2}(2)) = A(prx_bdy{i}(2),uphiz_bdy{2}(2)) ...
        - Lval(12,1+(i-1)*6);
    A(prx_bdy{i}(2),urx_bdy{3}(2)) = A(prx_bdy{i}(2),urx_bdy{3}(2)) - ...
        Lval(13,1+(i-1)*6);
    A(prx_bdy{i}(2),ury_bdy{3}(2)) = A(prx_bdy{i}(2),ury_bdy{3}(2)) - ...
        Lval(14,1+(i-1)*6);
    A(prx_bdy{i}(2),urz_bdy{3}(2)) = A(prx_bdy{i}(2),urz_bdy{3}(2)) - ...
        Lval(15,1+(i-1)*6);
    A(prx_bdy{i}(2),uphix_bdy{3}(2)) = A(prx_bdy{i}(2),uphix_bdy{3}(2)) ...
        - Lval(16,1+(i-1)*6);
    A(prx_bdy{i}(2),uphiy_bdy{3}(2)) = A(prx_bdy{i}(2),uphiy_bdy{3}(2)) ...
        - Lval(17,1+(i-1)*6);
    A(prx_bdy{i}(2),uphiz_bdy{3}(2)) = A(prx_bdy{i}(2),uphiz_bdy{3}(2)) ...
```

```
        - Lval(18,1+(i-1)*6);

    A(pry_bdy{i}(2),urx_bdy{1}(2)) = A(pry_bdy{i}(2),urx_bdy{1}(2)) - ...
        Lval(1,2+(i-1)*6);
    A(pry_bdy{i}(2),ury_bdy{1}(2)) = A(pry_bdy{i}(2),ury_bdy{1}(2)) - ...
        Lval(2,2+(i-1)*6);
    A(pry_bdy{i}(2),urz_bdy{1}(2)) = A(pry_bdy{i}(2),urz_bdy{1}(2)) - ...
        Lval(3,2+(i-1)*6);
    A(pry_bdy{i}(2),uphix_bdy{1}(2)) = A(pry_bdy{i}(2),uphix_bdy{1}(2)) ...
        - Lval(4,2+(i-1)*6);
    A(pry_bdy{i}(2),uphiy_bdy{1}(2)) = A(pry_bdy{i}(2),uphiy_bdy{1}(2)) ...
        - Lval(5,2+(i-1)*6);
    A(pry_bdy{i}(2),uphiz_bdy{1}(2)) = A(pry_bdy{i}(2),uphiz_bdy{1}(2)) ...
        - Lval(6,2+(i-1)*6);
    A(pry_bdy{i}(2),urx_bdy{2}(2)) = A(pry_bdy{i}(2),urx_bdy{2}(2)) - ...
        Lval(7,2+(i-1)*6);
    A(pry_bdy{i}(2),ury_bdy{2}(2)) = A(pry_bdy{i}(2),ury_bdy{2}(2)) - ...
        Lval(8,2+(i-1)*6);
    A(pry_bdy{i}(2),urz_bdy{2}(2)) = A(pry_bdy{i}(2),urz_bdy{2}(2)) - ...
        Lval(9,2+(i-1)*6);
    A(pry_bdy{i}(2),uphix_bdy{2}(2)) = A(pry_bdy{i}(2),uphix_bdy{2}(2)) ...
        - Lval(10,2+(i-1)*6);
    A(pry_bdy{i}(2),uphiy_bdy{2}(2)) = A(pry_bdy{i}(2),uphiy_bdy{2}(2)) ...
        - Lval(11,2+(i-1)*6);
    A(pry_bdy{i}(2),uphiz_bdy{2}(2)) = A(pry_bdy{i}(2),uphiz_bdy{2}(2)) ...
        - Lval(12,2+(i-1)*6);
    A(pry_bdy{i}(2),urx_bdy{3}(2)) = A(pry_bdy{i}(2),urx_bdy{3}(2)) - ...
        Lval(13,2+(i-1)*6);
    A(pry_bdy{i}(2),ury_bdy{3}(2)) = A(pry_bdy{i}(2),ury_bdy{3}(2)) - ...
        Lval(14,2+(i-1)*6);
    A(pry_bdy{i}(2),urz_bdy{3}(2)) = A(pry_bdy{i}(2),urz_bdy{3}(2)) - ...
        Lval(15,2+(i-1)*6);
    A(pry_bdy{i}(2),uphix_bdy{3}(2)) = A(pry_bdy{i}(2),uphix_bdy{3}(2)) ...
        - Lval(16,2+(i-1)*6);
    A(pry_bdy{i}(2),uphiy_bdy{3}(2)) = A(pry_bdy{i}(2),uphiy_bdy{3}(2)) ...
        - Lval(17,2+(i-1)*6);
    A(pry_bdy{i}(2),uphiz_bdy{3}(2)) = A(pry_bdy{i}(2),uphiz_bdy{3}(2)) ...
        - Lval(18,2+(i-1)*6);

    A(prz_bdy{i}(2),urx_bdy{1}(2)) = A(prz_bdy{i}(2),urx_bdy{1}(2)) - ...
        Lval(1,3+(i-1)*6);
    A(prz_bdy{i}(2),ury_bdy{1}(2)) = A(prz_bdy{i}(2),ury_bdy{1}(2)) - ...
        Lval(2,3+(i-1)*6);
    A(prz_bdy{i}(2),urz_bdy{1}(2)) = A(prz_bdy{i}(2),urz_bdy{1}(2)) - ...
        Lval(3,3+(i-1)*6);
    A(prz_bdy{i}(2),uphix_bdy{1}(2)) = A(prz_bdy{i}(2),uphix_bdy{1}(2)) ...
        - Lval(4,3+(i-1)*6);
    A(prz_bdy{i}(2),uphiy_bdy{1}(2)) = A(prz_bdy{i}(2),uphiy_bdy{1}(2)) ...
        - Lval(5,3+(i-1)*6);
    A(prz_bdy{i}(2),uphiz_bdy{1}(2)) = A(prz_bdy{i}(2),uphiz_bdy{1}(2)) ...
        - Lval(6,3+(i-1)*6);
```

```
A(prz_bdy{i}(2),urx_bdy{2}(2)) = A(prz_bdy{i}(2),urx_bdy{2}(2)) - ... Lval(7,3+(i-1)*6);
A(prz_bdy{i}(2),ury_bdy{2}(2)) = A(prz_bdy{i}(2),ury_bdy{2}(2)) - ... Lval(8,3+(i-1)*6);
A(prz_bdy{i}(2),urz_bdy{2}(2)) = A(prz_bdy{i}(2),urz_bdy{2}(2)) - ... Lval(9,3+(i-1)*6);
A(prz_bdy{i}(2),uphix_bdy{2}(2)) = A(prz_bdy{i}(2),uphix_bdy{2}(2)) ... - Lval(10,3+(i-1)*6);
A(prz_bdy{i}(2),uphiy_bdy{2}(2)) = A(prz_bdy{i}(2),uphiy_bdy{2}(2)) ... - Lval(11,3+(i-1)*6);
A(prz_bdy{i}(2),uphiz_bdy{2}(2)) = A(prz_bdy{i}(2),uphiz_bdy{2}(2)) ... - Lval(12,3+(i-1)*6);
A(prz_bdy{i}(2),urx_bdy{3}(2)) = A(prz_bdy{i}(2),urx_bdy{3}(2)) - ... Lval(13,3+(i-1)*6);
A(prz_bdy{i}(2),ury_bdy{3}(2)) = A(prz_bdy{i}(2),ury_bdy{3}(2)) - ... Lval(14,3+(i-1)*6);
A(prz_bdy{i}(2),urz_bdy{3}(2)) = A(prz_bdy{i}(2),urz_bdy{3}(2)) - ... Lval(15,3+(i-1)*6);
A(prz_bdy{i}(2),uphix_bdy{3}(2)) = A(prz_bdy{i}(2),uphix_bdy{3}(2)) ... - Lval(16,3+(i-1)*6);
A(prz_bdy{i}(2),uphiy_bdy{3}(2)) = A(prz_bdy{i}(2),uphiy_bdy{3}(2)) ... - Lval(17,3+(i-1)*6);
A(prz_bdy{i}(2),uphiz_bdy{3}(2)) = A(prz_bdy{i}(2),uphiz_bdy{3}(2)) ... - Lval(18,3+(i-1)*6);

A(pphix_bdy{i}(2),urx_bdy{1}(2)) = A(pphix_bdy{i}(2),urx_bdy{1}(2)) ... - Lval(1,4+(i-1)*6);
A(pphix_bdy{i}(2),ury_bdy{1}(2)) = A(pphix_bdy{i}(2),ury_bdy{1}(2)) ... - Lval(2,4+(i-1)*6);
A(pphix_bdy{i}(2),urz_bdy{1}(2)) = A(pphix_bdy{i}(2),urz_bdy{1}(2)) ... - Lval(3,4+(i-1)*6);
A(pphix_bdy{i}(2),uphix_bdy{1}(2)) = ... A(pphix_bdy{i}(2),uphix_bdy{1}(2)) - Lval(4,4+(i-1)*6);
A(pphix_bdy{i}(2),uphiy_bdy{1}(2)) = ... A(pphix_bdy{i}(2),uphiy_bdy{1}(2)) - Lval(5,4+(i-1)*6);
A(pphix_bdy{i}(2),uphiz_bdy{1}(2)) = ... A(pphix_bdy{i}(2),uphiz_bdy{1}(2)) - Lval(6,4+(i-1)*6);
A(pphix_bdy{i}(2),urx_bdy{2}(2)) = A(pphix_bdy{i}(2),urx_bdy{2}(2)) ... - Lval(7,4+(i-1)*6);
A(pphix_bdy{i}(2),ury_bdy{2}(2)) = A(pphix_bdy{i}(2),ury_bdy{2}(2)) ... - Lval(8,4+(i-1)*6);
A(pphix_bdy{i}(2),urz_bdy{2}(2)) = A(pphix_bdy{i}(2),urz_bdy{2}(2)) ... - Lval(9,4+(i-1)*6);
A(pphix_bdy{i}(2),uphix_bdy{2}(2)) = ... A(pphix_bdy{i}(2),uphix_bdy{2}(2)) - Lval(10,4+(i-1)*6);
A(pphix_bdy{i}(2),uphiy_bdy{2}(2)) = ... A(pphix_bdy{i}(2),uphiy_bdy{2}(2)) - Lval(11,4+(i-1)*6);
A(pphix_bdy{i}(2),uphiz_bdy{2}(2)) = ... A(pphix_bdy{i}(2),uphiz_bdy{2}(2)) - Lval(12,4+(i-1)*6);
A(pphix_bdy{i}(2),urx_bdy{3}(2)) = A(pphix_bdy{i}(2),urx_bdy{3}(2)) ... - Lval(13,4+(i-1)*6);
```

```
A(pphix_bdy{i}(2),ury_bdy{3}(2)) = A(pphix_bdy{i}(2),ury_bdy{3}(2)) ...
    - Lval(14,4+(i-1)*6);
A(pphix_bdy{i}(2),urz_bdy{3}(2)) = A(pphix_bdy{i}(2),urz_bdy{3}(2)) ...
    - Lval(15,4+(i-1)*6);
A(pphix_bdy{i}(2),uphix_bdy{3}(2)) = ...
    A(pphix_bdy{i}(2),uphix_bdy{3}(2)) - Lval(16,4+(i-1)*6);
A(pphix_bdy{i}(2),uphiy_bdy{3}(2)) = ...
    A(pphix_bdy{i}(2),uphiy_bdy{3}(2)) - Lval(17,4+(i-1)*6);
A(pphix_bdy{i}(2),uphiz_bdy{3}(2)) = ...
    A(pphix_bdy{i}(2),uphiz_bdy{3}(2)) - Lval(18,4+(i-1)*6);

A(pphiy_bdy{i}(2),urx_bdy{1}(2)) = A(pphiy_bdy{i}(2),urx_bdy{1}(2)) ...
    - Lval(1,5+(i-1)*6);
A(pphiy_bdy{i}(2),ury_bdy{1}(2)) = A(pphiy_bdy{i}(2),ury_bdy{1}(2)) ...
    - Lval(2,5+(i-1)*6);
A(pphiy_bdy{i}(2),urz_bdy{1}(2)) = A(pphiy_bdy{i}(2),urz_bdy{1}(2)) ...
    - Lval(3,5+(i-1)*6);
A(pphiy_bdy{i}(2),uphix_bdy{1}(2)) = ...
    A(pphiy_bdy{i}(2),uphix_bdy{1}(2)) - Lval(4,5+(i-1)*6);
A(pphiy_bdy{i}(2),uphiy_bdy{1}(2)) = ...
    A(pphiy_bdy{i}(2),uphiy_bdy{1}(2)) - Lval(5,5+(i-1)*6);
A(pphiy_bdy{i}(2),uphiz_bdy{1}(2)) = ...
    A(pphiy_bdy{i}(2),uphiz_bdy{1}(2)) - Lval(6,5+(i-1)*6);
A(pphiy_bdy{i}(2),urx_bdy{2}(2)) = A(pphiy_bdy{i}(2),urx_bdy{2}(2)) ...
    - Lval(7,5+(i-1)*6);
A(pphiy_bdy{i}(2),ury_bdy{2}(2)) = A(pphiy_bdy{i}(2),ury_bdy{2}(2)) ...
    - Lval(8,5+(i-1)*6);
A(pphiy_bdy{i}(2),urz_bdy{2}(2)) = A(pphiy_bdy{i}(2),urz_bdy{2}(2)) ...
    - Lval(9,5+(i-1)*6);
A(pphiy_bdy{i}(2),uphix_bdy{2}(2)) = ...
    A(pphiy_bdy{i}(2),uphix_bdy{2}(2)) - Lval(10,5+(i-1)*6);
A(pphiy_bdy{i}(2),uphiy_bdy{2}(2)) = ...
    A(pphiy_bdy{i}(2),uphiy_bdy{2}(2)) - Lval(11,5+(i-1)*6);
A(pphiy_bdy{i}(2),uphiz_bdy{2}(2)) = ...
    A(pphiy_bdy{i}(2),uphiz_bdy{2}(2)) - Lval(12,5+(i-1)*6);
A(pphiy_bdy{i}(2),urx_bdy{3}(2)) = A(pphiy_bdy{i}(2),urx_bdy{3}(2)) ...
    - Lval(13,5+(i-1)*6);
A(pphiy_bdy{i}(2),ury_bdy{3}(2)) = A(pphiy_bdy{i}(2),ury_bdy{3}(2)) ...
    - Lval(14,5+(i-1)*6);
A(pphiy_bdy{i}(2),urz_bdy{3}(2)) = A(pphiy_bdy{i}(2),urz_bdy{3}(2)) ...
    - Lval(15,5+(i-1)*6);
A(pphiy_bdy{i}(2),uphix_bdy{3}(2)) = ...
    A(pphiy_bdy{i}(2),uphix_bdy{3}(2)) - Lval(16,5+(i-1)*6);
A(pphiy_bdy{i}(2),uphiy_bdy{3}(2)) = ...
    A(pphiy_bdy{i}(2),uphiy_bdy{3}(2)) - Lval(17,5+(i-1)*6);
A(pphiy_bdy{i}(2),uphiz_bdy{3}(2)) = ...
    A(pphiy_bdy{i}(2),uphiz_bdy{3}(2)) - Lval(18,5+(i-1)*6);

A(pphiz_bdy{i}(2),urx_bdy{1}(2)) = A(pphiz_bdy{i}(2),urx_bdy{1}(2)) ...
    - Lval(1,6+(i-1)*6);
A(pphiz_bdy{i}(2),ury_bdy{1}(2)) = A(pphiz_bdy{i}(2),ury_bdy{1}(2)) ...
```

```
        - Lval(2,6+(i-1)*6);
    A(pphiz_bdy{i}(2),urz_bdy{1}(2)) = A(pphiz_bdy{i}(2),urz_bdy{1}(2)) ...
        - Lval(3,6+(i-1)*6);
    A(pphiz_bdy{i}(2),uphix_bdy{1}(2)) = ...
        A(pphiz_bdy{i}(2),uphix_bdy{1}(2)) - Lval(4,6+(i-1)*6);
    A(pphiz_bdy{i}(2),uphiy_bdy{1}(2)) = ...
        A(pphiz_bdy{i}(2),uphiy_bdy{1}(2)) - Lval(5,6+(i-1)*6);
    A(pphiz_bdy{i}(2),uphiz_bdy{1}(2)) = ...
        A(pphiz_bdy{i}(2),uphiz_bdy{1}(2)) - Lval(6,6+(i-1)*6);
    A(pphiz_bdy{i}(2),urx_bdy{2}(2)) = A(pphiz_bdy{i}(2),urx_bdy{2}(2)) ...
        - Lval(7,6+(i-1)*6);
    A(pphiz_bdy{i}(2),ury_bdy{2}(2)) = A(pphiz_bdy{i}(2),ury_bdy{2}(2)) ...
        - Lval(8,6+(i-1)*6);
    A(pphiz_bdy{i}(2),urz_bdy{2}(2)) = A(pphiz_bdy{i}(2),urz_bdy{2}(2)) ...
        - Lval(9,6+(i-1)*6);
    A(pphiz_bdy{i}(2),uphix_bdy{2}(2)) = ...
        A(pphiz_bdy{i}(2),uphix_bdy{2}(2)) - Lval(10,6+(i-1)*6);
    A(pphiz_bdy{i}(2),uphiy_bdy{2}(2)) = ...
        A(pphiz_bdy{i}(2),uphiy_bdy{2}(2)) - Lval(11,6+(i-1)*6);
    A(pphiz_bdy{i}(2),uphiz_bdy{2}(2)) = ...
        A(pphiz_bdy{i}(2),uphiz_bdy{2}(2)) - Lval(12,6+(i-1)*6);
    A(pphiz_bdy{i}(2),urx_bdy{3}(2)) = A(pphiz_bdy{i}(2),urx_bdy{3}(2)) ...
        - Lval(13,6+(i-1)*6);
    A(pphiz_bdy{i}(2),ury_bdy{3}(2)) = A(pphiz_bdy{i}(2),ury_bdy{3}(2)) ...
        - Lval(14,6+(i-1)*6);
    A(pphiz_bdy{i}(2),urz_bdy{3}(2)) = A(pphiz_bdy{i}(2),urz_bdy{3}(2)) ...
        - Lval(15,6+(i-1)*6);
    A(pphiz_bdy{i}(2),uphix_bdy{3}(2)) = ...
        A(pphiz_bdy{i}(2),uphix_bdy{3}(2)) - Lval(16,6+(i-1)*6);
    A(pphiz_bdy{i}(2),uphiy_bdy{3}(2)) = ...
        A(pphiz_bdy{i}(2),uphiy_bdy{3}(2)) - Lval(17,6+(i-1)*6);
    A(pphiz_bdy{i}(2),uphiz_bdy{3}(2)) = ...
        A(pphiz_bdy{i}(2),uphiz_bdy{3}(2)) - Lval(18,6+(i-1)*6);
end

syms x
for i = 1 : 3
    z{i} = ulocal{i};
    zNum{i} = double(subs(z{i},x01));
    dz{i} = diff(z{i},x);
    dzNum{i} = double(subs(dz{i},x01));
    dzNum{i}(:,end) = double(subs(dz{i}, x, 0.999999));
    dzNum{i}(:,1) = double(subs(dz{i}, x, 0.000001));

    fx(i) = q{i}(1);
    dfx{i} = diff(fx(i),x);
    dfxNum{i} = double(subs(dfx{i},x01));
    dfxNum{i}(:,end) = double(subs(dfx{i}, x, 0.999999));
    dfxNum{i}(:,1) = double(subs(dfx{i}, x, 0.000001));

    fy(i) = q{i}(2);
```

```
    dfy{i} = diff(fy(i),x);
    dfyNum{i} = double(subs(dfy{i},x01));
    dfyNum{i}(:,end) = double(subs(dfy{i}, x, 0.999999));
    dfyNum{i}(:,1) = double(subs(dfy{i}, x, 0.000001));

    fz(i) = q{i}(3);
    dfz{i} = diff(fz(i),x);
    dfzNum{i} = double(subs(dfz{i},x01));
    dfzNum{i}(:,end) = double(subs(dfz{i}, x, 0.999999));
    dfzNum{i}(:,1) = double(subs(dfz{i}, x, 0.000001));

    mx(i) = q{i}(4);
    dmx{i} = diff(mx(i),x);
    dmxNum{i} = double(subs(dmx{i},x01));
    dmxNum{i}(:,end) = double(subs(dmx{i}, x, 0.999999));
    dmxNum{i}(:,1) = double(subs(dmx{i}, x, 0.000001));

    my(i) = q{i}(5);
    dmy{i} = diff(my(i),x);
    dmyNum{i} = double(subs(dmy{i},x01));
    dmyNum{i}(:,end) = double(subs(dmy{i}, x, 0.999999));
    dmyNum{i}(:,1) = double(subs(dmy{i}, x, 0.000001));

    mz(i) = q{i}(6);
    dmz{i} = diff(mz(i),x);
    dmzNum{i} = double(subs(dmz{i},x01));
    dmzNum{i}(:,end) = double(subs(dmz{i}, x, 0.999999));
    dmzNum{i}(:,1) = double(subs(dmz{i}, x, 0.000001));
end

bold = zeros(size(A,1),1);

for i = 1 : 3
    bold(urx_bdy{i}(1):urx_bdy{i}(4)) = A4*[zNum{i}(1,:) ...
        dzNum{i}(1,:)*hmax]';
    bold(ury_bdy{i}(1):ury_bdy{i}(4)) = A4*[zNum{i}(2,:) ...
        dzNum{i}(2,:)*hmax]';
    bold(urz_bdy{i}(1):urz_bdy{i}(4)) = A4*[zNum{i}(3,:) ...
        dzNum{i}(3,:)*hmax]';
    bold(uphix_bdy{i}(1):uphix_bdy{i}(4)) = A4*[zNum{i}(4,:) ...
        dzNum{i}(4,:)*hmax]';
    bold(uphiy_bdy{i}(1):uphiy_bdy{i}(4)) = A4*[zNum{i}(5,:) ...
        dzNum{i}(5,:)*hmax]';
    bold(uphiz_bdy{i}(1):uphiz_bdy{i}(4)) = A4*[zNum{i}(6,:) ...
        dzNum{i}(6,:)*hmax]';

    bold(prx_bdy{i}(1):prx_bdy{i}(4)) = A4*[double(subs(fx(i),x01)) ...
        dfxNum{i}*hmax]';
    bold(pry_bdy{i}(1):pry_bdy{i}(4)) = A4*[double(subs(fy(i),x01)) ...
        dfyNum{i}*hmax]';
    bold(prz_bdy{i}(1):prz_bdy{i}(4)) = A4*[double(subs(fz(i),x01)) ...
```

```
        dfzNum{i}*hmax]';
    bold(pphix_bdy{i}(1):pphix_bdy{i}(4)) = A4*[double(subs(mx(i),x0l)) ...
        dmxNum{i}*hmax]';
    bold(pphiy_bdy{i}(1):pphiy_bdy{i}(4)) = A4*[double(subs(my(i),x0l)) ...
        dmyNum{i}*hmax]';
    bold(pphiz_bdy{i}(1):pphiz_bdy{i}(4)) = A4*[double(subs(mz(i),x0l)) ...
        dmzNum{i}*hmax]';
end

b = bold;


% bounday condition
for i = 1 : 3
    A(urx_bdy{i}(1),:) = 0;
    A(urx_bdy{i}(1),urx_bdy{i}(1)) = 1;
    b(urx_bdy{i}(1)) = 0;

    A(ury_bdy{i}(1),:) = 0;
    A(ury_bdy{i}(1),ury_bdy{i}(1)) = 1;
    b(ury_bdy{i}(1)) = 0;

    A(urz_bdy{i}(1),:) = 0;
    A(urz_bdy{i}(1),urz_bdy{i}(1)) = 1;
    b(urz_bdy{i}(1)) = u0(i);

    A(uphix_bdy{i}(1),:) = 0;
    A(uphix_bdy{i}(1),uphix_bdy{i}(1)) = 1;
    b(uphix_bdy{i}(1)) = u0(i);

    A(uphiy_bdy{i}(1),:) = 0;
    A(uphiy_bdy{i}(1),uphiy_bdy{i}(1)) = 1;
    b(uphiy_bdy{i}(1)) = u0(i);

    A(uphiz_bdy{i}(1),:) = 0;
    A(uphiz_bdy{i}(1),uphiz_bdy{i}(1)) = 1;
    b(uphiz_bdy{i}(1)) = u0(i);

    A(prx_bdy{i}(1),:) = 0;
    A(prx_bdy{i}(1),prx_bdy{i}(1)) = 1;
    b(prx_bdy{i}(1)) = 0;

    A(pry_bdy{i}(1),:) = 0;
    A(pry_bdy{i}(1),pry_bdy{i}(1)) = 1;
    b(pry_bdy{i}(1)) = 0;

    A(prz_bdy{i}(1),:) = 0;
    A(prz_bdy{i}(1),prz_bdy{i}(1)) = 1;
    b(prz_bdy{i}(1)) = u0(i);

    A(pphix_bdy{i}(1),:) = 0;
```

```
    A(pphix_bdy{i}(1),pphix_bdy{i}(1)) = 1;
    b(pphix_bdy{i}(1)) = u0(i);

    A(pphiy_bdy{i}(1),:) = 0;
    A(pphiy_bdy{i}(1),pphiy_bdy{i}(1)) = 1;
    b(pphiy_bdy{i}(1)) = u0(i);

    A(pphiz_bdy{i}(1),:) = 0;
    A(pphiz_bdy{i}(1),pphiz_bdy{i}(1)) = 1;
    b(pphiz_bdy{i}(1)) = u0(i);

    A(prz_bdy{i}(1),:) = 0;
    A(prz_bdy{i}(1),prz_bdy{i}(1)) = 1;
    b(prz_bdy{i}(1)) = u0(i);

end

solu = A\b;

for i = 1 : 3
    urx{i} = solu(urx_bdy{i}(1):urx_bdy{i}(2));
    ury{i} = solu(ury_bdy{i}(1):ury_bdy{i}(2));
    urz{i} = solu(urz_bdy{i}(1):urz_bdy{i}(2));
    uphix{i} = solu(uphix_bdy{i}(1):uphix_bdy{i}(2));
    uphiy{i} = solu(uphiy_bdy{i}(1):uphiy_bdy{i}(2));
    uphiz{i} = solu(uphiz_bdy{i}(1):uphiz_bdy{i}(2));

    prx{i} = solu(prx_bdy{i}(1):prx_bdy{i}(2));
    pry{i} = solu(pry_bdy{i}(1):pry_bdy{i}(2));
    prz{i} = solu(prz_bdy{i}(1):prz_bdy{i}(2));
    pphix{i} = solu(pphix_bdy{i}(1):pphix_bdy{i}(2));
    pphiy{i} = solu(pphiy_bdy{i}(1):pphiy_bdy{i}(2));
    pphiz{i} = solu(pphiz_bdy{i}(1):pphiz_bdy{i}(2));

    vrx{i} = prx{i} + subs(fx(i),x01)';
    vry{i} = pry{i} + subs(fy(i),x01)';
    vrz{i} = prz{i} + subs(fz(i),x01)';
    vphix{i} = pphix{i} + subs(mx(i),x01)';
    vphiy{i} = pphiy{i} + subs(my(i),x01)';
    vphiz{i} = pphiz{i} + subs(mz(i),x01)';
end

J = 0;

for i = 1 : 3
    tmp = norm(urx{i}-double(subs(z{i}(1),x01))')^2*hmax;
    J = J + tmp;
    tmp = norm(ury{i}-double(subs(z{i}(2),x01))')^2*hmax;
    J = J + tmp;
    tmp = norm(urz{i}-double(subs(z{i}(3),x01))')^2*hmax;
    J = J + tmp;
```

```
    tmp = norm(uphix{i}-double(subs(z{i}(4),x01))')^2*hmax;
    J = J + tmp;
    tmp = norm(uphiy{i}-double(subs(z{i}(5),x01))')^2*hmax;
    J = J + tmp;
    tmp = norm(uphiz{i}-double(subs(z{i}(6),x01))')^2*hmax;
    J = J + tmp;
    tmp = norm(prx{i})^2*hmax+norm(pry{i})^2*hmax+norm(prz{i})^2*hmax;
    J = J + tmp;
    tmp = ...
        norm(pphix{i})^2*hmax+norm(pphiy{i})^2*hmax+norm(pphiz{i})^2*hmax;
    J = J + tmp;
end

J = J/2;
end

function Ei1 = E(t)
    Ei1 = [cos(t) sin(t) 0;
          -sin(t) cos(t) 0;
           0 0 1];
end
```

A.3 Wave Equation

In this section, we delve into the application of dynamic optimization techniques to the wave equation. Specifically, we explore example 1 in section 5.4.1, focusing on optimizing boundary control while verifying the Turnpike property.

```
% Example 1
% y0 = x;y1=0;

% Function to compute convergence using quadprog
% T: Time period
% J/T: Convergence ratio

function J = Example1(T)

% Set default value for T if not provided
if nargin == 0
    T = 10;
end

% Parameter and time step
gamma = 0.1;
dt = 0.1;

% Number of time steps
M = T/dt;
```

```
% Number of expandsion terms
N = 20;

% Gauss point and weights
s_ref = [-sqrt(3/5),0,sqrt(3/5)];
wt = [5,8,5]/9;

% Function handle for alpha0(x, n) = y0(x) * phin(x, n)
alpha0 = @(x, n) y0(x) .* phin(x,n);

% //////// V sin /////////
VM = cell(M,1);
Vk = cell(M,1);

VMint = zeros(M,M);
Vkint = zeros(M,1);

tic;
for k = 1 : M
   tk1 = dt*(k-1);
   tk = dt*k;
   VM{k} = @(t,n) ...
       1/(pi/2+n*pi)*(cos((pi/2+n*pi)*(t-tk))-cos((pi/2+n*pi)*(t-tk1)));
   Vk{k} = @(t,n) 1/(pi/2+n*pi)*(1-cos((pi/2+n*pi)*(t-tk1)));
end
elapsed_time = toc;
fprintf('Assembly of VMsin code block execution time: %.4f s\n', ...
    elapsed_time);

% //////// V cos /////////
VMc = cell(M,1);
Vkc = cell(M,1);

VMcint = zeros(M,M);
Vkcint = zeros(M,1);

tic;
for k = 1 : M
   tk1 = dt*(k-1);
   tk = dt*k;
   VMc{k} = @(t,n) ...
       1/(pi/2+n*pi)*(-sin((pi/2+n*pi)*(t-tk))+sin((pi/2+n*pi)*(t-tk1)));
   Vkc{k} = @(t,n) 1/(pi/2+n*pi)*(sin((pi/2+n*pi)*(t-tk1)));
end
elapsed_time = toc;
fprintf('Assembly of VMcos code block execution time: %.4f s\n', ...
    elapsed_time);

%% ============= q ===============
% ///// q1 /////
```

```
tic;

for j = 1 : M
   tj1 = dt*(j-1);
   tj = dt*j;
   t = (tj1+tj)/2;
   for n = 0 : N
       alpha0int = integral(@(x) alpha0(x, n), 0, 1);
       for k = 1 : j-1
          tmp = (-1)^n*2*sqrt(2)*alpha0int/(pi/2+n*pi)
          *cos((pi/2+n*pi)*(t)).*VM{k}(t,n)*dt;
          VMint(k,j) = VMint(k,j) + tmp;
       end
         tmp1 = (-1)^n*2*sqrt(2)*alpha0int/(pi/2+n*pi)
         *cos((pi/2+n*pi)*(t)).*Vk{j}(t,n)*dt;
       Vkint(j) = Vkint(j) + tmp1;
       VMint(j,j) = Vkint(j);
   end
end
elapsed_time = toc;
fprintf('Assembly of q1 sub-vector code block execution time: %.4f ...
    s\n', elapsed_time);

tic;
q1 = sum(VMint,2);
elapsed_time = toc;
fprintf('Generation of q1 code block execution time: %.4f s\n', ...
    elapsed_time);

% ///// q2 /////
tic;
for j = 1 : M
   tj1 = dt*(j-1);
   tj = dt*j;
   s = (tj-tj1)/2*s_ref + (tj+tj1)/2;
   jac = (tj-tj1)/2;
   for n = 0 : N
       alpha0int = integral(@(x) alpha0(x, n), 0, 1);
       for k = 1 : j-1
            tmp = (-1)^(n+1)*2*sqrt(2)*alpha0int*(pi/2+n*pi)
            *sum(wt.*sin((pi/2+n*pi)*(s)).*VMc{k}(s,n))*jac;
            VMcint(k,j) = VMcint(k,j) + tmp;
       end
       tmp1 = (-1)^(n+1)*2*sqrt(2)*alpha0int*(pi/2+n*pi)
       *sum(wt.*sin((pi/2+n*pi)*(s)).*Vkc{j}(s,n))*jac;
       Vkcint(j) = Vkcint(j) + tmp1;
       VMcint(j,j) = Vkcint(j);
   end
end
elapsed_time = toc;
fprintf('Assembly of q2 sub-vector code block execution time: %.4f ...
```

```
    s\n', elapsed_time);

tic;
q2 = sum(VMcint,2);
elapsed_time = toc;
fprintf('Generation of q2 code block execution time: %.4f s\n', ...
    elapsed_time);


%% ============ Q ============
% ///// Q1 /////
tic;
Q1 = zeros(M,M);

tic

for k = 1 : M
   tk1 = (k-1)*dt;
   tk = k*dt;
   s = (tk-tk1)/2*s_ref + (tk+tk1)/2;
   jac = (tk-tk1)/2;
   Qeint = zeros(M,M);
   tic
   for i = 1:k-1
      for j = 1:i-1
         for n = 0 : N
            tmp2 = 2/(pi/2+n*pi)^2*sum(wt.*VM{i}(s,n)*(VM{j}(s,n)'))*jac;
            Qeint(i, j) = Qeint(i, j) + tmp2;
         end
         Qeint(j, i) = Qeint(i, j);
      end
      for n = 0 : N
         tmp = 2/(pi/2+n*pi)^2*sum(wt.*VM{k}(s,n)*(VM{k}(s,n)'))*jac;
         Qeint(i, i) = Qeint(i, i) + tmp;
      end
   end
   for n = 0 : N
      tmp = 2/(pi/2+n*pi)^2*sum(wt.*Vk{k}(s,n)*(Vk{k}(s,n)'))*jac;
      Qeint(k, k) = Qeint(k, k) + tmp;
   end
   Q1 = Q1 + Qeint;

   disp(['k = ', num2str(k)]);
   elapsed_time = toc;
   fprintf('Assembly of Qe{k} integral code block execution time: ...
      %.4f s\n', elapsed_time);
end
elapsed_time = toc;
fprintf('Assembly of Qeint{k} integral code block execution time: ...
    %.4f s\n', elapsed_time);

```

```
% ///// Q2 /////
tic;

Q2 = zeros(M,M);

tic
for k = 1 : M
   tk1 = (k-1)*dt;
   tk = k*dt;
   s = (tk-tk1)/2*s_ref + (tk+tk1)/2;
   jac = (tk-tk1)/2;
   Qeint2 = zeros(M,M);
   tic
   for i = 1:k-1
      for j = 1:i-1
         for n = 0 : N
            tmp1 = 2*sum(wt.*VMc{i}(s,n)*(VMc{j}(s,n)'))*jac;
            Qeint2(i, j) = Qeint2(i, j) + tmp1;
         end
         Qeint2(j, i) = Qeint2(i, j);
      end
      for n = 0 : N
         tmp = 2*sum(wt.*VMc{k}(s,n)*(VMc{k}(s,n)'))*jac;
         Qeint2(i, i) = Qeint2(i, i) + tmp;
      end
   end
   for n = 0 : N
      tmp = 2*sum(wt.*Vkc{k}(s,n)*(Vkc{k}(s,n)'))*jac;
      Qeint2(k, k) = Qeint2(k, k) + tmp;
   end
   Q2 = Q2 + Qeint2;

   disp(['k = ', num2str(k)]);
   elapsed_time = toc;
   fprintf('Assembly of Qe2{k} integral code block execution time: ...
      %.4f s\n', elapsed_time);
end
elapsed_time = toc;
fprintf('Assembly of Qeint{k} integral code block execution time: ...
   %.4f s\n', elapsed_time);

% ////// Q3 //////
Q3 = dt*eye(M);

%% ==== W =====
% Calculate W1
W1 = 0;
for k = 1 : M
   tk1 = dt*(k-1);
   tk = dt*k;
     s = (tk-tk1)/2*s_ref + (tk+tk1)/2;
```

```
    jac = (tk-tk1)/2;
    for n = 0 : N
        alpha0int = integral(@(x) alpha0(x, n), 0, 1);
        tmp = alpha0int^2*sum(wt.*cos((pi/2+n*pi)*(s)).^2)*jac;
        W1 = W1 + tmp;
    end
end

% Calculate W2
W2 = 0;
for k = 1 : M
    tk1 = dt*(k-1);
    tk = dt*k;
      s = (tk-tk1)/2*s_ref + (tk+tk1)/2;
    jac = (tk-tk1)/2;
    for n = 0 : N
        alpha0int = integral(@(x) alpha0(x, n), 0, 1);
        tmp = ...
            (pi/2+n*pi)^2*alpha0int^2*sum(wt.*sin((pi/2+n*pi)*(s)).^2)*jac;
        W2 = W2 + tmp;
    end
end

% Formulate matrices Q, q, and W
Q = Q3 + Q1 + gamma*Q2;
q = q1 + gamma*q2;
W = W1 + gamma*W2;

% Solve quadratic optimization problem
U = quadprog(2*Q,q);

% Plot control w.r.t time
t = 0:dt:T;
figure
plot(t(1:end-1),U,"LineWidth",1)
xlabel('t', 'FontSize', 14);
ylabel('control', 'FontSize', 14);
filename = strcat('u_', num2str(T), '.mat');
save(filename,"U","t");

% Compute cost function
J = U'*Q*U + q'*U + W;

J = J/2;
ratio = J/T;

% Display results
fprintf(['T= ' num2str(T), ' J= ' num2str(J), ' Ratio=' ...
    num2str(ratio) '\n']);
end
```

```
function y = y0(x)
   y = x;
end

function y = y1(x)
   y = 0*x;
end

function phi = phin(x,n)
   phi = sqrt(2).* sin((pi/2 + n * pi) .* x);
end
```

This is the code for the optimal control problems for one cycle.

```
function [J,u,y1r,y2r,y3r,z1r,z2r,z3r] = OptControl(eps)
% Check if input argument 'eps' is provided, otherwise set default value
if nargin == 0
   eps = 0.5;
end

%% ==== setting Parameters ===========
eps0 = 0.5;
epsmax = 1;

% The length of each edge.
L = 2;
Leps = [L-eps,L-eps,L-eps,eps,eps,eps];
Lopt = [L-eps0,L-eps0,L-eps0,eps0,eps0,eps0];
hmax = 0.01;

% Calculate number of intervals
l = L-epsmax;
v = 1;
N0l = ceil(l/hmax);
Nlz = ceil((epsmax-eps0)/hmax);
Nly = ceil((epsmax-eps)/hmax);

% Generate node coordinates
x0l = linspace(0,L-epsmax,N0l+1);
xlz = linspace(L-epsmax,L-eps0,Nlz+1);
xly = linspace(L-epsmax,L-eps,Nly+1);

xlz(1) = [];
xly(1) = [];

% node coordinates
xz = [x0l xlz];
xy = [x0l xly];
% cell sizes
hz = diff(xz);
hy = diff(xy);
```

```
% number of cells
Nxy = length(hy);
Nxz = length(hz);

% cell indices for objective functional
E = 1:N0l+1;

% Parameters for small cycle
Neps = 10;
heps = eps/Neps;
hepsOpt = eps0/Neps;
xepsopt= linspace(0,eps0,Neps+1);

% number of basis functions
Nfy = 2*(N0l+1);
Nfz = 2*(Nxz+1);
Nfeps = 2*(Neps+1);

%% compuet z
Nz = [length(hz),length(hz),length(hz),Neps,Neps,Neps];
z = cell(6,1);
for i = 1 : 6
   z{i} = zeros(2*Nz(i)+2,1);
end

% Solve equations for coefficients of z
syms a2z a3z a4z a5z a6z b1z b4z b5z b6z
eq1 = b1z - b5z == -v*Lopt(1);
eq2 = a4z*eps0 +b4z -b5z == 0;
eq3 = v + a4z -a5z == 0;
eq4 = a5z*eps0 + b5z -b6z == 0;
eq5 = a2z*Lopt(2) - b6z == 0;
eq6 = a2z +a5z -a6z == 0;
eq7 = b4z -a6z*eps0 -b6z == 0;
eq8 = a3z*Lopt(3) - b4z == 0;
eq9 = a3z + a6z -a4z == 0;
solu = solve(eq1,eq2,eq3,eq4,eq5,eq6,eq7,eq8,eq9);

az = [v; solu.a2z; solu.a3z; solu.a4z; solu.a5z; solu.a6z];
Bz = [solu.b1z; 0; 0; solu.b4z; solu.b5z; solu.b6z; ];

for i = 1 : 3
   z{i}(1:Nz(i)+1) = az(i)*xz + Bz(i);
   z{i}(Nz(i)+2:2*Nz(i)+2) = az(i)*hmax;
end

for i = 4 : 6
   z{i}(1:Nz(i)+1) = az(i)*xepsopt + Bz(i);
   z{i}(Nz(i)+2:2*Nz(i)+2) = az(i)*hepsOpt;
end

```

```
%% boundary condition
Nf = [repmat(Nfy,1,6)];
index = cumsum(Nf);
index = [0 index];
for i = 1 : 3
    % index of [y(0),y(L),y'(0),y'(L)];
    bdy{i} = index(2*i-1)+[1,Nf(2*i-1)/2,Nf(2*i-1)/2+1,Nf(2*i-1)];
    % index of [p(0),p(L),p'(0),p'(L)];
    bdp{i} = index(2*i)+[1,Nf(2*i)/2,Nf(2*i)/2+1,Nf(2*i)];
end

%% FEM matrices
[As,Am] = graph1D_matirx_Hermite(N0l,x0l);

As_array = repmat({As}, 1, 6);
A = blkdiag(As_array{:});

A(Nfy+1:2*Nfy,1:Nfy) = Am;
A(3*Nfy+1:4*Nfy,2*Nfy+1:3*Nfy) = Am;
A(5*Nfy+1:6*Nfy,4*Nfy+1:5*Nfy) = Am;

A(1,Nfy+1) = A(1,Nfy+1)-1;

A(bdy{1}(2),bdy{1}(2)) = A(bdy{1}(2),bdy{1}(2)) - 2/(2*eps-3);
A(bdy{1}(2),bdy{2}(2)) = A(bdy{1}(2),bdy{2}(2)) + 1/(2*eps-3);
A(bdy{1}(2),bdy{3}(2)) = A(bdy{1}(2),bdy{3}(2)) + 1/(2*eps-3);

A(bdy{2}(2),bdy{1}(2)) = A(bdy{2}(2),bdy{1}(2)) + 1/(2*eps-3);
A(bdy{2}(2),bdy{2}(2)) = A(bdy{2}(2),bdy{2}(2)) - 2/(2*eps-3);
A(bdy{2}(2),bdy{3}(2)) = A(bdy{2}(2),bdy{3}(2)) + 1/(2*eps-3);

A(bdy{3}(2),bdy{1}(2)) = A(bdy{3}(2),bdy{1}(2)) + 1/(2*eps-3);
A(bdy{3}(2),bdy{2}(2)) = A(bdy{3}(2),bdy{2}(2)) + 1/(2*eps-3);
A(bdy{3}(2),bdy{3}(2)) = A(bdy{3}(2),bdy{3}(2)) - 2/(2*eps-3);

A(bdp{1}(2),bdp{1}(2)) = A(bdp{1}(2),bdp{1}(2)) - 2/(2*eps-3);
A(bdp{1}(2),bdp{2}(2)) = A(bdp{1}(2),bdp{2}(2)) + 1/(2*eps-3);
A(bdp{1}(2),bdp{3}(2)) = A(bdp{1}(2),bdp{3}(2)) + 1/(2*eps-3);

A(bdp{2}(2),bdp{1}(2)) = A(bdp{2}(2),bdp{1}(2)) + 1/(2*eps-3);
A(bdp{2}(2),bdp{2}(2)) = A(bdp{2}(2),bdp{2}(2)) - 2/(2*eps-3);
A(bdp{2}(2),bdp{3}(2)) = A(bdp{2}(2),bdp{3}(2)) + 1/(2*eps-3);

A(bdp{3}(2),bdp{1}(2)) = A(bdp{3}(2),bdp{1}(2)) + 1/(2*eps-3);
A(bdp{3}(2),bdp{2}(2)) = A(bdp{3}(2),bdp{2}(2)) + 1/(2*eps-3);
A(bdp{3}(2),bdp{3}(2)) = A(bdp{3}(2),bdp{3}(2)) - 2/(2*eps-3);

b = zeros(size(A,1),1);
b(1) = b(1)-v;
b(Nfy+1:2*Nfy) = Am*z{1}([E,E+Nfz/2]);
b(3*Nfy+1:4*Nfy) = Am*z{2}([E,E+Nfz/2]);
```

```
b(5*Nfy+1:6*Nfy) = Am*z{3}([E,E+Nfz/2]);

%% boundary condition
% y2(0) = 0;
A(bdy{2}(1),:) = 0;
A(bdy{2}(1),bdy{2}(1)) = 1;
b(bdy{2}(1)) = 0;

% y3(0) = 0;
A(bdy{3}(1),:) = 0;
A(bdy{3}(1),bdy{3}(1)) = 1;
b(bdy{3}(1)) = 0;

% p2(0) = 0;
A(bdp{2}(1),:) = 0;
A(bdp{2}(1),bdp{2}(1)) = 1;
b(bdp{2}(1)) = 0;

% p3(0) = 0;
A(bdp{3}(1),:) = 0;
A(bdp{3}(1),bdp{3}(1)) = 1;
b(bdp{3}(1)) = 0;

% Solve linear system
solu = A\b;

for i = 1 : 3
   y{i} = solu(bdy{i}(1):bdy{i}(2));
   p{i} = solu(bdp{i}(1):bdp{i}(2));
   dy{i} = solu(bdy{i}(3):bdy{i}(4))/hmax;
   dp{i} = solu(bdp{i}(3):bdp{i}(4))/hmax;
end

% Compute objective function J
J = 0;
for i = 1 : 3
   tmp = norm(y{i}(E)-z{i}(E))^2*hmax;
   J = J + tmp;
end

% Compute control variable u
u = v - p{1}(1)
J = J + (u-v)^2;
J = J/2

y1r = y{1}(E);
y2r = y{2}(E);
y3r = y{3}(E);

z1r = z{1}(E);
z2r = z{2}(E);
```

```
z3r = z{3}(E);
end
```