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Relaxation and Optimal Control of Age- and Space-structured Epidemic Models

submitted by
Nicolas A. Schlosser

at the

Universität Konstanz

Mathematisch-Naturwissenschaftliche Sektion
Fachbereich Mathematik und Statistik

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Vorsitzender der Prüfungskommission: Prof. Dr. Robert Denk
Gutachter und mündliche Prüfer: Prof. Dr. Reinhard Racke
Jun.-Prof. Dr. Behzad Azmi

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Contents

1 Introduction

The discipline of mathematical epidemiology is over a hundred years old, and the recent Covid-19 epidemics has shown that it can have a great impact on everybody's life. Over time, a vast number of models has been developed, surpassing each other in complexity. The earliest example is the *SIR* model

$$\begin{aligned} \dot{S} &= -kIS, \\ \dot{I} &= kIS - \gamma I, \\ \dot{R} &= \gamma I, \end{aligned} \tag{SIR}$$

introduced in [KM91], where S is the number of *susceptible* individuals, who can possibly get infected, I is the *infected* or *infectious* population and R the *recovered* or *removed* population, the removal being by immunity or death by infection. The positive constants k and γ are the infection rate and the recovery rate, respectively. Starting with [McK25], [SL11] and later continued by [Web08], [Ian95] and many more, further variables have been added to epidemic models in order to increase their accuracy and applicability to real-world problems.

In real-world applications, infections behave differently on individuals with varying ages. For example, during the recent Covid-19 pandemic, elder individuals were at much higher risk to die of the infection than younger ones (cf. [KT20]). This suggests that including an age variable in epidemic models might produce more realistic results and can show new effects. Another interesting addition are variables for the location of individuals. This allows to model the spread of an infection over a country or a continent. Also, if one wishes to contain the epidemic by means of e.g. vaccinations, the inclusion of age and space allow for different vaccination strategies for any age group as well as for the vaccine to be only administered in certain areas. In most cases, movement of the population through space is modeled by diffusion, which is captured in the governing equations with a term containing the Laplace operator. Considering more than only time as an independent variable in a model generally leads to partial differential equations (PDEs for short).

While PDE models are able to capture reality with great precision, their mathematical analysis often poses difficulties. The main obstacles in the context of epidemic modeling with age and space structure are as follows.

- As can already be seen in eq. (SIR), equations that model a disease typically are nonlinear. This is particularly true for those terms in the model that capture the infection process, since there is more than one party involved in that process, one infectious and one susceptible. Also, infection can occur across age and space, meaning that susceptible individuals can catch the disease from infectious individuals no matter their individual age, and the probability that an infection will occur typically decreases with the spatial distance between susceptible and infective individual. This necessitates the use of *nonlocal* terms, typically involving integration over a kernel that models the infection rates between various ages and locations.

- In age-dependent models, in addition to the epidemic processes, one also has to account for the process of birth. Given a birth rate, which may depend on age as well, the offspring generated from the population can be calculated with a simple integral expression. Since newborns have age zero, the resulting equation can be interpreted as a boundary condition for the model, and since the value on the boundary depends on the state of the population inside the age domain, this boundary condition is of *implicit* nature.

- The inclusion of both age and space in a model further has the effect that the resulting model does not fall into well-established classes of PDEs, namely, it is neither parabolic nor hyperbolic, in a strict sense. This prevents the use of classical existence results, rather one has to restrict the equation onto sets (the so-called *characteristics*) where a unique solution can be guaranteed, and then, in a second step, reassemble the various solutions on characteristics to a solution on the whole domain.

After some short preliminaries in Chapter 2, we discuss the general type of epidemic models in Chapter 3. If we assume that the population moves in a spatial domain Ω and has a maximal age of $a_{\max}$, most epidemic models with age and space structure can be written in the form

$$(\partial_t + \partial_a)y + L(a,x)y + \Lambda(a,x,y)y + K(u)y = \sigma(a)\Delta y, \tag{1.1a}$$

$$y(t=0) = y_0, \quad \partial_\nu y(x \in \partial\Omega) = 0, \tag{1.1b}$$

$$y(a=0) = \int_0^{a_{\max}} \beta(\alpha,x)y(t,\alpha,x)\,\mathrm{d}\alpha, \tag{1.1c}$$

where t is time, $a \in [0, a_{\max}]$ represents age, and $x \in \Omega$ is the space variable. Further, $y \in \mathbb{R}^n$ is a vector consisting of the n compartments of the model, L gathers the linear terms, $\Lambda(y)$ the nonlinear nonlocal terms that, for example, capture infection processes, and $K(u)$ the terms that depend on an input control u, for example a vaccine. The matrix $\Lambda(y) \in \mathbb{R}^{n\times n}$ is defined in such a way that every entry $\Lambda(y)^{hi}$ (h, $i = 1,\dots,n$) can be obtained by integrating y over a kernel k^{hi}, that is to say

$$\Lambda(a,x,y)^{hi} = \int_0^{a_{\max}} \int_\Omega k^{hij}(a,x,\alpha,\xi)y_j(\alpha,\xi)\,\mathrm{d}\xi\,\mathrm{d}\alpha = \left\langle k^{hi}(a,x,\cdot,\cdot), y \right\rangle_{L^2((0,a_{\max})\times\Omega)^n}, \tag{1.1d}$$

where we use Einstein's summation convention over the index j. Here, the function $k^{hij}(a,x,\alpha,\xi)$ describes how many susceptible individuals from compartment y_i having age a and being located at position x are infected from infectives y_j with age α and location ξ and subsequently transition into the infected class y_h. This formulation allows the inclusion of multiple classes of susceptible and infective compartments, which allows the model to be applied to many different infectious diseases. Furthermore, $\beta(\alpha,x) \in \mathbb{R}^{n\times n}$ is a matrix of birth rates, the entry β^{hi} describing the contribution of compartment y_i to the newborn individuals in compartment y_h. Thus, the model can capture things such as vertical transmission (i.e. infection from a mother to her offspring at birth) or the infection having an impact on fertility. To sum up, a wide range of epidemiological phenomena can be covered by our general model.

Models of this or a related kind are abundantly found in the literature: [BvdDW08], [LYM20], [Per15] and [Web08] offer a comprehensible introduction to the topic of biological models with age and spatial diffusion. The study [WZK21] examines a space-structured model with Dirichlet boundary conditions. The works [CGMR20] discusses nonlocal models with age and space first-order transport-equation-like terms, and [KO15] discusses a similar model with diffusion terms. In [Wal23], a model incorporating both spatial structure and infection age is studied, where only the infected population depends on the age variable. [BDKW23] and [KR21] address age-structured models that include nonlocal diffusion terms to account for long-distance travel. [Fra05] considers linear age- and space-dependent population models where an additional delay term is present in the birth equation. However, the general question of existence and uniqueness for models with age and space structure and nonlocal terms has not been addressed yet. Building on the thesis [Sch21] and similar to [AS25], in Chapter 3 we introduce the concept of *weak solutions* to the model (1.1), and after some intermediate steps prove the first main result, Theorem 3.20, which shows the existence and uniqueness of weak solutions to our general model class, thus providing a mathematical base for many epidemiological models from past and future. The results in this chapter are a blueprint to Chapter 4, which in some parts is similar in structure, and are needed again in Chapter 5, where we analyze optimal control questions where the state is governed by the model.

However, diffusion is not the only possible way to model the movement of a population through a spatial domain, and it might not be suitable for every situation. This is mostly due to an inherent property of diffusion models called *infinite propagation speed*, which can be stated as follows: Consider a country with a population where at some fixed time, infectious individuals are only present in one city. If the infection behaved like a diffusion process, then at any later time, be it the mere fraction of a second, the percentage of infectious individuals would be nonzero all over the country. In other words, the disease has spread from the city it initially was contained in over the whole country infinitely fast. While this observation does not completely invalidate diffusion models (the share of infectious individuals far away from the original disease hotspot might be insignificantly small), it motivates the search for alternatives. One possibility, proposed by [Cat48] and [Max67], is the so-called *relaxation* of the model. The starting point for this technique is to introduce a formal delay time τ to the model, which prevents an infinite propagation speed since due to the delay, the model "needs some time" to react. Since time and age elapse in the same way, in our case we have to introduce the delay τ in both the time and the age variables, which turns eq. (1.1a) without control terms u into

$$\begin{aligned}(\partial_t + \partial_a)y(t+\tau, a+\tau, x) + L(a+\tau, x)y(t+\tau, a+\tau, x)& \\ +\Lambda(a+\tau, x, y)y(t+\tau, a+\tau, x) = \sigma(a)\Delta y(t, a, x).&\end{aligned} \tag{1.2}$$

Unfortunately, as shown in [DQR09] or [Rac12], delay models can be ill-posed. However, one can get around this by approximating the delay by means of a formal Taylor expansion. The resulting equation is no longer a diffusion-type equation, it is rather a damped wave equation, sometimes also called telegrapher's equation. Applying a formal Taylor expansion to eq. (1.2) yields

$$\begin{aligned}\Big(1 + \tau(\partial_t + \partial_a)\Big)\Big((\partial_t + \partial_a)y(t, a, x) + L(a, x)y(t, a, x)& \\ +\Lambda(a, x, y)y(t, a, x)\Big) = \sigma(a)\Delta y(t, a, x)&\end{aligned} \tag{1.3}$$

So far, age-dependent wave equations have not been used for epidemic modeling, and a main goal of this work is to establish an extensive theory of relaxed epidemic models with age and space structure. This work is done in Chapter 4. An additional difficulty of relaxed models with age structure, apart from their regularity which in general is worse than that of unrelaxed ones, is the need for an additional boundary condition for newborns. After a rigorous but somewhat classical motivation of eq. (1.3), and inspired by the works [AP18], [BP21] and [Hol93] which develop similar models for a one-dimensional space domain, we derive the relaxed model from eq. (1.3) in a different way, which allows us to deduce such a necessary condition. (In fact, the relaxed equation turns out to be an approximation of a bigger system of equations, which we discuss at length in Section 4.8.) We find that in addition to eq. (1.1b) and eq. (1.1c), we have to equip a relaxed epidemic model with the additional first-order initial condition

$$(\partial_t + \partial_a)y(t = 0) = y_1$$

and either also explicitly prescribe the value of $(\partial_t + \partial_a)y$ at $a = 0$, calculate it implicitly by

$$P(a = 0) = \int_0^{a_{\max}} \beta_1(\alpha)P(\alpha)\,\mathrm{d}\alpha \tag{1.4}$$

where we let

$$P(t,a,x) = (\partial_t + \partial_a)y(t,a,x) + L(a,x)y(t,a,x) + \Lambda(a,x,y)y(t,a,x), \tag{1.5}$$

or use a combination of the two. To achieve even greater generality, we also allow the addition of an explicit term in eq. (1.1c). Using the shorthand $\delta := \partial_t + \partial_a$, we arrive at the complete relaxed model

$$\begin{gathered}
(1 + \tau\delta)(\delta y + Ly + \Lambda(y)y) = \sigma(a)\Delta y, \\
y(t = 0) = y_0, \quad \delta y(t = 0) = y_1, \quad \partial_\nu y(x \in \partial\Omega) = 0, \\
y(t, a = 0, x) = \int_0^{a_{\max}} \beta_0(\alpha, x)y(t, \alpha, x)\,\mathrm{d}\alpha + g_0(t, x), \\
(\delta y + Ly + \Lambda(y)y)(t, a = 0, x) = \int_0^{a_{\max}} \beta_1(\alpha, x)(\delta y + Ly + \Lambda(y)y)(t, \alpha, x)\,\mathrm{d}\alpha + g_1(t, x).
\end{gathered} \tag{1.6}$$

By adapting the strategy from Chapter 3, we are able to show, in Theorem 4.20, the existence and uniqueness of weak solutions to the system (1.6).

Considering the new complexities that arise with the relaxation, one might wonder if the procedure is really worth the effort. Questions of this kind are also addressed in Chapter 4. In Theorem 4.22 and Theorem 4.23, we show that the smaller the relaxation parameter τ is, the closer solutions of the relaxed model (1.6) are to solutions of the unrelaxed one from eq. (1.1). This theoretical observation is further supported by numerical calculations, illustrated in Figure 4 which shows the expected behavior. From Figure 4 we can also infer that the relaxation of the model does in fact prevent infinite propagation speed, and it can clearly be seen that the infection travels through a spatial domain with a finite velocity. Thus, relaxed models can provide an alternative, perhaps more suitable way to model real-world epidemics.

In the case of an epidemic, researchers and government agencies not only wish to model the spread of a disease, but rather take active control measures in order to reduce the severity of the epidemic and to prevent hospitals from overcrowding and people from dying. Of course, given a control strategy u, its effect can be incorporated into existing models and its outcome can be studied, cf. the term $K(u)$ in eq. (1.1). The question that arises naturally is what control strategy u to choose for an optimal outcome. Mathematically speaking, the state y of an epidemic can be interpreted as a function $y(u)$ depending on the input control u, and the objective is to minimize a given target functional $J = J(u,y)$ that may depend on both the control and the state. For example, a target functional for eq. (1.1a) might take the form

$$J(u,y) = \frac{1}{2}\int_0^T \int_0^{a_{\max}} \int_\Omega \left|g \cdot y(t,a,x)\right|^2 \mathrm{d}x\,\mathrm{d}a\,\mathrm{d}t + \frac{\alpha}{2}\int_0^T \int_0^{a_{\max}} \int_\Omega |u(t,a,x)|^2 \mathrm{d}x\,\mathrm{d}a\,\mathrm{d}t. \tag{1.7}$$

Here, the first term represents number of infective individuals in the state vector y, weighted by g. For example, if $y = (S,I,R)^\mathsf{T}$ as in eq. (SIR), one can choose $g = (0,1,0)^\mathsf{T}$ to only minimize the number of infected individuals. The second term, involving u, represents the costs attached to our control process, e.g. the costs of producing vaccines and maintain vaccination centers. In applications, there are further constraints on the control, we can for example impose box constraints

$$0 \le u(t,a,x) \le \bar{u} \quad \text{for all } t,\ a,\ x, \tag{1.8}$$

where $\bar{u}$ is an upper bound the control is not allowed to exceed. Now the question of existence of an optimal control problem of the form

$$\text{Find } \min_u J(u,y) \quad \text{where} \quad (y,u) \text{ satisfies eq. (1.1) and eq. (1.8)} \tag{1.9}$$

can be answered mathematically. Given the recent Covid-19 pandemics, this topic has been frequently addressed in the literature: The works [SM17] and [YB12] discuss the optimal control of epidemic models formulated using ordinary differential equations, and [CF25] considers optimal control of an age-structured model. The paper [AW23] discusses an age-structured Covid-19 model and addresses the question which age group to prefer when the supply of the vaccine is limited. The book [AAC11] provides numerous examples of optimal control problems, including results specific to age-structured population models. Additionally, the articles [ARS25, ACG⁺23, BPW24, CGMR24, ZXL19] investigate optimal control problems in reaction-diffusion epidemic models that incorporate spatial structure but not age structure.

However, existence of an optimal control of an epidemic model where both age and space structure is present has not been established yet. In Chapter 5 we consider the problem eq. (1.9) and present two existence results, Theorem 5.1 and Remark 5.2. It turns out that the presence of both variables poses a huge challenge to the existence proofs. This is due to the fact that solutions of the PDE model are not regular enough for well-known compactness results to hold. Our results present two ways around this obstacle. The first one, applied in the proof of Theorem 5.1 is to assume that the control has a certain form, namely fixed age groups and locations where the vaccine is administered. This approach, in addition to being a more realistic type of control, reduces the complexity of the problem and allows an existence proof using the direct method of variational calculus. The second approach, discussed at length in Remark 5.2, is to use different forms of control and assuming that infections also occur nonlocal in time (i.e. happen via germs in the environment). Under these assumptions, compactness results are not necessary, therefore existence of an optimal control can be shown again. The optimal control obtained in Theorem 5.1 can furthermore be characterized by first-order optimality conditions, a result we present in Corollary 5.7.

We finish the chapter by performing explicit computations on a simple example of our model. After giving an explicit numerical scheme and explaining the algorithm, we present our results and provide some interpretation. The results we present in Example 5.8 suggest that in a very simple variant of eq. (SIR) equipped with a compartment for vaccinated individuals, it is beneficial to prefer vaccinating young individuals over older ones, and to start the vaccination process early in order to avoid a huge outbreak of the disease. After altering the maximal vaccination rate in Example 5.9, we also can infer that the more individuals can be vaccinated simultaneously, the lower the target functional can get. This suggests that it might be worth the effort to maintain a higher rate of vaccinations, even if there are costs attached to that.

Our numerical experiments show that further investigation of optimal control theory in the context of age- and space-structured epidemic modeling might give helpful results that could be used in future pandemics. Further research could be, for example:

- Testing more complex models with real-world data, and compare the optimal results with actual control strategies.
- Include even more variables in the model, for example so-called *class-age*, which tracks how long individuals remain in a compartment. Class age can for example be used to account for incubation times or gestation periods.
- To obtain a long-lasting effect, so called *receding horizon control* can be employed, as described in [AK16]. This technique can yield controls that cover any time span, no matter how large, and could be used to actually eradicate the disease by driving the model into a disease-free equilibrium.
- The question of long-time existence for the general unrelaxed and relaxed models with nonlocal nonlinearities is open.
- Relaxed models can show undesired properties, such as nonpositivity of the solutions. However, in our simulations such an effect could not be observed, so the question remains if and under which circumstances the solution can become negative.
- Instead of equipping the relaxed model with two boundary conditions for the newborns, it might be a good idea to prescribe one condition for newborns and one for individuals that have reached the maximum age and are no longer considered by the model. This is both motivated from reality and from the unrelaxed model, where in simple cases it can be shown that, under certain assumptions on the mortality rate, the population at maximum age is zero. It is not clear if one can always choose a boundary condition for a relaxed model in such a way that its population attains a desired state.
- In Chapter 4, the wave-like model is shown to be an approximation of a bigger model. This bigger model is not fully understood, in particular when it comes to the exact formulation of boundary conditions. In addition, one could consider better approximations of this model and analyze their properties.

- Finally, so far we only considered optimal control of unrelaxed models. It is interesting to see if similar results to those from Chapter 5 also hold if the state equation is governed by the class of relaxed models we considered in Chapter 4, and if optimal controls of relaxed models converge to optimal controls of unrelaxed ones if the relaxing parameter tends to zero.

2 Preliminaries

In this chapter we collect useful facts and theorems that are frequently used in the following. We cite them without giving a detailed proof in every case, rather we give the source where additional information can be found.

2.1 Some inequalities

Here we collect some basic inequalities.

2.1 Lemma. Let x, $y \in \mathbb{R}$. Then we have the inequalities

- $x \cdot y \leq \frac{1}{2}\left(\varepsilon x^2 + \varepsilon^{-1} y^2\right)$ for any $\varepsilon > 0$ (*Young's inequality*),
- $(x+y)^2 \leq 2(x^2+y^2)$.

Proof: Both inequalities follow from the inequality $2xy \leq x^2 + y^2$, which can be derived from noting that $(x-y)^2 \geq 0$. The first one then follows from writing $xy = \varepsilon x \cdot y/\varepsilon$ and the second from $(x+y)^2 = x^2 + 2xy + y^2 \leq x^2 + 2\left(\frac{x^2}{2} + \frac{y^2}{2}\right) + y^2 = 2(x^2+y^2)$. □

2.2 Lemma (Cauchy–Schwarz inequality). Let X be a vector space and $\langle \cdot, \cdot \rangle$ an inner product on X. Let $\|\cdot\|$ be the norm associated to the inner product on X. Then for any x, $y \in X$ there is

$$\left|\langle x, y \rangle\right| \leq \|x\| \cdot \left\|y\right\|.$$

2.3 Corollary (special case of Jensen's inequality). Let $T > 0$ and $f : [0,T] \to \mathbb{R}$. Then it holds

$$\left(\int_0^T f(x)\,\mathrm{d}x\right)^2 \leq T \int_0^T f(x)^2\,\mathrm{d}x.$$

Proof: Using the Cauchy-Schwarz inequality in $L^2((0,T))$ with f and the constant function $\varphi \equiv 1$ yields

$$\int_0^T f(x)\,\mathrm{d}x = \langle f, \varphi \rangle_{L^2((0,T))} \leq \|f\|_{L^2((0,T))} \|\varphi\|_{L^2((0,T))} = \sqrt{T} \cdot \left(\int_0^T f(x)^2\,\mathrm{d}x\right)^{1/2}.$$

This concludes the proof. □

2.4 Lemma (Gronwall's inequality). Let $a < b \in \mathbb{R}$ and $\alpha, \Phi \in C([a,b],\mathbb{R})$ where $\alpha \geq 0$. Further let $\beta \in \mathbb{R}$ and assume there is an inequality of the form

$$\Phi(t) \leq \beta + \int_a^t \alpha(s)\Phi(s)\,\mathrm{d}s$$

for all $t \in [a,b]$. Then there is an explicit estimate of the form

$$\Phi(t) \leq \beta \exp\left(\int_a^t \alpha(s)\,\mathrm{d}s \right)$$

for all $t \in [a,b]$. If $\alpha(t) \equiv \alpha$ is constant, this further simplifies to

$$\Phi(t) \leq \beta e^{\alpha(t-a)}.$$

Proof: [Gro19]. □

To conclude this section, we present some inequalities in the context of Lebesgue spaces.

2.5 Lemma (Hölder's inequality). Let $\mathcal{I} = (a,b) \subseteq \mathbb{R}$ an interval and let $p, q \in [1,\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in L^p(\mathcal{I})$, $g \in L^q(\mathcal{I})$. Then the product $f \cdot g$ lies in $L^1(\mathcal{I})$ and there is the estimate

$$\|f \cdot g\|_{L^1(\mathcal{I})} \leq \|f\|_{L^p(\mathcal{I})}\|g\|_{L^q(\mathcal{I})}.$$

Note that the Cauchy–Schwarz inequality is a special case of Hölder's inequality.

2.6 Lemma (Embedding of Lebesgue spaces). Let $\mathcal{I} = (a,b) \subseteq \mathbb{R}$ a bounded interval, X a Banach space. Further let $1 \leq p_1 \leq p_2 \leq \infty$. Then there is a continuous embedding $L^{p_2}(\mathcal{I},X) \hookrightarrow L^{p_1}(\mathcal{I},X)$ and for any $f \in L^{p_2}(\mathcal{I},X)$ there is the estimate

$$\|f\|_{L^{p_1}(\mathcal{I},X)} \leq (b-a)^{\frac{1}{p_1}-\frac{1}{p_2}} \|f\|_{L^{p_2}(\mathcal{I},X)}.$$

Proof: For $p_2 = \infty$ the claim is obvious, so without loss of generality we assume that $p_2 < \infty$. Hölder's inequality directly gives

$$\|f\|_{L^{p_1}(\mathcal{I},X)}^{p_1} = \int_a^b 1 \cdot \|f(t)\|_X^{p_1}\,\mathrm{d}t \leq \left(\int_a^b \|f(t)\|_X^{qp_1}\,\mathrm{d}t \right)^{1/q} \left(\int_a^b 1^{q'}\,\mathrm{d}t \right)^{1/q'}$$

for any $q \in (1,\infty)$, where q' is defined as $\frac{1}{q} + \frac{1}{q'} = 1$. Letting $q = \frac{p_2}{p_1} \in (1,\infty)$ gives $\frac{1}{q'} = 1 - \frac{p_1}{p_2} = \frac{p_2-p_1}{p_2}$ and hence

$$\|f\|_{L^{p_1}(\mathcal{I},X)} \leq (b-a)^{\frac{p_2-p_1}{p_1p_2}} \|f\|_{L^{p_2}(\mathcal{I},X)},$$

which concludes the proof. □

2.7 Lemma (Multipliers in L^p). Let $\mathcal{I} = (a,b) \subseteq \mathbb{R}$ an interval, X a Banach space and $p \in [1,\infty]$. Then for any $f \in L^p(\mathcal{I},X)$ and $\varphi \in L^\infty(\mathcal{I})$, the product φf is in $L^p(\mathcal{I},X)$ again and satisfies $\|\varphi f\|_{L^p(\mathcal{I},X)} \leq \|\varphi\|_{L^\infty(\mathcal{I})}\|f\|_{L^p(\mathcal{I},X)}$. In particular, the multiplication operator $M_\varphi : f \mapsto \varphi f$ is a bounded operator from $L^p(\mathcal{I},X)$ into itself.

Proof: Again the claim is evident for $p = \infty$. For $p < \infty$ we have

$$\|\varphi f\|^p_{L^p(\mathcal{I},X)} = \int_a^b \|\varphi(t)f(t)\|^p_X \,\mathrm{d}t = \int_a^b |\varphi(t)|^p \|f(t)\|^p_X \,\mathrm{d}t \leq \operatorname*{ess\,sup}_{t\in(a,b)} (|\varphi(t)|)^p \|f\|^p_{L^p(\mathcal{I},X)}. \qquad \square$$

We refer to Lemma 4.10 for the more complex case when the space L^p is replaced by a Sobolev space.

2.8 Theorem (Sobolev). Let $\Omega \subseteq \mathbb{R}^n$ be a smooth domain, let $j \geq 0$ and $m \geq 1$ integers and $1 \leq p < \infty$. Then the following hold:

1. If $mp > n$ or $m = n$ and $p = 1$ hold, then the space $W^{j+m,p}(\Omega)$ embeds continuously into $C^j(\bar{\Omega})$. Also it embeds into all $W^{j,q}(\Omega)$ where $p \leq q \leq \infty$.
2. If $mp = n$ holds, then $W^{j+m,p}(\Omega)$ embeds continuously into $W^{j,q}(\Omega)$ for all $p \leq q < \infty$.
3. If $mp < n$ holds, then $W^{j+m,p}(\Omega)$ embeds continuously into $W^{j,q}(\Omega)$ for all $p \leq q \leq \frac{p}{n-mp}$.

Proof: The original proofs appeared in [Sob38], [Sob08] and [Mor40]. The version we present here is taken from [AF03, Thm. 4.12]. □

2.2 Variational methods for partial differential equations

In the following, we present a classical method to obtain solutions for partial differential equations with time-dependent coefficients in Hilbert spaces. We only state the most important results here; more extensive treatments of the matter can be found in [DL92], [Eva10], [HPUU09] or [RR04].

2.9 Definition. A *Gelfand triple* is a triple (V,H,V') of Hilbert spaces where $V \subseteq H$ and the inclusion is continuous and dense and V' is the dual space of V. Duality is declared via the identification of H and its dual H', by which we mean that for any $v \in V$ and $h \in H$, we have $\langle h,v\rangle_{V'\times V} = \langle h,v\rangle_H$. It follows that also the inclusion $H \subseteq V'$ is continuous and dense. For any $v \in V$, $v' \in V'$ we let

$$\left\langle v',v\right\rangle_{V'\times V} := v'(v).$$

The norm on V' is the operator norm for functionals on V, as a consequence we have $|\langle v',v\rangle_{V'\times V}| \leq \|v'\|_{V'}\|v\|_V$ for all $v \in V$, $v' \in V'$.

2.10 Definition. Let $t_0 < t_1 \in \mathbb{R}$ and $V \subseteq H \subseteq V'$ be a Gelfand triple. We define the space

$$W((t_0,t_1),V,V') := L^2((t_1,t_2),V) \cap H^1((t_1,t_2),V').$$

This space, equipped with the norm

$$\left\|y\right\|_{W((t_0,t_1),V,V')} = \left(\left\|y\right\|^2_{L^2((t_0,t_1),V)} + \left\|y_t\right\|^2_{L^2((t_0,t_1),V')}\right)^{1/2},$$

is a Banach space.

2.11 Lemma. There is a continuous embedding from $W((t_0,t_1),V,V')$ into $C([t_0,t_1],H)$.

Proof: [DL92, Thm. XVIII.1.1]. □

2.12 Lemma. For any u, $v \in W((0,T),V,V')$ it holds that

$$\int_{t_1}^{t_2} \left\langle u'(t), v(t)\right\rangle_{V'\times V} + \left\langle v'(t), u(t)\right\rangle_{V'\times V} \,\mathrm{d}t = \Big[\langle u(t), v(t)\rangle_H\Big|_{t_1}^{t_2}.$$

Proof: [DL92, Thm. XVIII.1.2]. □

The general idea of variational equations is to interpret differential operators, such as the Laplacian, as maps from V to V'.

2.13 Theorem (First-order equations). Let $T > 0$ and $V \subseteq H \subseteq V'$ be a Gelfand triple. Assume there is a time-dependent operator $A \in C([0,T], L(V,V'))$ which is uniformly bounded, i.e. there is a constant M independent of t such that

$$\|A(t)u\|_{V'} \leq M\|u\|_V \quad \forall u \in V$$

holds. Furthermore, assume that the corresponding bilinear form $a(t,u,v) := -\langle A(t)u, v\rangle_{V'\times V}$ is *coercive*, i.e. there are constants α, $\beta > 0$ with

$$a(t,v,v) \geq \alpha\|v\|_V^2 - \beta\|v\|_H^2$$

for all $v \in V$. Further let $f \in L^2((0,T),V')$ and $u_0 \in H$. Then the initial value problem

$$u'(t) = A(t)u + f(t), \quad u(t=0) = u_0$$

has a unique weak solution $u \in W((0,T),V,V')$. Here, *weak solution* means that for all $v \in V$ and almost all $t \in [0,T]$ the equation

$$\left\langle u'(t), v\right\rangle_{V'\times V} + a(t,u(t),v) = \left\langle f(t), v\right\rangle_{V'\times V}$$

is satisfied and the initial conditions holds in the sense of $W((0,T),V,V') \subseteq C([0,T],H)$.

Proof: [DL92, Thms. XVIII.3.1 and 3.2]. □

2.14 Corollary. The solution u from the previous theorem satisfies an energy estimate of the form

$$\left\|u'\right\|^2_{L^2((0,T),V')} + \|u\|^2_{C([0,T],H)} + \|u\|^2_{L^2((0,T),V)} \leq c\left(\|u_0\|_H^2 + \|f\|^2_{L^2((0,T),V')}\right)$$

with a constant c that does not depend on $\|u_0\|_H$ or $\|f\|_{L^2((0,T),V')}$.

Proof: [HPUU09, Thm. 1.35]. □

2.15 Corollary (Higher regularity). In the situation of Theorem 2.13 assume further that $A \in C^1([0,T], L(V,V'))$ and $f \in H^1((0,T),V') \cap L^2((0,T),V)$ and $u_0 \in D(A(0))$, i.e. there is $w \in H$ such that $a(t,u,v) = -\langle w,v\rangle_H$ for all $v \in V$. Then the solution u of the differential equation satisfies $u \in H^1((0,T),V) \cap H^2((0,T),V')$ and $A(\cdot)u \in H^1((0,T),V') \cap L^2((0,T),V)$.

Proof: [RR04, Sec. 11.1.4]. □

2.16 Theorem (Second-order equations). Let $T > 0$ and $V \subseteq H \subseteq V'$ a Gelfand triple. We assume to be given

- an operator family $A(t) = A_0(t) + A_1(t)$ where $A_0 \in C^1([0,T], L(V,V'))$ is Hermitian (i.e. $a_0(t,u,v) = \overline{a_0(t,v,u)}$ for all $t \in [0,T]$ and $u, v \in V$) and coercive (as defined in Theorem 2.13) and $A_1 \in L^\infty((0,T), L(V,H)) \cap L^\infty((0,T), L(H,V'))$,
- an operator family $B \in C^1([0,T], L(H,H))$,
- an operator family $C \in C^1([0,T], L(H,H))$ where for all t the operator $C(t)$ is Hermitian and for all $u \in H$ we have $\langle C(t)u,u\rangle_H \geq \gamma \|u\|_H^2$ for some $\gamma > 0$ independently of t.

Furthermore, let $u_0 \in V$, $u_1 \in H$, $f \in L^2((0,T),H)$. Then the second-order initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}(C(t)u'(t)) + B(t)u'(t) + A(t)u(t) = f(t), \quad u(t=0) = u_0, \quad u'(t=0) = u_1$$

has a unique weak solution $u \in C([0,T],V) \cap C^1([0,T],H)$ with $C(\cdot)u' \in H^1([0,T],V')$. Here, *weak solution* means that for all $v \in V$ and almost all $t \in [0,T]$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle C(t)u'(t),v\right\rangle_H + \left\langle B(t)u'(t),v\right\rangle_H + \langle A(t)u(t),v\rangle_{V'\times V} = \left\langle f(t),v\right\rangle_H.$$

Proof: [DL92, Thm. XVIII.5.3 and 5.4]. □

2.17 Corollary. The solution u from the previous theorem satisfies the energy equality

$$\begin{aligned}
&\left\langle C(t)u'(t),u'(t)\right\rangle_H + \langle A_0(t)u(t),u(t)\rangle_{V'\times V} + 2\mathfrak{Re}\int_0^t \left\langle B(s)u'(s),u'(s)\right\rangle_H + \left\langle A_1(s)u(s),u'(s)\right\rangle_H \mathrm{d}s \\
&\quad = \langle C(0)u_1,u_1\rangle_H + \langle A_0(0)u_0,u_0\rangle_{V'\times V} \\
&\qquad + \int_0^t -\left\langle C'(s)u'(s),u'(s)\right\rangle_H + \left\langle A_0'(s)u(s),u(s)\right\rangle_{V'\times V} + 2\mathfrak{Re}\left\langle f(s),u'(s)\right\rangle_H \mathrm{d}s.
\end{aligned}$$

Proof: [DL92, Lem. XVIII.5.7]. Note that the equality can formally be derived by multiplying the equation with u' and integrating by parts. However, since u' does not lie in V, this is not allowed, and thus the proof requires more intricate calculations. □

2.3 Volterra equations

A *Volterra* equation is an integral equation of the form

$$u(t) = F(t) + \int_0^t A(s)u(t-s)\,\mathrm{d}s.$$

They often occur in mathematical biology where birth processes are involved, and in this context they are frequently called *renewal equations*, cf. [BvdDW08, eq. 9.32] and [Ian95, eq. (3.4)]. In the following we only need a basic existence result from the theory of Volterra equations. More extensive treatments of this topic can be found in [GLS90] and [Prü93].

2.18 Lemma. Let $A \in L^\infty(\mathcal{I}, L(X))$. and $F \in L^p(\mathcal{I}, X)$. Then the Volterra equation

$$u(t) = F(t) + \int_0^t A(s)u(t-s)\,\mathrm{d}s \quad \text{for almost all } t \in \bar{\mathcal{I}}$$

has a unique solution $u \in L^p(\mathcal{I}, X)$ with $\|u\|_{L^p(\mathcal{I},X)} \leq c\|F\|_{L^p(\mathcal{I},X)}$, with a constant c that only depends on $\|A\|_{L^\infty(\mathcal{I},L(X))}$.

Proof: First assume that F is an element of $C(\bar{\mathcal{I}}, X)$ with $F(0) = 0$. By noting that $A \in L^1(\mathcal{I}, L(X))$ and $A(s)\,\mathrm{d}s$ is an operator-valued Radon measure, existence of a solution $u \in C(\mathcal{I}, X)$ follows from [Prü93, Cor. 0.2]. This work also yields the norm estimate $\|u\|_{L^p(\mathcal{I},X)} \leq (1 + \ell(T))\|F\|_{L^p(\mathcal{I},X)}$, where ℓ is a solution of the scalar Volterra equation

$$\ell(t) = \int_0^t \|A(s)\|_{L(X)}\,\mathrm{d}s + \int_0^t \|A(t-s)\|_{L(X)}\ell(s)\,\mathrm{d}s.$$

An application of Gronwall's inequality shows that

$$\ell(t) \leq t\|A\|_{L^\infty(\mathcal{I},L(X))} \exp\left(\int_0^t \|A(t-s)\|_{L(X)}\,\mathrm{d}s\right) \leq t\|A\|_{L^\infty(\mathcal{I},L(X))} e^{t\|A\|_{L^\infty(\mathcal{I},L(X))}},$$

from which we conclude that

$$\|u\|_{L^p(\mathcal{I},X)} \leq \left(1 + T\|A\|_{L^\infty(\mathcal{I},L(X))} e^{T\|A\|_{L^\infty(\mathcal{I},L(X))}}\right)\|F\|_{L^p(\mathcal{I},X)}. \tag{2.1}$$

For general $F \in L^p(\mathcal{I}, X)$ we can find a sequence $(F_n)_{n\in\mathbb{N}} \subseteq C(\bar{\mathcal{I}}, X)$ that converges to F in $L^p(\mathcal{I}, X)$. If we let $u_n \in C(\bar{\mathcal{I}}, X)$ be the corresponding solutions of the Volterra equations, the estimate

$$\|u_m - u_n\|_{L^p(\mathcal{I},X)} \leq \left(1 + T\|A\|_{L^\infty(\mathcal{I},L(X))} e^{T\|A\|_{L^\infty(\mathcal{I},L(X))}}\right)\|F_m - f_n\|_{L^p(\mathcal{I},X)},$$

which follows from eq. (2.1), shows that there is some $u \in L^p(\mathcal{I}, X)$ with $u_n \to u$ in $L^p(\mathcal{I}, X)$. By invoking the dominated convergence theorem (cf. [HvVW16, Prop. 1.2.5]) and noting that the sequences $(F_n)_n$ and $(u_n)_n$ converge pointwise almost everywhere, we infer that the Volterra equation $u(t) = F(t) + \int_0^t A(s)u(t-s)\,\mathrm{d}s$ actually holds for F and u. This concludes the proof. □

2.4 Numerical methods for ordinary differential equations

In this short section we present some methods for the numerical approximation of solutions to the ordinary differential equation $\dot{x}(t) = f(t, x(t))$ where $t \in [0, T]$ with initial value $x(t_0) = x_0$. A general reference to this topic is [HNW93]. We choose some time step Δt and define an equidistant time grid $t_n := t_0 + n\Delta t$ where $n = 0, \ldots, N$ and N is small enough to ensure $t_N \leq T$. In Table 2.1 we present various methods that yield approximations for $x_n \approx x(t_n)$, starting with x_0 given by the initial value, and then calculate x_{n+1} for $n \in \{0, \ldots, N-1\}$. The values for the orders are taken from [HNW93, Table III.2.1]. The definition of *order* of a numerical scheme can be found in [HNW93, Def. III.2.3]; for the sake of simplicity we do not repeat it here. Typically, models of higher order produce better approximations but are more expensive to calculate.

Name	Formula	Type	Order
Explicit Euler	$x_{n+1} = x_n + \Delta t f(t_n, x_n)$	explicit	1
Implicit Euler	$x_{n+1} = x_n + \Delta t f(t_{n+1}, x_{n+1})$	implicit	1
Trapezoidal rule	$x_{n+1} = x_n + \frac{\Delta t}{2}(f(t_n, x_n) + f(t_{n+1}, x_{n+1}))$	implicit	2
Second-order Adams–Bashforth	$x_{n+1} = x_n + \frac{\Delta t}{2}(3f(t_n, x_n) - f(t_{n-1}, x_{n-1}))$ where we set $t_{-1} = t_0$ and $x_{-1} = x_0$	explicit	2

Table 2.1: Some numerical methods for ordinary differential equations

The Explicit Euler method was first published in [Eul68] and the Adams–Bashforth method in [BA83]. Typically, the implicit methods have better stability properties at the cost of being more complex to compute. We note that for $n = 0$, the formulae for the Euler method and the Adams–Bashforth method coincide.

In the context of partial differential equations, where, for example, the function f involves approximations of a Laplacian, the implicit trapezoidal rule is often called the *Crank–Nicolson* method, cf. [LP99, Section 4.3.5], which first appeared in [CN47]. We can unify three of the above-mentioned schemes by introducing a parameter $\vartheta \in [0, 1]$: Letting

$$x_{n+1} = x_n + \Delta t((1 - \vartheta) f(t_n, x_n) + \vartheta f(t_{n+1}, x_{n+1})),$$

(this is the variable-weight implicit method from [LP99, Section 4.3.6], which we call *theta Method* for short), we obtain a large class of ODE solvers, which cover the Explicit Euler ($\vartheta = 0$), Implicit Euler ($\vartheta = 1$), and Crank–Nicolson method ($\vartheta = \frac{1}{2}$).

2.5 Optimal control theory

The contents of this section are taken from [HPUU09, Section 1.7.2], which covers the subject in great detail. We restrict ourselves to the results which are most important for us. Given Banach spaces U, Y, Z and functions $J : U \times Y \to \mathbb{R}$ and $e : Y \times U \to Z$, we consider a problem of the form

$$\text{Find} \min_{y \in Y, u \in U} J(u, y) \quad \text{subject to} \quad e(y, u) = 0 \text{ and } u \in U^{\text{ad}} \tag{2.2}$$

where the set of admissible controls U^{ad} is a nonempty, closed and convex subset of U. Often, we call y the *state* variable and u the *control*. In many cases e encodes a differential equation for y where a term involving u appears. We assume the following:

2.19 Assumption.

- J and e are continuously Fréchet differentiable.
- For all $u \in V$, where V is some neighborhood of U^{ad} in U, there is a unique solution $y = y(u)$ to the equation $e(y,u) = 0$.
- For all $u \in V$, the linear map $e_y(y(u),u) \in L(Y,Z)$ has a bounded inverse.

From the implicit function theorem (cf. [HPUU09, Thm. 1.41]) it follows that the map $u \mapsto y(u)$ is differentiable with $y_u = -(e_y \circ y)^{-1}(e_u \circ y)$.

2.20 Definition. The *adjoint state* $p(u) \in Z'$ corresponding to the control u is defined as a solution to the *adjoint equation*

$$e_y(y(u),u)'p(u) = -J_y(u,y(u)) \quad \text{in } Y'. \tag{2.3}$$

Here, $e_y(y,u)' \in L(Z',Y')$ denotes the (Banach) adjoint operator to $e_y(y,u) \in L(Y,Z)$, as defined in Section 2.6.

2.21 Definition. The *Lagrange function* $\mathcal{L} : U \times Y \times Z' \to \mathbb{R}$ associated to J and e is defined as

$$L(u,y,p) := J(u,y) + \langle p, e(y,u)\rangle_{Z' \times Z}.$$

Simple calculations show that

$$\begin{aligned} \mathcal{L}_p(u,y,p) &= e(y,u) \in Z, \\ \mathcal{L}_y(u,y,p) &= J_y(u,y) + \left\langle p, e_y(y,u)\right\rangle_{Z' \times Z} \in Y', \\ \mathcal{L}_u(u,y,p) &= J_u(u,y) + \langle p, e_u(y,u)\rangle_{Z' \times Z} \in U'. \end{aligned} \tag{2.4}$$

We see that $\mathcal{L}_y(u,y(u),p) = 0$, and if $p = p(u)$ is chosen as the adjoint state, then also $\mathcal{L}_y(u,y(u),p(u)) = 0$. Using that the inverse of an adjoint operator is the adjoint of its inverse, together with the chain rule we have

$$\begin{aligned} \mathcal{L}_u(u,y(u),p(u)) &= J_u(u,y(u)) - \left\langle (e_y(y(u),u)^*)^{-1} J_y(u,y(u)), e_u(y,u)\right\rangle_{Z' \times Z} \\ &= J_u(u,y(u)) - \left\langle J_y(u,y(u)), e_y(y(u),u)^{-1} e_u(y,u)\right\rangle_{Y' \times Y} \\ &= J_u(u,y(u)) + \left\langle J_y(u,y(u)), y_u(u)\right\rangle_{Y' \times Y} \\ &= \frac{\mathrm{d}}{\mathrm{d}u} J(u,y(u)). \end{aligned} \tag{2.5}$$

This property is useful in optimization algorithms. Furthermore, we have a characterization of optimal solutions to eq. (2.2):

2.22 Theorem. Let $u^* \in U$ and $y^* := y(u^*)$ be chosen in such a way that (y^*, u^*) is a solution to eq. (2.2), and let p^* be the adjoint state corresponding to u^*. Then the following first-order optimality conditions hold:

$$\langle J_u(u^*,y^*) + e_u(y^*,u^*)'p^*, u - u^*\rangle_{U' \times U} \geq 0 \quad \forall u \in U^{\mathrm{ad}}.$$

Using the Lagrange function, this can be written more compactly as

$$\begin{gathered} \mathcal{L}_p(u^*,y^*,p^*) = 0, \quad \mathcal{L}_y(u^*,y^*,p^*) = 0, \\ \langle \mathcal{L}_u((u^*,y^*,p^*)), u - u^*\rangle_{U' \times U} \geq 0 \quad \forall u \in U^{\mathrm{ad}}. \end{gathered}$$

Proof: Directly from [HPUU09, Cor. 1.3]. □

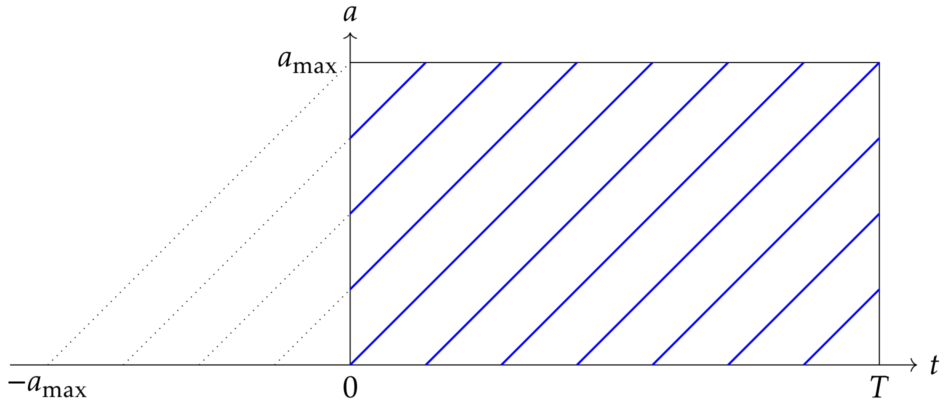


Figure 1: Schematic image of the characteristics (blue lines). For every $t_0 \in [-a_{\max}, T]$ we obtain a characteristic: If $t_0 \geq 0$, it starts in $(t, a) = (t_0, 0)$, for $t_0 < 0$ it starts in $(t, a) = (0, -t_0)$.

2.6 Notation

In this section, we introduce notations and basic definitions we will use throughout the work. For any set Ω and any subset $\Omega_0 \subseteq \Omega$ we denote by $\mathbb{1}_{\Omega_0} : \Omega \to \{0,1\}$ the indicator function which maps $\omega \in \Omega$ to 1 if $\omega \in \Omega_0$ and to 0 otherwise. By $x^+ := \max\{x, 0\}$, we denote the positive part of a real number x.

For any open subset $\Omega \subseteq \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$, the integral of f over Ω is denoted $\int_\Omega f(x)\,\mathrm{d}x$. If no confusion arises, we may drop the variable x altogether and just write $\int_\Omega f$. If $g : \partial\Omega \to \mathbb{R}$ is a function, where $\partial\Omega$ is the boundary of Ω which we assume to be sufficiently smooth, we denote the surface integral of g over $\partial\Omega$ by $\oint_{\partial\Omega} g(x)\,\mathrm{d}A(x)$ or just $\oint_{\partial\Omega} g$ for short.

For a Banach space X, we denote the norm of X by $\|\cdot\|_X$. For another Banach space Y, the notation $L(X, Y)$ denotes the bounded linear operators from X to Y. Further we let $L(X) := L(X, X)$ and $X' := L(X, \mathbb{R})$. For any $A \in L(X, Y)$ let $A' \in L(Y', X')$ the adjoint map, defined as $(A'y')(x) := y'(Ax)$ for any $y' \in Y'$, $x \in X$. If X is even a Hilbert space, its inner product is denoted by $\langle \cdot, \cdot \rangle_X$ In this case, one can show, using the Riesz-Fréchet representation theorem (cf. [Eva10, Thm. D.2]) that for any $A \in L(X)$, the adjoint operator A' can also be characterized as follows: for any x_1, $x_2 \in X$, it holds $\langle Ax_1, x_2 \rangle_X = \langle x_1, A'x_2 \rangle$. All spaces are assumed to be real, until stated otherwise.

In the following we fix a smoothly bounded domain $\Omega \subseteq \mathbb{R}^d$ (in applications, typically d equals two) and define the spaces

$$H := L^2(\Omega)^n, \quad V := H^1(\Omega)^n,$$

where n is the number of compartments in the epidemic model. By defining V' as the dual of V via the dual pairing induced by the inner product on H, it is easy to see that $V \subseteq H \subseteq V'$ is a Gelfand triple in the sense of Definition 2.9. Hence, we can also define the spaces $W((0, T), V, V')$ as in Definition 2.10. By $\Delta := \Delta_x$ we denote the Laplace operator acting on the space variables in Ω, i.e.

$$\Delta y := \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} y.$$

Partial derivatives are often denoted by an index, for example we may write $y_t := \frac{\partial y}{\partial t}$. Higher-order derivatives are written in a similar way, we can for example write $\Delta y = \sum_i y_{x_i x_i}$.

The age variable is assumed to be bounded by a finite maximal age, denoted by $a_{\max} > 0$. For brevity, we define $\mathcal{I} := (0, a_{\max})$. We will frequently work in the spaces

$$\mathcal{H} := L^2(\mathcal{I}, H) = L^2(\mathcal{I} \times \Omega)^n, \quad \mathcal{V} := L^2(\mathcal{I}, V), \quad \mathcal{V}' := L^2(\mathcal{I}, V').$$

To facilitate our analysis, we fix a time horizon $0 < T < \infty$. To keep the notation short, we write

$$\delta y := (\partial_t + \partial_a) y,$$

the expression being interpreted as a directional derivative. It is often useful to partition the variable space $[0,T] \times \overline{\mathcal{I}}$ into sets of the form

$$\operatorname{char}(t_0) := \{(t_0 + h, h) \mid 0 \le h \le a_{\max}\} \cap ([0,T] \times \overline{\mathcal{I}})$$

where $t_0 \in [-a_{\max}, T]$. These sets represent the so-called *characteristic lines* and are illustrated in Figure 1. Often we restrict functions φ defined on $[0,T] \times \bar{\mathcal{I}}$ on these characteristics, for which we will use the notation $\varphi|_{\operatorname{char}(t_0)}(h) := \varphi(t_0 + h, h)$, for all parameters $h \in \overline{\mathcal{I}_{t_0}}$. For brevity we let

$$\mathcal{I}_{t_0} := (\max\{-t_0, 0\}, \min\{T - t_0, a_{\max}\})$$

and for any $f : [0,T] \times \bar{\mathcal{I}}$ we write

$$\int_{\operatorname{char}(t_0)} f(h) \, \mathrm{d}h := \int_{\mathcal{I}_{t_0}} f(t_0 + h, h) \, \mathrm{d}h$$

In estimates, we frequently use the notation $A \lesssim B$ for an inequality of the form $A \le cB$, where c is a generic constant independent of the quantities to be estimated.

3 The unrelaxed equation

In this chapter we discuss unrelaxed epidemic models with control terms written in the form of eq. (1.1):

$$\delta y + L(a,x)y + \Lambda(a,x,y)y + K(u)y = \sigma(a)\Delta y, \tag{3.1a}$$

$$y(t=0) = y_0, \quad \partial_\nu y(x \in \partial\Omega) = 0, \tag{3.1b}$$

$$y(a=0) = \int_0^{a_{\max}} \beta(\alpha,x)y(t,\alpha,x)\,\mathrm{d}\alpha\,, \tag{3.1c}$$

$$\Lambda(a,x,y) = \int_0^{a_{\max}} \int_\Omega k(a,x,\alpha,\xi)y_I(t,\alpha,\xi)\,\mathrm{d}\xi\,\mathrm{d}\alpha\,. \tag{3.1d}$$

We start with a brief motivation and introduce the general model in eq. (3.1). The following sections discuss the question of existence and uniqueness of solutions to this model, which requires several intermediate steps, and one of the main theorems in this chapter is Theorem 3.20, which answers this question in the affirmative. This chapter extends and generalizes the work [Sch21] with more details and a better approach to the nonlinear equation. Parts of the chapter have already been published in [AS25, Sec. 2].

3.1 Motivation

The simplest epidemic models are systems of nonlinear ordinary differential equations. The easiest example is the *SIR* model we already saw in eq. (SIR). As a motivation we consider the slightly more complex model

$$\begin{aligned}
\dot S(t) &= cV - (u(t) + kI(t))S(t),\\
\dot V(t) &= u(t)S(t) - (c + \varphi_1 kI(t))V(t),\\
\dot I(t) &= kI(t)(S(t) + \varphi_1 V(t) + \varphi_2 R(t)) - (\gamma + \mu_I)I(t),\\
\dot R(t) &= \gamma I(t) - \varphi_2 kI(t)R(t),
\end{aligned} \tag{3.2}$$

which is a simplified version of [ARS25, eq. (1)], where S, I, R are as in Equation (SIR) and we have added a V compartment for the *vaccinated* individuals. The parameters u, k, c, φ_1, φ_2, γ and μ_I are explained as follows.

The model can be derived by observing the flux between compartments: Assume that per time step Δt an average percentage of $\gamma\Delta t$ infectives recover from their infection. During the same time, we assume that a share of $u(t)\Delta t$ susceptible individuals receive a vaccine and for a fraction $c\Delta t$ of vaccinated individuals, the vaccine expires and they become susceptible again. If the disease is fatal, then an additional percentage $\mu_I\Delta t$ of infected individuals die. Since u is the only parameter that can directly be influenced by doctors, governments, etc., we assume that it is the only time-dependent parameter and that all other coefficients in the model remain constant over time.

In order to model the infection process, we assume that individuals encounter, on average, some percentage $q\Delta t$ of the total population N, and that any contact of a susceptible person with an infectious one leads to an infection with probability p_S. By noting that the probability by which an individual is infective is given by $\frac{I}{N}$, the total infection rate for susceptible individuals can be expressed as $Nq\Delta t \cdot \frac{I}{N} \cdot p_S = kI\Delta t$ with $k := p_S q$. Further we assume that the vaccine lowers the probability of an infection to $p_V = \varphi_1 p_S$ with some $\varphi_1 \in (0,1)$, and that having recovered from the disease also provides some protection from getting infected again, the probability being $p_R = \varphi_2 p_S$ for $\varphi_2 \in (0,1)$. Then the change of susceptible, vaccinated, infective, and removed individuals during the time $(t, t+\Delta t)$ can be expressed as

$$\begin{aligned} S(t+\Delta t) - S(t) &= cV(t)\Delta t - (u(t) + kI(t))S(t)\Delta t, \\ V(t+\Delta t) - V(t) &= u(t)S(t)\Delta t - (c + \varphi_1 kI(t))V(t)\Delta t, \\ I(t+\Delta t) - I(t) &= kI(t)\Delta t \cdot (S(t) + \varphi_1 V(t) + \varphi_2 R(t)) - (\gamma + \mu_I)I(t)\Delta t, \\ R(t+\Delta t) - R(t) &= \gamma I(t)\Delta t - \varphi_2 kI(t)R(t)\Delta t. \end{aligned} \tag{3.3}$$

Dividing by Δt and letting $\Delta t \to 0$ then gives eq. (3.2). In a similar way one can derive a vast number of epidemic models of varying complexity.

The model (3.2) can easily be equipped with an age structure. If in eq. (3.3) we assume that the compartments and coefficients depend on an age variable a as well and note that age and time elapse the same way, we directly arrive at

$$\begin{aligned} S(t+\Delta t, a+\Delta t) - S(t,a) &= c(a)V(t,a)\Delta t - (u(t,a) + k(a)I(t,a))S(t,a)\Delta t, \\ V(t+\Delta t, a+\Delta t) - V(t,a) &= u(t,a)S(t,a)\Delta t - (c(a) + \varphi_1 k(a)I(t,a))V(t,a)\Delta t, \\ I(t+\Delta t, a+\Delta t) - I(t,a) &= k(a)I(t,a)\Delta t \cdot (S(t,a) + \varphi_1 V(t,a) + \varphi_2 R(t,a)) - (\gamma(a) + \mu_I(a))I(t,a)\Delta t, \\ R(t+\Delta t, a+\Delta t) - R(t,a) &= \gamma I(t,a)\Delta t - \varphi_2 k(a)I(t,a)R(t,a)\Delta t. \end{aligned}$$

and by letting $\Delta t \to 0$ as before, we arrive at a variant of the so-called *Lotka–McKendrick model*, cf. [BvdDW08, eq. 9.4]:

$$\begin{aligned} \delta S(t,a) &= c(a)V(t,a) - (u(t,a) + k(a)I(t,a))S(t,a), \\ \delta V(t,a) &= u(t,a)S(t,a) - (c(a) + \varphi_1 k(a)I(t,a))V(t,a), \\ \delta I(t,a) &= k(a)I(t,a)(S(t,a) + \varphi_1 V(t,a) + \varphi_2 R(t,a)) - (\gamma(a) + \mu_I(a))I(t,a), \\ \delta R(t,a) &= \gamma I(t,a) - \varphi_2 k(a)I(t,a)R(t,a), \end{aligned}$$

where we let $\delta := \partial_t + \partial_a$. However, in this model an infection can only occur between individuals of the same age. For a more realistic model we need to take individuals of all age groups into account, and this amounts to replacing the terms $k \cdot I$ by $\Lambda(I)$ which is defined by

$$\Lambda(a, I(t,\cdot)) := \int_0^{a_{\max}} k(a,\alpha) I(t,\alpha)\,\mathrm{d}\alpha\,.$$

Here $k(a,\alpha)$ is a weight describing how infective individuals at age α are to susceptible individuals at age a.

In order for the model to be well-posed, we need to equip it with suitable initial and boundary conditions. The condition at $t = 0$ is simply the initial state of the population. The condition in $a = 0$ describes the number of newborn individuals. Typically, newborns are created by the population itself through birth. Let $\tilde{\beta} = \tilde{\beta}(a)$ the age-dependent birth rate that describes the fertility of the age group a. If we assume that all newborns are susceptible, we arrive at

$$S(t, a = 0) = \int_0^{a_{\max}} \tilde{\beta}(a)(S + V + I + R)(t, \alpha)\,\mathrm{d}\alpha,$$
$$V(a = 0) = I(a = 0) = R(a = 0) = 0.$$

We also need to account for death of the population, thus we add an age-dependent death rate μ to all compartments. The complete age-structured epidemic model then reads (for better readability we avoid the variables t and a in the notation)

$$\begin{aligned}\delta S &= cV - (\mu + u + \Lambda(I))S,\\ \delta V &= uS - (\mu + c + \varphi_1 \Lambda(I))V,\\ \delta I &= \Lambda(I)(S + \varphi_1 V + \varphi_2 R) - (\mu + \gamma + \mu_I)I,\\ \delta R &= \gamma I - (\mu + \varphi_2 \Lambda(I))R,\end{aligned} \tag{3.4}$$

This model can be shown to be well-posed, cf. [BvdDW08, Section 9.4], or [Ian95, Thm. 1.4.2] for a simpler model.

We now also want to incorporate spatial movement in the population. Let $\Omega \subseteq \mathbb{R}^2$ be a domain with sufficiently smooth boundary $\partial\Omega$ and outer normal vector ν, and let u be some time- and space-dependent quantity. Then the change of u is modeled by the continuity equation

$$u_t + \operatorname{div} Q = f, \tag{3.5}$$

where Q is the flux of u, and f is a source term (cf. [LMPS22, eq. (1.1)]). If we assume that the flux is proportional to the negative gradient of the quantity, i.e. the quantity moves from higher to lower concentrations, and the bigger the difference the faster the movement, the flux takes the form

$$Q = -\sigma \nabla u \tag{3.6}$$

where $\sigma > 0$ is some diffusion coefficient, and the model decouples to the diffusion equation

$$u_t - f = -\operatorname{div}(-\sigma \nabla u) = \sigma \Delta u.$$

A common boundary condition is

$$\nu \cdot Q = 0, \tag{3.7}$$

i.e. there is no flux into or out of Ω. In absence of source terms f, this condition leads to conservation of quantity, since by the divergence theorem

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u(x)\,\mathrm{d}x = -\int_\Omega \operatorname{div} Q(x)\,\mathrm{d}x = -\oint_{\partial\Omega} \nu(x) \cdot Q(x)\,\mathrm{d}A(x) = 0.$$

Multiplying eq. (3.6) by ν and using $\nu \cdot Q = 0$ yields $0 = \nu \cdot (-\sigma \nabla u)$, which is equivalent to the Neumann condition $\partial_\nu u = 0$.

In our example, the quantity that moves through space is the number of individuals that are susceptible, vaccinated, etc. Hence, we can model the movement of individuals by adding diffusion terms to the model (3.4). This yields the system

$$\begin{aligned} \delta S - \sigma_S \Delta S &= cV - [u + \mu + \Lambda(I)]\, S, \\ \delta V - \sigma_V \Delta V &= uS - [\mu + c + \varphi_1 \Lambda(I)]\, V, \\ \delta I - \sigma_I \Delta I &= \Lambda(I)(S + \varphi_1 V + \varphi_2 R) - (\mu + \mu_I + \gamma) I, \\ \delta R - \sigma_R \Delta R &= \gamma I - [\mu + \varphi_2 \Lambda(I)]\, R \end{aligned} \tag{SVIR}$$

with the diffusion coefficients $\sigma_S, \sigma_V, \sigma_I, \sigma_R$. Note that now we also have to change the definition of Λ to

$$\Lambda(a, x, I(t, \cdot, \cdot)) = \int_0^{a_{\max}} \int_\Omega \lambda(a, x, \alpha, \xi) I(t, \alpha, \xi) \, \mathrm{d}\xi \, \mathrm{d}\alpha .$$

This additionally allows the kernel k to take into account infections that happen between different points in space. It should be noted that other processes can be modeled with this kind of terms too, such as noncompliance to Covid-19 rules as in [BPW24].

By letting

$$y = \begin{pmatrix} S \\ V \\ I \\ R \end{pmatrix}, \quad \sigma = \begin{pmatrix} \sigma_S & & & \\ & \sigma_V & & \\ & & \sigma_I & \\ & & & \sigma_R \end{pmatrix}, \quad \beta = \tilde{\beta} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix},$$

$$L = -\begin{pmatrix} -\mu & c & & \\ & -(\mu + c) & & \\ & & -(\mu + \mu_I + \gamma) & \\ & & \gamma & -\mu \end{pmatrix}, \quad K(u) = -u \cdot \begin{pmatrix} -1 & & & \\ 1 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix},$$

$$\Lambda(a, x, y) = -\langle \lambda(a, x, \cdot, \cdot), y_3 \rangle_{L^2((0, a_{\max}) \times \Omega)} \cdot \begin{pmatrix} -1 & & & \\ & -\varphi_1 & & \\ 1 & \varphi_1 & 0 & \varphi_2 \\ & & & -\varphi_2 \end{pmatrix}, \tag{3.8}$$

the model (SVIR) can be written as a special case of eq. (3.1). This definition of Λ amounts to

$$k^{hij} = \begin{cases} \lambda \cdot \begin{pmatrix} -1 & & & \\ & -\varphi_1 & & \\ 1 & \varphi_1 & 0 & \varphi_2 \\ & & & -\varphi_2 \end{pmatrix}^{hi}, & j = 3, \\ 0, & j \neq 3. \end{cases}$$

3.2 The linearized equation with fixed birth numbers

In the following sections we prove existence and uniqueness of weak solutions to the general model (3.1). We first consider a linearized equation where Λ and K are zero and instead of the implicit birth law from eq. (3.1c) we are given an explicit number $B = B(t,x)$ of newborns. Then, for given f and y_0, the equation takes the form

$$\begin{gathered}\delta y + L(a,x)y - \sigma(a)\Delta y = f,\\ y(t=0) = y_0, \quad y(a=0) = B,\\ \partial_\nu y(x \in \partial\Omega) = 0.\end{gathered} \tag{3.9}$$

We make the following assumptions on the coefficients:

3.1 Assumption. Suppose that:

1. $B \in L^2((0,T),H)$, $f \in L^2((0,T),\mathcal{V}')$ and $y_0 \in \mathcal{H}$, the spaces being defined as in Section 2.6.
2. $L \in C(\overline{\mathcal{I}}, L^\infty(\Omega))^{n\times n}$ and its entries are uniformly bounded away from zero with respect to both age and space.
3. $\sigma \in C(\overline{\mathcal{I}}, \mathbb{R}^{n\times n})$ is a diagonal matrix whose entries are uniformly bounded away from zero with respect to age.

The linearized equation (3.9) is most effectively solved along characteristic lines, see e.g. [Wal13] and [Web08, Sec. 1.3]. By fixing a birth date $t_0 \in [-a_{\max}, T]$ and setting $v(h) = y\big|_{\mathrm{char}(t_0)}(h) = y(t_0 + h, h)$ for $h \in \mathcal{I}_{t_0}$, we obtain the system

$$\begin{aligned}\partial_h v + L(h,x)v - \sigma(h)\Delta v &= f|_{\mathrm{char}(t_0)},\\ v(h = \max\{0,-t_0\}, x) = \begin{cases} B(t_0,x), & t_0 > 0,\\ y_0(-t_0,x), & t_0 < 0,\end{cases} &=: v_0,\\ \partial_\nu v(x \in \partial\Omega) &= 0.\end{aligned} \tag{3.10}$$

Note that the structure of the equation is independent of t_0, only the initial conditions and inhomogeneous terms depend on it. In order to obtain v_0, we have to evaluate an L^2 function which in Assumption 3.1 we assumed to take values in H. This evaluation can be done by choosing representatives, and later, Corollary 3.8 will show that this does not cause any problems.

We can formulate eq. (3.10) in a weak sense: Using the fact that, formally, for $w \in V$ we have

$$\int_\Omega \sigma(h)(\Delta y)(h,x)w(x)\,\mathrm{d}x = -\int_\Omega \sigma(h)\nabla y(h,x)\cdot\nabla w(x)\,\mathrm{d}x + \oint_{\partial\Omega} \sigma(h)w(x)\partial_\nu y(h,x)\,\mathrm{d}S(x) \tag{3.11}$$

and the last term vanishes because we assumed Neumann boundary conditions, we arrive at the weak formulation

$$\langle v'(h), w\rangle_{V'\times V} + \langle L(h)v, w\rangle_H + \langle \sigma(h)\nabla v, \nabla w\rangle_{H^d} = \langle f(h), w\rangle_{V'\times V}. \tag{3.12}$$

With this formulation we can define *weak solutions* to eq. (3.10) as in Theorem 2.13.

3.2 Theorem. Under Assumption 3.1, the equation on characteristics (3.10) has a unique weak solution $v \in W((\max\{0,-t_0\}, a_{\max}), V, V') \hookrightarrow C(\mathcal{I}_{t_0}, H)$. The solution satisfies the estimate

$$\|v\|^2_{W((\max\{0,-t_0\},a_{\max}),V,V')} \lesssim \|v_0\|_H^2 + \left\| f|_{\mathrm{char}(t_0)}\right\|^2_{L^2(\mathrm{char}(t_0),V')}.$$

Proof: The assumptions directly give that $v_0 \in H$. We note that

- for every h, the map $(u,v) \in H^2 \mapsto \langle L(h)u,v\rangle_H$ is well defined and continuous, since by Lemma 2.7 we can estimate
$$\left|\langle L(h)u,v\rangle_H\right| \leq \|L(h)u\|_H \|v\|_H \lesssim \|L\|_{L^\infty}\|u\|_H\|v\|_H.$$
It is also continuous in time, as can be seen by
$$\left|\langle L(h_1)u,v\rangle_H - \langle L(h_2)u,v\rangle_H\right| \leq \|L(h_1)-L(h_2)\|_{L^\infty}\|u\|_H\|v\|_H \overset{h_1\to h_2}{\longrightarrow} 0.$$
- For every h, the map $(u,v) \in V^2 \mapsto \langle \sigma(h)\nabla u, \nabla v\rangle_{H^d}$ is well-defined and continuous, with essentially the same proof as above.
- Finally, because of the boundedness away from zero, the two maps we just defined are coercive:
$$\langle L(h)u,u\rangle_H \gtrsim -\|L\|_{L^\infty}\|u\|_H^2, \quad \langle \sigma(h)\nabla u, \nabla u\rangle_H \geq c_\sigma \|u\|_V^2. \tag{3.13}$$

This shows that the form $a(u,v) := \langle L(h)u,v\rangle_H + \langle \sigma(h)\nabla u, \nabla v\rangle_{H^d}$ satisfies the conditions of Theorem 2.13, which in turn guarantees the existence of a unique weak solution of eq. (3.10). □

3.3 Definition. For any $0 \leq s \leq t \leq a_{\max}$, we define the evolution operator $U(t,s)v_0$ as the solution of the homogeneous equation (3.10) on characteristics at time t that takes the initial value v_0 at time s. Similarly, the solution of the inhomogeneous equation with initial values zero is denoted by $S(t,s)\, f|_{\mathrm{char}(t_0)}$. Let $U'(t,s)u_0$, $S'(t,s)\, f|_{\mathrm{char}(t_0)}$ be the respective weak derivative with respect to the t variable.

3.4 Corollary. Let $0 \leq s \leq t \leq a_{\max}$, then we have the following results for the evolution operators U, S, U' and S':

(i) $U(t,s)U(s,r) = U(t,r)$ and $U'(t,s)U'(s,r) = U'(t,r)$ for any $0 \leq r \leq s$, and the same for S,

(ii) $U(\cdot,s) \in L(H, W((s,a_{\max}),V,V'))$ and $S(\cdot,s) \in L(L^2((s,a_{\max}),V'), W((s,a_{\max}),V,V'))$,

(iii) $U'(\cdot,s) \in L(H, L^2((s,a_{\max}),V'))$ and $S'(\cdot,s) \in L(L^2((s,a_{\max}),V'), L^2((s,a_{\max}),V'))$,

(iv) $U(t,s) \in L(H,H)$ and $S(t,s) \in L(L^2((s,t),V'),H)$, and both are continuous in t.

Proof: This follows directly from Theorem 3.2: The estimates are clear, and property (i) is a consequence of the uniqueness of solutions. □

3.5 Remark. Note that in the case where even $f \in L^2((\max\{0,-t_0\},a_{\max}),H)$, we can use Duhamel's principle (variation of constants) to show that a solution to eq. (3.10) with initial condition $v(s) = v_0 \in H$ can be expressed as
$$v(t) = U(t,s)v_0 + \int_s^t U(t,r)f(r)\,\mathrm{d}r.$$

In other words, we have $S(t,s)f = \int_s^t U(t,r)f(r)\,\mathrm{d}r$.

3.6 Lemma. The operators U, S, U' and S' are uniformly bounded in s and t, i.e. the norms of all operators in Corollary 3.4 can be estimated by a constant that does not depend on t and s.

Proof: We adapt the arguments given in [Eva10, Thm. 7.1.2]: Let y the solution of eq. (3.10). Multiplying the differential equation with v in H and partial integration yields

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}h}\|v(h)\|_H^2 &= \langle v_h(h), v(h)\rangle_{V'\times V}\\ &= \langle f(h), v(h)\rangle_{V'\times V} - \langle L(h)v(h), v(h)\rangle_H - \langle \sigma(h)\nabla v(h), \nabla v(h)\rangle_H.\end{aligned}$$

Using eq. (3.13) we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}h}\|v(h)\|_H^2 + c_\sigma\|v(h)\|_V^2 \lesssim \langle f(h), v(h)\rangle_{V'\times V} + \|L\|_{L^\infty}\|v(h)\|_H^2.$$

Integrating from s to t and using Young's inequality yields for any $\varepsilon > 0$

$$\begin{aligned}\frac{1}{2}\|v(t)\|_H^2 + c_\sigma\|v\|_{L^2((s,t),V)}^2 \lesssim{}& \frac{1}{2}\|v(s)\|_H^2 + \|L\|_{L^\infty}\|v\|_{L^2((s,t),H)}^2\\ &+ \frac{1}{2\varepsilon}\|f\|_{L^2((s,t),V')}^2 + \frac{\varepsilon}{2}\|v\|_{L^2((s,t),V)}^2.\end{aligned}$$

For sufficiently small $\varepsilon < 2c_\sigma$ (independently of t, s), then we have

$$\|v(t)\|_H^2 + (2c_\sigma - \varepsilon)\|v\|_{L^2((s,t),V)}^2 \lesssim \|v(s)\|_H^2 + 2\|L\|_{L^\infty}\|v\|_{L^2((s,t),H)}^2 + \varepsilon^{-1}\|f\|_{L^2((s,t),V')}^2. \tag{3.14}$$

Gronwall (without the second term on the left-hand side) then yields

$$\|v(t)\|_H^2 \le \left(\|v(s)\|_H^2 + \varepsilon^{-1}\|f\|_{L^2((s,t),V')}^2\right)e^{2\|L\|_{L^\infty}(t-s)}. \tag{3.15}$$

Together with eq. (3.14), where we now ignore the first term on the left-hand side, we obtain

$$\begin{aligned}(2c_\sigma - \varepsilon)\|v\|_{L^2((s,t),V)}^2 &\lesssim \|v(s)\|_H^2 + \varepsilon^{-1}\|f\|_{L^2((s,t),V')}^2 + 2\|L\|_{L^\infty}\int_s^t \|v(r)\|_H^2\,\mathrm{d}r\\ &\lesssim \|v(s)\|_H^2 + \varepsilon^{-1}\|f\|_{L^2((s,t),V')}^2 + \int_s^t 2\|L\|_{L^\infty}\left(\|v(s)\|_H^2 + \varepsilon^{-1}\|f\|_{L^2((s,t),V')}^2\right)e^{2\|L\|_{L^\infty}(r-s)}\,\mathrm{d}r\\ &= \|v(s)\|_H^2 + \varepsilon^{-1}\|f\|_{L^2((s,t),V')}^2 + \left(\|v(s)\|_H^2 + \varepsilon^{-1}\|f\|_{L^2((s,t),V')}^2\right)e^{2\|L\|_{L^\infty}(t-s)}\\ &= \left(1 + e^{2\|L\|_{L^\infty}(t-s)}\right)\left(\|v(s)\|_H^2 + \varepsilon^{-1}\|f\|_{L^2((s,t),V')}^2\right).\end{aligned} \tag{3.16}$$

Next, the weak formulation (3.12) yields for any $w \in V$:

$$\begin{aligned}\left|\langle v'(h), w\rangle_{V'\times V}\right| &\le \left|\langle f(h), w\rangle_{V'\times V}\right| + \left|\langle L(h)v(h), w\rangle_H\right| + \left|\langle \sigma(h)\nabla v(h), \nabla w\rangle_H\right|\\ &\le \left(\|f(h)\|_{V'} + \|L\|_{L^\infty}\|v(h)\|_H + \|\sigma\|_{L^\infty}\|\nabla v(h)\|_H\right)\|w\|_V,\end{aligned}$$

leading to

$$\left\|v'(h)\right\|_{V'} \lesssim \|f(h)\|_{V'} + \|v(h)\|_V,$$

and after integration with respect to time we arrive at

$$\left\|v'\right\|^2_{L^2((s,t),V')} \lesssim 2\|f\|^2_{L^2((s,t),V')} + \|v\|^2_{L^2((s,t),V)}.$$

This estimate, combined with eq. (3.15) and eq. (3.16), yields

$$\|v(t)\|^2_H + \|v\|^2_{L^2((s,t),V)} + \left\|v'\right\|^2_{L^2((s,t),V')} \lesssim \left(1 + e^{2\|L\|_{L^\infty}(t-s)}\right)\left(\|v(s)\|^2_H + \|f\|^2_{L^2((s,t),V')}\right),$$

and using $t - s \le a_{\max}$ concludes the proof. □

For a fixed number of births, the solution of the "full" equation (3.9) can now be expressed as

$$y(t,a,x) := \begin{cases} U(a,0)B(t-a) + S(a,0)\, f|_{\mathrm{char}(t-a)}, & t > a, \\ U(a,a-t)y_0(a-t) + S(a,a-t)\, f|_{\mathrm{char}(t-a)}, & t \le a \end{cases} \quad \text{for all } t \ge 0,\, a \in \mathcal{I}. \tag{3.17}$$

Our next step is to show that δy is in fact given by

$$v(t,a,x) := \begin{cases} U'(a,0)B(t-a) + S'(a,0)\, f|_{\mathrm{char}(t-a)}, & t > a, \\ U'(a,a-t)y_0(a-t) + S'(a,a-t)\, f|_{\mathrm{char}(t-a)}, & t \le a \end{cases} \quad \text{for all } t \ge 0,\, a \in \mathcal{I}. \tag{3.18}$$

However, this representation requires the evaluation of the L^2 functions B and y_0, as well as restricting the L^2 function f on characteristics. This requires the choice of representatives, which in turn makes it possible that y and v are not well defined. Corollary 3.8 shows that this is not the case when y and v are only interpreted modulo null sets as well.

3.7 Lemma. The following statements hold:

1. Let $N_1 \subseteq [0,T] \times \bar{\mathcal{I}}$ be a null set. Then the set

   $$\mathcal{N}_1 := \{t_0 \in [-a_{\max}, T] \mid N_1 \cap \mathrm{char}(t_0) \text{ has nonzero measure in} \,\mathrm{char}(t_0)\}$$

   is a null set in $[-a_{\max}, T]$.

2. Let $N_2 \subseteq [-a_{\max}, T]$ be a null set. Then the set

   $$\mathcal{N}_2 := \bigcup_{t_0 \in N_2} \mathrm{char}(t_0)$$

   is a null set in $[0,T] \times \bar{\mathcal{I}}$.

Proof: Both claims follow using Fubini's theorem and the fact that for any function φ we have

$$\int\limits_{[0,T]\times\mathcal{I}} \varphi(t,a)\,\mathrm{d}(t,a) = \int\limits_{-a_{\max}}^{T} \int\limits_{\mathrm{char}(t_0)} \varphi|_{\mathrm{char}(t_0)}(h)\,\mathrm{d}h\,\mathrm{d}t_0.$$

Due to

$$0 = \int\limits_{[0,T]\times\mathcal{I}} \mathbb{1}_{N_1}(t,a)\,\mathrm{d}(t,a) = \int\limits_{-a_{\max}}^{T} \int\limits_{\mathrm{char}(t_0)} \mathbb{1}_{N_1 \cap \mathrm{char}(t_0)}(h)\,\mathrm{d}h\,\mathrm{d}t_0,$$

we can conclude that $\int_{\mathrm{char}(t_0)} \mathbb{1}_{N_1\cap\mathrm{char}(t_0)}(h)\,\mathrm{d}h = 0$ for almost all $t_0 \in (-a_{\max}, T)$, which shows the first claim. Further, the second claim follows from the fact that

$$\int\limits_{[0,T]\times\mathcal{I}} \mathbb{1}_{\mathcal{N}_2}(t,a)\,\mathrm{d}(t,a) = \int\limits_{N_2}\int\limits_{\mathrm{char}(t_0)} 1\,\mathrm{d}h\,\mathrm{d}t_0 = 0. \qquad \square$$

3.8 Corollary. Under Assumption 3.1, let B^1, B^2 be two representatives of B; y_0^1 and y_0^2 representatives of y_0 and f^1, f^2 representatives of f. Denote by y^1 and y^2 the solutions obtained from eq. (3.17) using the corresponding representatives. Then $y^1 = y^2$ holds for almost all (t,a).

Proof: We first assume that $f^1 = f^2$ and consider the general case later. Let

$$N_2 = \left\{t_0 \in (-a_{\max}, 0) \;\middle|\; y_0^1(-t_0) \neq y_0^2(-t_0)\right\} \cup \left\{t_0 \in (0,T) \;\middle|\; B^1(t_0) \neq B^2(t_0)\right\},$$

then N_2 is a null set. Define $\mathcal{N}_2$ as in Lemma 3.7, then from the construction of the y^i $(i = 1,2)$ we see that y^1 and y^2 will differ on at most $\mathcal{N}_2$, which is a null set. Hence, the claim holds.

Now consider the general case. Define $N_1 = \left\{(t,a) \in (0,T)\times\mathcal{I} \mid f^1 \neq f^2\right\}$ and $\mathcal{N}_1$ as in Lemma 3.7. For any $t_0 \in (-a_{\max}, T)$ not contained in $\mathcal{N}_1$, the set $N_1 \cap \mathrm{char}(t_0)$ is a null set in $\mathrm{char}(t_0)$. This means that $f^1\big|_{\mathrm{char}(t_0)}$ and $f^2\big|_{\mathrm{char}(t_0)}$ represent the same element in $L^2(\mathrm{char}(t_0), V')$, and thus we have $y^1\big|_{\mathrm{char}(t_0)} = y^2\big|_{\mathrm{char}(t_0)}$ for these t_0. Hence, if we let $\mathcal{N}_3 := \bigcup_{t_0\in\mathcal{N}_1} \mathrm{char}(t_0)$, then y^1 and y^2 will differ at most on $\mathcal{N}_1 \cup \mathcal{N}_3$, which according to Lemma 3.7 is a null set again. This concludes the proof. $\square$

3.9 Lemma. Under Assumption 3.1 we have $y \in L^2((0,T),\mathcal{V}) \cap L^\infty((0,T),\mathcal{H})$ and $v \in L^2((0,T),\mathcal{V}')$ and the estimate

$$\begin{aligned}
&\left\|y\right\|^2_{L^2((0,T),\mathcal{V})} + \left\|y\right\|^2_{L^\infty((0,T),\mathcal{H})} + \|v\|^2_{L^2((0,T),\mathcal{V}')} \\
&\lesssim \left(\left\|y_0\right\|^2_{\mathcal{H}} + \|B\|^2_{L^2((0,T),H)} + \|f\|^2_{L^2((0,T),\mathcal{V}')}\right),
\end{aligned} \tag{3.19}$$

with a constant independent of y_0, f and B.

Proof: By transforming on characteristic lines and applying Fubini's theorem we obtain

$$\begin{aligned}
\left\|y\right\|^2_{L^2((0,T),\mathcal{V})} + \|v\|^2_{L^2((0,T),\mathcal{V}')} &= \int\limits_0^T \int\limits_0^{a_{\max}} \left\|y(t,a)\right\|^2_V + \|v(t,a)\|^2_{V'}\,\mathrm{d}a\,\mathrm{d}t \\
&= \int\limits_{-a_{\max}}^T \int\limits_{\max\{-t_0,0\}}^{\min\{a_{\max},T-t_0\}} \left\|y(t_0+h,h)\right\|^2_V + \|v(t_0+h,h)\|^2_{V'}\,\mathrm{d}h\,\mathrm{d}t_0.
\end{aligned} \tag{3.20}$$

Splitting the integral into $\int_{-a_{\max}}^0$ and $\int_0^T$ yields

$$\begin{aligned}\left\|y\right\|^2_{L^2((0,T),\mathcal{V})} + \|v\|^2_{L^2((0,T),\mathcal{V}')} = \int_{-a_{\max}}^{0} \int_{-t_0}^{\min\{a_{\max},T-t_0\}} & \left\|U(h,-t_0)y_0(-t_0) + S(h,-t_0)\, f|_{\mathrm{char}(t_0)}\right\|^2_V \\ & + \left\|U'(h,-t_0)y_0(-t_0) + S'(h,-t_0)\, f|_{\mathrm{char}(t_0)}\right\|^2_{V'} \,\mathrm{d}h\,\mathrm{d}t_0 \\ + \int_0^T \int_0^{\min\{a_{\max},T-t_0\}} & \left\|U(h,0)B(t_0) + S(h,0)\, f|_{\mathrm{char}(t_0)}\right\|^2_V \\ & + \left\|U'(h,0)B(t_0) + S'(h,0)\, f|_{\mathrm{char}(t_0)}\right\|^2_{V'} \,\mathrm{d}h\,\mathrm{d}t_0 .\end{aligned}$$

Applying the results from Lemma 3.6 allows us to further estimate

$$\begin{aligned}\left\|y\right\|^2_{L^2((0,T),\mathcal{V})} + \|v\|^2_{L^2((0,T),\mathcal{V}')} &\lesssim \int_{-a_{\max}}^{0} \left\|y_0(-t_0)\right\|^2_H + \left\|f|_{\mathrm{char}(t_0)}\right\|^2_{L^2((-t_0,\min\{a_{\max},T-t_0\}),V')} \mathrm{d}t_0 \\ &\quad + \int_0^T \|B(t_0)\|^2_H + \left\|f|_{\mathrm{char}(t_0)}\right\|^2_{L^2((0,\min\{a_{\max},T-t_0\}),V')} \mathrm{d}t_0 \\ &\lesssim \left\|y_0\right\|^2_{\mathcal{H}} + \|B\|^2_{L^2((0,T),H)} + \|f\|^2_{L^2((0,T),\mathcal{V}')} < \infty. \end{aligned} \tag{3.21}$$

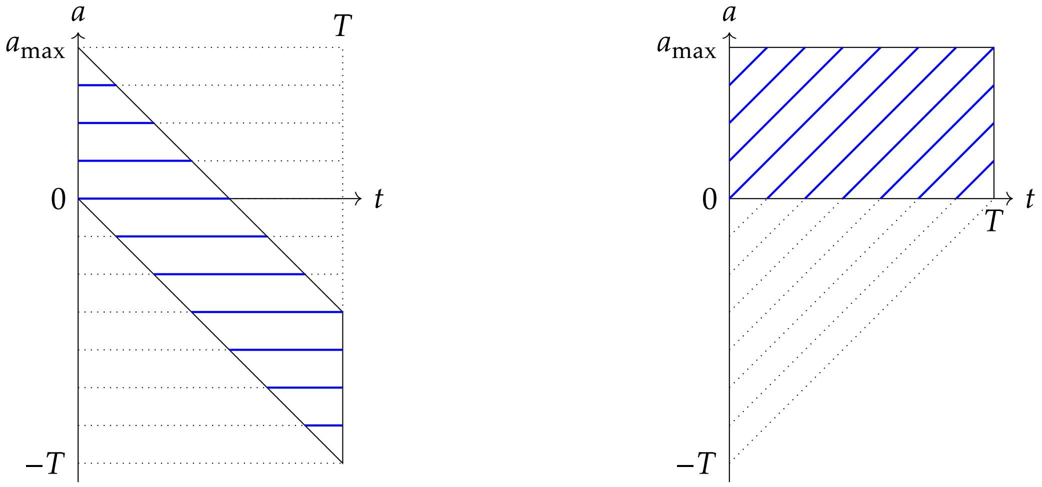
For the L^∞ norms we proceed in a similar way:

$$\begin{aligned}\left\|y(t)\right\|^2_{\mathcal{H}} &= \int_0^{a_{\max}} \left\|y(t,a)\right\|^2_H \mathrm{d}a \\ &= \int_0^{\min(t,a_{\max})} \left\|U(a,0)B(t-a) + S(a,0)\, f|_{\mathrm{char}(t-a)}\right\|^2_H \mathrm{d}a \\ &\quad + \int_{\min(t,a_{\max})}^{a_{\max}} \left\|U(a,a-t)y_0(a-t) + S(a,a-t)\, f|_{\mathrm{char}(t-a)}\right\|^2_H \mathrm{d}a \\ &\lesssim \int_0^t \|B(t-a)\|^2 \mathrm{d}a + \int_0^t \int_0^{a_{\max}} \|f(s,a)\|^2 \mathrm{d}a\,\mathrm{d}s + \int_0^{a_{\max}} \left\|y_0(a)\right\|^2 \mathrm{d}a < \infty.\end{aligned}$$

Together with eq. (3.21), this estimate concludes the proof. □

3.10 Theorem. The function $t \mapsto y(t,\cdot,\cdot)$ is a continuous mapping from $[0,T]$ into $\mathcal{H}$, i.e. $y \in C([0,T],\mathcal{H})$, and we have

$$\left\|y\right\|^2_{C([0,T],\mathcal{H})} \lesssim \left\|y_0\right\|^2_{\mathcal{H}} + \|B\|^2_{L^2((0,T),H)} + \|f\|^2_{L^2((0,T),\mathcal{V}')}.$$



(a) Definition of $\tilde{y}$ (b) Definition of y

Figure 2: Illustration of the relationship between $\tilde{y}$ and y, and the operation of the shift and restriction operators.

Proof: First consider the function $\tilde{y} : [0,T] \times [-T, a_{\max}] \to H$ defined by

$$\tilde{y}(t,a) := \begin{cases} U(a+t,0)B(-a) + S(a+t,0)\, f|_{\operatorname{char}(-a)} & a < 0, \\ U(a+t,a)y_0(a) + S(a+t,a)\, f|_{\operatorname{char}(-a)} & a \geq 0, \end{cases}$$

where we generalize $U(t,s) := U(\min\{t, a_{\max}\}, \max\{s, 0\})$ and proceed with S in an analogous way. Note that also with the new definition we have $t \mapsto U(t,s)\varphi \in C([s,\infty), H)$ for all t, $s \in \mathbb{R}$ and all $\varphi \in H$, and by Lemma 3.6 we can estimate $\|U(t,s)\|_{L(H,H)}$ by a uniform upper bound independent of t and s. Similar statements hold for S. For a motivation on how to define $\tilde{y}$, we refer to Figure 2. Figure 2a shows how $\tilde{y}$ is defined: the desired evolution operators are applied on the blue lines, and their definition is extended onto the dashed lines to ensure a pointwise continuous function. Later in the proof we apply a the shift operator and restrict the domain for a, by which we obtain Figure 2b, which represents the desired function y (cf. Figure 1).

Since for all $t \in [0,T]$ we have, using Lemma 3.6,

$$\left\|\tilde{y}(t,a)\right\|_H^2 \lesssim \left\| f|_{\operatorname{char}(-a)}\right\|^2_{L^2(\operatorname{char}(-a),V')} + \begin{cases} \|B(-a)\|_H^2, & a < 0, \\ \left\|y_0(a)\right\|_H^2, & a \geq 0, \end{cases} \tag{3.22}$$

we conclude $\left\|\tilde{y}(t)\right\|^2_{L^2((-T,a_{\max}),H)} \lesssim \left\|y_0\right\|^2_{\mathcal{H}} + \|B\|^2_{L^2((0,T),H)} + \|f\|^2_{L^2((0,T),\mathcal{V}')}$. This allows us to conclude that $\tilde{y} \in L^\infty((0,T), L^2((-T,a_{\max}),H))$. We claim that $\tilde{y}$ is in fact an element of $C([0,T], L^2((-T,a_{\max}),H))$. To show this, let $(t_n)_{n\in\mathbb{N}}$ be a sequence in $[0,T]$ that converges to some $t^* \in [0,T]$. From the continuity of U and S it follows that $\tilde{y}(t_n,a) \to \tilde{y}(t^*,a)$ pointwise for all $a \in [-T, a_{\max}]$. Since in estimate (3.22) the right-hand side is an element of $L^1((-T,a_{\max}),H)$ when read as a function of a, we can invoke the dominated convergence theorem (cf. [HvVW16, Prop. 1.2.5]) to obtain

$$\lim_{n\to\infty} \int_{-T}^{a_{\max}} \left\|\tilde{y}(t_n,a) - \tilde{y}(t^*,a)\right\|_H^2 \,\mathrm{d}a = 0.$$

This just means that $\tilde{y}(t_n) \to \tilde{y}(t^*)$ in $L^2((-T,a_{\max}),H))$, and hence shows the continuity of $\tilde{y}$.

Next, define the restriction operator

$$R \in L(L^2((-T,a_{\max}),H),L^2((0,a_{\max}),H)), \quad \varphi \mapsto \varphi|_{(0,a_{\max})}$$

and for any $t \in [0,T]$ the age-shift operator

$$\mathcal{S}(t) \in L(L^2((-T,a_{\max}),H)), \quad (\mathcal{S}(t)\varphi)(a) = \begin{cases} \varphi(a-t), & a-t \geq -T, \\ 0, & a-t < -T. \end{cases}$$

It is well known that R and $\mathcal{S}(t)$ for any t are bounded, and that $\mathcal{S}$ is continuous with respect to t. Our next step is to show that

$$y(t) = R\mathcal{S}(t)\tilde{y}(t),$$

which concludes the proof since we have written y as a composition of continuous functions. In fact, for any $t \in (0,T)$ and $a \in (0,a_{\max})$ we have $a-t \geq -T$ and hence

$$\begin{aligned} (R\mathcal{S}(t)\tilde{y}(t))(a) &= \begin{cases} U((a-t)+t,0)B(-(a-t)) + S((a-t)+t,0)\, f|_{\mathrm{char}(-(a-t))}, & a-t<0, \\ U((a-t)+t,a)y_0(a-t) + S((a-t)+t,a)\, f|_{\mathrm{char}(-(a-t))}, & a-t \geq 0, \end{cases} \\ &= \begin{cases} U(a,0)B(t-a) + S(a,0)\, f|_{\mathrm{char}(t-a)}, & t>a, \\ U(a,a-t)y_0(a-t) + S(a,a-t)\, f|_{\mathrm{char}(t-a)}, & t \leq a, \end{cases} \end{aligned}$$

which is just the expression for y from eq. (3.17). The norm estimate has already been shown in Lemma 3.9. □

3.11 Remark. Tracking the convergence for $t_n \to 0$ shows that, in fact, $y(t=0,\cdot,\cdot) = y_0$, where by Theorem 3.10 the left-hand side is a valid expression. By swapping the roles of t and a in the previous proof we can also show that $a \mapsto y(\cdot,a,\cdot)$ is a continuous mapping from $\bar{\mathcal{I}}$ into $L^2((0,T),H)$ and it holds that $y(\cdot,a=0,\cdot) = B$.

3.12 Remark. Proving the continuity separately cannot be avoided, since an analogon of Lemma 2.11 stating that for all $y \in L^2((0,T),\mathcal{V})$ with $\delta y \in L^2((0,T),\mathcal{V}')$ there is $y \in C([0,T],\mathcal{H})$ does not hold. However, one can show embeddings into spaces of continuous functions on smaller time intervals, cf. Section 3.5.

3.13 Theorem. The function v is indeed the weak age-space derivative of y, i.e. we have $v = \delta y = \partial_t y + \partial_a y$, in the sense of V'-valued functions on $[0,T]\times\mathcal{I}$.

Proof: Let $\varphi \in C_c^\infty([0,T]\times\mathcal{I},V')$, then the same transformation as in eq. (3.20) yields

$$\begin{aligned} &\int_0^T \int_0^{a_{\max}} \langle \delta\varphi(t,a), y(t,a)\rangle_{V'} + \langle \varphi(t,a), v(t,a)\rangle_{V'} \,\mathrm{d}a\,\mathrm{d}t \\ &\quad = \int_{-a_{\max}}^{T} \int_{\max\{-t_0,0\}}^{\min\{a_{\max},T-t_0\}} \langle \delta\varphi(t_0+h,h), y(t_0+h,h)\rangle_{V'} + \langle \varphi(t_0+h,h), v(t_0+h,h)\rangle_{V'} \,\mathrm{d}h\,\mathrm{d}t_0. \end{aligned}$$

Splitting the integral into $\int_{-a_{\max}}^0 + \int_0^T$ and using eq. (3.17) gives the following expression:

$$\begin{aligned}
&\int_0^T \int_0^{a_{\max}} \left\langle \delta\varphi(t,a), y(t,a)\right\rangle_{V'} + \left\langle \varphi(t,a), v(t,a)\right\rangle_{V'} \mathrm{d}a\,\mathrm{d}t \\
&= \int_{-a_{\max}}^{0} \int_{-t_0}^{\min\{a_{\max}, T-t_0\}} \left\langle \frac{\mathrm{d}}{\mathrm{d}h}\varphi(t_0+h,h), U(h,-t_0)y_0(-t_0)\right\rangle_{V'} + \left\langle \varphi(t_0+h,h), U'(h,-t_0)y_0(-t_0)\right\rangle_{V'} \\
&\quad + \left\langle \frac{\mathrm{d}}{\mathrm{d}h}\varphi(t_0+h,h), S(h,-t_0)\, f|_{\mathrm{char}(t_0)}\right\rangle_{V'} + \left\langle \varphi(t_0+h,h), S'(h,-t_0)\, f|_{\mathrm{char}(t_0)}\right\rangle_{V'} \mathrm{d}h\,\mathrm{d}t_0 \\
&\quad + \int_0^T \int_0^{\min\{a_{\max}, T-t_0\}} \left\langle \frac{\mathrm{d}}{\mathrm{d}h}\varphi(t_0+h,h), U(h,0)B(t_0)\right\rangle_{V'} + \left\langle \varphi(t_0+h,h), U'(h,0)B(t_0)\right\rangle_{V'} \\
&\quad + \left\langle \frac{\mathrm{d}}{\mathrm{d}h}\varphi(t_0+h,h), S(h,0)\, f|_{\mathrm{char}(t_0)}\right\rangle_{V'} + \left\langle \varphi(t_0+h,h), S'(h,0)\, f|_{\mathrm{char}(t_0)}\right\rangle_{V'} \mathrm{d}h\,\mathrm{d}t_0 \\
&= 0,
\end{aligned}$$

the last equality holding because of the definition of weak derivatives in intervals. Note that $h \mapsto \varphi(t_0+h,h)$ is a test function again for all values of t_0: The infinite differentiability is clear, and from

$$\begin{aligned}
\varphi(t_0+h,h)\Big|_{h=\max\{-t_0,0\}} &= \begin{cases} \varphi(0,-t_0), & t_0 < 0, \\ \varphi(t_0,0), & t_0 \geq 0, \end{cases} \\
\varphi(t_0+h,h)\Big|_{h=\min\{a_{\max},T-t_0\}} &= \begin{cases} \varphi(t_0+a_{\max},a_{\max}), & a_{\max} < T-t_0, \\ \varphi(T,T-t_0), & a_{\max} \geq T-t_0, \end{cases}
\end{aligned}$$

we conclude that the restriction of φ on characteristics is indeed zero at the endpoints, and thus is compactly supported on the characteristics. □

3.14 Corollary. The function y defined in eq. (3.17) lies in $L^2((0,T),\mathcal{V}) \cap C([0,T],\mathcal{H}) \cap C(\bar{\mathcal{I}}, L^2((0,T),H))$ and we have $\delta y \in L^2((0,T),\mathcal{V}')$. Further, y is a weak solution of eq. (3.9) in the sense that it holds for almost all $(t,a) \in (0,T)\times\mathcal{I}$ that

$$\left\langle \delta y(t,a), v\right\rangle_{V'\times V} + \langle L(a)u(t,a), v\rangle_H + \langle \sigma(a)\nabla u(t,a), \nabla v\rangle_{H^d} = \left\langle f(t,a), v\right\rangle_{V'\times V}$$

as well as the initial conditions $y(t=0,\cdot,\cdot) = y_0$ and $y(\cdot, a=0,\cdot) = B$.

Proof: The claim follows directly by Lemma 3.9, Remark 3.11 and Theorem 3.13. □

3.3 The linearized equation with implicit birth law

Up to now, the number B of newborn individuals was fixed. Our next goal is to implement an implicit birth condition from eq. (3.1c) instead. This yields

$$\begin{aligned}
\delta y + L(a,x)y - \sigma(a)\Delta y &= f, \\
y(t=0) = y_0, \quad \partial_\nu y(x \in \partial\Omega) &= 0, \\
y(a=0) &= \int_0^{a_{\max}} \beta(\alpha,x) y(t,\alpha,x)\,\mathrm{d}\alpha =: B.
\end{aligned} \tag{3.23}$$

3.15 Assumption. For the remainder of this section, we adopt the same assumptions as stated in Assumption 3.1, except for the first line. In more concrete terms we assume:

1. $\beta \in L^\infty(\mathcal{I}, L^\infty(\Omega))^{n\times n}$, $f \in L^2((0,T),\mathcal{V}')$ and $y_0 \in \mathcal{H}$.
2. $L \in C(\overline{\mathcal{I}}, L^\infty(\Omega))^{n\times n}$ and its entries are uniformly bounded away from zero with respect to both age and space.
3. $\sigma \in C(\overline{\mathcal{I}}, \mathbb{R}^{n\times n})$ is a diagonal matrix whose entries are uniformly bounded away from zero with respect to age.

Formally plugging (3.17) into the implicit birth law yields (we suppress x in the notation)

$$B(t) = \int_0^{a_{\max}} \beta(\alpha) \begin{cases} U(\alpha,0)B(t-a) + S(\alpha,0)\, f|_{\mathrm{char}(t-\alpha)}, & t > \alpha \\ U(\alpha,\alpha-t)y_0(\alpha-t) + S(\alpha,\alpha-t)\, f|_{\mathrm{char}(t-\alpha)}, & t \le \alpha \end{cases} \mathrm{d}\alpha.$$

Splitting the integral into $\int_0^{\min(t,a_{\max})} + \int_{\min(t,a_{\max})}^{a_{\max}}$ yields

$$\begin{aligned} B(t) = &\int_0^{\min(t,a_{\max})} \beta(\alpha,x)U(\alpha,0)B(t-\alpha)\,\mathrm{d}\alpha + \int_0^{\min(t,a_{\max})} \beta(\alpha)S(\alpha,0)\, f|_{\mathrm{char}(t-\alpha)}\,\mathrm{d}\alpha \\ &+ \int_{\min(t,a_{\max})}^{a_{\max}} \beta(\alpha)U(\alpha,\alpha-t)y_0(\alpha-t)\,\mathrm{d}\alpha + \int_{\min(t,a_{\max})}^{a_{\max}} \beta(\alpha)S(a,a-t)\, f|_{\mathrm{char}(t-a)}\,\mathrm{d}\alpha, \end{aligned} \tag{3.24}$$

which is a fixed point Volterra equation for B.

3.16 Theorem. Suppose that Assumption 3.15 holds. Then there exists a unique solution $B \in L^2((0,T),H)$ to the Volterra equation (3.24), satisfying the estimate

$$\|B\|_{L^2((0,T),H)} \lesssim \left\|y_0\right\|_{\mathcal{H}} + \|f\|_{L^2((0,T),\mathcal{V}')}$$

with a constant independent of y_0 and f.

Proof: We want to verify the conditions of Lemma 2.18. For $t \in (0,T)$ let

$$A(t) := \begin{cases} \beta(t)U(t,a), & t \in \mathcal{I}, \\ 0, & \text{else} \end{cases}$$

and

$$F(t) := \int_0^{a_{\max}} \beta(\alpha,x)S(\alpha,(\alpha-t)^+)\, f|_{\mathrm{char}(t-\alpha)}\,\mathrm{d}\alpha + \int_{\min(t,a_{\max})}^{a_{\max}} \beta(\alpha,x)U(\alpha,\alpha-t)y_0(\alpha-t)\,\mathrm{d}\alpha,$$

then our goal is to verify that $A \in L^\infty((0,T),L(H))$ and $F \in L^2((0,T),H)$. With the help of Lemma 3.6, we can directly calculate

$$\|A\|_{L^\infty((0,T),L(H))} = \operatorname*{ess\,sup}_{\alpha\in\mathcal{I}} \left\|\beta(\alpha)U(\alpha,0)\right\|_{L(H)} < \infty.$$

For F, again using Lemma 3.6 and the substitutions $t_0 = t - \alpha$, $s = \alpha - t$ we calculate

$$\begin{aligned}
\|F\|^2_{L^2((0,T),H)} &\leq \left\| \int_0^{a_{\max}} \beta(\alpha,x) S(\alpha,(\alpha-t)^+)\, f|_{\mathrm{char}(t-\alpha)} \,\mathrm{d}\alpha \right\|^2_{L^2((0,T),H)} \\
&\quad + \left\| \int_{\min(t,a_{\max})}^{a_{\max}} \beta(\alpha,x) U(\alpha,\alpha-t) y_0(\alpha-t) \,\mathrm{d}\alpha \right\|^2_{L^2((0,T),H)} \\
&\lesssim \int_0^T \int_0^{a_{\max}} \left\| f|_{\mathrm{char}(t-\alpha)} \right\|^2_{L^2(\mathrm{char}(t-\alpha),V')} \,\mathrm{d}\alpha\,\mathrm{d}t + \int_0^T \int_{\min\{t,a_{\max}\}}^{a_{\max}} \left\| y_0(\alpha-t) \right\|^2_H \,\mathrm{d}\alpha\,\mathrm{d}t \\
&= \int_0^T \int_{t-a_{\max}}^{t} \left\| f|_{\mathrm{char}(t_0)} \right\|^2_{L^2(\mathrm{char}(t_0),V')} \,\mathrm{d}t_0\,\mathrm{d}t + \int_0^T \int_{(a_{\max}-t)^+}^{a_{\max}-t} \left\| y_0(s) \right\|^2_H \,\mathrm{d}s\,\mathrm{d}t \\
&\lesssim T \int_{-a_{\max}}^{T} \left\| f|_{\mathrm{char}(t_0)} \right\|^2_{L^2(\mathrm{char}(t_0),V')} \,\mathrm{d}t_0 + T \int_0^{a_{\max}} \left\| y_0(s) \right\|^2_H \,\mathrm{d}s \\
&\lesssim \|f\|^2_{L^2((0,T),\mathcal{V}')} + \left\| y_0 \right\|^2_{\mathcal{H}}. \qquad \square
\end{aligned}$$

3.17 Remark. Suppose that $f = 0$, and for given $y_0 \in L^2(\mathcal{I},H)$, let $B := B(y_0,0)$ as in Theorem 3.16. Then, if we define

$$(\mathcal{T}(t)y_0)(a) := \begin{cases} U(a,0)B(t-a), & t > a, \\ U(a,a-t)y_0(a-t), & t \leq a, \end{cases}$$

one can show that $(\mathcal{T}(t))_{t\geq 0}$ is a strongly continuous semigroup of operators on $\mathcal{H}$ corresponding to the Volterra equation (3.24), see e.g. [Web08, Thm. 4]. It is possible to solve the inhomogeneous or, respectively, the nonlinear equation with this semigroup and Duhamel's formula. However, we proceed in a different manner, because for the relaxed equation, semigroup methods turn out to be suboptimal (cf. Remark 4.15), and for better comparability we wish to solve both equations using similar techniques.

3.18 Theorem. Suppose that Assumption 3.15 holds. Then there exists a unique weak solution $y \in L^2((0,T),\mathcal{V}) \cap C([0,T],\mathcal{H}) \cap C(\bar{\mathcal{I}}, L^2((0,T),H))$ with $\delta y \in L^2((0,T),\mathcal{V}')$ to eq. (3.23) in the sense given in Corollary 3.14. Furthermore, y satisfies the estimate

$$\left\|y\right\|^2_{L^2((0,T),\mathcal{V})} + \left\|y\right\|^2_{C([0,T],\mathcal{H})} + \left\|\delta y\right\|^2_{L^2((0,T),\mathcal{V}')} \lesssim \left\|y_0\right\|^2_{\mathcal{H}} + \|f\|^2_{L^2((0,T),\mathcal{V}')}. \tag{3.25}$$

Proof: Combine the estimates from eq. (3.19) and Theorem 3.10 with the results for B from Theorem 3.16. □

3.4 The nonlinear equation

In this section, we investigate the well-posedness of the original nonlinear model (3.1). To achieve this, we make the following assumptions.

3.19 Assumption. 1. Assumption 3.15 still holds. That is,

   1.1. $\beta \in L^\infty(\mathcal{I}, L^\infty(\Omega)^{n\times n})$, $f \in L^2((0,T),\mathcal{V}')$ and $y_0 \in \mathcal{H}$.

   1.2. $L \in C(\overline{\mathcal{I}}, L^\infty(\Omega))^{n\times n}$ and its entries are uniformly bounded away from zero with respect to both age and space.

   1.3. $\sigma \in C(\overline{\mathcal{I}}, \mathbb{R}^{n\times n})$ is a diagonal matrix whose entries are uniformly bounded away from zero with respect to age.

2. For every $u \in L^\infty((0,T)\times\mathcal{I}\times\Omega)$, the control input $K(u) \in L^\infty((0,T)\times\mathcal{I}\times\Omega)^{n\times n}$ depends linearly on u while not explicitly depending on time, age and space. Also, as in eq. (1.8), we assume the control u to be positive and bounded above by some $\bar{u} > 0$.

3. All infection kernels k^{hi} from eq. (1.1d) are elements of $L^\infty(\mathcal{I}\times\Omega,\mathcal{H})$.

Under these assumptions, Λ is a well-defined operator, since for all h, i and all $v \in \mathcal{H}$ the map

$$(a,x) \in \mathcal{I}\times\Omega \mapsto \left\langle k^{hi}(a,x,\cdot,\cdot), v\right\rangle_{\mathcal{H}}$$

is well-defined and belongs to $L^\infty(\mathcal{I}\times\Omega)$. This in turn means that $\Lambda(v)$ defined by eq. (1.1d) is an element of $L^\infty((0,T)\times\mathcal{I}\times\Omega)^{n\times n}$, which can be multiplied with elements from $\mathcal{H}$, the product again being an element of $\mathcal{H}$. Furthermore, this structure allows an estimate of the form

$$\|\Lambda(v_1)v_2\|_{\mathcal{H}} \le c(k)\|v_1\|_{\mathcal{H}}\|v_2\|_{\mathcal{H}} \tag{3.26}$$

for some constant c depending on k. Integrating this estimate on Λ with y_1, $y_2 \in L^2((0,T),\mathcal{H})$ over the interval $[0,t]$ where $t \in [0,T]$ yields

$$\left\|\Lambda(y_1)y_2\right\|_{L^2((0,t),\mathcal{H})} \lesssim \left\|y_1\right\|_{L^\infty((0,t),\mathcal{H})}\left\|y_2\right\|_{L^2((0,t),\mathcal{H})}, \tag{3.27a}$$

or, respectively,

$$\left\|\Lambda(y_1)y_2\right\|_{L^2((0,t),\mathcal{H})} \lesssim \left\|y_1\right\|_{L^2((0,t),\mathcal{H})}\left\|y_2\right\|_{L^\infty((0,t),\mathcal{H})}. \tag{3.27b}$$

This shows that for $y \in C([0,T],\mathcal{H})$, there is $\Lambda(y)y \in L^2((0,T),\mathcal{H})$. According to Theorem 3.18 this is regular enough for f in eq. (3.23) to be replaced by $\Lambda(y)y$. Formally this is done with a fixed-point argument.

3.20 Theorem. Suppose that Assumption 3.19 holds. Then there exists a $T^* > 0$ such that for almost every $T \in (0,T^*)$ the state equation

$$\begin{aligned}
\delta y + L(a,x)y + \Lambda(a,x,y)y + K(u)y &= \sigma(a)\Delta y,\\
y(t=0) &= y_0,\\
y(a=0) &= \int_0^{a_{\max}} \beta(\alpha,x)y(t,\alpha,x)\,\mathrm{d}\alpha,\\
\partial_\nu y(x\in\partial\Omega) &= 0
\end{aligned}$$

has a unique weak solution $y \in L^2((0,T),\mathcal{V}) \cap C([0,T],\mathcal{H}) \cap C(\bar{\mathcal{I}}, L^2((0,T),H))$ with $\delta y \in L^2((0,T),\mathcal{V}')$ satisfying the weak formulation

$$\left\langle \delta y, v\right\rangle_{V' \times V} + \left\langle Ly, v\right\rangle_H + \left\langle \Lambda(y)y, v\right\rangle_H + \left\langle K(u)y, v\right\rangle_H + \left\langle \sigma(a)\nabla y, \nabla v\right\rangle_{H^d} = 0 \tag{3.28}$$

for all $v \in V$ and almost all t and a. More precisely, one can choose

$$T^* = C\left(\left\|y_0\right\|_{\mathcal{H}}^2 + \bar{u}^2\right)^{-1}$$

with a constant C independent of y_0 and $\bar{u}$. Furthermore, we have an energy estimate of the form

$$\left\|y\right\|_{L^2((0,T),\mathcal{V})}^2 + \left\|y\right\|_{C([0,T],\mathcal{H})}^2 + \left\|\delta y\right\|_{L^2((0,T),\mathcal{V}')}^2 \lesssim \left\|y_0\right\|_{\mathcal{H}}^2. \tag{3.29}$$

Proof: The proof is based on Banach's fixed-point argument and is inspired by [Smo83, Thm. 14.2 and Lem. 14.3]. We start by showing uniqueness. If y^1 is a solution of the nonlinear equation to the initial value y_0^1 and the control u_1 and y^2 a solution to the initial value y_0^2 and control u_2, then the difference $w := y^1 - y^2$ satisfies

$$\begin{aligned}
\delta w + L(a,x)w &= \sigma(a)\Delta w + \Lambda(y^2)y^2 - \Lambda(y^1)y^1 + K(u_2)y^2 - K(u_1)y^1, \\
w(t=0) &= y_0^1 - y_0^2, \\
w(a=0) &= \int_0^{a_{\max}} \beta(\alpha,x)w(t,\alpha,x)\,\mathrm{d}\alpha, \\
\partial_\nu w(x \in \partial\Omega) &= 0.
\end{aligned}$$

Thus, by Theorem 3.18 and using eq. (3.27), we can write the estimate

$$\begin{aligned}
&\|w\|_{L^2((0,T),\mathcal{V})}^2 + \|w(t)\|_{\mathcal{H}}^2 + \|\delta w\|_{L^2((0,T),\mathcal{V}')}^2 \\
&\quad\lesssim \left\|y_0^1 - y_0^2\right\|_{\mathcal{H}}^2 + \left\|\Lambda(y^2)y^2 - \Lambda(y^1)y^1\right\|_{L^2((0,T),\mathcal{V}')}^2 + \left\|K(u_2)y^2 - K(u_1)y^1\right\|_{L^2((0,T),\mathcal{V}')}^2 \\
&\quad\lesssim \left\|y_0^1 - y_0^2\right\|_{\mathcal{H}}^2 + \left\|\Lambda(y^2)(y^2 - y^1)\right\|_{L^2((0,T),\mathcal{V}')}^2 + \left\|\Lambda(y^2 - y^1)y^1\right\|_{L^2((0,T),\mathcal{V}')}^2 \\
&\qquad + \left\|K(u_2)(y^2 - y^1)\right\|_{L^2((0,T),\mathcal{V}')}^2 + \left\|K(u_2 - u_1)y^1\right\|_{L^2((0,T),\mathcal{V}')}^2 \\
&\quad\lesssim \left\|y_0^1 - y_0^2\right\|_{\mathcal{H}}^2 + \max\left\{\left\|y^1\right\|_{L^\infty((0,T),\mathcal{H})}^2, \left\|y^2\right\|_{L^\infty((0,T),\mathcal{H})}^2\right\} \|w\|_{L^2((0,T),\mathcal{H})}^2 \\
&\qquad + \|u_2\|_{L^\infty}^2 \|w\|_{L^2((0,T),\mathcal{H})}^2 + \|u_2 - u_1\|_{L^\infty}^2 \left\|y^1\right\|_{L^2((0,T),\mathcal{H})}^2.
\end{aligned} \tag{3.30}$$

If we assume $\left\|y^i\right\|_{L^\infty((0,T),\mathcal{H})}$ $(i = 1,2)$ to be bounded above by $S > 0$, Gronwall's lemma yields the estimate

$$\|w(t)\|_{\mathcal{H}}^2 \lesssim \left(\left\|y_0^1 - y_0^2\right\|_{\mathcal{H}}^2 + S^2\|u_2 - u_1\|_{L^\infty}^2\right)\exp\left(\max\left\{S^2, \bar{u}^2\right\} T\right).$$

This estimate establishes the local (w.r.t the $L^\infty((0,T),\mathcal{H})$-norm) Lipschitz continuity of the solution operator with respect to the control and the initial function, and for $y_0^1 = y_0^2$ and $u_1 = u_2$ yields uniqueness in $L^\infty((0,T),\mathcal{H})$.

Next we prove the existence of a solution. For this purpose, we define the set

$$\Gamma := \left\{ y \in C([0,T],\mathcal{H}) \;\middle|\; \left\|y - \bar{y}\right\|_{L^\infty((0,T),\mathcal{H})}^2 \le \left\|y_0\right\|_{\mathcal{H}}^2 \right\}$$

where $\bar{y}$ is the solution of the corresponding linear equation, i.e.

$$\begin{gathered}\delta\bar{y} + L(a,x)\bar{y} - \sigma(a)\Delta\bar{y} = 0,\\ \bar{y}(t=0) = y_0, \quad \bar{y}(a=0) = \int\limits_0^{a_{\max}} \beta(\alpha,x)y(t,\alpha,x)\,\mathrm{d}\alpha,\\ \partial_\nu\bar{y}(x \in \partial\Omega) = 0.\end{gathered}$$

According to Theorem 3.18, $\bar{y}$ satisfies the estimate

$$\left\|\bar{y}\right\|^2_{L^2((0,T),\mathcal{V})} + \left\|\bar{y}\right\|^2_{C([0,T],\mathcal{H})} + \left\|\delta\bar{y}\right\|^2_{L^2((0,T),\mathcal{V}')} \lesssim \left\|y_0\right\|^2_{\mathcal{H}}. \tag{3.31}$$

Note that Γ is a closed subset of $L^\infty((0,T),\mathcal{H}))$. For any $y \in \Gamma$ we can estimate

$$\left\|y(t)\right\|^2_{\mathcal{H}} \le \left\|y(t) - \bar{y}\right\|^2_{\mathcal{H}} + \left\|\bar{y}\right\|^2_{\mathcal{H}} \lesssim \left\|y_0\right\|^2_{\mathcal{H}}, \tag{3.32}$$

which shows that Γ is also a bounded subset of $C([0,T],\mathcal{H})$.

We also define the mapping $\Phi : \Gamma \to \Gamma$, which maps any $y \in \Gamma$ to the solution $v := \Phi(y)$ of

$$\begin{gathered}\delta v + L(a,x)v + \Lambda(a,x,y)y + K(u)y = \sigma(a)\Delta v,\\ v(t=0) = y_0, \quad v(a=0) = \int\limits_0^{a_{\max}} \beta(\alpha,x)v(t,\alpha,x)\,\mathrm{d}\alpha,\\ \partial_\nu v(x \in \partial\Omega) = 0.\end{gathered}$$

Note that $\bar{y} = \Phi(0)$.

Now, for any y^1, $y^2 \in \Gamma$, we have for $w := \Phi(y^1) - \Phi(y^2)$ with similar computations as in eq. (3.30) that

$$\begin{aligned}&\|w\|^2_{L^\infty((0,T),\mathcal{H})}\\ &\lesssim \max\left\{\left\|y^1\right\|^2_{L^\infty((0,T),\mathcal{H})}, \left\|y^2\right\|^2_{L^\infty((0,T),\mathcal{H})}\right\} \left\|y^2 - y^1\right\|^2_{L^2((0,T),\mathcal{H})} + \|u\|_{L^\infty}\left\|y^2 - y^1\right\|^2_{L^2((0,T),\mathcal{H})}\\ &\lesssim \left(\left\|y_0\right\|^2_{\mathcal{H}} + \bar{u}^2\right) T \left\|y^1 - y^2\right\|^2_{L^\infty((0,T),\mathcal{H})}.\end{aligned}$$

In the last line, we have used eq. (3.32) for y^1, $y^2 \in \Gamma$. Letting $y^2 = 0$ shows that for sufficiently small T^* proportional to $\left(\left\|y_0\right\|^2_{\mathcal{H}} + \bar{u}^2\right)^{-1}$, the image of the map Φ is again contained in Γ, since then we have

$$\left\|\Phi(y^1) - \bar{y}\right\|^2_{L^\infty((0,T),\mathcal{H})} \lesssim \left(\left\|y_0\right\|^2_{\mathcal{H}} + \bar{u}^2\right) T \left\|y^1\right\|^2_{L^\infty((0,T),\mathcal{H})} \lesssim \left(\left\|y_0\right\|^2_{\mathcal{H}} + \bar{u}^2\right) T \left\|y_0\right\|^2_{\mathcal{H}}.$$

It also shows that for sufficiently small $T^* \le T$, which again can be chosen proportional to $\left(\left\|y_0\right\|^2_{\mathcal{H}} + \bar{u}^2\right)^{-1}$, the mapping Φ is a contraction. Consequently, the existence of a fixed point $\bar{y} \in \Gamma$ follows from Banach's fixed-point theorem.

Furthermore, we can deduce even higher regularity and the corresponding energy estimates for the solution. Using eq. (3.31), eq. (3.30) and the fact that $\Phi(y) = y \in \Gamma$

yields

$$
\begin{aligned}
&\|y\|^2_{L^2((0,T),\mathcal{V})} + \|y\|^2_{C([0,T],\mathcal{H})} + \|\delta y\|^2_{L^2((0,T),\mathcal{V}')} \\
&\quad \lesssim \|y-\bar{y}\|^2_{L^2((0,T),\mathcal{V})} + \|y-\bar{y}\|^2_{C([0,T],\mathcal{H})} + \|\delta y - \bar{y}\|^2_{L^2((0,T),\mathcal{V}')} + \|y_0\|^2_{\mathcal{H}} \\
&\quad \lesssim \max\left\{\|y\|^2_{L^\infty((0,T),\mathcal{H})}, \|\bar{y}\|^2_{L^\infty((0,T),\mathcal{H})}, \bar{u}^2\right\} \|y-\bar{y}\|^2_{L^2((0,T),\mathcal{H})} + \|y_0\|^2_{\mathcal{H}} \\
&\quad \lesssim \max\left\{\|y_0\|^2_{\mathcal{H}}, \bar{u}^2\right\} T \|y-\bar{y}\|^2_{L^\infty((0,T),\mathcal{H})} + \|y_0\|^2_{\mathcal{H}},
\end{aligned}
$$

and by choosing T proportional to $\left(\|y_0\|^2_{\mathcal{H}} + \bar{u}^2\right)^{-1}$ and using $y \in \Gamma$ we finally arrive at

$$
\|y\|^2_{L^2((0,T),\mathcal{V})} + \|y\|^2_{C([0,T],\mathcal{H})} + \|\delta y\|^2_{L^2((0,T),\mathcal{V}')} \lesssim \|y_0\|^2_{\mathcal{H}},
$$

the energy estimate we wanted to show. □

3.21 Remark. The following integrated weak formulation is often more useful for optimal control purposes: For any $v \in L^2((0,T),\mathcal{V})$ we have

$$
\begin{aligned}
\langle \delta y, v\rangle_{L^2((0,T),\mathcal{V}')\times L^2((0,T),\mathcal{V})} + \langle Ly, v\rangle_{L^2((0,T),\mathcal{H})} + \langle \Lambda(y)y, v\rangle_{L^2((0,T),\mathcal{H})} \\
+ \langle K(u)y, v\rangle_{L^2((0,T),\mathcal{H})} + \langle \sigma(a)\nabla y, \nabla v\rangle_{L^2((0,T),\mathcal{H})^d} = 0.
\end{aligned}
\tag{3.33}
$$

The proof that this is equivalent to eq. (3.28) can be done in a similar fashion to [HPUU09, Thm. 1.33].

3.5 Addendum: Analysis of the state space of the unrelaxed equation

In Theorem 3.13, we established that the solution y to eq. (3.9) lies in the space

$$
\mathbb{V} := \left\{ f \in L^2((0,T),\mathcal{V}) \;\middle|\; \exists \delta f \in L^2((0,T),\mathcal{V}') \right\}.
$$

In this short section we analyze this space in more detail, prove a trace theorem and an analogon to Lemma 2.11. Similar to [Eva10, Sec. 5.3], we can show that the space

$$
\mathbb{V} \cap C^1([0,T]\times[0,a_{\max}],V)
$$

is dense in $\mathbb{V}$.

3.22 Theorem. Let $\varepsilon > 0$. Then there exists a bounded trace operator $\operatorname{tr} : \mathbb{V} \to L^2((0,T-\varepsilon),H)$ with the property that for every $f \in C^1([0,T]\times[0,a_{\max}],V)$ it holds that $\operatorname{tr} f = f(\cdot, a=0)|_{[0,T-\varepsilon]}$.

Proof: The proof is adapted from [Eva10, Thm. 5.5.1]. First assume $f \in \mathbb{V} \cap C^1([0,T] \times [0,a_{\max}], V)$ and let $\varphi_\varepsilon : [0,\varepsilon] \to \mathbb{R}$, $s \mapsto 1 - \varepsilon^{-1}s$, then from Lemma 2.12 we infer that

$$\begin{aligned}
\|f(\cdot,0)\|^2_{L^2((0,T-\varepsilon),H)} &= \int_0^{T-\varepsilon} \|f(t,0)\|_H^2 \,\mathrm{d}t \\
&= -\int_0^{T-\varepsilon} \big\langle \varphi_\varepsilon(\varepsilon) f(t+\varepsilon,\varepsilon), f(t+\varepsilon,\varepsilon)\big\rangle_H - \big\langle \varphi_\varepsilon(0) f(t,0), f(t,0)\big\rangle_H \,\mathrm{d}t \\
&= -\int_0^{T-\varepsilon}\int_0^{\varepsilon} \left\langle \frac{\mathrm{d}}{\mathrm{d}a}(\varphi_\varepsilon(a) f(t+a,a)), f(t+a,a)\right\rangle_{V'\times V} \\
&\quad + \left\langle \frac{\mathrm{d}}{\mathrm{d}a} f(t+a,a), \varphi_\varepsilon(a) f(t+a,a)\right\rangle_{V'\times V} \mathrm{d}a\,\mathrm{d}t \\
&= -\int_0^{T-\varepsilon}\int_0^{\varepsilon} \big\langle \varphi_\varepsilon'(a) f(t+a,a) + \varphi_\varepsilon(a)\delta f(t+a,a), f(t+a,a)\big\rangle_{V'\times V} \\
&\quad + \big\langle \delta f(t+a,a), \varphi_\varepsilon(a) f(t+a,a)\big\rangle_{V'\times V} \,\mathrm{d}a\,\mathrm{d}t \\
&\leq \int_0^{T-\varepsilon}\int_0^{\varepsilon} \varepsilon^{-1}\|f(t+a,a)\|_H^2 + 2\|f(t+a,a)\|_V \|\delta f(t+a,a)\|_{V'} \,\mathrm{d}a\,\mathrm{d}t \\
&\leq 2\varepsilon^{-1}\|f\|^2_{L^2((0,T),\mathcal{V})} + \varepsilon\|\delta f\|^2_{L^2((0,T),\mathcal{V}')}.
\end{aligned} \tag{3.34}$$

The general case then follows by approximating elements $f \in \mathbb{V}$ with sequences $(f_n)_{n\in\mathbb{N}} \subseteq \mathbb{V} \cap C^1([0,T]\times[0,a_{\max}],V)$ and letting $\operatorname{tr} f = \lim_{n\to\infty} \operatorname{tr} f_n$. This is well-defined: If $(\tilde{f}_n)_{n\in\mathbb{N}}$ is another sequence approximating f, we have the estimate

$$\left\|\operatorname{tr} f_n - \operatorname{tr} \tilde{f}_n\right\|^2_{L^2((0,T-\varepsilon),H)} \leq 2\varepsilon^{-1}\left\|f_n - \tilde{f}_n\right\|^2_{L^2((0,T),\mathcal{V})} + \varepsilon\left\|\delta f_n - \delta \tilde{f}_n\right\|^2_{L^2((0,T),\mathcal{V}')} \to 0.$$

This shows that the definition of $\operatorname{tr} f$ does not depend on the approximating sequence. Furthermore, the same estimate $\|\operatorname{tr} f\|^2_{L^2((0,T-\varepsilon),H)} \leq 2\varepsilon^{-1}\|f\|^2_{L^2((0,T),\mathcal{V})} + \varepsilon\|\delta f\|^2_{L^2((0,T),\mathcal{V}')}$ holds. □

In a similar way we can define a trace operator from $\mathbb{V}$ to $L^2((0,a_{\max}-\varepsilon),H)$ that maps any continuous f to $f(t=0,\cdot)$. The additional parameter ε is necessary to ensure that $t+a$ is still bounded by T if we let a vary from 0 to ε. Intuitively, this problem arises from the fact that in the corner at $(t,a)=(T,0)$ (cf. Figure 1), the length of the characteristics approaches zero. This would require the function φ_ε to become infinitely steep, increasing the value of its derivative and rendering an estimate of the form $|\varphi_\varepsilon'| < \varepsilon^{-1}$ impossible. However, it is possible to avoid the need of introducing ε by, for example, assuming that it holds $f(t=T)\equiv 0$ instead: simply extend f by zero on the strip $[T,T+\varepsilon]$.

The trace result can be used to obtain a result on embeddings:

3.23 Theorem. For every ε the space $\mathbb{V}$ embeds continuously into $C([\varepsilon,T-\varepsilon],\mathcal{H})$.

Proof: We mimic the arguments of [RR04, Lem. 11.4]. It is sufficient to show the claim on strips with width $a_{\max}/2$ and iterating the argument. For simplicity, we show the claim on the strip $[0, a_{\max}/2]$. Let $f \in \mathbb{V} \cap C^1([0, a_{\max}/2] \times [0, a_{\max}], V)$, and let $0 < t^* \le t < a_{\max}/2$, then with eq. (3.34) it follows

$$\begin{aligned}
\|f(t,\cdot)\|_{\mathcal{H}}^2 &= \int_0^{a_{\max}} \|f(t,a)\|_H^2 \,\mathrm{d}a = \int_0^{t-t^*} \|f(t,a)\|_H^2 \,\mathrm{d}a + \int_{t-t^*}^{a_{\max}} \|f(t,a)\|_H^2 \,\mathrm{d}a \\
&= \int_0^{t-t^*} \Bigg(\|f(t-a,0)\|_H^2 + \int_0^a \big\langle \delta f(t-a+h,h), f(t-a+h,h)\big\rangle_{V'\times V} \,\mathrm{d}h \Bigg) \mathrm{d}a \\
&\quad + \int_{t-t^*}^{a_{\max}} \Bigg(\|f(t^*, a-t+t^*)\|_H^2 \\
&\quad + \int_0^{t-t^*} \big\langle \delta f(t^*+h, a-t+t^*+h), f(t^*+h, a-t+t^*+h)\big\rangle_{V'\times V} \,\mathrm{d}h \Bigg) \mathrm{d}a \\
&= \|f(\cdot,0)\|^2_{L^2((t^*,t),H)} + \|f(t^*,\cdot)\|^2_{L^2((0,a_{\max}-t+t^*),H)} \\
&\quad + \int_0^{a_{\max}} \int_0^{\min(a,t-t^*)} \langle \delta f(t-\min(a,t-t^*)+h, a-\min(a,t-t^*)+h), \\
&\qquad\qquad f(t-\min(a,t-t^*)+h, a-\min(a,t-t^*)+h)\rangle_{V'\times V} \,\mathrm{d}h\,\mathrm{d}a \\
&\le \|f(\cdot,0)\|^2_{L^2((t^*,t),H)} + \|f(t^*,\cdot)\|^2_{\mathcal{H}} + \|\delta f\|_{L^2((t^*,t),\mathcal{V}')} \|f\|_{L^2((t^*,t),\mathcal{V})}.
\end{aligned}$$

With the intermediate value theorem we now choose t^* in such a way that the equation

$$\|f(t^*,\cdot)\|^2_{\mathcal{H}} = t^{-1} \int_0^t \|f(s,\cdot)\|^2_{\mathcal{H}} \,\mathrm{d}s = t^{-1} \|f\|^2_{L^2((0,t),\mathcal{H}}.$$

is satisfied. Furthermore, using eq. (3.34), we estimate

$$\|f(\cdot,0)\|^2_{L^2((t^*,t),H)} \le 2\varepsilon^{-1} \|f\|^2_{L^2((0,a_{\max}/2+\varepsilon),\mathcal{V})} + \varepsilon \|\delta f\|^2_{L^2((0,a_{\max}/2+\varepsilon),\mathcal{V}')}.$$

Putting these estimates together we obtain, using Young's inequality,

$$\begin{aligned}
\|f(t,\cdot)\|^2_{\mathcal{H}} &\le \|f(\cdot,0)\|^2_{L^2((t^*,t),H)} + \|f(t^*,\cdot)\|^2_{\mathcal{H}} + \|\delta f\|_{L^2((t^*,t),\mathcal{V}')} \|f\|_{L^2((t^*,t),\mathcal{V})} \\
&\le 2\varepsilon^{-1} \|f\|^2_{L^2((0,a_{\max}/2+\varepsilon),\mathcal{V})} + \varepsilon \|\delta f\|^2_{L^2((0,a_{\max}/2+\varepsilon),\mathcal{V}')} + t^{-1} \|f\|^2_{L^2((0,t),\mathcal{H}} \\
&\quad + \frac{\varepsilon}{2} \|\delta f\|^2_{L^2((t^*,t),\mathcal{V}')} + \frac{1}{2\varepsilon} \|f\|^2_{L^2((t^*,t),\mathcal{V})}.
\end{aligned}$$

Assuming that $t \ge \varepsilon$, i.e. t remains strictly away from zero, this leads to an estimate of the form

$$\|f(t,\cdot)\|^2_{\mathcal{H}} \lesssim \varepsilon^{-1} \|f\|^2_{L^2((0,a_{\max}/2+\varepsilon),\mathcal{V})} + \varepsilon \|\delta f\|^2_{L^2((0,a_{\max}/2+\varepsilon),\mathcal{V}')}.$$

A similar approximation argument as in the proof of Theorem 3.22 shows the claim for general f, which concludes the proof. □

4 The relaxed equation

After the discussion of the unrelaxed equation where no second-order time or age derivative is present, we are now ready to introduce a relaxation parameter τ, which yields the relaxed equation

$$(1+\tau\delta)(\delta y + Ly + \Lambda(y)y) = \sigma(a)\Delta y, \tag{4.1a}$$

$$y(t=0) = y_0, \quad \delta y(t=0) = y_1, \quad \partial_\nu y(x \in \partial\Omega) = 0, \tag{4.1b}$$

$$y(t,a=0) = \int_0^{a_{\max}} \beta_0(\alpha,x)y(t,\alpha,x)\,\mathrm{d}\alpha + g_0(t,x), \tag{4.1c}$$

$$(\delta y + Ly + \Lambda(y)y)(t,a=0,x) = \int_0^{a_{\max}} \beta_1(\alpha,x)(\delta y + Ly + \Lambda(y)y)(t,\alpha,x)\,\mathrm{d}\alpha + g_1(t,x) \tag{4.1d}$$

presented in eq. (1.6). In order to establish the existence of a solution, we proceed in a similar fashion to the previous chapter: pass to characteristic lines, solve a Volterra equation, obtain a solution to the nonlinear equation with Banach's fixed-point theorem. However, we have to be more careful with the regularities of spaces and initial conditions, and we also have to precisely track every occurrence of τ, since we also want to determine a convergence rate for $\tau \to 0$.

4.1 Motivation

The diffusion models discussed in the previous chapter have the property of *infinite propagation speed*, meaning that if initially the quantity is solely concentrated in a subdomain of Ω and zero everywhere else, for any positive time, no matter how small, the concentration is nonzero over the whole of Ω. While this may be a suitable model for e.g. heat dissipation, this property is unrealistic when the quantity u describes things like moving populations that clearly move at finite speeds. A common way to bypass this problem is to introduce a delay, which amounts to modifying the flux equation from eq. (3.6) to

$$Q(t+\tau) = -\sigma\nabla u(t),$$

where $\tau > 0$ is the time the quantity needs to perceive the gradient and move accordingly. Unfortunately, it has been shown (cf. [DQR09], or [Rac12] for a more complex model) that this model is not well-posed. However, by formally applying the first-order Taylor expansion $Q(t+\tau) \approx Q(t) + \tau Q_t(t)$, together with eq. (3.5) the complete model becomes

$$\begin{cases} u_t + \operatorname{div} Q = f \\ (1+\tau\partial_t)Q = -\sigma\nabla u \end{cases} \quad \Rightarrow \quad (1+\tau\partial_t)(u_t - f) = -(1+\tau\partial_t)\operatorname{div} Q = \sigma\Delta u, \tag{4.2}$$

which is a damped wave equation (also sometimes called telegrapher's equation). This now hyperbolic equation can be shown to be well-posed (e.g. with Theorem 2.16) and in addition to possess a finite speed of propagation, cf. [Rac15, Rem. at Fig. 3.2]. This property is favorable in the kind of model we consider in this work. As in eq. (3.7), we require the boundary condition $\nu \cdot Q = 0$, then since also $\nu \cdot Q_t = 0$ we can conclude that

$$0 = \nu \cdot (1 + \tau\partial_t)Q = -\sigma\nu \cdot \nabla u = -\sigma\partial_\nu u,$$

and we again have Neumann boundary conditions.

If we want to apply the relaxation from eq. (4.2) to eq. (1.1a), we have to take into account that the population ages with time. Hence, τ has to be introduced in both the time and the age variable, and a formal Taylor equation yields yields

$$(1 + \tau\delta)\big(\delta y(t,a,x) + L(a,x)y(t,a,x) + \Lambda(a,x,y(t,\cdot))y(t,a,x)\big) = \sigma(a)\Delta y(t,a,x),$$

the same as eq. (4.1b). Since this is now a second-order equation, we need additional initial/boundary conditions where t or a is zero. For the boundary at $t = 0$, we can simply impose another initial condition on δy. The $a = 0$-boundary is more complex because of the implicit condition

$$y(t,a=0,x) = \int_0^{a_{\max}} \beta(\alpha,x)y(t,\alpha,x)\,\mathrm{d}\alpha =: B_0(t,x) \tag{4.3}$$

we saw in eq. (1.1c). However, it can be used to derive a similar implicit condition for $B_1 := \delta y(a = 0)$: Looking back to eq. (1.1a) (neglecting the control parameter u) and setting a to zero gives

$$B_1(t,x) + L(0,x)B_0(t,x) + \Lambda(0,x,y(t,\cdot))B_0(t,x) = -\sigma(a)\Delta B_0(t,x). \tag{4.4}$$

While this holds for the unrelaxed equation, this exact formulation does not work for the relaxed equation due to regularity issues (in particular, the spatial regularity of ΔB_0 will always be worse than that of B_1). However, there are two ways around this:

(1) One simple workaround is to choose the birth number of the *unrelaxed* model in the boundary condition for the *relaxed* model. If y^0 is a solution of eq. (1.1) and $B_0^0 := y^0(a = 0)$, then one can choose

$$B_1(t,x) + L(0,x)B_0(t,x) + \Lambda(0,x,y^0(t,\cdot))B_0(t,x) = -\sigma(0)\Delta B_0^0(t,x) \tag{4.5}$$

to determine B_1. This essentially means that instead of determining the value of δy at $a = 0$ implicitly, we just prescribe it in an explicit way. This boundary condition works great when comparing the two model solutions y and y^0, but is a little artificial.

(2) For the second possibility we assume that β does not depend on space. Using eq. (4.3) in the term with the Laplacian of eq. (4.4) yields

$$\begin{aligned} &B_1(t,x) + L(0,x)B_0(t,x) + \Lambda(0,x,y(t,\cdot))B_0(t,x) \\ &= -\sigma(0)\Delta \int_0^{a_{\max}} \beta(\alpha)y(t,\alpha,x)\,\mathrm{d}\alpha \end{aligned}$$

$$
\begin{aligned}
&= -\sigma(0) \int_0^{a_{\max}} \beta(\alpha)\sigma(\alpha)^{-1}\sigma(\alpha)\Delta y(t,\alpha,x)\,\mathrm{d}\alpha \\
&= \int_0^{a_{\max}} \sigma(0)\beta(\alpha)\sigma(\alpha)^{-1}(\delta y(t,\alpha,x) + L(\alpha,x)y(t,\alpha,x) + \Lambda(\alpha,x,y(t,\cdot)))\,\mathrm{d}\alpha. \qquad (4.6)
\end{aligned}
$$

This condition naturally holds for the unrelaxed model, and it turns out to be a valid choice for the relaxed one. Letting $P(t,a,x) := \delta y(t,a,x) + L(a,x)y(t,a,x) + \Lambda(a,x,y(t,\cdot))$ as in eq. (1.5), this condition is written more compactly as

$$
P(t,0,x) = \int_0^{a_{\max}} \sigma(0)\beta(\alpha)\sigma(\alpha)^{-1}P(t,\alpha,x)\,\mathrm{d}\alpha, \qquad (4.7)
$$

similar to eq. (1.4). This condition is more natural, but due to its nonlinearity poses more problems in what follows.

A third choice for a boundary condition would be to choose B_1 in such a way that $y(t,a_{\max},x) = 0$ almost everywhere. This property can be shown to hold for simple unrelaxed models, cf. [AAC11, Thm. 4.2], and would also come in handy later in Section 4.5. However, it is unclear whether this is always possible, and thus is not pursued here any further.

To allow for more generality, and later to be able to prove the convergence results from Section 4.6, we allow a combination of implicit and explicit conditions for $\delta y(a=0)$. Instead of only allowing terms of the form $\sigma(0)\Delta B_0^0$ as in eq. (4.5), we allow general functions g_1, and instead of $\sigma(0)\beta\sigma^{-1}$ from eq. (4.6) we allow general rates β_1 that also may depend on x. Combined, we obtain the condition

$$
\begin{aligned}
&B_1(t,x) + L(0,x)B_0(t,x) + \Lambda(0,x,y^0(t,\cdot))B_0(t,x) \\
&= \int_0^{a_{\max}} \beta_1(\alpha,x)(\delta y + Ly + \Lambda(y)y)(t,\alpha,x)\,\mathrm{d}\alpha + g_1(t,x),
\end{aligned}
$$

which is an equivalent formulation of eq. (4.1d). By also adding an explicit term g_0 to eq. (1.1c), which will be needed in Section 4.6), we obtain the general model (4.1).

4.2 Population movement in two dimensions

This section is motivated by [AP18], [BP21, Sec. 2.1] and [Hol93, Appendix A], which discuss the one-dimensional case. It serves as an alternative motivation for the relaxed equation and provides some new insights.

Consider an age-structured population moving in two-dimensional space $\mathbb{R}^2$, which we identify with $\mathbb{C}$ in the canonical way. For any $\varphi \in [0,2\pi)$ let $p^{\varphi}(t,a,z)$ be the density of the population that at time t has age a, is located in $z = x + iy$ and moves in the direction $e^{i\varphi}$. The speed $c = c(a)$ of the individuals is allowed to change with age. We further assume that individuals change their angle of direction by the amount ψ at the rate $\lambda_{\psi} \geq 0$ and that, in addition, there is a semilinear reaction term $f(p)p^{\varphi}$ which may depend on the total number $p = \int_0^{2\pi} p^\varphi \,\mathrm{d}\varphi$ of individuals. Then for small times Δt one obtains the equation

$$p^\varphi(t+\Delta t, a+\Delta t, z+c(a)e^{i\varphi}\Delta t) - p^\varphi(t,a,z) = \Delta t\left(\int\limits_0^{2\pi} \lambda_\psi (p^{\varphi-\psi} - p^\varphi)\,\mathrm{d}\psi + f(p)p^\varphi\right).$$

Division by Δt and then letting $\Delta t \to 0$ leads us to the following differential equation, where for brevity we suppress the variables t, a, z:

$$(\partial_t + \partial_a + c\cos(\varphi)\partial_x + c\sin(\varphi)\partial_y)p^\varphi = \int\limits_0^{2\pi} \lambda_\psi (p^{\varphi-\psi} - p^\varphi)\,\mathrm{d}\psi + f(p)p^\varphi, \tag{4.8}$$

which is an uncountably-infinite hyperbolic system. Note, however, that φ is 2π-periodic, and it turns out that calculating the Fourier series over φ yields a simpler structure. To this end, for any $j \in \mathbb{Z}$ let

$$\hat{p}^j := \int\limits_0^{2\pi} e^{ij\varphi} p^\varphi \,\mathrm{d}\varphi, \quad \hat{\lambda}_j := \int\limits_0^{2\pi} e^{ij\varphi}\lambda_\psi \,\mathrm{d}\psi. \tag{4.9}$$

From now on we assume that $\lambda_{-\psi} = \lambda_\psi$, i.e. individuals are just as likely to turn to the left as to the right, the turn rate only depends on the magnitude of the angle. This leads to

$$\hat{\lambda}_{-j} = \hat{\lambda}_j. \tag{4.10}$$

Since λ_ψ is a real function, we also have $\hat{\lambda}_{-j} = \overline{\hat{\lambda}_j}$, thus all Fourier coefficients of λ are real numbers as well.

Multiplying eq. (4.8) with $e^{ij\varphi}$ ($j \in \mathbb{Z}$) yields

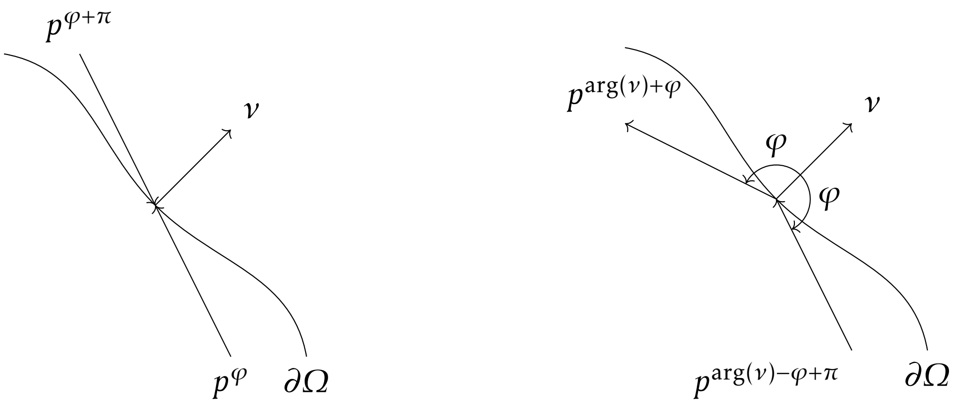
$$\begin{aligned}&\int\limits_0^{2\pi} p_t^\varphi e^{ij\varphi} + p_a^\varphi e^{ij\varphi} + c\cos(\varphi)e^{ij\varphi}p_x^\varphi + c\sin(\varphi)e^{ij\varphi}p_y^\varphi \,\mathrm{d}\varphi \\ &= \int\limits_0^{2\pi}\int\limits_0^{2\pi} \lambda_\psi e^{ij\varphi}(p^{\varphi-\psi} - p^\varphi)\,\mathrm{d}\psi\,\mathrm{d}\varphi + \int\limits_0^{2\pi} e^{ij\varphi} f(p) p^\varphi \,\mathrm{d}\varphi.\end{aligned}$$

Using

$$\sin(\varphi)e^{ij\varphi} = \frac{e^{i(j+1)\varphi} - e^{i(j-1)\varphi}}{2i}, \quad \cos(\varphi)e^{ij\varphi} = \frac{e^{i(j+1)\varphi} + e^{i(j-1)\varphi}}{2},$$

and the substitution $\varphi \rightsquigarrow \varphi + \psi$ in the integral involving $p^{\varphi-\psi}$, we obtain

$$\int\limits_0^{2\pi} p_t^\varphi e^{ij\varphi} + p_a^\varphi e^{ij\varphi} + \frac{c}{2}\left(e^{i(j+1)\varphi} + e^{i(j-1)\varphi}\right)p_x^\varphi + \frac{c}{2i}\left(e^{i(j+1)\varphi} - e^{i(j-1)\varphi}\right)p_y^\varphi \,\mathrm{d}\varphi$$



(a) U-turn (b) Reflection

Figure 3: Possible boundary conditions for p^{φ}.

$$= \int_0^{2\pi}\int_0^{2\pi} \lambda_{\psi} e^{ij\varphi}(e^{ij\psi}-1)p^{\varphi}\,\mathrm{d}\psi\,\mathrm{d}\varphi + f(p)\cdot\int_0^{2\pi} e^{ij\varphi}p^{\varphi}\,\mathrm{d}\varphi,$$

which translates into

$$\hat{p}_t^j + \hat{p}_a^j + \frac{c}{2}\left(\hat{p}_x^{j+1} + \hat{p}_x^{j-1} - i\hat{p}_y^{j+1} + i\hat{p}_y^{j-1}\right) + (\hat{\lambda}_0 - \hat{\lambda}_j)\hat{p}^j = f(\hat{p}^0)\hat{p}^j \tag{4.11}$$

for all $j \in \mathbb{Z}$. Note that p equals $\hat{p}^0$, and that because of eq. (4.10) we have

$$\hat{\lambda}_0 - \hat{\lambda}_j = \mathfrak{Re}(\hat{\lambda}_0 - \hat{\lambda}_j) = \int_0^{2\pi}(1-\cos(j\psi))\lambda(\psi)\,\mathrm{d}\psi \geq 0, \tag{4.12}$$

since λ was assumed to be nonnegative. Introducing the so-called *Wirtinger derivatives* (which are well-known objects in the theory of complex variables, see for example [Ahl79, p. 27])

$$\partial f := \frac{1}{2}(\partial_x f - i\partial_y f), \quad \bar{\partial} f := \frac{1}{2}(\partial_x f + i\partial_y f),$$

eq. (4.11) can be written more compactly:

$$\hat{p}_t^j + \hat{p}_a^j + c\partial\hat{p}^{j+1} + c\bar{\partial}\hat{p}^{j-1} + (\hat{\lambda}_0 - \hat{\lambda}_j)\hat{p}^j = f(\hat{p}^0)\hat{p}^j. \tag{4.13}$$

This is still a hyperbolic system with infinitely many coupled equations, but the number of equations is now countable and the coupling term for $\hat{p}^j$ now only involves the adjacent values $\hat{p}^{j\pm 1}$. Neglecting the reaction term, it is possible to prove a Cauchy–Kovalevskaya type existence result for eq. (4.13), cf. Theorem 4.24.

In case the movement of the population is limited to a space domain $\Omega \subseteq \mathbb{C}$, we need to consider boundary conditions. Let $\nu : \partial\Omega \to \{z \in \mathbb{C} \mid |z| = 1\}$ the outer normal of Ω. Two natural boundary conditions are illustrated in Figure 3. The U-turn condition

$$\mathfrak{Re}(\nu e^{-i\varphi})(p^{\varphi} - p^{\varphi+\pi}) = 0$$

models that upon hitting the boundary, individuals do a 180-degree turn, except when moving in a tangential direction, in which case the term $\mathfrak{Re}(\nu e^{-i\varphi})$ becomes zero. The other, perhaps more natural condition is the reflection condition

$$p^{\arg(\nu)+\varphi} - p^{\arg(\nu)-\varphi+\pi} = 0$$

which means that incoming individuals are reflected like light on a mirror, the angle of incidence being equal to the angle of reflection with respect to the normal vector. For any $j \in \mathbb{Z}$, calculating the Fourier series yields the transformed conditions

$$\begin{aligned}
0 &= \int_0^{2\pi} e^{ij\varphi}(\nu e^{-i\varphi} + \bar{\nu} e^{i\varphi})(p^{\varphi} - p^{\varphi+\pi})\,\mathrm{d}\varphi \\
&= \int_0^{2\pi} (\nu e^{i(j-1)\varphi} + \bar{\nu} e^{i(j+1)\varphi}) p^{\varphi}\,\mathrm{d}\varphi - \int_0^{2\pi} (\nu e^{i(j-1)(\varphi-\pi)} + \bar{\nu} e^{i(j+1)(\varphi-\pi)}) p^{\varphi}\,\mathrm{d}\varphi \\
&= (1 + (-1)^j)(\nu \hat{p}^{j-1} + \bar{\nu}\hat{p}^{j+1})
\end{aligned} \tag{4.14}$$

for the U-turn condition and

$$\begin{aligned}
0 &= \int_0^{2\pi} e^{ij\varphi}(p^{\arg(\nu)+\varphi} - p^{\arg(\nu)-\varphi+\pi})\,\mathrm{d}\varphi \\
&= \int_0^{2\pi} e^{ij(\varphi-\arg(\nu))} p^{\varphi}\,\mathrm{d}\varphi - \int_0^{2\pi} e^{ij(\varphi+\arg(\nu)+\pi)} p^{-\varphi}\,\mathrm{d}\varphi \\
&= \bar{\nu}^j \hat{p}^j - (-\nu)^j \hat{p}^{-j}
\end{aligned} \tag{4.15}$$

for the reflection condition. If we neglect the age structure, a unique solution to eq. (4.13) with boundary condition eq. (4.14) can be constructed with the theory of semigroups, cf. Corollary 4.34.

Another important boundary is the one where $a = 0$. Similar to eq. (4.3), we want to impose implicit birth conditions. Assuming that individuals at age a give birth to a number of $\beta(a,\psi)$ newborns whose angle of movement differ to the one of the parent by ψ, we obtain the boundary condition

$$p^{\varphi}(t, a = 0, z) = \int_0^{2\pi} \int_0^{a_{\max}} \beta(\alpha, \varphi - \psi) p^{\psi}(t, \alpha, z)\,\mathrm{d}\alpha\,\mathrm{d}\psi.$$

Calculating the Fourier series gives

$$\begin{aligned}
\hat{p}^j(a = 0) &= \int_0^{2\pi} e^{ij\varphi} p^{\varphi}(a = 0)\,\mathrm{d}\varphi \\
&= \int_0^{2\pi} e^{ij\varphi} \int_0^{2\pi} \int_0^{a_{\max}} \beta(\alpha, \varphi - \psi) p^{\psi}(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}\psi\,\mathrm{d}\varphi
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{2\pi}\int_0^{2\pi}\int_0^{a_{\max}} e^{ij(\varphi-\psi)}\beta(\alpha,\varphi-\psi)\cdot e^{ij\psi}p^{\psi}(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}\varphi\,\mathrm{d}\psi \\
&\overset{*}{=} \int_0^{a_{\max}} \left(\int_0^{2\pi} e^{ij\varphi}\beta(\alpha,\varphi)\,\mathrm{d}\varphi\right)\cdot\left(\int_0^{2\pi} e^{ij\psi}p^{\psi}(\alpha)\,\mathrm{d}\psi\right)\mathrm{d}\alpha \\
&= \int_0^{a_{\max}} \hat{\beta}_j(\alpha)\hat{p}^j(\alpha)\,\mathrm{d}\alpha,
\end{aligned}
\tag{4.16}
$$

where in the marked step we used the substitutiion $\varphi \mapsto \varphi + \psi$. The result is a similar condition to eq. (4.3). We mention two special cases:

- If the direction of newborns is completely independent of the direction of the parents, i.e. $\beta(\alpha,\psi) = \beta(\alpha)/2\pi$, the transformed condition reads

$$
\hat{\beta}_j(\alpha) = \begin{cases} \beta(\alpha), & j = 0, \\ 0, & j \in \mathbb{Z}\setminus\{0\}. \end{cases} \tag{4.17}
$$

- In the case where the direction of newborns matches the direction of their parents, i.e. $\beta(\alpha,\psi) = \beta(\alpha)\delta_0(\psi)/2\pi$, using the Dirac distribution, a slight abuse of notation, we get

$$
\hat{\beta}_j(\alpha) = \beta(\alpha) \quad \text{for all } j \in \mathbb{Z}. \tag{4.18}
$$

Similar to λ, we will assume that $\beta(\alpha,-\varphi) = \beta(\varphi)$, i.e. the direction of newborns deviates from the direction of the parents to the right with equal probability than to the left. This assumption leads to

$$
\hat{\beta}_{-j} = \hat{\beta}_j. \tag{4.19}
$$

Since $\beta(\alpha,\varphi)$ is real, we also deduce that $\hat{\beta}_{-j} = \overline{\hat{\beta}_j}$, thus all Fourier coefficients of β are again real numbers.

While eq. (4.13) with initial and birth conditions is a nice and complete model, it may be difficult to be applied in practice because of its complexity: In order to do simulations or predictions, one has to prescribe the direction of every single individual in every single point. The Fourier series model allows for simplifications, by truncating the system after a certain number of terms. This can be justified by the fact that we are mainly interested in the total population number $p = \hat{p}^0$, and if p^{φ} is square-integrable with respect to φ, the $\hat{p}^j$ converge to zero as $j \to \pm\infty$, so there is hope that we do not need all of the terms to get a good approximation for reality. Ignoring every term but the one for $j = 0$ in eq. (4.13) yields

$$
\delta p = f(p)p,
$$

which is a valid model (we saw it already in eq. (3.4)), but does not contain space variables at all. By truncating later, taking into account the terms where $j \in \{-1,0,1\}$ and defining $\lambda = \hat{\lambda}_0 - \hat{\lambda}_1$ (note that from eq. (4.12) we have $\lambda \geq 0$) we arrive at

$$
\begin{aligned}
\delta p + c(\partial\hat{p}^1 + \bar{\partial}\hat{p}^{-1}) &= f(p)p, \\
\delta\hat{p}^1 + c\bar{\partial}p + \lambda\hat{p}^1 &= f(p)\hat{p}^1, \\
\delta\hat{p}^{-1} + c\partial p + \lambda\hat{p}^{-1} &= f(p)\hat{p}^{-1},
\end{aligned}
\tag{4.20}
$$

a similar system to eq. (4.2) if one interpretes $\hat{p}^{\pm 1}$ as the flux Q and the Wirtinger derivatives as gradients resp. divergences. Applying $\delta + \lambda - f(p)$ to the first equation and noting that $\partial\bar{\partial} = \bar{\partial}\partial = \Delta/4$, the system decouples into

$$(\delta + \lambda - f(p))(\delta p - f(p)p) = -c(\delta + \lambda - f(p))(\partial\hat{p}^1 + \bar{\partial}\hat{p}^{-1}) = c^2(\partial\bar{\partial} + \bar{\partial}\partial)p = \tfrac{c^2}{2}\Delta p.$$

By dividing by λ, letting $\tau = \lambda^{-1}$ and $\sigma(a) := \frac{c(a)^2}{2\lambda}$ and ignoring the leftmost $f(p)$-term, we arrive at

$$(1 + \tau\delta)(\delta p - f(p)p) = \sigma\Delta p,$$

the same telegrapher's equation as eq. (4.2).

It is quite remarkable that the U-turn condition (4.14) and the reflection condition (4.15) both yield the very same boundary conditions, namely none for $j = 0$ and

$$\bar{\nu}\hat{p}^1 + \nu\hat{p}^{-1} = 0$$

for $j = \pm 1$, the latter being similar in structure to eq. (3.7). This implies Neumann conditions, since from eq. (4.20) we have that

$$\begin{aligned} 0 &= (\delta + \lambda - f(p))(\bar{\nu}\hat{p}^1 + \nu\hat{p}^{-1}) \\ &= \bar{\nu}(\delta + \lambda - f(p))\hat{p}^1 + \nu(\delta + \lambda - f(p))\hat{p}^{-1} \\ &= -c(\bar{\nu}\bar{\partial} + \nu\partial)p, \end{aligned}$$

and $\bar{\nu}\bar{\partial} + \nu\partial = 2\mathfrak{Re}\,\bar{\nu}\bar{\partial} = (\mathfrak{Re}\,\nu, \mathfrak{Im}\,\nu)^\mathsf{T} \cdot (\partial_x, \partial_y)^\mathsf{T} = \partial_\nu$.

Next we investigate what conditions we obtain for $a = 0$. In eq. (4.16) we already have deduced the condition $\hat{p}^j(a = 0) = \int_0^{a_\max} \hat{\beta}_j(\alpha)\hat{p}^j(\alpha)\,\mathrm{d}\alpha$, which for $j = 0$ immediately leads to

$$p(a = 0) = \int_0^{a_\max} \hat{\beta}_0(\alpha)p(\alpha)\,\mathrm{d}\alpha,$$

as seen before in eq. (4.3). Letting $a = 0$ in eq. (4.20) and using eq. (4.16) for $j = \pm 1$ now leads to

$$\begin{aligned} (\delta p - f(p)p)(a = 0) &= -c(0)(\partial\hat{p}^1 + \bar{\partial}\hat{p}^{-1})(a = 0) \\ &= -c(0)\int_0^{a_\max} \hat{\beta}_1(\alpha)\partial\hat{p}^1(\alpha) + \hat{\beta}_{-1}(\alpha)\bar{\partial}\hat{p}^{-1}(\alpha)\,\mathrm{d}\alpha. \end{aligned}$$

Using eq. (4.19), this can be simplified to

$$\begin{aligned} (\delta p - f(p)p)(a = 0) &= -c(0)\int_0^{a_\max} \hat{\beta}_1(\alpha)\Big(\partial\hat{p}^1(\alpha) + \bar{\partial}\hat{p}^{-1}(\alpha)\Big)\,\mathrm{d}\alpha \\ &= \int_0^{a_\max} c(0)\hat{\beta}_1(\alpha)c(\alpha)^{-1}(\delta p - f(p)p)(\alpha)\,\mathrm{d}\alpha, \end{aligned}$$

which is similar to eq. (4.6). By letting $P = \delta p - f(p)p$ as in eq. (1.5) we obtain

$$P(a=0) = \int_0^{a_{\max}} c(0)\hat{\beta}_1(\alpha)c(\alpha)^{-1}P(\alpha)\,\mathrm{d}\alpha,$$

which is similar to eq. (4.7) if we assume $\hat{\beta}_1(\alpha) = \frac{c(0)}{c(\alpha)}\hat{\beta}_0(\alpha)$.

4.1 Remark. We can derive a similar system by starting with the *Kolmogorov equation*

$$p_t(t,x,v) + v \cdot \nabla_x p(t,x,v) - \Delta_v p(t,x,v) = 0, \quad \text{for } t \geq 0,\ x,\ v \in \mathbb{R}^d$$

(cf. [Kol34], and [NZ21] for an extensive mathematical treatment) where $u = u(t,x,v)$ describes the state of a population at time t and position x that has velocity v. For simplicity we only consider the homogeneous case. We proceed as above: let $d = 2$ and restrict the velocity v on a circle around the origin with radius c. With the usual identification of $\mathbb{C}$ and $\mathbb{R}^2$ and the convention $p^\varphi := p(\cdot,\cdot,ce^{i\varphi})$ we arrive at

$$p_t^\varphi + c(\cos\varphi, \sin\varphi)\nabla p^\varphi - \frac{1}{c^2}\frac{\partial^2}{\partial\varphi^2}p^\varphi = 0,$$

where we used the Laplace operator

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$$

in polar coordinates, and dropped the terms involving derivatives of r. Defining $\hat{p}$ as in eq. (4.9) and using results on Fourier coefficients of derivatives, similar calculations to those that gave eq. (4.13) yield

$$\hat{p}_t^j + c\partial\hat{p}^{j+1} + c\bar{\partial}\hat{p}^{j-1} + \frac{j^2}{c^2}\hat{p}^j = 0.$$

This is exactly eq. (4.13) with $\hat{\lambda}_j - \hat{\lambda}_0 = j^2/c^2$, thus we can derive the wave equation in exactly the same way. Note, however, that the results from Section 4.8, especially Theorem 4.24 and Corollary 4.34, are not applicable for this variant of the system because the terms j^2/c^2 are not uniformly bounded.

4.3 The linearized equation with fixed birth numbers

This section, and the two following ones, lead to a proof of existence and uniqueness of solutions to eq. (4.1). Similar to Section 3.2, we start by considering a linearized, inhomogeneous version of eq. (4.1) given by

$$(1+\tau\delta)(\delta y(t,a,x) + L(a,x)y(t,a,x)) = \sigma(a)\Delta y(t,a,x) + f(t,a,x) \tag{4.21a}$$

with the usual boundary conditions

$$\partial_\nu y(x \in \partial\Omega) = 0, \tag{4.21b}$$

the desired initial values

$$y(t=0) = y_0, \quad \delta y(t=0) = y_1, \tag{4.21c}$$

and explicit birth numbers

$$y(a=0) = B_0, \quad \delta y(a=0) = B_1. \tag{4.21d}$$

Let $t_0 \in (-a_{\max}, T)$. By introducing characteristics as in eq. (3.10) and letting $v := y\big|_{\mathrm{char}(t_0)}$, we obtain the second-order system

$$\begin{aligned}
(1+\tau\partial_h)(v_h(h,x) + L(h,x)v(h,x)) = \sigma(h)\Delta v(h,x) + f(t_0+h,h,x) \text{ in } [(-t_0)^+, a_{\max}] \times \Omega,\\
v(h=(-t_0)^+) = \begin{cases} y_0(-t_0), & t_0 < 0 \\ B_0(t_0), & t_0 > 0 \end{cases} =: v_0 \text{ in } \Omega,\\
v_h(h=(-t_0)^+) = \begin{cases} y_1(-t_0), & t_0 < 0 \\ B_1(t_0), & t_0 > 0 \end{cases} =: v_1 \text{ in } \Omega,\\
\partial_\nu v(h) = 0 \text{ in } [(-t_0)^+, a_{\max}] \times \partial\Omega.
\end{aligned} \tag{4.22}$$

An application of the chain rule yields

$$(1+\tau\partial_h)(v_h + L(h)v) = \tau v_{hh} + (1+\tau L(h))v_h + (L(h) + \tau L_h(h))v, \tag{4.23}$$

which indicates that we need higher regularity assumptions on L than in Assumption 3.1, since merely requesting $L \in C(\mathcal{I}, L^\infty(\Omega)^{n\times n})$ does not guarantee the existence of L_h.

4.2 Assumption. We assume

1. $\tau \in (0, \tau_{\max})$ for some $0 < \tau_{\max} < \infty$.
2. $y_0 \in \mathcal{V}$, $y_1 \in \mathcal{H}$, $B_0 \in L^2((0,T),V)$, $B_1 \in L^2((0,T),H)$ and $f \in L^2((0,T),\mathcal{H})$.
3. $L \in C^1(\bar{\mathcal{I}}, L^\infty(\Omega))^{n\times n}$ and its entries are uniformly bounded away from zero with respect to both age and space.
4. $\sigma \in C^1(\bar{\mathcal{I}}, \mathbb{R})^{n\times n}$ is a diagonal matrix whose entries are uniformly bounded away from zero with respect to age.

With eq. (3.11) and eq. (4.23) we arrive at the following weak formulation for eq. (4.22): for all $w \in V$ we require that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}h}\langle \tau v_h(h), w\rangle_H + \langle v_h(h) + \tau L(h)v_h(h), w\rangle_H + \langle L(h)v(h) + \tau L_h(h)v(h), w\rangle_H\\
+\langle \sigma(h)\nabla v(h), \nabla w\rangle_{H^d} = \big\langle f(t_0+h,h), w\big\rangle_H
\end{aligned}$$

holds for almost all $h \in ((-t_0)^+, a_{\max})$.

4.3 Theorem. Under Assumption 4.2, for every $t_0 \in (-a_{\max}, T)$ there exists a unique weak solution $v \in C([(-t_0)^+, a_{\max}], V) \cap C^1([(-t_0)^+, a_{\max}], H) \cap C^2([(-t_0)^+, a_{\max}], V')$ of the relaxed equation (4.22) on characteristics.

Proof: We verify the conditions of Theorem 2.16, which will then yield the result. To this end, we introduce the operators

$$\mathcal{A}_0 := -\sigma\Delta, \qquad \mathcal{A}_1 := L + \tau L_h, \qquad \mathcal{B} := I + \tau L, \qquad \mathcal{C} := \tau I,$$

where I denotes the identity operator. Note that none of these operators depends on t_0. The assumptions ensure that $v_0 \in V$, $V_1 \in H$ and $f|_{\mathrm{char}(t_0)} \in L^2(H)$. Further, they show that $\mathcal{A}_0 \in C^1([0,a_{\max}], L(V,V'))$, $\mathcal{A}_1 \in C([0,a_{\max}], L(H,H))$ (which is stronger than necessary), and $\mathcal{B}, \mathcal{C} \in C^1([0,a_{\max}], L(H,H))$. The fact that the form generated by $\mathcal{A}_0$ is hermitian and coercive can be seen as in the proof of Theorem 3.2. Further, we note that $\langle \mathcal{C}(t)u, u\rangle_H = \tau\|u\|_H^2$, and the form generated by $\mathcal{C}$ is obviously hermitian. Since we now have verified all conditions of Theorem 2.16, the claim follows. □

It will become very important later, in particular when considering convergence issues in Theorem 4.23, to precisely track the τ-dependence in energy estimates. The following lemma is the basis to this.

4.4 Lemma. Let $s := (-t_0)^+$ and $\tau_{\max} > 0$. Then for all $h \in [s, a_{\max}]$ and all $\tau \in (0, \tau_{\max})$ the solution v from Theorem 4.3 satisfies the estimate

$$\tau\|v_h(h)\|_H^2 + \|v(h)\|_V^2 \lesssim \tau\|v_1\|_H^2 + \|v_0\|_V^2 + \|f\|_{L^2((s,a_{\max}),H)}^2$$

with a constant that does not depend on τ or s, but possibly on $\tau_{\max}$.

Proof: From Corollary 2.17 we have the energy equality

$$\begin{aligned}\tau\|v_h(h)\|_H^2 + \langle\sigma(h)\nabla v(h), \nabla v(h)\rangle_H &= \tau\|v_1\|_H^2 + \langle\sigma(s)\nabla v_0, \nabla v_0\rangle_H \\ &\quad + \int_s^h \big\langle\sigma_h(\eta)\nabla v(\eta), \nabla v(\eta)\big\rangle_H + 2\big\langle f(\eta), v_h(\eta)\big\rangle_H \,\mathrm{d}\eta \\ &\quad - 2\int_s^h \big\langle(1+\tau L(\eta))v_h(\eta), v_h(\eta)\big\rangle_H + \big\langle(L(\eta)+\tau L_h(\eta))v(\eta), v_h(\eta)\big\rangle_H \,\mathrm{d}\eta\,.\end{aligned}$$

Using the properties of σ and L from Assumption 4.2 we estimate (ε and ζ are to be defined later):

$$\begin{aligned}\tau\|v_h(h)\|_H^2 + c_\sigma\|\nabla v(h)\|_H^2 &\le \tau\|v_1\|_H^2 + \|\sigma(s)\|_{L^\infty}\|\nabla v_0\|_H^2 \\ &\quad + \int_s^h \big\|\sigma'\big\|_{L^\infty}\big\|\nabla v(\eta)\big\|_H^2 + \varepsilon^{-1}\big\|f(\eta)\big\|_H^2 + \varepsilon\big\|v_h(\eta)\big\|_H^2 \,\mathrm{d}\eta \\ &\quad + \int_s^h -2\big\|v_h(\eta)\big\|_H^2 + 2\tau\|L\|\big\|v_h(\eta)\big\|_H^2 + \zeta^{-1}\|L+\tau L_h\|^2\big\|v(\eta)\big\|_H^2 + \zeta\big\|v_h(\eta)\big\|_H^2 \,\mathrm{d}\eta\,.\end{aligned} \tag{4.24}$$

Furthermore, from the fundamental theorem of calculus we infer the estimate

$$\gamma\|v(h)\|_H^2 \le 2\gamma\|v_0\|_H^2 + 2\gamma h\int_s^h \big\|v_h(\eta)\big\|_H^2 \,\mathrm{d}\eta$$

for any $\gamma > 0$. Adding this to eq. (4.24) yields

$$\begin{aligned}\tau\|v_h(h)\|_H^2 + \gamma\|v(h)\|_H^2 + c_\sigma\|\nabla v(h)\|_H^2 &\le \tau\|v_1\|_H^2 + 2\gamma\|v_0\|_H^2 + \|\sigma(s)\|_{L^\infty}\|\nabla v_0\|_H^2 \\ &\quad + \int_s^h \big\|\sigma'\big\|_{L^\infty}\big\|\nabla v(\eta)\big\|_H^2 + \varepsilon^{-1}\big\|f(\eta)\big\|_H^2 + \zeta^{-1}\|L+\tau L_h\|^2\big\|v(\eta)\big\|_H^2 \,\mathrm{d}\eta \\ &\quad + \int_s^h (2\gamma h + \varepsilon - 2 + 2\tau\|L\| + \zeta)\big\|v_h(\eta)\big\|_H^2 \,\mathrm{d}\eta\,.\end{aligned}$$

By choosing small enough values of γ, ζ, ε independently of τ we can make sure that $2\gamma h+\varepsilon-2+\zeta<0$ for all values of $h\in[s,a_{\max}]$, thus the respective term can be neglected in the estimate, leaving us with

$$\begin{aligned}\tau\|v_h(h)\|_H^2+\gamma\|v(h)\|_H^2+c_\sigma\|\nabla v(h)\|_H^2 &\le \tau\|v_1\|_H^2+2\gamma\|v_0\|_H^2+\|\sigma(s)\|_{L^\infty}\|\nabla v_0\|_H^2\\ &+\int_s^h \left\|\sigma'\right\|_{L^\infty}\left\|\nabla v(\eta)\right\|_H^2+\varepsilon^{-1}\left\|f(\eta)\right\|_H^2+\zeta^{-1}\|L+\tau L_h\|^2\left\|v(\eta)\right\|_H^2+2\tau\|L\|\left\|v_h(\eta)\right\|_H^2\,\mathrm{d}\eta\,.\end{aligned}$$

We now can apply Gronwall's lemma, where in Lemma 2.4 we choose

$$\begin{aligned}\Phi&=\tau\|v_h(h)\|_H^2+\gamma\|v(h)\|_H^2+c_\sigma\|\nabla v(h)\|_H^2,\\ \beta&=\tau\|v_1\|_H^2+\|v_0\|_H^2+\|\nabla v_0\|_H^2+\int_s^h\left\|f(\eta)\right\|_H^2\mathrm{d}\eta\,,\end{aligned}$$

and α the maximum of all the remaining constants. Further estimating $\|L+\tau L_h\|$ by $\|L\|+\tau_{\max}\|L_h\|$ yields the estimate

$$\tau\|v_h(h)\|_H^2+\|v(h)\|_H^2+\|\nabla v(h)\|_H^2\lesssim\tau\|v_1\|_H^2+\|v_0\|_H^2+\|\nabla v_0\|_H^2+\int_s^h\left\|f(\eta)\right\|_H^2\mathrm{d}\eta\,,$$

where the constant does not depend on τ. This concludes the proof. □

4.5 Definition. In the same spirit as in Definition 3.3, let

- $M(t,s)v_0$ the value $v(t)$ of the solution v to eq. (4.22) with initial conditions $v(s)=v_0$, $v_h(s)=0$ and $f\equiv 0$.
- $N(t,s)v_1$ the value $v(t)$ of the solution v to eq. (4.22) with initial conditions $v(s)=0$, $v_h(s)=v_1$ and $f\equiv 0$.
- $P(t,s)f$ the value $v(t)$ of the solution v to eq. (4.22) with initial conditions $v(s)=0$, $v_h(s)=0$.

By M', N', P', we denote the respective time derivative. Note that the names M and N are standard, cf. [EN00, p. 383], [MF88] or [Zhe94].

4.6 Corollary. Let $s\in(0,a_{\max})$, $v_0\in V$, $v_1\in H$, $f\in L^2((s,a_{\max}),H)$. Then we have

$$\begin{aligned}M(\cdot,s)v_0&\in C([s,a_{\max}],V)\cap C^1([s,a_{\max}],H), & \|M(t,s)v_0\|_V^2+\tau\left\|M'(t,s)v_0\right\|_H^2&\lesssim\|v_0\|_V^2,\\ N(\cdot,s)v_1&\in C([s,a_{\max}],V)\cap C^1([s,a_{\max}],H), & \|N(t,s)v_1\|_V^2+\tau\left\|N'(t,s)v_1\right\|_H^2&\lesssim\tau\|v_1\|_H^2,\\ P(\cdot,s)f&\in C([s,a_{\max}],V)\cap C^1([s,a_{\max}],H), & \|P(t,s)f\|_V^2+\tau\left\|P'(t,s)f\right\|_H^2&\lesssim\|f\|_{L^2((s,a_{\max}),H)}^2,\end{aligned}$$

and all constants can be chosen independently of τ, t or s.

Proof: This is a direct consequence of Lemma 4.4. □

In a similar fashion to eq. (3.17), we can now reassemble the solution on the various characteristics. This gives the expression

$$y(t,a) = \begin{cases} M(a,a-t)y_0(a-t) + N(a,a-t)y_1(a-t) + P(a,a-t)\, f|_{\mathrm{char}(t-a)}, & t \le a, \\ M(a,0)B_0(t-a) + N(a,0)B_1(t-a) + P(a,0)\, f|_{\mathrm{char}(t-a)}, & t \ge a. \end{cases} \tag{4.25}$$

A simlar proof to Corollary 3.8 shows that this expression is in fact well-defined and independent of the choice of representatives for f, y_0, y_1, B_0 and B_1.

As before, we have to verify that the function y defined by eq. (4.25) is indeed a weak solution to eq. (4.21). To this end, and similar to eq. (3.18), we define a new function v by

$$v(t,a) = \begin{cases} M'(a,a-t)y_0(a-t) + N'(a,a-t)y_1(a-t) + P'(a,a-t)\, f|_{\mathrm{char}(t-a)}, & t \le a, \\ M'(a,0)B_0(t-a) + N'(a,0)B_1(t-a) + P'(a,0)\, f|_{\mathrm{char}(t-a)}, & t \ge a. \end{cases} \tag{4.26}$$

4.7 Theorem. Assume that Assumption 4.2 holds and define y by eq. (4.25) and v by eq. (4.26). Then we have $y \in L^\infty((0,T),\mathcal{V})$, $v \in L^\infty((0,T),\mathcal{H})$, and the estimate

$$\tau\|v(t)\|_{\mathcal{H}}^2 + \left\|y(t)\right\|_{\mathcal{V}}^2 \lesssim \|B_0\|_{L^2((0,t),V)}^2 + \tau\|B_1\|_{L^2((0,t),H)}^2 + \left\|y_0\right\|_{\mathcal{V}}^2 + \tau\left\|y_1\right\|_{\mathcal{H}}^2 + \|f\|_{L^2((0,t),\mathcal{H})}^2$$

for all $t \in [0,T]$.

Proof: Using the representations of y and v and Fubini's theorem, we calculate

$$\begin{aligned}
\tau\|v(t)\|_{\mathcal{H}}^2 + \left\|y(t)\right\|_{\mathcal{V}}^2 &= \int_0^{a_{\max}} \tau\|v(t,a)\|_H^2 + \left\|y(t,a)\right\|_V^2 \,\mathrm{d}a \\
&\lesssim \int_0^{\min(t,a_{\max})} \|B_0(t-a)\|_V^2 + \tau\|B_1(t-a)\|_H^2 \,\mathrm{d}a + \int_{\min(t,a_{\max})}^{a_{\max}} \left\|y_0(a-t)\right\|_V^2 + \tau\left\|y_1(a-t)\right\|_H^2 \mathrm{d}a \\
&\quad + \int_0^{a_{\max}} \left\|f|_{\mathrm{char}(t-a)}\right\|_{L^2(H)}^2 \mathrm{d}a \\
&= \|B_0\|_{L^2(((t-a_{\max})^+,t),V)}^2 + \tau\|B_1\|_{L^2(((t-a_{\max})^+,t),H)}^2 + \left\|y_0\right\|_{L^2((-(t-a_{\max})^+,a_{\max}-t),V)}^2 \\
&\quad + \tau\left\|y_1\right\|_{L^2((-(t-a_{\max})^+,a_{\max}-t),H)}^2 + \int_0^{a_{\max}} \int_{(a-t)^+}^{a} \|f(t-a+r,r)\|_H^2 \,\mathrm{d}r\,\mathrm{d}a \\
&\le \|B_0\|_{L^2((0,t),V)}^2 + \tau\|B_1\|_{L^2((0,t),H)}^2 + \left\|y_0\right\|_{\mathcal{V}}^2 + \tau\left\|y_1\right\|_{\mathcal{H}}^2 \\
&\quad + \int_0^{a_{\max}} \int_r^{\min(r+t,a_{\max})} \|f(t-a+r,r)\|_H^2 \,\mathrm{d}a\,\mathrm{d}r.
\end{aligned}$$

With the substitution $s = a - r$ we obtain

$$\begin{aligned}
\tau\|v(t)\|_{\mathcal{H}}^2 + \left\|y(t)\right\|_{\mathcal{V}}^2 &\lesssim \|B_0\|_{L^2((0,t),V)}^2 + \tau\|B_1\|_{L^2((0,t),H)}^2 + \left\|y_0\right\|_{\mathcal{V}}^2 + \tau\left\|y_1\right\|_{\mathcal{H}}^2 \\
&\quad + \int\limits_0^{a_{\max}} \int\limits_0^{\min(t,a_{\max}-r)} \|f(t-s,r)\|_H^2 \,\mathrm{d}s\,\mathrm{d}r \\
&\leq \|B_0\|_{L^2((0,t),V)}^2 + \tau\|B_1\|_{L^2((0,t),H)}^2 + \left\|y_0\right\|_{\mathcal{V}}^2 + \tau\left\|y_1\right\|_{\mathcal{H}}^2 + \|f\|_{L^2((0,t),\mathcal{H}}^2.
\end{aligned}$$

This concludes the proof. □

4.8 Theorem. Assume that Assumption 4.2 holds. Then the function v from eq. (4.26) equals the weak derivative δy, where y is defined by eq. (4.25). Furthermore we have

$$\begin{aligned}
y &\in C([0,T],\mathcal{V}) \cap C(\bar{\mathcal{I}}, L^2((0,T),V)), \\
\delta y &\in C([0,T],\mathcal{H}) \cap C(\bar{\mathcal{I}}, L^2((0,T),H)),
\end{aligned}$$

and the energy estimate

$$\begin{aligned}
\tau\left\|\delta y\right\|_{C([0,T],\mathcal{H})}^2 + \left\|y\right\|_{C([0,T],\mathcal{V})}^2 &\lesssim \|B_0\|_{L^2((0,T),V)}^2 + \tau\|B_1\|_{L^2((0,T),H)}^2 \\
&\quad + \left\|y_0\right\|_{\mathcal{V}}^2 + \tau\left\|y_1\right\|_{\mathcal{H}}^2 + \|f\|_{L^2((0,T),\mathcal{H})}^2.
\end{aligned} \tag{4.27}$$

Also we have

$$y(t=0) = y_0, \quad \delta y(t=0) = y_1, \quad y(a=0) = B_0, \quad \delta y(a=0) = B_1.$$

Proof: By replacing U with M or N, and S with P, respectively, proving that v is the weak derivative of y can be done in a fashion similar to Theorem 3.13. The energy estimate then follows directly from Theorem 4.7. Because of Corollary 4.6, a proof similar to Theorem 3.10 with the same replacements for U and S shows the continuity of y and $v = \delta y$ and confirms the initial values for t and a. □

Combining Theorem 4.8 with the definition of a weak solution on characteristics allows us to conclude the following existence theorem for the linearized system (4.21).

4.9 Theorem. Assume that Assumption 4.2 holds. Then the function y defined by eq. (4.25) is a weak solution of eq. (4.21) in the sense that for almost all $t \in (0,T)$, $a \in \mathcal{I}$ and all $v \in V$ the weak formulation

$$\begin{aligned}
\left\langle \tau\delta^2 y(t,a), v\right\rangle_{V'\times V} + \left\langle (1+\tau L(a))\delta y(t,a), v\right\rangle_H + \left\langle (L(a)+\tau L_a(a))y(t,a), v\right\rangle_H \\
= \left\langle \sigma(a)\nabla y(t,a), \nabla v\right\rangle_H + \left\langle f(t,a), v\right\rangle_H
\end{aligned}$$

and the initial conditions

$$\begin{aligned}
y(0,a,x) &= y_0(a,x), \quad \delta y(0,a,x) = y_1(a,x), \\
y(t,0,x) &= B_0(t,x), \quad \delta y(t,0,x) = B_1(t,x)
\end{aligned}$$

hold, the latter expressions being defined by Theorem 4.8.

4.4 The linearized equation with implicit birth laws

The next step is to incorporate implicit birth conditions into model (4.21). We leave eq. (4.1c) unchanged, and instead of the nonlinear condition (4.1d), we assume that the value of δy in $a = 0$ depends implicitly on both y and δy. This yields the system

$$(1+\tau\delta)(\delta y(t,a,x) + L(a,x)y(t,a,x)) = \sigma(a)\Delta y(t,a,x) + f(t,a,x), \tag{4.28a}$$

$$y(t=0) = y_0, \quad \delta y(t=0) = y_1, \quad \partial_\nu y(x \in \partial\Omega) = 0, \tag{4.28b}$$

$$y(a=0) = \int_0^{a_{\max}} \beta_0(\alpha) y(\alpha)\,\mathrm{d}\alpha + g_0 =: B_0(t), \tag{4.28c}$$

$$(\delta y)(a=0) = \int_0^{a_{\max}} \beta_L(\alpha)y(\alpha) + \beta_1(\alpha)(\delta y)(\alpha)\,\mathrm{d}\alpha + g_1 =: B_1(t) \tag{4.28d}$$

with some given “birth rates” β_0, β_1, β_L. The name β_L results from the fact that later, in eq. (4.41), we will choose $\beta_L(\alpha) = \beta_1(\alpha)L(\alpha) - L(0)\beta_0(\alpha)$.

In order to make the following easier to read, we introduce a more compact notation of our results so far. We introduce the space $\mathbb{V}^\tau := V \times H$ with the norm

$$\left\|(u_0, u_1)^\mathsf{T}\right\|_{\mathbb{V}^\tau} = \left(\|u_0\|_V^2 + \tau\|u_1\|_H^2\right)^{1/2}$$

(this is obviously a Banach space) and let

$$Y := \begin{pmatrix} y \\ \delta y \end{pmatrix}, \quad \mathcal{U} := \begin{pmatrix} M & N \\ M' & N' \end{pmatrix}, \quad \mathcal{S} := \begin{pmatrix} P \\ P' \end{pmatrix},$$
$$Y_0 := \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}, \quad \mathcal{B} := \begin{pmatrix} B_0 \\ B_1 \end{pmatrix}, \quad \mathfrak{b} := \begin{pmatrix} \beta_0 & 0 \\ \beta_L & \beta_1 \end{pmatrix},$$

then from eq. (4.25) we have

$$Y(t,a) = \begin{cases} \mathcal{U}(a, a-t)Y_0(a-t), & t \le a \\ \mathcal{U}(a,0)\mathcal{B}(t-a), & t \ge a \end{cases} + \mathcal{S}(a,(a-t)^+)\, f|_{\mathrm{char}(t-a)},$$

where

$$\mathcal{B}(t) = \int_0^{a_{\max}} \mathfrak{b}(\alpha) Y(t,\alpha)\,\mathrm{d}\alpha + \begin{pmatrix} g_0 \\ g_1 \end{pmatrix}.$$

In this context, we can reformulate eq. (4.27) to

$$\|Y\|^2_{C([0,T],L^2(\mathcal{I},\mathbb{V}^\tau))} \lesssim \|\mathcal{B}\|^2_{L^2((0,T),\mathbb{V}^\tau)} + \|Y_0\|^2_{L^2(\mathcal{I},\mathbb{V}^\tau)} + \|f\|^2_{L^2((0,T),\mathcal{H})}. \tag{4.29}$$

A similar calculation to the one giving eq. (3.24), where we replace B by $\mathcal{B}$, y by Y, U by $\mathcal{U}$, S by $\mathcal{S}$, β by $\mathfrak{b}$, and y_0 by Y_0 yields

$$\begin{aligned} \mathcal{B}(t) = &\int_0^{\min(t,a_{\max})} \mathfrak{b}(\alpha)\mathcal{U}(\alpha,0)\mathcal{B}(t-\alpha)\,\mathrm{d}\alpha + \int_{\min(t,a_{\max})}^{a_{\max}} \mathfrak{b}(\alpha)\mathcal{U}(\alpha,\alpha-t)Y_0(\alpha-t)\,\mathrm{d}\alpha \\ &+ \int_0^{a_{\max}} \mathfrak{b}(\alpha)\mathcal{S}(\alpha,(\alpha-t)^+)\, f|_{\mathrm{char}(t-\alpha)}\,\mathrm{d}\alpha + \begin{pmatrix} g_0 \\ g_1 \end{pmatrix}. \end{aligned} \tag{4.30}$$

In contrast to the unrelaxed problem, this time we have to solve a Volterra equation of a function that in the first component takes values in V. This means that multiplication of an element $v \in V$ with $\beta_0(a)$ ($a \in \mathcal{I}$) must again yield an element of V. In Assumption 3.15 we assumed $\beta(a) \in L^\infty(\Omega)$, but this is not sufficient here. Rather we need the results from the following lemma:

4.10 Lemma. Let $\varphi \in H^1(\Omega)$ where $\Omega \subseteq \mathbb{R}^d$ is an open and bounded domain and $d \geq 3$. Then for any $\rho \in W_b^{1,d}(\Omega) := W^{1,d}(\Omega) \cap L^\infty(\Omega)$ we have $\rho\varphi \in H^1(\Omega)$ and $\|\rho\varphi\|_{H^1} \leq \|\varphi\|_{H^1}\|\rho\|_{W_b^{1,d}(\Omega)}$. The space $W_b^{1,d}(\Omega)$ is complete with respect to the norm $\|\rho\|_{W_b^{1,d}(\Omega)} = \|\rho\|_{L^\infty(\Omega)} + \|\nabla\rho\|_{L^d(\Omega)^d}$. If $d = 2$, the same results hold if we replace the space $W_b^{1,d}$ by $W_b^{1,d+\varepsilon} := W^{1,d+\varepsilon}(\Omega) \cap L^\infty(\Omega)$ for some $\varepsilon > 0$.

Proof: First let $d > 2$. If $\rho\varphi$ is an element of $H^1(\Omega)$ then the product rule $\nabla(\rho\varphi) = \varphi\nabla\rho + \rho\nabla\varphi$ holds. This means that in order to show the claim we need to show that both summands are contained in $L^2(\Omega)$ again. The second one does so provided $\rho \in L^\infty$. The first one, $\varphi\nabla\rho$, needs a little more reasoning. According to case 3 of Sobolev's Embedding Theorem 2.8, we have $H^1(\Omega) \hookrightarrow L^q(\Omega)$ for any $2 \leq q \leq \frac{2d}{d-2}$. For these q, Hölder's inequality yields ($i = 1, \dots, d$)

$$\|\varphi \cdot \partial_i\rho\|_{L^2}^2 = \left\|\varphi^2(\partial_i\rho)^2\right\|_{L^1} \leq \left\|\varphi^2\right\|_{L^{q/2}}\left\|(\partial_i\rho)^2\right\|_{L^{(q/2)'}}, \tag{4.31}$$

where $(q/2)'$ denotes the conjugate exponent to $q/2$, i.e. it holds $\frac{1}{q/2} + \frac{1}{(q/2)'} = 1$. In the special case $q = \frac{2d}{d-2} \geq 2$ we have $\left(\frac{q}{2}\right)' = \frac{d}{2}$ and therefore $\|\varphi \cdot \partial_i\rho\|_{L^2}^2 \leq \|\partial_i\rho\|_{L^d}^2\|\varphi\|_{L^q}^2$. Thus, we have shown that $\nabla(\rho\varphi) \in L^2(\Omega)$, and together with $\rho \in L^\infty(\Omega)$ it follows that $\rho\varphi \in H^1(\Omega)$. Finally, the completeness of the space follows from $L^\infty(\Omega) \hookrightarrow L^d(\Omega)$.

If $d = 2$, case 2 from Sobolev's Theorem 2.8 shows that $H^1(\Omega)$ embeds into $L^q(\Omega)$ for all $q \in [2, \infty)$. Hence, the above proof works if for q in eq. (4.31) we choose the value $q = 2 + \frac{4}{\varepsilon}$. In this case we have $\left(\frac{q}{2}\right)' = \frac{d+\varepsilon}{2}$, and we can conclude as in the case where $d > 2$. □

4.11 Remark. From Sobolev's Theorem 2.8 it follows that for any $q > d$ there is an embedding $W^{1,q}(\Omega) \hookrightarrow W_b^{1,d}(\Omega)$.

4.12 Assumption. For the following section we assume that

1. $\beta_0 \in L^\infty(\mathcal{I}, W_b^{1,d}(\Omega))$ resp. in $W_b^{1,d+\varepsilon}$ for some $\varepsilon > 0$ if $d = 2$, and $\beta_L, \beta_1 \in L^\infty(\mathcal{I} \times \Omega)$.
2. $g_0 \in L^2((0,T), V)$, $g_1 \in L^2((0,T), H)$.
3. The rest of Assumption 4.2 holds:
    - 3.1. $\tau \in (0, \tau_{\max})$ for some $0 < \tau_{\max} < \infty$.
    - 3.2. $y_0 \in \mathcal{V}$, $y_1 \in \mathcal{H}$ and $f \in L^2((0,T), \mathcal{H})$.
    - 3.3. $L \in C^1(\bar{\mathcal{I}}, L^\infty(\Omega))^{n\times n}$ and its entries are uniformly bounded away from zero with respect to both age and space.
    - 3.4. $\sigma \in C^1(\bar{\mathcal{I}}, \mathbb{R})^{n\times n}$ is a diagonal matrix whose entries are uniformly bounded away from zero with respect to age.

4.13 Theorem. The Volterra equation (4.30) has a unique solution $\mathcal{B} = \mathcal{B}(Y_0, f, g_0, g_1) \in L^2((0,T), \mathbb{V}^\tau)$ which satisfies the estimate

$$\|\mathcal{B}\|^2_{L^2((0,T),\mathbb{V}^\tau)} \lesssim \|Y_0\|^2_{L^2(\mathcal{I},\mathbb{V}^\tau)} + \|f\|^2_{L^2((0,T),\mathcal{H})} + \|g_0\|^2_{L^2((0,T),V)} + \tau\|g_1\|^2_{L^2((0,T),H)}, \tag{4.32}$$

the constant not depending on τ.

Proof: We want to verify the conditions of Lemma 2.18. To this end define

$$A(t) = \mathbb{1}_{\mathcal{I}}(t)\mathfrak{b}(t)\mathcal{U}(t,0),$$

$$F(t) = \int\limits_{\min(t,a_{\max})}^{a_{\max}} \mathfrak{b}(\alpha)\mathcal{U}(\alpha,\alpha-t)Y_0(\alpha-t)\,\mathrm{d}\alpha + \int\limits_0^{a_{\max}} \mathfrak{b}(\alpha)\mathcal{S}(\alpha,(\alpha-t)^+)\,f|_{\mathrm{char}(t-\alpha)}\,\mathrm{d}\alpha.$$

First we calculate, using Corollary 4.6

$$\begin{aligned}
\left\|\mathfrak{b}(\alpha)(u_0,u_1)^\mathsf{T}\right\|_{\mathbb{V}^\tau} &= \left\|\beta_0(\alpha)u_0\right\|_V^2 + \tau\left\|\beta_L(\alpha)u_0 + \beta_1(\alpha)u_1\right\|_H^2 \\
&\lesssim \|u_0\|_V^2 + \tau\left(\|u_0\|_H^2 + \|u_1\|_H^2\right) \\
&= \left\|(u_0,u_1)^\mathsf{T}\right\|_{\mathbb{V}^\tau}, \\
\left\|\mathcal{U}(t,s)(u_0,u_1)^\mathsf{T}\right\|_{\mathbb{V}^\tau} &= \|M(t,s)u_0 + N(t,s)u_1\|_V^2 + \tau\left\|M'(t,s)u_0 + N'(t,s)u_1\right\|_V^2 \\
&\lesssim \|u_0\|_V^2 + \tau\|u_1\|_H^2 + \|u_0\|_V^2 + \tau\|u_1\|_H^2 \\
&= \left\|(u_0,u_1)^\mathsf{T}\right\|_{\mathbb{V}^\tau}, \\
\|\mathcal{S}(t,s)f\|_{\mathbb{V}^\tau}^2 &= \|P(t,s)f\|_V^2 + \tau\left\|P'(t,s)f\right\|_H^2 \\
&\lesssim \|f\|^2_{L^2((s,t),H)}
\end{aligned}$$

with constants independent of τ, s, t or α. Now, calculations similar to those in the proof of Theorem 3.16 (where we replace H by $\mathbb{V}^\tau$, β by $\mathfrak{b}$, y_0 by Y_0, U by $\mathcal{U}$ and S by $\mathcal{S}$) show that $A \in L^\infty((0,T), L(\mathbb{V}^\tau))$, its norm not depending on τ, and that

$$\|F(t)\|_{L^2((0,T),\mathbb{V}^\tau)} \lesssim \|f\|^2_{L^2((0,T),H)} + \|Y_0\|^2_{L^2(\mathcal{I},\mathbb{V}^\tau)}$$

where the constant again does not depend on τ. It is now easy to see that

$$\left\|F(t) + (g_0,g_1)^\mathsf{T}\right\|_{L^2((0,T),\mathbb{V}^\tau)} \lesssim \|f\|^2_{L^2((0,T),H)} + \|Y_0\|^2_{L^2(\mathcal{I},\mathbb{V}^\tau)} + \|(g_0,g_1)\|^2_{L^2((0,T),\mathbb{V}^\tau)}.$$

Now the claim and the estimate follow directly by Lemma 2.18. □

4.14 Theorem. There exists a unique weak solution $y \in C([0,T],\mathcal{V}) \cap C(\bar{\mathcal{I}}, L^2((0,T),V))$ with $\delta y \in C([0,T],\mathcal{H}) \cap C(\bar{\mathcal{I}}, L^2((0,T),H))$, in the sense given in Theorem 4.9, to the system (4.28). This solution satisfies

$$\begin{aligned}
\tau\left\|\delta y\right\|^2_{C([0,T],\mathcal{H})} + \left\|y\right\|^2_{C([0,T],\mathcal{V})} \lesssim \left\|y_0\right\|^2_{\mathcal{V}} + \tau\left\|y_1\right\|^2_{\mathcal{H}} + \|f\|^2_{L^2((0,T),\mathcal{H})} \\
+ \|g_0\|^2_{L^2((0,T),V)} + \tau\|g_1\|^2_{L^2((0,T),H)}.
\end{aligned} \tag{4.33}$$

Proof: We obtain y by using the solution $\mathcal{B}$ from Theorem 4.13 in the construction from Theorem 4.8. Plugging eq. (4.32) into eq. (4.29) yields

$$\|Y\|^2_{C([0,T],L^2(\mathcal{I},\mathbb{V}^\tau))} \lesssim \|Y_0\|^2_{L^2(\mathcal{I},\mathbb{V}^\tau)} + \|f\|^2_{L^2((0,T),\mathcal{H})} + \|(g_0,g_1)\|^2_{L^2((0,T),\mathbb{V}^\tau)},$$

which is just a reformulation of the claimed estimate. □

4.15 Remark. If f, g_0, g_1 are all zero and we define $\mathcal{B} = \mathcal{B}(Y_0, 0, 0, 0)$ as in Theorem 4.13, then one can show that $(\mathcal{T}(t))_{t\geq 0}$ defined by

$$(\mathcal{T}(t)Y_0)(a) := \begin{cases} \mathcal{U}(a, a-t)Y_0(a-t), & t \leq a \\ \mathcal{U}(a, 0)\mathcal{B}(t-a), & t \geq a \end{cases}$$

is a strongly continuous semigroup of operators that formally solves eq. (4.28). Compare this to Remark 3.17, where a similar formula appears. Formally applying Duhamel's principle shows that for any f_1, f_2, the function

$$V(t) = (v_1, v_2)(t)^{\mathsf{T}} := \int_0^t \mathcal{T}(t-s)(f_1, f_2)^{\mathsf{T}}(s)\,\mathrm{d}s$$

yields a solution to

$$\begin{gathered} (1+\tau\delta)(\delta v_1 + L v_1 - f_1) = \sigma\Delta v_1 + \tau(L f_1 + f_2), \\ v_1(t=0) = 0, \quad \delta v_1(t=0) = f_1(t=0), \\ v_1(a=0) = \int_0^{a_{\max}} \beta_0(\alpha) v_1(\alpha)\,\mathrm{d}\alpha, \\ (\delta v_1 - f_1)(a=0) = \int_0^{a_{\max}} \beta_L v_1 + \beta_1(\delta v_1 - f_1)\,\mathrm{d}a. \end{gathered}$$

The choice $f_1 = 0$, $f_2 = \tau^{-1} f$ gives a solution to eq. (4.28) without g_0 and g_1, but the extra factor of τ^{-1} remains present in estimates, which would yield worse convergence rates in Theorem 4.23. Another choice is $f_1 = f$, $f_2 = -Lf$ and $\beta_L = \beta_1 L - L(0)\beta_0$, which captures the structure of eq. (4.1) better, but has the disadvantage that it is unclear in what sense this construction actually represents a solution to the problem. In addition, the semigroup method provides less flexibility, for example it does not allow for constant terms in the birth equations. For this reason we do not pursue it any further.

4.5 The nonlinear equation

In this section we are ready to reintroduce the nonlinear terms to the relaxed equation to obtain model (4.1) again:

$$(1+\tau\delta)(\delta y + L y + \Lambda(y) y) = \sigma(a)\Delta y + f, \tag{4.34a}$$

$$y(t=0) = y_0, \quad \delta y(t=0) = y_1, \quad \partial_\nu y(x \in \partial\Omega) = 0, \tag{4.34b}$$

$$y(t, a=0) = \int_0^{a_{\max}} \beta_0(\alpha, x) y(t, \alpha, x)\,\mathrm{d}\alpha + g_0(t, x), \tag{4.34c}$$

$$(\delta y + L y + \Lambda(y) y)(t, a=0, x) = \int_0^{a_{\max}} \beta_1(\alpha, x)(\delta y + L y + \Lambda(y) y)(t, \alpha, x)\,\mathrm{d}\alpha + g_1(t, x). \tag{4.34d}$$

Note that we keep an inhomogeneous term f in eq. (4.34a), for later use in Section 4.6. The presence of nonlinear terms both in the equation and the implicit boundary conditions makes this section harder than the corresponding part for the unrelaxed equation in Section 3.4. We start with some assumptions.

4.16 Assumption. For the rest of this chapter we assume

1. For all $h, i, j = 1, \dots, n$, we have $k^{hij} \in C^1(\bar{I} \times \bar{\Omega} \times \bar{I} \times \bar{\Omega})$ and it holds $k^{hij}(a, x, a_{\max}, \xi) = 0$ for all $a \in \mathcal{I}$ and $x, \xi \in \Omega$. In other words, individuals of maximal age are not infectious at all.
2. $g_0 \in L^\infty((0,T), H) \cap L^2((0,T), V)$.
3. All of Assumption 4.12 still holds:
   - 3.1. $\tau \in (0, \tau_{\max})$ for some $0 < \tau_{\max} < \infty$.
   - 3.2. $y_0 \in \mathcal{V}$, $y_1 \in \mathcal{H}$, $f \in L^2((0,T), \mathcal{H})$ and $g_1 \in L^2((0,T), H)$.
   - 3.3. $L \in C^1(\bar{\mathcal{I}}, L^\infty(\Omega))^{n \times n}$ and its entries are uniformly bounded away from zero with respect to both age and space.
   - 3.4. $\sigma \in C^1(\bar{\mathcal{I}}, \mathbb{R})^{n \times n}$ is a diagonal matrix whose entries are uniformly bounded away from zero with respect to age.
   - 3.5. $\beta_0 \in L^\infty(\mathcal{I}, W_b^{1,d}(\Omega))$ resp. in $W_b^{1,d+\varepsilon}$ for some $\varepsilon > 0$ if $d = 2$ (for the definition of these spaces we refer to Lemma 4.10) and $\beta_1 \in L^\infty(\mathcal{I} \times \Omega)$.

From eq. (1.1d) we recall the definition

$$\Lambda(a, x, w)^{hi} = \int_0^{a_{\max}} \int_\Omega k^{hij}(a, x, \alpha, \xi) w_j(\alpha, \xi) \, \mathrm{d}\xi \, \mathrm{d}\alpha = \left\langle k^{hi}(a, x, \cdot, \cdot), w \right\rangle_{\mathcal{H}},$$

where $w \in \mathcal{H}$, $h, i = 1, \dots, n$ and we sum over the index j. In what follows, to keep the notation short we will drop the indices h, i, j when no confusion arises, and interpret the term kw as tensor contraction. Further we use Einstein's convention over every index that appears twice. This allows us to write eq. (1.1d) as

$$\Lambda(a, x, w) = \int_0^{a_{\max}} \int_\Omega k(a, x, \alpha, \xi) w(\alpha, \xi) \, \mathrm{d}\xi \, \mathrm{d}\alpha \,. \tag{4.35}$$

Assumption 4.16 permit the following expression for $\Lambda(a = 0)$ we need in birth condition (4.34d):

$$\Lambda(a = 0, x, w) = \int_0^{a_{\max}} \int_\Omega k(0, x, \alpha, \xi) w(\alpha, \xi) \, \mathrm{d}\xi \, \mathrm{d}\alpha \,. \tag{4.36}$$

Note that in contrast to the relaxed equation, not only does the equation contain a term $\Lambda(y)y$, but also one of the form $\tau\delta(\Lambda(y)y)$, which requires some clarification. From the definition of Λ from eq. (4.35) we formally have

$$\begin{aligned} \delta(\Lambda(a, x, y(t, \cdot, \cdot))y(t, a, x)) = {} & \Lambda(a, x, y(t, \cdot, \cdot))\delta y(t, a, x) \\ & + \Lambda_a(a, x, y(t, \cdot, \cdot))y(t, a, x) + \Lambda(a, x, y_t(t, \cdot, \cdot))y(t, a, x), \end{aligned}$$

which is inconvenient since Theorem 4.14 only provides information on $\delta y = y_t + y_a$, not on y_t alone. However, since $\Lambda(y)$ is linear in y we can write $\Lambda(y_t) = \Lambda(\delta y) - \Lambda(y_a)$, and integrating the term $\Lambda(y_a)$ by parts yields

$$\begin{aligned}\Lambda(y_a) &= \int_\Omega \int_0^{a_{\max}} k(a,x,\alpha,\xi) y_a(t,\alpha,\xi)\,\mathrm{d}\alpha\,\mathrm{d}\xi \\ &= \left[\int_\Omega k(a,x,\alpha,\xi)y(t,\alpha,\xi)\,\mathrm{d}\xi\right|_{\alpha=0}^{a_{\max}} - \int_\Omega\int_0^{a_{\max}} k_\alpha(a,x,\alpha,\xi)y(t,\alpha,\xi)\,\mathrm{d}\alpha\,\mathrm{d}\xi\,.\end{aligned}$$

Because of the condition $k(a,x,a_{\max},\xi) = 0$, the term where we evaluate the integral at $a_{\max}$ vanishes. (Note that this condition would not be necessary if one could show that $y(a = a_{\max}) = 0$, as in [AAC11, Thm. 4.2].) Using the implicit birth condition (4.34c) we can rewrite the $(\alpha = 0)$-term as

$$\begin{aligned}\int_\Omega k(a,x,0,\xi)y(t,0,\xi)\,\mathrm{d}\xi &= \int_\Omega k(a,x,0,\xi)\left(\int_0^{a_{\max}} \beta_0(\alpha,\xi)y(t,\alpha,\xi)\,\mathrm{d}\alpha + g_0(t,\xi)\right)\mathrm{d}\xi \\ &= \int_\Omega\int_0^{a_{\max}} \big(k(a,x,0,\xi)\beta_0(\alpha,\xi)\big)y(t,\alpha,\xi)\,\mathrm{d}\alpha\,\mathrm{d}\xi + \int_\Omega k(a,x,0,\xi)g_0(t,\xi)\,\mathrm{d}\xi\,,\end{aligned}$$

which shows that $\delta\Lambda$ will also depend on g_0, in a spatially nonlocal way. Hence, for v, $w \in L^\infty((0,T),\mathcal{H})$ we introduce the notation $\delta\Lambda(v,g_0)w$ defined by

$$\delta\Lambda(v,g_0)w(t,a,x) := \Lambda(v)\delta w + \Lambda(\delta v)w + \Lambda_1(v)w + \Lambda_2(g_0(t))w, \tag{4.37}$$

where, with a similar notation as in eq. (4.35) we let $(h,\, i = 1,\dots,n)$

$$\begin{aligned}\Lambda_1(v)^{hi} &:= \int_\Omega\int_0^{a_{\max}} \big(k_a^{hij}(a,x,\alpha,\xi) + k_\alpha^{hij}(a,x,\alpha,\xi) + k^{hi\ell}(a,x,0,\xi)\beta_0^{\ell j}(\alpha,\xi)\big)v_j(\alpha,\xi)\,\mathrm{d}\alpha\,\mathrm{d}\xi\,, \\ \Lambda_2(g_0(t))^{hi} &:= \int_\Omega k^{hij}(a,x,0,\xi)(g_0)_j(t,\xi)\,\mathrm{d}\xi\,.\end{aligned}$$

This will be the expression we use for $\delta\Lambda$ in eq. (4.34a). Furthermore we write

$$(1+\tau\delta)\Lambda(v,g_0)w := \Lambda(v)w + \tau\delta\Lambda(v,g_0)w, \tag{4.38}$$

with $\delta\Lambda$ defined as in eq. (4.37). In total, eq. (4.34a) takes the following form:

$$(1+\tau\delta)(\delta y + Ly) + \Lambda(y)y + \tau\Lambda(y)\delta y + \tau\Lambda(\delta y)y + \tau\Lambda_1(y)y + \tau\Lambda_2(g_0)y = \sigma\Delta y + f.$$

Assumption 4.16 yields that $k_a + k_\alpha + k(\alpha = 0)\beta_0 \in L^\infty(\mathcal{I}\times\Omega,\mathcal{H})$. Thus we can estimate

$$\begin{aligned}\|\delta\Lambda(v(t),g_0(t))w(t)\|_{\mathcal{H}} \lesssim{}& \|v(t)\|_{\mathcal{H}}\|w(t)\|_{\mathcal{H}} + \|\delta v(t)\|_{\mathcal{H}}\|w(t)\|_{\mathcal{H}} \\ &+ \|v(t)\|_{\mathcal{H}}\|\delta w(t)\|_{\mathcal{H}} + \|g_0(t)\|_{H}\|v(t)\|_{\mathcal{H}}\end{aligned} \tag{4.39}$$

This shows that, contrary to Assumption 4.12, assuming $g_0 \in L^2((0,T),H)$ is not enough, we need the stronger assumption $g_0 \in L^\infty((0,T),H)$. Combining eq. (3.26) with eq. (4.39) and integration with respect to the time variable yields

$$\begin{aligned}\|(1+\tau\delta)\Lambda(v,g_0)w\|_{L^2((0,T),\mathcal{H})} &\lesssim \|v\|_{L^\infty((0,T),\mathcal{H})}\|w\|_{L^2((0,T),\mathcal{H})} + \tau\|\delta v\|_{L^\infty((0,T),\mathcal{H})}\|w\|_{L^2((0,T),\mathcal{H})} \\ &\quad + \tau\|v\|_{L^\infty((0,T),\mathcal{H})}\|\delta w\|_{L^2((0,T),\mathcal{H})} + \tau\|g_0\|_{L^\infty((0,T),H)}\|w\|_{L^2((0,T),\mathcal{H})}, \end{aligned} \tag{4.40a}$$

or alternatively

$$\begin{aligned}\|(1+\tau\delta)\Lambda(v,g_0)w\|_{L^2((0,T),\mathcal{H})} &\lesssim \|v\|_{L^2((0,T),\mathcal{H})}\|w\|_{L^\infty((0,T),\mathcal{H})} + \tau\|\delta v\|_{L^2((0,T),\mathcal{H})}\|w\|_{L^\infty((0,T),\mathcal{H})} \\ &\quad + \tau\|v\|_{L^2((0,T),\mathcal{H})}\|\delta w\|_{L^\infty((0,T),\mathcal{H})} + \tau\|g_0\|_{L^\infty((0,T),H)}\|w\|_{L^2((0,T),\mathcal{H})}. \end{aligned} \tag{4.40b}$$

Our next task is to interpret the nonlinear first-order birth equation

$$(\delta y + Ly + \Lambda(y)y)(a=0) = \int_0^{a_{\max}} \beta_1(\alpha)(\delta y + Ly + \Lambda(y)y)(\alpha)\,\mathrm{d}\alpha + g_1(t,x)$$

from eq. (4.34d). Using eq. (4.34c), this can be rewritten to

$$\begin{aligned}\delta y(a=0) + L(a=0)\left(\int_0^{a_{\max}} \beta_0(\alpha)y(\alpha)\,\mathrm{d}\alpha + g_0\right) + \Lambda(a=0,y)\left(\int_0^{a_{\max}} \beta_0(\alpha)y(\alpha)\,\mathrm{d}\alpha + g_0\right) \\ = \int_0^{a_{\max}} \beta_1(\alpha)(\delta y + Ly + \Lambda(y)y)(\alpha)\,\mathrm{d}\alpha + g_1.\end{aligned}$$

Restructuring yields

$$\begin{aligned}\delta y(a=0) = &\int_0^{a_{\max}} \beta_1(\alpha)\delta y(\alpha)\,\mathrm{d}\alpha \\ &+ \int_0^{a_{\max}} (\beta_1(\alpha)L(\alpha) - L(a=0)\beta_0(\alpha))y(\alpha)\,\mathrm{d}\alpha \\ &+ \int_0^{a_{\max}} (\beta_1(\alpha)\Lambda(\alpha,y)(\alpha) - \Lambda(a=0,y)\beta_0(\alpha))y(\alpha)\,\mathrm{d}\alpha \\ &- \Lambda(a=0,y)g_0 - L(a=0)g_0 + g_1.\end{aligned}$$

We define

$$\beta_L(\alpha) := \beta_1(\alpha)L(\alpha) - L(a=0)\beta_0(\alpha), \tag{4.41}$$

$$G(v)(w,g_0) := \int_0^{a_{\max}} (\beta_1(\alpha)\Lambda(\alpha,v) - \Lambda(a=0,v)\beta_0(\alpha))w(\alpha)\,\mathrm{d}\alpha - \Lambda(a=0,v)g_0 \tag{4.42}$$

for v, $w \in L^\infty((0,T)),\mathcal{H}$, where $\Lambda(a=0)$ is defined as in eq. (4.36). Then we interpret eq. (4.34d) as

$$\delta y(a=0) = \int_0^{a_{\max}} \beta_1(\alpha)\delta y(\alpha) + \beta_L(\alpha) y(\alpha)\,\mathrm{d}\alpha + G(y)(y,g_0) - L(a=0)g_0 + g_1. \tag{4.43}$$

From Assumption 4.16 we can conclude that $\beta_L \in L^\infty(\mathcal{I}\times\Omega)$ and $L(a=0)\in L^\infty(\Omega)^{n\times n}$. This directly yields the estimate

$$\|L(a=0)g_0\|_{L^2((0,T),H)} \lesssim \|g_0\|_{L^2((0,T),H)}. \tag{4.44}$$

Note that G from eq. (4.42) is nonlocal in v and linear in both of its arguments v and (w,g_0), hence the seemingly strange notation. We can estimate

$$\begin{aligned}\|G(v)(w,g_0)\|_{L^2((0,T),H)} &\lesssim \left\|\beta_1\Lambda(v)w\right\|_{L^2((0,T),\mathcal{H})}\\ &\quad + \left\|\Lambda(a=0,v)\beta_0 w\right\|_{L^2((0,T),\mathcal{H})} + \|\Lambda(a=0,v)g_0\|_{L^2((0,T),H)}\\ &\lesssim \|v\|_{L^\infty((0,T),\mathcal{H})}\|w\|_{L^2((0,T),\mathcal{H})} + \|v\|_{L^2((0,T),\mathcal{H})}\|g_0\|_{L^\infty((0,T),H)},\end{aligned} \tag{4.45a}$$

or alternatively

$$\|G(v)(w,g_0)\|_{L^2((0,T),H)} \lesssim \|v\|_{L^2((0,T),\mathcal{H})}\|w\|_{L^\infty((0,T),\mathcal{H})} + \|v\|_{L^2((0,T),\mathcal{H})}\|g_0\|_{L^\infty((0,T),H)}. \tag{4.45b}$$

Altogether, eq. (4.34) can be reformulated as

$$\begin{aligned}(1+\tau\delta)(\delta y + Ly) &= \sigma\Delta y - \Lambda(y)y - \tau\Lambda(y)\delta y - \tau\Lambda(\delta y)y - \tau\Lambda_1(y)y - \tau\Lambda_2(g_0)y + f,\\ y(t=0) &= y_0, \quad \delta y(t=0) = y_1, \quad \partial_\nu y = 0 \text{ in } \partial\Omega,\\ y(a=0) &= \int_0^{a_{\max}} \beta_0(\alpha)y(\alpha)\,\mathrm{d}\alpha + g_0,\\ \delta y(a=0) &= \int_0^{a_{\max}} \beta_1(\alpha)\delta y(\alpha) + \beta_L(\alpha)y(\alpha)\,\mathrm{d}\alpha + G(y)(y,g_0) - L(a=0)g_0 + g_1.\end{aligned} \tag{4.46}$$

The strategy of finding a solution to Equation (4.46) is as in Theorem 3.20: find fixed points of the map Φ which maps a given w to the weak solution y (in the sense of Theorem 4.14) of

$$\begin{aligned}(1+\tau\delta)(\delta y + Ly) &= \sigma\Delta y - (1+\tau\delta)\Lambda(w,g_0)w + f,\\ y(t=0) &= y_0, \quad \delta y(t=0) = y_1, \quad \partial_\nu y = 0 \text{ in } \partial\Omega,\\ y(a=0) &= \int_0^{a_{\max}} \beta_0(\alpha)y(\alpha)\,\mathrm{d}\alpha + g_0,\\ \delta y(a=0) &= \int_0^{a_{\max}} \beta_1(\alpha)\delta y(\alpha) + \beta_L(\alpha)y(\alpha)\,\mathrm{d}\alpha + G(w)(w,g_0) - L(a=0)g_0 + g_1.\end{aligned} \tag{4.47}$$

Using eq. (4.40), eq. (4.44) and eq. (4.45) shows that under Assumption 4.16, for any $w \in L^\infty((0,T),\mathcal{H})$ with $\delta w \in L^\infty((0,T),\mathcal{H})$, the conditions of Theorem 4.14 are satisfied.

Thus, eq. (4.33) together with the fact that $\tau \le \tau_{\max}$ yields an estimate for a solution of eq. (4.47):

$$
\begin{aligned}
\tau\left\|\delta y(t)\right\|_{\mathcal{H}}^2 + \left\|y(t)\right\|_{\mathcal{V}}^2 &\lesssim \left\|y_0\right\|_{\mathcal{V}}^2 + \tau\left\|y_1\right\|_{\mathcal{H}}^2 + \|f\|_{L^2((0,T),\mathcal{H})}^2 + \|(1+\tau\delta)\Lambda(w,g_0)w\|_{L^2((0,t),\mathcal{H})^2}^2 \\
&\quad + \|g_0\|_{L^2((0,T),V)}^2 + \tau\|G(w)(w,g_0) - L(0)g_0 + g_1\|_{L^2((0,t),H)}^2 \\
&\lesssim \left\|y_0\right\|_{\mathcal{V}}^2 + \tau\left\|y_1\right\|_{\mathcal{H}}^2 + \|g_0\|_{L^2((0,T),V)}^2 + \|f\|_{L^2((0,T),\mathcal{H})}^2 \\
&\quad + \Big(\|w\|_{L^2((0,T),\mathcal{H})}\|w\|_{L^\infty((0,T),\mathcal{H})} + \tau\|\delta w\|_{L^2((0,T),\mathcal{H})}\|w\|_{L^\infty((0,T),\mathcal{H})} \\
&\quad + \tau\|g_0\|_{L^\infty((0,T),H)}\|w\|_{L^2((0,T),\mathcal{H})}\Big)^2 + \tau\Big(\|w\|_{L^2((0,T),\mathcal{H})}\|w\|_{L^\infty((0,T),\mathcal{H})} \\
&\quad + \|w\|_{L^2((0,T),\mathcal{H})}\|g_0\|_{L^\infty((0,T),H)} + \|L(0)g_0\|_{L^2((0,t),H)} + \|g_1\|_{L^2((0,t),H)}\Big)^2 \\
&\lesssim \left\|y_0\right\|_{\mathcal{V}}^2 + \tau\left\|y_1\right\|_{\mathcal{H}}^2 + \|f\|_{L^2((0,T),\mathcal{H})}^2 + \|g_0\|_{L^2((0,T),V)}^2 + \tau\|g_0\|_{L^2((0,t),H)}^2 + \tau\|g_1\|_{L^2((0,t),H)}^2 \\
&\quad + \|w\|_{L^2((0,T),\mathcal{H})}^2\|w\|_{L^\infty((0,T),\mathcal{H})}^2 + \tau^2\|\delta w\|_{L^2((0,T),\mathcal{H})}^2\|w\|_{L^\infty((0,T),\mathcal{H})}^2 \\
&\quad + \tau\|g_0\|_{L^\infty((0,T),H)}^2\|w\|_{L^2((0,T),\mathcal{H})}^2.
\end{aligned}
\tag{4.48}
$$

Now we can proceed as in Theorem 3.20, resp. in [Smo83, Thm. 14.2 and Lem. 14.3].

4.17 Lemma. $(1+\tau\delta)\Lambda$ and G are locally Lipschitz continuous.

Proof: Let v_i, $w_i \in L^\infty((0,T),\mathcal{H})$ and $g_0^i \in L^2((0,T),H)$ ($i \in \{1,2\}$). If also δv_i, $\delta w_i \in L^\infty((0,T),\mathcal{H})$, eq. (4.40) yields

$$
\begin{aligned}
&\left\|(1+\tau\delta)[\Lambda(v_1,g_0^1)w_1] - (1+\tau\delta)[\Lambda(v_2,g_0^2)w_2]\right\|_{L^2((0,T),\mathcal{H})} \\
&\lesssim \left\|(1+\tau\delta)[\Lambda(v_1,g_0^1)(w_1-w_2)]\right\|_{L^2((0,T),\mathcal{H})} + \left\|(1+\tau\delta)[\Lambda(v_1-v_2,g_0^1-g_0^2)w_2]\right\|_{L^2((0,T),\mathcal{H})} \\
&\lesssim \|v_1\|_{L^\infty((0,T),\mathcal{H})}\|w_1-w_2\|_{L^2((0,T),\mathcal{H})} + \tau\|\delta v_1\|_{L^\infty((0,T),\mathcal{H})}\|w_1-w_2\|_{L^2((0,T),\mathcal{H})} \\
&\quad + \tau\|v_1\|_{L^\infty((0,T),\mathcal{H})}\|\delta w_1-\delta w_2\|_{L^2((0,T),\mathcal{H})} + \tau\left\|g_0^1\right\|_{L^\infty((0,t),H)}\|w_1-w_2\|_{L^2((0,T),\mathcal{H})} \\
&\quad + \|v_1-v_2\|_{L^2((0,T),\mathcal{H})}\|w_2\|_{L^\infty((0,T),\mathcal{H})} + \tau\|\delta v_1-\delta v_2\|_{L^2((0,T),\mathcal{H})}\|w_2\|_{L^\infty((0,T),\mathcal{H})} \\
&\quad + \tau\|v_1-v_2\|_{L^2((0,T),\mathcal{H})}\|\delta w_2\|_{L^\infty((0,T),\mathcal{H})} + \tau\left\|g_0^1-g_0^2\right\|_{L^\infty((0,t),H)}\|w_2\|_{L^2((0,T),\mathcal{H})} \\
&\lesssim \max\left\{\|v_1\|_{L^\infty((0,T),\mathcal{H})}, \tau\|\delta v_1\|_{L^\infty((0,T),\mathcal{H})}, \|w_2\|_{L^\infty((0,T),\mathcal{H})}, \tau\|\delta w_2\|_{L^\infty((0,T),\mathcal{H})}, \tau\left\|g_0^1\right\|_{L^\infty((0,t),H)}\right\} \\
&\quad \cdot \Big(\|v_1-v_2\|_{L^2((0,T),\mathcal{H})} + \|w_1-w_2\|_{L^2((0,T),\mathcal{H})} + \tau\|\delta v_1-\delta v_2\|_{L^2((0,T),\mathcal{H})} \\
&\qquad + \tau\|\delta w_1-\delta w_2\|_{L^2((0,T),\mathcal{H})} + \tau\left\|g_0^1-g_0^2\right\|_{L^\infty((0,T),H)}\Big).
\end{aligned}
$$

Similarly, for G we have, using eq. (4.45):

$$
\begin{aligned}
&\left\|G(v_1)(w_1,g_0^1) - G(v_2)(w_2,g_0^2)\right\|_{L^2((0,T),H)} \\
&\le \left\|G(v_1-v_2)(w_1,g_0^1)\right\|_{L^2((0,T),H)} + \left\|G(v_2)(w_1-w_2,g_0^1-g_0^2)\right\|_{L^2((0,T),H)} \\
&\lesssim \max\left\{\|v_1\|_{L^\infty((0,T),\mathcal{H})}, \|w_2\|_{L^\infty((0,T),\mathcal{H})}, \left\|g_0^1\right\|_{L^\infty((0,T),H)}\right\} \\
&\quad \cdot \Big(\|v_1-v_2\|_{L^2((0,T),\mathcal{H})} + \|w_1-w_2\|_{L^2((0,T),\mathcal{H})} + \left\|g_0^1-g_0^2\right\|_{L^2((0,T),H)}\Big).
\end{aligned}
$$

□

4.18 Lemma. Let y^1 and y^2 be two solutions of eq. (4.46) to the respective data y_0^i, y_1^i, f^i, g_0^i, g_1^i ($i \in \{1,2\}$) and let $y := y^1 - y^2$. Further assume that $\|y^i\|_{L^\infty((0,T),\mathcal{H})}$, $\tau\|\delta y^i\|_{L^\infty((0,T),\mathcal{H})}$ and $\tau\|g_0^i\|_{L^\infty((0,T),H)}$ are bounded above by some S for $i \in \{1,2\}$. Then we have an estimate of the form

$$\begin{aligned}\tau\|\delta y\|^2_{C([0,T],\mathcal{H})} + \|y(t)\|^2_{C([0,T],\mathcal{V})} \lesssim{}& \|y_0^1 - y_0^2\|^2_{\mathcal{V}} + \tau\|y_1^1 - y_1^2\|^2_{\mathcal{H}} + \|f^1 - f^2\|^2_{L^2((0,T),\mathcal{H})} \\ &+ \|g_0^1 - g_0^2\|^2_{L^2((0,T),V)} + \tau\|g_1^1 - g_1^2\|^2_{L^2((0,T),H)}.\end{aligned}$$

Proof: The difference y is a solution to

$$\begin{aligned}(1+\tau\delta)(\delta y + Ly)y &= \sigma(a)\Delta y - (1+\tau\delta)(\Lambda(y^1,g_0^1)y^1 - \Lambda(y^2,g_0^2)y^2) + f^1 - f^2, \\ y(t=0) &= y_0^1 - y_0^2, \quad \delta y(t=0) = y_1^1 - y_1^2, \\ \partial_\nu y &= 0 \text{ in } \partial\Omega, \\ y(a=0) &= \int_0^{a_{\max}} \beta_0(\alpha)y(\alpha)\,\mathrm{d}\alpha + g_0^1 - g_0^2, \\ (\delta y)(a=0) &= \int_0^{a_{\max}} \beta_1(\alpha)\delta y(\alpha) + \beta_L(\alpha)y(\alpha)\,\mathrm{d}\alpha \\ &\quad + G(y^1)(y^1,g_0^1) - G(y^2)(y^2,g_0^2) - L(0)(g_0^1 - g_0^2) + g_1^1 - g_1^2.\end{aligned}$$

By Theorem 4.14 and Lemma 4.17 we obtain

$$\begin{aligned}&\tau\|\delta y(t)\|^2_{\mathcal{H}} + \|y(t)\|^2_{\mathcal{V}} \\ &\lesssim \|y_0^1 - y_0^2\|^2_{\mathcal{V}} + \tau\|y_1^1 - y_1^2\|^2_{\mathcal{H}} + \|(1+\tau\delta)(\Lambda(y^1,g_0^1)y^1 - \Lambda(y^2,g_0^2)y^2)\|^2_{L^2((0,t),\mathcal{H})} \\ &\quad + \|f^1 - f^2\|^2_{L^2((0,t),\mathcal{H})} + \|g_0^1 - g_0^2\|^2_{L^2((0,t),V)} + \tau\|g_1^1 - g_1^2\|^2_{L^2((0,t),H)} + \tau\|L(0)(g_0^1 - g_0^2)\|^2_{L^2((0,t),H)} \\ &\quad + \tau\|G(y^1)(y^1,g_0^1) - G(y^2)(y^2,g_0^2)\|^2_{L^2((0,t),H)} \\ &\lesssim \|y_0^1 - y_0^2\|^2_{\mathcal{V}} + \tau\|y_1^1 - y_1^2\|^2_{\mathcal{H}} + \|f^1 - f^2\|^2_{L^2((0,t),\mathcal{H})} \\ &\quad + \|g_0^1 - g_0^2\|^2_{L^2((0,t),V)} + \tau\|g_1^1 - g_1^2\|^2_{L^2((0,t),H)} + \tau\|g_0^1 - g_0^2\|^2_{L^2((0,t),H)} \\ &\quad + \max\left\{\|y^1\|^2_{L^\infty((0,t),\mathcal{H})}, \tau\|\delta y^1\|^2_{L^\infty((0,t),\mathcal{H})}, \tau\|g_0^1\|^2_{L^\infty((0,t),H)}, \|y^2\|^2_{L^\infty((0,t),\mathcal{H})}, \tau\|\delta y^2\|^2_{L^\infty((0,t),\mathcal{H})}\right\} \\ &\quad \cdot \left(\|y^1 - y^2\|^2_{L^2((0,t),\mathcal{H})} + \tau\|\delta y^1 - \delta y^2\|^2_{L^2((0,t),\mathcal{H})} + \tau\|g_0^1 - g_0^2\|^2_{L^2((0,t),H)}\right).\end{aligned} \tag{4.49}$$

Estimating by S yields

$$\begin{aligned}\tau\|\delta y(t)\|^2_{\mathcal{H}} + \|y(t)\|^2_{\mathcal{V}} \lesssim{}& \|y_0^1 - y_0^2\|^2_{\mathcal{V}} + \tau\|y_1^1 - y_1^2\|^2_{\mathcal{H}} + \|f^1 - f^2\|^2_{L^2((0,t),\mathcal{H})} \\ &+ \|g_0^1 - g_0^2\|^2_{L^2((0,t),V)} + \tau\|g_1^1 - g_1^2\|^2_{L^2((0,t),H)} + \tau\|g_0^1 - g_0^2\|^2_{L^2((0,t),H)} \\ &+ S^2\left(\|y^1 - y^2\|_{L^2((0,T),\mathcal{H})} + \tau\|\delta y^1 - \delta y^2\|_{L^2((0,T),\mathcal{H})} + \tau\|g_0^1 - g_0^2\|_{L^2((0,t),H)}\right)^2.\end{aligned}$$

Gronwall's lemma, where in Lemma 2.4 we choose $\Phi(t) = \tau\left\|\delta y(t)\right\|_{\mathcal{H}}^2 + \left\|y(t)\right\|_{\mathcal{V}}^2$, yields the estimate

$$\begin{aligned}\tau\left\|\delta y(t)\right\|_{\mathcal{H}}^2 + \left\|y(t)\right\|_{\mathcal{V}}^2 \lesssim{}& \left\|y_0^1 - y_0^2\right\|_{\mathcal{V}}^2 + \tau\left\|y_1^1 - y_1^2\right\|_{\mathcal{H}}^2 + \left\|f^1 - f^2\right\|_{L^2((0,t),\mathcal{H})}^2 \\ &+ \left\|g_0^1 - g_0^2\right\|_{L^2((0,t),V)}^2 + \tau\left\|g_1^1 - g_1^2\right\|_{L^2((0,t),H)}^2 + \tau\left\|g_0^1 - g_0^2\right\|_{L^2((0,t),H)}^2 \\ &+ \tau^2 S^2\left\|g_0^1 - g_0^2\right\|_{L^2((0,t),H)}^2,\end{aligned}$$

and estimating τ by $\tau_{\max}$ and norms on $(0,t)$ by those on $(0,T)$ shows the claim. □

Letting $y_0^1 = y_0^2$ etc. in the previous theorem directly yields

4.19 Corollary. There is at most one weak solution $y \in L^\infty((0,T),\mathcal{H})$ to the system (4.46).

4.20 Theorem. Under Assumption 4.16 there exists a $T^* > 0$ such that for every $T \in (0,T^*)$ there is a weak solution $y \in C([0,T],\mathcal{V}) \cap C(\bar{\mathcal{I}}, L^2((0,T),V))$ with $\delta y \in C([0,T],\mathcal{H}) \cap C(\bar{\mathcal{I}}, L^2((0,T),H))$ to the system (4.46). That means, for all $v \in V$ the weak formulation

$$\begin{aligned}\left\langle \tau\delta^2 y(t,a), v\right\rangle_{V'\times V} + \left\langle (1+\tau L(a))\delta y(t,a), v\right\rangle_H + \left\langle (L(a) + \tau L_a(a)) y(t,a), v\right\rangle_H \\ = \left\langle \sigma(a)\nabla y(t,a), \nabla v\right\rangle_H + \left\langle f(t,a), v\right\rangle_H - \left\langle (1+\tau\delta)\Lambda(y,g_0)y, v\right\rangle_H,\end{aligned}$$

where $(1+\tau\delta)\Lambda$ is defined as in eq. (4.38), is satisfied and the initial conditions (4.34b), (4.34c) and (4.43) hold. In addition, y satisfies the estimate

$$\left\|y\right\|_{C([0,T],\mathcal{V})}^2 + \tau\left\|\delta y\right\|_{C([0,T],\mathcal{H})}^2 \lesssim \mathcal{K}$$

where

$$\mathcal{K} := \left\|y_0\right\|_{\mathcal{V}}^2 + \tau\left\|y_1\right\|_{\mathcal{H}}^2 + \|f\|_{L^2((0,t),\mathcal{H})}^2 + \|g_0\|_{L^2((0,t),V)}^2 + \tau\|g_1\|_{L^2((0,T),H)}^2.$$

More precisely, T^* can be chosen as $C\cdot(\mathcal{K} + \tau\|g_0\|_{L^\infty((0,T),H)}^2)^{-1}$ with a constant C independent of y_0, y_1, f, g_0, g_1, or τ, but possibly depending on $\tau_{\max}$.

Proof: Let $X := \{u \in C([0,T],\mathcal{V}) \mid \exists \delta u \in C([0,T],\mathcal{H})\}$ endowed with the norm

$$\|u\|_{X,\tau} := \left(\|u\|_{C([0,T],\mathcal{V})}^2 + \tau\|\delta u\|_{C([0,T],\mathcal{H})}^2\right)^{1/2},$$

then it is easy to see that X is a Banach space. Further let Φ as in eq. (4.47). Estimate (4.48) shows that Φ is a well-defined map in X. Furthermore let $\bar{y} := \Phi(0)$ the solution to the linear equation

$$\begin{aligned}(1+\tau\delta)(\delta\bar{y} + L\bar{y})y &= \sigma(a)\Delta\bar{y} + f, \\ \bar{y}(t=0) &= y_0, \quad \delta\bar{y}(t=0) = y_1, \\ \partial_\nu\bar{y} &= 0 \text{ in } \partial\Omega, \\ \bar{y}(a=0) &= \int_0^{a_{\max}} \beta_0(\alpha)\bar{y}(\alpha)\,\mathrm{d}\alpha + g_0, \\ (\delta\bar{y})(a=0) &= \int_0^{a_{\max}} \beta_1(\alpha)\delta\bar{y}(\alpha) + \beta_L(\alpha)\bar{y}(\alpha)\,\mathrm{d}\alpha + g_1.\end{aligned}$$

From eq. (4.33) we have the estimate

$$\left\|\bar{y}\right\|_{X,\tau}^2 \lesssim \left\|y_0\right\|_{\mathcal{V}}^2 + \tau\left\|y_1\right\|_{\mathcal{H}}^2 + \|f\|_{L^2((0,t),\mathcal{H})}^2 + \|g_0\|_{L^2((0,T),V)}^2 + \tau\|g_1\|_{L^2((0,T),H)}^2 \lesssim \mathcal{K}.$$

Finally let

$$\Gamma^\tau = \left\{ v \in X \;\middle|\; \left\|v - \bar{y}\right\|_{X,\tau}^2 \le \mathcal{K} \right\}.$$

Obviously, Γ^τ is a nonempty and closed set in X. Since for any $v \in \Gamma^\tau$ we have

$$\|v\|_{X,\tau}^2 \lesssim \left\|v - \bar{y}\right\|_{X,\tau}^2 + \left\|\bar{y}\right\|_{X,\tau}^2 \lesssim \mathcal{K}, \tag{4.50}$$

the set Γ^τ is also bounded in X.

Now let y^1 and y^2 both in Γ^τ and let $y := \Phi(y^2) - \Phi(y^2)$, then y satisfies

$$\begin{aligned}
(1+\tau\delta)(\delta y + Ly)y &= \sigma(a)\Delta y - (1+\tau\delta)(\Lambda(y^1, g_0)y^1 - \Lambda(y^2, g_0)y^2),\\
y(t=0) &= 0, \quad \delta y(t=0) = 0,\\
\partial_\nu y &= 0 \text{ in } \partial\Omega,\\
y(a=0) &= \int_0^{a_{\max}} \beta_0(\alpha) y(\alpha)\,\mathrm{d}\alpha\\
(\delta y)(a=0) &= \int_0^{a_{\max}} \beta_1(\alpha)\delta y(\alpha) + \beta_L(\alpha) y(\alpha)\,\mathrm{d}\alpha,\\
&\quad + G(y^1)(y^1, g_0) - G(y^2)(y^2, g_0).
\end{aligned}$$

Thus, a similar calculation to eq. (4.49) combined with eq. (4.50) yields

$$\begin{aligned}
\left\|y\right\|_{X,\tau}^2 \lesssim \max\Big\{ &\left\|y^1\right\|_{L^\infty((0,T),\mathcal{H})}^2, \tau\left\|\delta y^1\right\|_{L^\infty((0,T),\mathcal{H})}^2, \tau\|g_0\|_{L^\infty((0,T),H)}^2, \left\|y^2\right\|_{L^\infty((0,T),\mathcal{H})}^2,\\
&\tau\left\|\delta y^2\right\|_{L^\infty((0,T),\mathcal{H})}^2 \Big\} \cdot \left(\left\|y^1 - y^2\right\|_{L^2((0,T),\mathcal{H})}^2 + \tau\left\|\delta y^1 - \delta y^2\right\|_{L^2((0,T),\mathcal{H})}^2 \right)\\
\lesssim \Big(&\mathcal{K} + \tau\|g_0\|_{L^\infty((0,T),H)}^2\Big) T \left\|y^1 - y^2\right\|_{X,\tau}.
\end{aligned}$$

Choosing T small enough, proportional to $\left(\mathcal{K} + \tau\|g_0\|_{L^\infty((0,T),H)}^2\right)^{-1}$, allows us to compensate for the constant in front of the expression, turning Φ into a contraction on Γ^τ. Together with eq. (4.50), this also shows that for any $y \in \Gamma^\tau$, the norm of $\Phi(y) - \bar{y} = \Phi(v) - \Phi(0)$ can be bounded above by

$$\left\|\Phi(y) - \bar{y}\right\|_{X,\tau} \lesssim \left(\mathcal{K} + \tau\|g_0\|_{L^\infty((0,T),H)}^2\right) \cdot T\left\|y\right\|_{X,\tau}^2 \lesssim \mathcal{K}\left(\mathcal{K} + \tau\|g_0\|_{L^\infty((0,T),H)}^2\right)T,$$

which shows that if T is chosen sufficiently small and proportional to the reciprocal of $\mathcal{K} + \tau\|g_0\|_{L^\infty((0,T),H)}^2$, we can ensure that Φ maps Γ^τ into itself. Thus, Banach's fixed-point theorem yields a unique fixed point of Φ, which is a solution to our differential equation. Since the fixed point lies in Γ^τ, the energy estimate follows from eq. (4.50), and the weak formulation and regularity are direct consequences of Theorem 4.14. □

4.6 Convergence for $\tau \to 0$

In this section we compare solutions y and y^τ to the unrelaxed model from eq. (1.1) or from eq. (3.1)

$$\begin{gathered}\delta y + L(a,x)y + \Lambda(a,x,y)y = \sigma(a)\Delta y,\\ y(t=0) = y_0, \quad \partial_\nu y(x \in \partial\Omega) = 0,\\ y(t,a=0,x) = \int_0^{a_{\max}} \beta(\alpha,x)y(t,\alpha,x)\,\mathrm{d}\alpha\end{gathered} \tag{4.51}$$

and the relaxed model from eq. (1.6) or eq. (4.34)

$$\begin{gathered}(1+\tau\delta)(\delta y^\tau + L(a,x)y^\tau + \Lambda(a,x,y^\tau)y^\tau) = \sigma(a)\Delta y^\tau,\\ y^\tau(t=0) = y_0, \quad \delta y^\tau(t=0) = y_1, \quad \partial_\nu y^\tau(x \in \partial\Omega) = 0,\\ y^\tau(t,a=0,x) = \int_0^{a_{\max}} \beta_0(\alpha,x)y^\tau(t,\alpha,x)\,\mathrm{d}\alpha + g_0,\\ (\delta y^\tau + Ly^\tau + \Lambda(y^\tau)y^\tau)(t,a=0,x) = \int_0^{a_{\max}} \beta_1(\alpha,x)(\delta y^\tau + Ly^\tau + \Lambda(y^\tau)y^\tau)(t,\alpha,x)\,\mathrm{d}\alpha + g_1.\end{gathered} \tag{4.52}$$

We always assume that Assumption 4.16 holds. From Theorem 3.20 and Theorem 4.20 we conclude that there exists a time T independent of τ such that $y \in L^2((0,T),\mathcal{V}) \cap C([0,T],\mathcal{H})$ and $\delta y \in L^2((0,T),\mathcal{V}')$ as well as $y^\tau \in C([0,T],\mathcal{V})$ and $\delta y^\tau \in C([0,T],\mathcal{H})$ for all $\tau \in (0,\tau_{\max})$.

4.21 Remark. For the following proofs to work, we need to assume a higher smoothness of y than we have proven so far. However, for the linearized equation (3.23) such a result seems plausible, as can be seen as follows: Let $t_0 \in (-a_{\max}, T)$ and v be a solution of the unrelaxed linearized equation (3.10) on $\mathrm{char}(t_0)$, and assume the assumptions 4.2 we imposed for the relaxed equation hold. They directly yield $v_0 \in V$ and $v_1 \in H$, and from Theorem 4.3 we can conclude that $L - \sigma\Delta \in C^1(\mathrm{char}(t_0), L(V,V'))$. From the fact that $v_t(t=0) \in H$ in the unrelaxed equation (3.10) on characteristics we can deduce that also $\sigma(0)\Delta v_0 = f(0,0) - v_1 - L(0)v_0 \in H$, i.e. $v_0 \in D(\sigma(0)\Delta)$. By imposing some stronger regularity on f, the conditions of Corollary 2.15 are satisfied, and thus we have $v \in H^1(\mathrm{char}(t_0), V) \cap H^2(\mathrm{char}(t_0), V')$. From case 1 of Sobolev's Embedding Theorem 2.8 (which also holds for Banach-space valued functions) and Lemma 2.11 it follows that $v \in C(\mathrm{char}(t_0), V)$ and $v' \in C(\mathrm{char}(t_0), H)$, which is the same regularity we obtained in Theorem 4.3 for the solution of the relaxed equation on characteristics. Therefore, similar proofs as in Section 4.3 can be carried out to show that in the case where Λ, β, β_0 and β_1 are all zero, y and y^τ have the same regularity.

For the convergence result to hold it is necessary that the relaxed and the unrelaxed equations have matching initial and boundary conditions. Let $q_1 \in \mathbb{R}$, then from

$$y(a=0) = q_1 y(a=0) + (1-q_1)y(a=0) = \int_0^{a_{\max}} q_1\beta(\alpha)y(\alpha)\,\mathrm{d}\alpha + (1-q_1)y(a=0)$$

we infer that the implicit birth conditions for y and y^τ match if we choose $\beta_0 := q_1\beta$ and $g_0 := (1-q_1)y(a=0)$. In other words, q_1 allows to switch between implicit birth conditions for the relaxed equation and explicitly given birth numbers from the unrelaxed model. By applying $(1+\tau\delta)$ to the first line of eq. (4.51), we conclude that y also satisfies

$$\begin{aligned} (1+\tau\delta)(\delta y + Ly + \Lambda(y)y) &= \sigma\Delta y + \tau\delta\sigma\Delta y, \\ y(t=0) = y_0, \quad \delta y(t=0) &= \sigma\Delta y_0 - (L+\Lambda(y_0))y_0, \\ y(a=0,x) &= \int_0^{a_{\max}} \beta_0(\alpha,x)y(t,\alpha,x)\,\mathrm{d}\alpha + g_0, \\ (\delta y + Ly + \Lambda(y)y)(a=0) &= \int_0^{a_{\max}} \beta_1(\alpha)(\delta y + Ly + \Lambda(y)y)(\alpha)\,\mathrm{d}\alpha + \tilde{g}_1, \\ \partial_\nu y(x\in\partial\Omega) &= 0, \end{aligned} \tag{4.53}$$

where we set

$$\tilde{g}_1 := (\delta y + Ly + \Lambda(y)y)(a=0) - \int_0^{a_{\max}} \beta_1(\alpha)(\delta y + Ly + \Lambda(y)y)(\alpha)\,\mathrm{d}\alpha.$$

The resulting equation has the form of eq. (4.34) with

$$\begin{aligned} f &= \tau\delta\sigma\Delta y, \\ y_1 &= \sigma\Delta y_0 - (L+\Lambda(y_0))y_0, \\ g_1 &= \tilde{g}_1. \end{aligned} \tag{4.54}$$

This allows a first convergence result:

4.22 Theorem. Under Assumption 4.16, let y^τ be a weak solution of eq. (4.52) and y a weak solution of eq. (4.51). Further we assume that $y \in C([0,T],\mathcal{V})$ with $\delta y \in C([0,T],\mathcal{H})$, that

$$\tau\delta\sigma\Delta y \in L^2((0,T),\mathcal{H}), \quad \sigma\Delta y_0 - (L+\Lambda(y_0))y_0 \in \mathcal{H},$$
$$(\delta y + Ly + \Lambda(y)y)(a=0) - \int_0^{a_{\max}} \beta_1(\alpha)(\delta y + Ly + \Lambda(y)y)(\alpha)\,\mathrm{d}\alpha \in L^2((0,T),H)$$

and that $\beta_0 = q_1\beta$ and $g_0 = (1-q_1)y(a=0) \in L^\infty((0,T),H) \cap L^2((0,T),V)$ for some $q_1 \in \mathbb{R}$. Then there is a constant C independent of τ (but possibly depending on y) such that

$$\tau\left\|\delta y^\tau - \delta y\right\|^2_{C([0,T],\mathcal{H})} + \left\|y^\tau - y\right\|^2_{C([0,T],\mathcal{V})} \le C\tau.$$

Hence, y^τ converges in $C([0,T],\mathcal{V})$ with rate $\sqrt{\tau}$ to y as $\tau \to 0$.

Proof: Applying Lemma 4.18 to eq. (4.52) and eq. (4.53) directly gives the estimate

$$\begin{aligned} &\tau\left\|\delta y^\tau - \delta y\right\|^2_{C([0,T],\mathcal{H})} + \left\|y^\tau - y\right\|^2_{C([0,T],\mathcal{V})} \\ &\qquad \lesssim \tau\left\|y_1 - \sigma\Delta y_0 + (L+\Lambda(y_0))y_0\right\|^2_{\mathcal{H}} + \left\|\tau\delta\sigma\Delta y\right\|^2_{L^2((0,T),\mathcal{H})} + \tau\|g_1 - \tilde{g}_1\|^2_{L^2((0,T),H)}. \end{aligned}$$

Thus, choosing C as a multiple of

$$\max\left\{\left\|y_1 - \sigma\Delta y_0 + (L + \Lambda(y_0))y_0\right\|_{\mathcal{H}}^2, \tau_{\max}\left\|\delta\sigma\Delta y\right\|_{L^2((0,T),\mathcal{H})}^2, \|g_1 - \tilde{g}_1\|_{L^2((0,T),H)}^2\right\}$$

concludes the proof. □

The convergence rate can be improved if the first-order initial and birth conditions for y and y^τ are compatible too. For this, we need to assume that β, β_0 and β_1 do not depend on space but only on age. Then, as we have seen in eq. (4.6) or eq. (4.5), y satisfies a first-order implicit birth condition of the form

$$(\delta y + Ly + \Lambda(y)y)(a = 0) = \int_0^{a_{\max}} \sigma(0)\beta(\alpha)\sigma(\alpha)^{-1}(\delta y + Ly + \Lambda(y)y)(\alpha)\,\mathrm{d}\alpha\,.$$

This allows for a similar trick as before: Let $q_2 \in \mathbb{R}$, then from

$$\begin{aligned}(\delta y + Ly + \Lambda(y)y)(a = 0) &= q_2(\delta y + Ly + \Lambda(y)y)(a = 0) + (1 - q_2)(\delta y + Ly + \Lambda(y)y)(a = 0)\\ &= \int_0^{a_{\max}} q_2\sigma(0)\beta(\alpha)\sigma(\alpha)^{-1}(\delta y + Ly + \Lambda(y)y)(\alpha)\,\mathrm{d}\alpha + (1 - q_2)(\delta y + Ly + \Lambda(y)y)(a = 0)\end{aligned}$$

we conclude that the first-order birth conditions for y^τ and y match if we choose

$$\begin{aligned}\beta_1(\alpha) &:= q_2\sigma(0)\beta(\alpha)\sigma(\alpha)^{-1},\\ g_1 &:= (1 - q_2)(\delta y + Ly + \Lambda(y)y)(a = 0).\end{aligned}$$

Further, we need to assume that the condition $y_1 = \sigma\Delta y_0 - (L + \Lambda(y_0))y_0$ from eq. (4.54) is satisfied. Under these conditions, y solves the system

$$\begin{aligned}(1 + \tau\delta)(\delta y + Ly + \Lambda(y)y) &= \sigma\Delta y + \tau\delta\sigma\Delta y,\\ y(t = 0) &= y_0, \quad \delta y(t = 0) = y_1,\\ y(a = 0, x) &= \int_0^{a_{\max}} \beta_0(\alpha, x)y(t, \alpha, x)\,\mathrm{d}\alpha + g_0,\\ (\delta y + Ly + \Lambda(y)y)(a = 0) &= \int_0^{a_{\max}} \beta_1(\alpha)(\delta y + Ly + \Lambda(y)y)(\alpha)\,\mathrm{d}\alpha + g_1,\\ \partial_\nu y(x \in \partial\Omega) &= 0.\end{aligned}$$

This formulation allows to prove the following theorem.

4.23 Theorem. In the situation of Theorem 4.22, assume that β, β_0 and β_1 do not depend on x. Let $q_2 \in \mathbb{R}$ and assume that the compatibility conditions

$$\begin{aligned}y_1 &= \sigma\Delta y_0 - (L + \Lambda(y_0))y_0,\\ \beta_0 &= q_1\beta,\\ \beta_1(\alpha) &= q_2\sigma(0)\beta(\alpha)\sigma(\alpha)^{-1},\end{aligned}$$

$$\begin{aligned} g_0 &= (1-q_1)y(a=0), \\ g_1 &= (1-q_2)(\delta y + Ly + \Lambda(y)y)(a=0) \end{aligned}$$

hold. Then we can estimate

$$\tau \left\| \delta y^\tau - \delta y \right\|^2_{C([0,T],\mathcal{H})} + \left\| y^\tau - y \right\|^2_{C([0,T],\mathcal{V})} \le C\tau^2$$

with a constant C independent of τ, but possibly depending on y. From this follows that y^τ converges in $C([0,T],\mathcal{V})$ to y with rate τ, and δy^τ converges to δy in $C([0,T],\mathcal{H})$ with rate $\sqrt{\tau}$.

Proof: Invoking Lemma 4.18 shows that

$$\tau \left\| \delta y^\tau - \delta y \right\|^2_{C([0,T],\mathcal{H})} + \left\| y^\tau - y \right\|^2_{C([0,T],\mathcal{V})} \lesssim \tau^2 \left\| \delta \sigma \Delta y \right\|^2_{L^2((0,T),\mathcal{H})}.$$

This concludes the proof. □

4.7 Numerical simulation

This section addresses some of the questions posed earlier, such as

- Does the relaxed model actually have finite propagation speed?
- Is it worth to consider a relaxed model, or are the solutions "close enough" to the simpler unrelaxed model that the extra effort is not worth it?
- Can we, in some way, visualize the convergence for τ getting close to zero?

Rather than giving universal results on our very general class of models, which is difficult due to the high complexity, we consider the simple special model (SVIR)

$$\begin{aligned} \delta S - \sigma_S \Delta S &= cV - [u + \mu + \Lambda(I)]\, S, \\ \delta V - \sigma_V \Delta V &= uS - [\mu + c + \varphi_1 \Lambda(I)]\, V, \\ \delta I - \sigma_I \Delta I &= \Lambda(I)(S + \varphi_1 V + \varphi_2 R) - (\mu + \delta + \gamma) I, \\ \delta R - \sigma_R \Delta R &= \gamma I - [\mu + \varphi_2 \Lambda(I)]\, R \end{aligned}$$

introduced in Section 3.1, where

$$\Lambda(a,x,I(t,\cdot,\cdot)) = \int_0^{a_{\max}} \int_\Omega \lambda(a,\alpha,x,\xi) I(t,\alpha,\xi)\, \mathrm{d}\xi\, \mathrm{d}\alpha\,.$$

With the conventions from eq. (3.8), we were able to write the model (SVIR) into the form of eq. (1.1). Using these conventions in eq. (1.6) (or in eq. (4.1)), we obtain a corresponding relaxed version of the SVIR model. We proceed by calculating numerical solutions to both equations and compare the results for varying values of τ to the solution of the unrelaxed equation where $\tau = 0$.

Detailed explanations on the numerics of how to obtain an approximate solution of the state equation will be explained later in Section 5.3. The space domain Ω is chosen as the interval, and for simplicity and to better illustrate the spread of the infection, we assume that no vaccine is administered. We note that the relaxed equation (1.1)

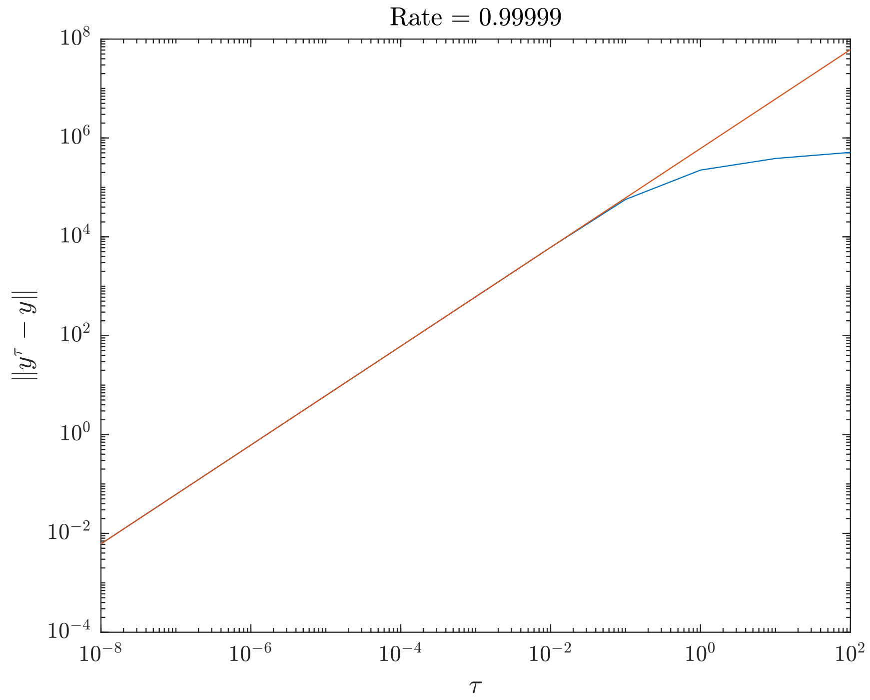
$$\begin{aligned}
(1+\tau\delta)(\delta y + Ly + \Lambda(y))y &= \sigma(a)\Delta y, \\
y(t=0) = y_0, \quad \delta y(t=0) = y_1, &\quad \partial_\nu y(x \in \partial\Omega) = 0, \\
y(t,a=0) &= \int_0^{a_{\max}} \beta_0(\alpha,x) y(t,\alpha,x)\,\mathrm{d}\alpha + g_0(t,x), \\
(\delta y + Ly + \Lambda(y)y)(t,a=0,x) &= \int_0^{a_{\max}} \beta_1(\alpha,x)(\delta y + Ly + \Lambda(y)y)(t,\alpha,x)\,\mathrm{d}\alpha + g_1(t,x)
\end{aligned} \tag{4.55}$$

can be rewritten as a system: Let $z := (y, \delta y + L(a,x)y + \Lambda(a,x,y)y)^{\mathsf{T}}$, then eq. (4.55) is equivalent to the system

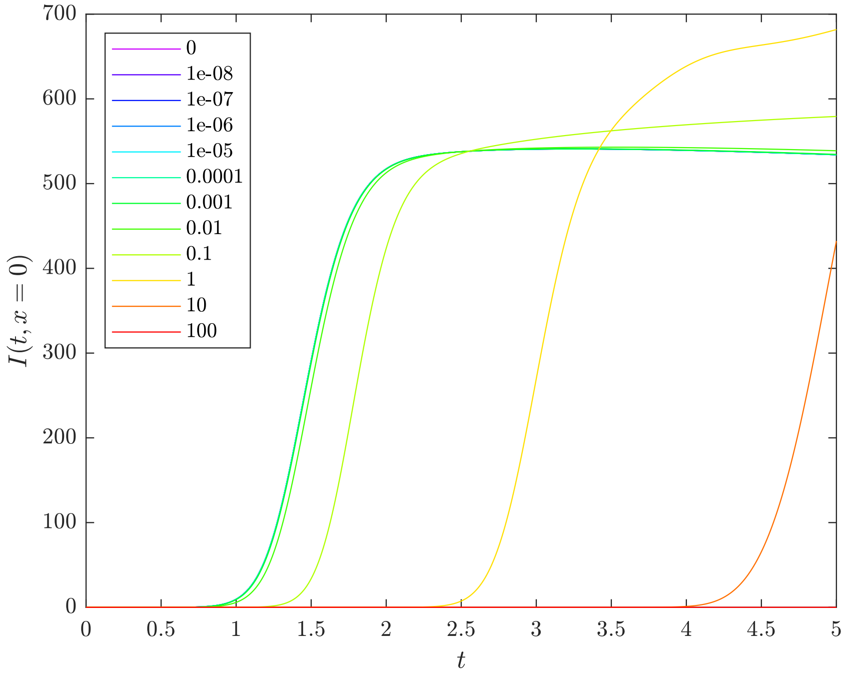
$$\begin{aligned}
\delta z &= \begin{pmatrix} -(L(a,x) + \Lambda(a,x,y)) & 1 \\ \frac{\sigma}{\tau}\Delta & \frac{-1}{\tau} \end{pmatrix} z, \\
z(t=0) &= (y_0, y_1 + Ly_0 + \Lambda(y_0)y_0)^{\mathsf{T}}, \\
z(a=0) &= \int_0^{a_{\max}} \begin{pmatrix} \beta_0 & \\ & \beta_1 \end{pmatrix}(\alpha) z(\alpha)\,\mathrm{d}\alpha,
\end{aligned}$$

which can be numerically calculated in a similar way to the unrelaxed model in section 5.3. (This transformation, however, does not allow existence proofs, because it does not admit energy estimates.) We choose the same parameters as in Table 5.1, but use different initial values: Rather than having infected individuals spread out evenly over the whole age and space domain, we instead assume they are concentrated on the rightmost border $x = 1$ of the space domain. This allows us to track the progression of the epidemic through space and to compare the differing propagation speed. We test various values for τ, namely all $\tau = 10^j$ for $j \in \{-8, \dots, 2\}$. Also, we choose $y_1 = 0$ and $\beta_1 = \beta_0 = \beta$, which, as explained in eq. (4.18), can be interpreted as follows: At $t = 0$, the same number of individuals move to the left and to the right, and newborn individuals move in the same direction as their parents. A simple alternative choice would be $\beta_1 = 0$, which according to eq. (4.17) means that the direction of newborns is determined at random. Thus, the compatibility conditions of Theorem 4.22 are satisfied with $q_1 = 1$.

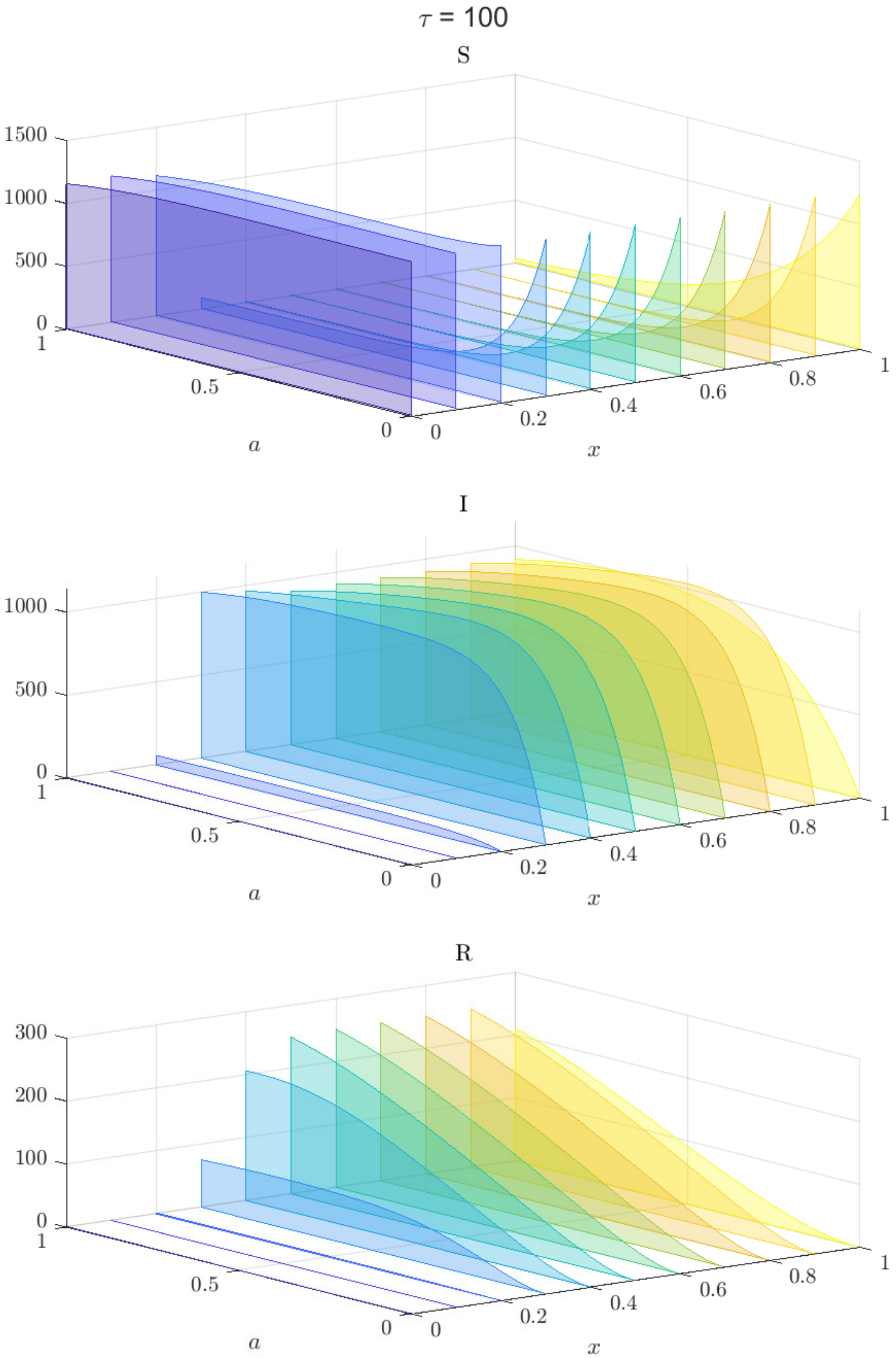
The results can be found in Figure 4. In Figure 4a, we numerically calculated the rate of convergence of the unrelaxed model towards the relaxed one. The figure shows a graph of $\|y^\tau - y\|_{L^\infty}$ over τ, and a polynomial fit of the data. The resulting slope, and hence the convergence rate, turns out to be 1, which is even better than the value of 0.5 we would expect from Theorem 4.22, and suggests that the results from Theorem 4.23 hold in higher generality. Figure 4b shows the total number of infective individuals at the $x = 0$ boundary, directly opposite to where all infectives have been in the beginning of the simulation. The lines corresponding to smaller values of τ all overlap, but the lines for $\tau = 0.1$, $\tau = 1$ and $\tau = 10$ are clearly distinct and show that the larger τ is, the slower the infection moves through the domain. The line corresponding to $\tau = 100$ is just flat at the bottom, the population of infective individuals has not yet fully traversed the interval. This can also be seen in Figure 4c, which shows the state for $\tau = 100$ at the



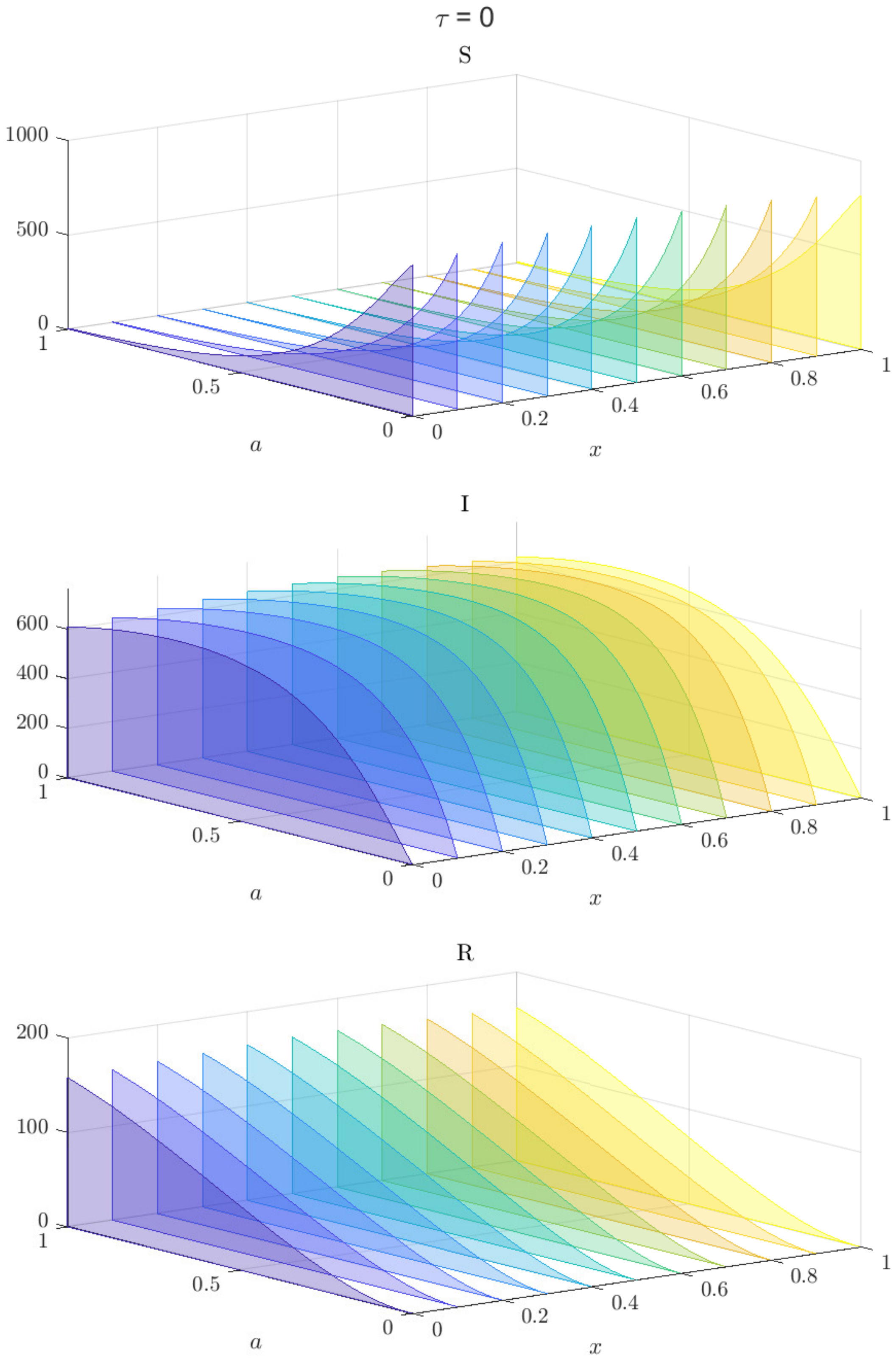
(a) Convergence rate



(b) Values of I at $x = 0$



(c) Final state of the relaxed equation



(d) Final state of the unrelaxed equation

Figure 4: Comparison of different values of τ

final time of the simulation. Here, the progress of the infection can directly be observed, it seems to have just arrived at the $x = 0.2$ mark. Compare this result to Figure 4d, which shows the final state of the solution for $\tau = 0$, where no real spatial difference can be seen. To summarize, our results suggest that the relaxed model does in fact show finite propagation speed and hence is very suitable to model the spread of an epidemic over a region, but choosing small values for τ yields results that do not differ very much from an unrelaxed model.

The graphs, especially Figure 4b, also show that the peak number of infectious individuals is higher for large values of tau. However, note that this is not only true for the infective population. In fact, for the values of τ larger than 0.01, the total population increases, while for smaller values it behaves like the unrelaxed model, i.e. it decreases slightly. This can also be seen from Figure 4c, where, for example, the number of susceptible numbers visibly exceeds the initial value of 1000 individuals, while in Figure 4d it remains below that number. This suggests that either large values of τ yield a model that deviates too much from the unrelaxed equation to be useful, or that the birth rates need to be adjusted for these values.

4.8 Addendum: The symmetric-hyperbolic system for moving populations

In this short section we consider the “full” model

$$\delta p^j + c\partial p^{j+1} + c\bar{\partial} p^{j-1} + \lambda_j p^j = f(p^0)p^j.$$

which we adopted from eq. (4.13), where for better readability we let $p^j := \hat{p}^j$ and $\lambda_j := \hat{\lambda}_j - \hat{\lambda}_0$. From eq. (4.12) we infer that $\lambda_j \geq 0$ for all $j \in \mathbb{Z}$. Truncating every but the 0th, 1st and -1st term gave us a wave-equation-like model. In this section, we rather focus on the whole homogeneous model on characteristics (i.e. we ignore f and the age structure), and give two small existence results, one of Cauchy–Kovalevskaya type and one in the context of semigroups. We start right away:

4.24 Theorem. We consider the problem

$$\begin{aligned} p^j_t + c\partial p^{j+1} + c\bar{\partial} p^{j-1} + \lambda_j p^j &= 0 \quad (j \in \mathbb{Z}), \\ p^j(t=0) &= p^j_0 \end{aligned}$$

where $c \in \mathbb{R}$ and $\lambda_j \geq 0$ $(j \in \mathbb{Z})$. Assume there exists $\lambda > 0$ such that $\lambda_j \leq \lambda$ for all j. Further assume that the p^j_0 are (real) analytic in a neighborhood around the origin, i.e. there are $a^j_{0,l,m} \in \mathbb{C}$ such that it holds

$$p^j_0(z) = \sum_{l,m=0}^{\infty} a^j_{0,l,m} z^l \bar{z}^m$$

for $z \in \mathbb{C}$ sufficiently close to zero. We also assume that there is $\mathcal{A} > 0$ such that the uniform bound $\left|a^j_{0,l,m}\right| \leq \mathcal{A}$ holds for all j, l, m. Then there exist $r > 0$ and $T > 0$ such that the system has a solution that is analytic in $t \in [0,T]$ and $z \in \mathbb{C}$ with $|z| < r$.

Proof: We start with the ansatz

$$p^j(t,z) = \sum_{k,l,m=0}^{\infty} a^j_{k,l,m} t^k z^l \bar{z}^m$$

for certain $a^j_{k,l,m} \in \mathbb{C}$. There is no collision of names, since plugging in $t = 0$ shows that the $a^j_{0,k,l}$ are in fact given by the initial value. From the differential equation and general results on power series we have

$$\begin{aligned}(a^j_{k,l,m})_{j\in\mathbb{Z}} &= \frac{1}{k!l!m!} \partial_t^k \partial^l \bar{\partial}^m (p^j)_{j\in\mathbb{Z}}\Big|_{t,z,\bar{z}=0} \\ &= \frac{(-1)^k}{k!l!m!} \partial^l \bar{\partial}^m (M_\lambda + c\partial S + c\bar{\partial} S^{-1})^k (p^j)_{j\in\mathbb{Z}}\Big|_{t,z,\bar{z}=0}\end{aligned}$$

where M_λ is the multiplication operator on $\ell^2_{\mathbb{Z}}$ that multiplies with λ^j, i.e. $M_\lambda(x^j)_{j\in\mathbb{Z}} = (\lambda_j x^j)_{j\in\mathbb{Z}}$, and S is the left-shift operator on $\ell^2_{\mathbb{Z}}$, that is $S(x^j)_{j\in\mathbb{Z}} = (x^{j+1})_{j\in\mathbb{Z}}$. Note the pointwise estimate $\left|M(x^j)_{j\in\mathbb{Z}}\right| \leq \lambda\left(\left|x^j\right|\right)_{j\in\mathbb{Z}}$. Unfortunately, M_λ and S do not commute, preventing us to simply use the binomial theorem. In the following we denote pointwise inequality by $\leq$. Estimating absolute values yields

$$\begin{aligned}\left(\left|a^j_{k,l,m}\right|\right)_{j\in\mathbb{Z}} &= \left|\frac{(-1)^k}{k!l!m!} \partial^l \bar{\partial}^m (M_\lambda + c\partial S + c\bar{\partial} S^{-1})^k (p^j)_{j\in\mathbb{Z}}\Big|_{t,z,\bar{z}=0}\right| \\ &\leq \frac{1}{k!l!m!} \left(\left|\partial^l \bar{\partial}^m (\lambda + c\partial S + c\bar{\partial} S^{-1})^k p^j\Big|_{t,z,\bar{z}=0}\right|\right)_{j\in\mathbb{Z}}.\end{aligned}$$

Using the equation

$$(A + B + C)^k = \sum_{\alpha=0}^{k} \sum_{\beta=0}^{\alpha} \frac{k!}{(k-\alpha)!(\alpha-\beta)!\beta!} A^\beta B^{\alpha-\beta} C^{k-\alpha}$$

which holds for all commuting A, B, C, we can further estimate

$$\begin{aligned}\left(\left|a^j_{k,l,m}\right|\right)_{j\in\mathbb{Z}} &= \frac{1}{k!l!m!} \left(\left|\partial^l \bar{\partial}^m \sum_{\alpha=0}^{k} \sum_{\beta=0}^{\alpha} \frac{k!}{(k-\alpha)!(\alpha-\beta)!\beta!} \lambda^\beta (c\partial S)^{\alpha-\beta} (c\bar{\partial} S^{-1})^{k-\alpha} p^j \Bigg|_{t,z,\bar{z}=0}\right|\right)_{j\in\mathbb{Z}} \\ &\leq \sum_{\alpha=0}^{k} \sum_{\beta=0}^{\alpha} \frac{\lambda^\beta c^{k-\beta}}{l!m!(k-\alpha)!(\alpha-\beta)!\beta!} \left(\left|\partial^{l+\alpha-\beta} \bar{\partial}^{m+k-\alpha} S^{2\alpha-\beta-k} p^j\Big|_{t,z,\bar{z}=0}\right|\right)_{j\in\mathbb{Z}} \\ &= \sum_{\alpha=0}^{k} \sum_{\beta=0}^{\alpha} \frac{\lambda^\beta c^{k-\beta} (l+\alpha-\beta)!(m+k-\alpha)!}{l!m!(k-\alpha)!(\alpha-\beta)!\beta!} \left(\left|a^{j+2\alpha-\beta-k}_{0,l+\alpha-\beta,m+k-\alpha}\right|\right)_{j\in\mathbb{Z}} \\ &= \sum_{\alpha=0}^{k} \sum_{\beta=0}^{\alpha} = \frac{\lambda^\beta c^{k-\beta}}{\beta!} \binom{l+\alpha-\beta}{l} \binom{m+k-\alpha}{m} \left(\left|a^{j+2\alpha-\beta-k}_{0,l+\alpha-\beta,m+k-\alpha}\right|\right)_{j\in\mathbb{Z}}. \qquad (4.56)\end{aligned}$$

We can now show that the series $p^j(t,z) = \sum_{k,l,m=0}^{\infty} a^j_{k,l,m} t^k z^l \bar{z}^m$ converges absolutely in a neighborhood of the origin. Using the formula

$$\sum_{k=0}^{\infty} \sum_{\alpha=0}^{k} \sum_{\beta=0}^{\alpha} a_\beta b_{\alpha-\beta} c_{k-\alpha} = \left(\sum_{k=0}^{\infty} a_k\right) \cdot \left(\sum_{k=0}^{\infty} b_k\right) \cdot \left(\sum_{k=0}^{\infty} c_k\right)$$

and writing $t^k = t^\beta t^{\alpha-\beta} t^{k-\alpha}$ and $c^{k-\beta} = c^{k-\alpha} c^{\alpha-\beta}$ we obtain with eq. (4.56) that

$$
\begin{aligned}
&\sum_{k,l,m=0}^{\infty} \left|a^j_{k,l,m}\right| t^k |z|^l |\bar{z}|^m \\
&\leq \sum_{k,l,m=0}^{\infty} \sum_{\alpha=0}^{k} \sum_{\beta=0}^{\alpha} \frac{\lambda^\beta c^{k-\beta}}{\beta!} \binom{l+\alpha-\beta}{l} \binom{m+k-\alpha}{m} \left|a^{j+2\alpha-\beta-k}_{0,l+\alpha-\beta,m+k-\alpha}\right| t^k |z|^l |\bar{z}|^m \\
&\leq \mathcal{A} \cdot \left(\sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \right) \cdot \left(\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (ct)^k \binom{l+k}{l} |z|^l \right) \cdot \left(\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} (ct)^k \binom{m+k}{m} |\bar{z}|^m \right). \qquad (4.57)
\end{aligned}
$$

By finally noting that for all $a, b \geq 0$ with $a + b < 1$ it holds

$$
\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a^k \binom{l+k}{l} b^l = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} a^\beta b^{\alpha-\beta} = \sum_{\alpha=0}^{\infty} (a+b)^\alpha = \frac{1}{1-a-b},
$$

eq. (4.57) yields the simple estimate

$$
\sum_{k,l,m=0}^{\infty} \left|a^j_{k,l,m}\right| t^k |z|^l |\bar{z}|^m \leq \frac{\mathcal{A} e^{\lambda t}}{(1-|z|-ct)^2}
$$

which holds for all $|z| + ct < 1$. This shows the required convergence and thus concludes the proof. □

We can even do better and show an existence result that covers entire domains, not only small neighborhoods of the origin. As we have seen in Section 4.2, it is convenient to complement the model

$$
\begin{gathered}
p^j_t + c\partial p^{j+1} + c\bar{\partial} p^{j-1} + \lambda_j p^j = 0 \quad (j \in \mathbb{Z}), \\
p^j(t=0) = p^j_0
\end{gathered} \qquad (4.58)
$$

with one of the following two boundary conditions : the U-turn condition (4.14)

$$
\nu p^{j-1} + \bar{\nu} p^{j+1} = 0 \quad \text{for even } j \in \mathbb{Z}, \qquad (4.59)
$$

or the reflection condition (4.15)

$$
\bar{\nu}^j p^j - (-\nu)^j p^{-j} = 0 \quad \text{for all } j \in \mathbb{Z}. \qquad (4.60)
$$

We first gather some easily verifiable results on Wirtinger derivatives:

4.25 Lemma. Let $\Omega \subseteq \mathbb{C}$ a bounded and smooth domain with outer normal ν and let f, $g \in C^1(\Omega, \mathbb{C})$ be (real) differentiable. Then the following statements hold:

Conjugation: $\overline{\partial f} = \bar{\partial} \bar{f}$ and $\overline{\bar{\partial} f} = \partial \bar{f}$, from which follows $\overline{\partial \bar{f}} = \bar{\partial} f$ and $\overline{\bar{\partial} \bar{f}} = \partial f$.

Product rule: $\partial(fg) = f\partial g + g\partial f$ and $\bar{\partial}(fg) = f\bar{\partial} g + g\bar{\partial} f$.

Divergence theorem: $\int_\Omega \partial f = \frac{1}{2}\oint_{\partial\Omega} \bar{\nu} f$ and $\int_\Omega \bar{\partial} f = \frac{1}{2}\oint_{\partial\Omega} \nu f$.

Integration by parts: $\int_\Omega f \cdot \partial g + g \cdot \partial f = \frac{1}{2}\oint_{\partial\Omega} \bar{\nu} f g$ and $\int_\Omega f \cdot \bar{\partial} g + g \cdot \bar{\partial} f = \frac{1}{2}\oint_{\partial\Omega} \nu f g$.

This allows us to define Wirtinger derivatives in a weak sense. In what follows, all functions are assumed to be complex-valued.

4.26 Definition. Let $\Omega \subseteq \mathbb{C}$ a bounded and smooth domain and $f \in L^2(\Omega)$. We call $g \in L^2(\Omega)$ the *weak Wirtinger derivative* ∂f if for all $\varphi \in C_c^\infty(\Omega)$ the equation

$$\int_\Omega f\,\partial\varphi + g\varphi = 0$$

holds. In an analogous way we can define $\bar\partial f$ in a weak sense.

Keeping the definition of weak Wirtinger derivatives in mind, we can now introduce a weak notion of the operator $(p^j)_{j\in\mathbb{Z}} \mapsto (\partial p^{j+1} + \bar\partial p^{j-1})$:

4.27 Definition. During this section let $\Omega \subseteq \mathbb{C}$ a bounded and smooth domain and $\mathbb{H} := \ell^2_{\mathbb{Z}}(L^2(\Omega))$ with the scalar product $\langle p, q\rangle_{\mathbb{H}} = \sum_{j\in\mathbb{Z}} \int_\Omega p\bar q$. Where there is no chance of confusion, the index of the scalar product may be dropped. Further let

$$\begin{aligned} D(\mathcal{A}) &:= \left\{ p \in \mathbb{H} \,\middle|\, \exists \tilde p \in \mathbb{H}\, \forall \varphi \in C_c^\infty(\Omega) \forall j \in \mathbb{Z} : \int_\Omega \tilde p^j \varphi + p^{j+1}\partial\varphi + p^{j-1}\bar\partial\varphi = 0 \right\}, \\ \mathcal{A}p &:= \tilde p. \end{aligned}$$

Using the fundamental lemma of variational calculus, one can show that such a $\tilde p$, if it exists, is unique. We now consider the boundary condition (4.59). To this end, note that for p, $q \in D(\mathcal{A})$ we formally obtain

$$\begin{aligned} \langle \mathcal{A}p, q\rangle + \langle p, \mathcal{A}q\rangle &= \sum_{j\in\mathbb{Z}} \int_\Omega (\partial p^{j+1} + \bar\partial p^{j-1})\bar q^j + p^j \overline{(\partial q^{j+1} + \bar\partial q^{j-1})} \\ &= \sum_{j\in\mathbb{Z}} \int_\Omega (\partial p^{j+1} + \bar\partial p^{j-1})\bar q^j + p^j (\bar\partial \bar q^{j+1} + \partial \bar q^{j-1}) \\ &\overset{*}{=} \sum_{j\in\mathbb{Z}} \int_\Omega \partial p^{j+1}\bar q^j + \bar\partial p^{j-1}\bar q^j + p^{j-1}\bar\partial \bar q^j + p^{j+1}\partial \bar q^j \\ &= \frac{1}{2} \sum_{j\in\mathbb{Z}} \oint_{\partial\Omega} (\bar\nu p^{j+1} + \nu p^{j-1})\bar q^j, \end{aligned}$$

where in the highlighted step we performed two index shifts in the last two summands. This shows that if p satisfies eq. (4.59), the sum vanishes if $p^j = 0$ holds in $\partial\Omega$ for all odd j. This gives rise to

4.28 Definition. The weak formulation of $(p^j)_{j\in\mathbb{Z}} \mapsto (\partial p^{j+1} + \bar\partial p^{j-1})$, where p satisfies the boundary condition 4.59, is given by

$$\begin{aligned} D(\mathcal{A}_1) &:= \left\{ p \in D(\mathcal{A}) \,\middle|\, \forall q \in D(\mathcal{A}),\, q^j \in C_c^\infty(\Omega) \text{ for odd } j : \langle \mathcal{A}p, q\rangle + \langle p, \mathcal{A}q\rangle = 0 \right\}, \\ \mathcal{A}_1 p &:= \mathcal{A}p. \end{aligned}$$

Our goal is to show that $\mathcal{A}_1$ is skew-adjoint. To this end, we start by gathering some helpful lemmas.

4.29 Definition. For $p \in \mathbb{H}$ we call $p^{\mathrm{ev}} \in \mathbb{H}$ defined by

$$(p^{\mathrm{ev}})^j = \begin{cases} p^j, & j \text{ even} \\ 0, & j \text{ odd} \end{cases}$$

the *even part* of p and $p^{\mathrm{odd}} := p - p^{\mathrm{ev}}$ the *odd part* of p.

4.30 Lemma. The following statements hold:

1. For all $p, q \in \mathbb{H}$ there is $\left\langle p^{\mathrm{ev}}, q^{\mathrm{odd}} \right\rangle = 0$.
2. For any $p \in D(\mathcal{A})$ also p^{ev}, $p^{\mathrm{odd}} \in D(\mathcal{A})$. Further we have $\mathcal{A}(p^{\mathrm{ev}}) = (\mathcal{A}p)^{\mathrm{odd}}$ and $\mathcal{A}(p^{\mathrm{odd}}) = (\mathcal{A}p)^{\mathrm{ev}}$.
3. An equivalent characterization of $D(\mathcal{A}_1)$ is given by

$$\left\{ p \in D(\mathcal{A}) \,\middle|\, \forall q \in D(\mathcal{A}) : \left\langle \mathcal{A}p, q^{\mathrm{ev}} \right\rangle + \left\langle p, \mathcal{A}(q^{\mathrm{ev}}) \right\rangle = 0 \right\}.$$

4. For $p \in D(\mathcal{A})$ it holds that $p^{\mathrm{ev}} \in D(\mathcal{A}_1)$.

Proof: 1. This is a direct consequence of the definition of the inner product $\langle \cdot, \cdot \rangle_{\mathbb{H}}$.

2. Let $p \in D(\mathcal{A})$ and $\varphi \in C_c^\infty(\Omega)$. Then for any $j \in \mathbb{Z}$ we have

$$\int_\Omega (p^{\mathrm{ev}})^{j+1} \partial\varphi + (p^{\mathrm{ev}})^{j-1} \bar{\partial}\varphi = \begin{cases} 0, & j \text{ even,} \\ -\int_\Omega (\mathcal{A}p)^j \varphi, & j \text{ odd.} \end{cases}$$

This shows that $(\mathcal{A}p)^{\mathrm{odd}}$ is a valid choice for $\tilde{p}$ in the definition 4.27 of $\mathcal{A}(p^{\mathrm{ev}})$. The other claim follows in a similar way.

3. Given $q \in D(\mathcal{A})$ with $q^j \in C_c^\infty(\Omega)$ for all odd j, we calculate

$$\begin{aligned} \left\langle \mathcal{A}p, q^{\mathrm{odd}} \right\rangle + \left\langle p, \mathcal{A}(q^{\mathrm{odd}}) \right\rangle &= \sum_{j \in \mathbb{Z}} \int_\Omega (\mathcal{A}p)^j \overline{(q^{\mathrm{odd}})^j} + p^j \overline{(\partial (q^{\mathrm{odd}})^{j+1} + \bar{\partial} (q^{\mathrm{odd}})^{j-1})} \\ &= \sum_{j \text{ odd}} \int_\Omega (\mathcal{A}p)^j \bar{q}^j + p^{j-1} \bar{\partial} \bar{q}^j + p^{j+1} \partial q^j \\ &= 0, \end{aligned}$$

where the last line follows from the definition of $D(\mathcal{A})$. Therefore, Definition 4.28 remains unchanged if we replace q by q^{ev}, and since all odd entries of q^{ev} are trivially compactly supported smooth functions, we then can as well drop the condition that $q^j \in C_c^\infty(\Omega)$ for all odd j. This shows the claim.

4. This follows directly from what we just proved: For any $p, q \in D(\mathcal{A})$ we have, using statements 1 and 2, that

$$\left\langle \mathcal{A}(p^{\mathrm{ev}}), q^{\mathrm{ev}} \right\rangle + \left\langle p^{\mathrm{ev}}, \mathcal{A}(q^{\mathrm{ev}}) \right\rangle = \left\langle (\mathcal{A}p)^{\mathrm{odd}}, q^{\mathrm{ev}} \right\rangle + \left\langle p^{\mathrm{ev}}, (\mathcal{A}q)^{\mathrm{odd}} \right\rangle = 0,$$

which by statement 3 shows that $p^{\mathrm{ev}} \in D(\mathcal{A}_1)$. □

4.31 Lemma. Let $\varphi \in C_c^\infty(\Omega)$ and $e_k = (\delta_{jk})_{j\in\mathbb{Z}}$ for some $k \in \mathbb{Z}$ and the Dirac delta δ. Then we have $p := \varphi e_k \in D(\mathcal{A}_1)$ and $\mathcal{A}_1 p = \mathcal{A}p = \partial\varphi e_{k-1} + \bar\partial\varphi e_{k+1}$.

Proof: Define $\tilde{p} := \partial\varphi e_{k-1} + \bar\partial\varphi e_{k+1}$, then obviously $\tilde{p} \in \mathbb{H}$. First let $\psi \in C_c^\infty$, then for any $j \in \mathbb{Z}$ it follows from the partial integration theorem in Lemma 4.25 that

$$\int_\Omega \tilde{p}^j \psi + p^{j+1}\partial\psi + p^{j-1}\bar\partial\psi = \begin{cases} \int_\Omega \partial\varphi\cdot\psi + \varphi\partial\psi, & j = k-1\\ \int_\Omega \bar\partial\varphi\cdot\psi + \varphi\bar\partial\psi, & j = k+1\\ 0, & \text{else}\end{cases}$$
$$= 0.$$

This shows that $p \in D(\mathcal{A})$ and $\mathcal{A}p = q$.

Now let $q \in D(\mathcal{A})$, then it follows that

$$\langle \mathcal{A}p, q\rangle + \langle p, \mathcal{A}q\rangle = \int_\Omega \partial\varphi\cdot\overline{q^{k-1}} + \bar\partial\varphi\cdot\overline{q^{k+1}} + \varphi\cdot\overline{(\mathcal{A}q)^k}.$$

This integral is the complex conjugate of

$$\int_\Omega \bar\partial\bar\varphi\cdot q^{k-1} + \partial\bar\varphi\cdot q^{k+1} + \bar\varphi\cdot(\mathcal{A}q)^k,$$

which is zero by the fact that $q \in D(\mathcal{A})$ (note that $\bar\varphi$ is a test function again). From this follows in particular that $p \in D(\mathcal{A}_1)$ (simply replace q by q^{ev}). This concludes the proof. □

4.32 Lemma. The operator $\mathcal{A}_1$ is densely defined.

Proof: The claim follows if we show that the orthogonal complement of $D(\mathcal{A}_1)$ is only the zero set. So let $q \in D(\mathcal{A}_1)^\perp$ and $p = \varphi e_k$ as in Lemma 4.31. Then from $\langle p, q\rangle = 0$ we conclude that $\int_\Omega \varphi\overline{x^k} = 0$. Since φ and k were arbitrary, we can infer that all x^k are zero from the fundamental lemma of variational calculus. This concludes the proof. □

4.33 Theorem. The operator $\mathcal{A}_1$ is skew-adjoint, i.e. we have $\mathcal{A}_1' = -\mathcal{A}_1$.

Proof: First let p, $q \in D(\mathcal{A}_1)$. Then we see that

$$\begin{aligned}\langle \mathcal{A}_1 p, q\rangle + \langle p, \mathcal{A}_1 q\rangle &= \langle \mathcal{A}_1 p, q^{\mathrm{ev}}\rangle + \left\langle \mathcal{A}_1 p, q^{\mathrm{odd}}\right\rangle + \langle p^{\mathrm{ev}}, \mathcal{A}_1 q\rangle + \left\langle p^{\mathrm{odd}}, \mathcal{A}_1 q\right\rangle\\ &= \left\langle \mathcal{A}_1(p^{\mathrm{odd}}), q^{\mathrm{ev}}\right\rangle + \left\langle \mathcal{A}_1(p^{\mathrm{ev}}), q^{\mathrm{odd}}\right\rangle + \left\langle p^{\mathrm{ev}}, \mathcal{A}_1(q^{\mathrm{odd}})\right\rangle + \left\langle p^{\mathrm{odd}}, \mathcal{A}_1(q^{\mathrm{ev}})\right\rangle\\ &= -\left\langle p^{\mathrm{odd}}, \mathcal{A}_1(q^{\mathrm{ev}})\right\rangle + \left\langle \mathcal{A}_1(p^{\mathrm{ev}}), q^{\mathrm{odd}}\right\rangle - \left\langle \mathcal{A}_1(p^{\mathrm{ev}}), q^{\mathrm{odd}}\right\rangle + \left\langle p^{\mathrm{odd}}, \mathcal{A}_1(q^{\mathrm{ev}})\right\rangle\\ &= 0,\end{aligned}$$

which means that $-\mathcal{A}_1 \subseteq \mathcal{A}_1'$, proving that $\mathcal{A}_1$ is skew-symmetric. Conversely, let $q \in D(\mathcal{A}_1')$, i.e. it holds that

$$\langle \mathcal{A}_1 p, q\rangle = \langle p, \mathcal{A}_1' q\rangle \quad \text{for all } p \in D(\mathcal{A}_1). \tag{4.61}$$

First let $p = \varphi e_k$ as in Lemma 4.31, then $p \in D(\mathcal{A}_1)$ and $\mathcal{A}_1 p = \mathcal{A}p = \partial\varphi e_{k-1} + \bar{\partial}\varphi e_{k+1}$. Then eq. (4.61) turns into

$$\int_\Omega \partial\varphi \cdot \bar{q}^{k-1} + \bar{\partial}\varphi \cdot \bar{q}^{k+1} = \int_\Omega \varphi\overline{(\mathcal{A}'q)^k}.$$

Complex conjugation and rearranging then yield

$$\int_\Omega -(\mathcal{A}'q)^k \bar{\varphi} + q^{k-1}\bar{\partial}\bar{\varphi} + q^{k+1}\partial\bar{\varphi} = 0.$$

Since φ and k were arbitrary, we have $q \in D(\mathcal{A})$ and $\mathcal{A}q = -\mathcal{A}'q$. Now let $p \in D(\mathcal{A})$ be given, then by Lemma 4.30 it holds that $p^{\mathrm{ev}} \in D(\mathcal{A}_1)$ and thus

$$\langle \mathcal{A}(p^{\mathrm{ev}}), q \rangle = \langle p^{\mathrm{ev}}, \mathcal{A}'q \rangle = -\langle p^{\mathrm{ev}}, \mathcal{A}q \rangle.$$

Rearranging and complex conjugation then shows

$$\langle \mathcal{A}q, p^{\mathrm{ev}} \rangle + \langle q, \mathcal{A}(p^{\mathrm{ev}}) \rangle = 0 \quad \text{for all } p \in D(\mathcal{A}).$$

Lemma 4.30 then yields that $q \in D(\mathcal{A}_1)$. Hence it holds that $\mathcal{A}_1' \subseteq -\mathcal{A}_1$, which concludes the proof. □

4.34 Corollary. Assume that $(\lambda_j)_{j\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ and $(p_j^0)_{j\in\mathbb{Z}} \in D(\mathcal{A}_1)$. Then there exists a unique solution $p \in C^1(\mathbb{R}, \mathbb{H}) \cap C(\mathbb{R}, D(\mathcal{A}_1))$ to eq. (4.58) with boundary conditions (4.59).

Proof: Lemma 4.32 and Theorem 4.33 show that the conditions of Stone's theorem [EN00, Thm. II.3.24] are satisfied. Thus we infer that the operator $\mathcal{A}_1$ generates a unitary group of operators, which is equivalent (cf. [EN00, Generation theorem for groups, p. 79]) to the fact that $\mathcal{A}_1$ and $-\mathcal{A}_1$ both generate strongly continuous semigroups. It is easy to see that

$$\mathcal{B} : \mathbb{H} \to \mathbb{H}, (p^j)_{j\in\mathbb{Z}} \mapsto (\lambda_j p^j)_{j\in\mathbb{Z}}$$

is a bounded operator. Hence, by a theorem about bounded perturbations of semigroups, [EN00, Thm. III.1.3], the operators $\pm\mathcal{A} \pm \mathcal{B}$ generates a strongly continuous semigroup. The rest of the proof follows by [EN00, Prop. II.6.2]. □

4.35 Remark. It is not clear, mainly because of the complex powers of the unit normal, how the condition (4.60) is to be formulated in a weak sense. However, formal calculations show that the operator $\mathcal{A}_2$, defined by

$$D(\mathcal{A}_2) := \{p \in \mathbb{H} \mid p \text{ satisfies eq. (4.60)}\}, \quad \mathcal{A}_2 p := \mathcal{A}p$$

is skew-symmetric, indicating that one can also obtain a unitary group of operators.

5 Optimal control of the unrelaxed model

In this chapter we consider the unrelaxed model with a control term we introduced in eq. (1.1) and eq. (3.1):

$$\begin{aligned} \delta y + L(a,x)y + \Lambda(a,x,y)y + K(u)y &= \sigma(a)\Delta y, \\ y(t=0) = y_0, \quad \partial_\nu y(x \in \partial\Omega) &= 0, \\ y(a=0) &= \int_0^{a_{\max}} \beta(\alpha,x)y(t,\alpha,x)\,\mathrm{d}\alpha\,, \\ \Lambda(a,x,y) &= \int_0^{a_{\max}}\int_\Omega k(a,x,\alpha,\xi)y_I(t,\alpha,\xi)\,\mathrm{d}\xi\,\mathrm{d}\alpha\,. \end{aligned} \tag{5.1}$$

The existence of a solution to this equation has already been established in Theorem 3.20. In this chapter, we examine the question whether there exists an optimal control that drives the target functional

$$J(u,y) = \frac{1}{2}\int_0^T\int_0^{a_{\max}}\int_\Omega \left|g\cdot y(t,a,x)\right|^2 \mathrm{d}x\,\mathrm{d}a\,\mathrm{d}t + \frac{\alpha}{2}\int_0^T\int_0^{a_{\max}}\int_\Omega |u(t,a,x)|^2\,\mathrm{d}x\,\mathrm{d}a\,\mathrm{d}t \tag{5.2}$$

from eq. (1.7) to its minimum. Here, $\alpha > 0$ balances the two contributions of the state and the control to the target functional, and $g \in \mathbb{R}^n$ is a vector of weights describing which compartments are infective and whose number we thus want to minimize. For example, if we consider model (SVIR) again, we have $y = (S,V,I,R)^\mathsf{T}$. We wish to minimize the number of infectious individuals, hence we choose $g = (0,0,1,0)^\mathsf{T}$. If the general model contains several classes of infectious individuals, the weight g allows for a grading between them. Most of this chapter has already been published in [AS25, Secs. 3 to 5]. For the entire chapter, we assume Assumption 3.19 to hold. In particular we impose the control constraint

$$0 \le u(t,a,x) \le \bar{u} \quad \text{for almost all } t,\ a,\ x \tag{5.3}$$

we already saw in eq. (1.8)

5.1 Existence of an optimal control

This section addresses the well-posedness of optimal control problems governed by eq. (5.1). For mathematical reasons and convenience in numerical experiments, we consider controls of the form

$$u(t,a,x) = \sum_{i=1}^{M}\sum_{j=1}^{N} u_{ij}(t)\mathbb{1}_{\Omega_i}(x)\mathbb{1}_{[a_{j-1},a_j]}(a) \quad \text{with } u(t) := (u_{ij}(t))_{i,j} \in \mathbb{R}^{M\times N}. \tag{5.4}$$

Here, the indicator functions $\mathbb{1}_{\Omega_i}(x)$, where $\Omega_i \subseteq \Omega$ $(i = 1,\ldots,M)$ are mutually disjoint, represent zones where the vaccine is administered, while the indicator functions $\mathbb{1}_{[a_{j-1},a_j]}(a)$, with ages $0 \le a_0 < a_1 < \cdots < a_N \le a_{\max}$, distinguish different age classes. Interpreted in a real-world setting, this means that the vaccine is administered in vaccination centers Ω_i and based on predefined age groups (a_{j-1},a_j), which is a realistic assumption. In this setting, the target functional of eq. (5.2) will take the form

$$J(u,y) = \frac{1}{2}\int_0^T\int_0^{a_{\max}}\int_\Omega \left|g\cdot y(t,a,x)\right|^2 \mathrm{d}x\,\mathrm{d}a\,\mathrm{d}t + \frac{\alpha}{2}\int_0^T \|u(t)\|^2_{\mathbb{R}^{M\times N}}\,\mathrm{d}t, \tag{5.5}$$

where g is a vector of weights. Further, the control constraints (5.3) can be rewritten as

$$0 \le u_{ij}(t) \le \bar{u} \qquad \text{for a.a } t \in (0,T) \text{ and all } i = 1,\ldots,N \text{ and } j = 1,\ldots,N. \tag{5.6}$$

The optimal control problem is then defined as in eq. (1.9):

$$\inf\{J(u,y) \mid (u,y) \text{ satisfy eq. (5.1) and eq. (5.6)}\}. \tag{OC}$$

A key challenge in working with nonlinear age- and space-structured models is the lack of compactness. Common compactness results, such as the Rellich–Kondrachov or Aubin–Lions theorems, are not applicable in this setting, as they require information on y, $\nabla_x y$, y_t, and y_a, ensuring that y belongs to an appropriate Sobolev space. However, we only have information on y, $\nabla_x y$ and δy.

To address this issue, we have to consider more regular controls, specifically $u \in W^{1,p}((0,T),\mathbb{R}^{M\times N})$ with $1 < p \le \infty$, whose age and space structure is fixed. This regularity is achieved by imposing additional control constraints or adding an extra control cost to the target functional.

5.1 Theorem. Let $1 < p \le \infty$ and suppose that one of the following conditions holds:

C1: The target functional in Equation (OC) is replaced by $J(u,y) + \frac{\alpha_d}{2}\|\partial_t u\|^2_{L^p((0,T),\mathbb{R}^{M\times N})}$ with some $\alpha_d > 0$.

C2: The following additional control constraint is imposed on the problem (OC):

$$\|\partial_t u\|_{L^p((0,T),\mathbb{R}^{M\times N})} \le \bar{u}_d.$$

Then there exists an optimal control u to problem (OC) of the form given in eq. (5.4).

Proof: The proof is based on the direct method in the calculus of variations (cf. [Giu03, Thm. 4.6]). Since J is bounded from below, its infimum exists and is nonnegative. This allows us to choose a minimizing sequence $(u_n)_{n\in\mathbb{N}}$ with corresponding states $(y_n)_{n\in\mathbb{N}}$ such that $J(u_n,y_n) \to \inf_u J(u,y(u))$, or in the case that C1 holds, that $J(u_n,y_n) + \frac{\alpha_d}{2}\|\partial_t u_n\|^2_{L^2((0,T),\mathbb{R}^{M\times N})}$ converges to its infimum as $n \to \infty$. We consider both cases C1 and C2. In the case where C1 holds, since the cost function is radially unbounded due to the control cost $\|\cdot\|^2_{W^{1,p}((0,T),\mathbb{R}^{M\times N})}$, we can infer that the sequence $(u_n)_n$ is bounded in $W^{1,p}((0,T),\mathbb{R}^{M\times N})$. If C2 holds, the boundedness of $(u_n)_n$ in the space $W^{1,p}((0,T),\mathbb{R}^{M\times N})$ follows directly. Thus, in either case, there exists a weakly convergent subsequence $u_n \rightharpoonup u^*$ with $u^* \in W^{1,p}((0,T),\mathbb{R}^{M\times N})$. We use the same notation for the sequence and its subsequence for convenience.

Since the space $W^{1,p}((0,T),\mathbb{R}^{M\times N})$ with $1 < p < \infty$ is compactly embedded into $C([0,T],\mathbb{R}^{M\times N})$ (see, e.g. [AF03, Thm. 6.3]), we can conclude that $u_n \to u^*$ strongly in $C([0,T],\mathbb{R}^{M\times N})$. Now, it remains to show that the subsequence $(y_n)_n$ associated with $(u_n)_n$ also converges to y^*, the state corresponding to u^*. In a similar manner to eq. (3.30), we can write for almost every $t \in (0,T)$ that

$$\begin{aligned}
&\left\|y_n - y^*\right\|^2_{L^2((0,T),\mathcal{V})} + \left\|y_n(t) - y^*(t)\right\|^2_{\mathcal{H}} + \left\|\delta y_n - \delta y^*\right\|^2_{L^2((0,T),\mathcal{V}')} \\
&\lesssim \max\left\{\left\|y_n\right\|^2_{L^\infty((0,T),\mathcal{H})}, \left\|y^*\right\|^2_{L^\infty((0,T),\mathcal{H})}\right\} \left\|y_n - y^*\right\|^2_{L^2((0,T),\mathcal{H})} \\
&\quad + \|u_n\|_{L^\infty((0,T),\mathbb{R}^{M\times N})} \left\|y_n - y^*\right\|^2_{L^2((0,T),\mathcal{H})} + \|u_n - u^*\|_{L^\infty((0,T),\mathbb{R}^{M\times N})} \left\|y^*\right\|^2_{L^2((0,T),\mathcal{H})}.
\end{aligned}$$

Together with the uniform boundedness of $\left(\|u_n\|_{L^\infty((0,T),\mathbb{R}^{M\times N})}\right)_n$ and $\left(\left\|y_n\right\|_{L^\infty((0,T),\mathbb{R}^{M\times N})}\right)_n$ (which follows from eq. (3.29)), and applying Gronwall's inequality, we obtain

$$\left\|y_n - y^*\right\|^2_{L^\infty((0,\infty),\mathcal{H})} \lesssim \|u_n - u^*\|_{L^\infty((0,T),\mathbb{R}^{M\times N})} \left\|y^*\right\|^2_{L^2((0,T),\mathcal{H})}.$$

This shows that $y_n \to y^*$ strongly in $C([0,T],\mathcal{H})$. This strong convergence allows us to conclude that $J(u_n, y_n) \to J(u^*, y^*) = \inf_u J(u, y(u))$, proving the existence of an optimal control in case C2. The existence for case C1 follows from the weak convergence $u_n \rightharpoonup u^*$ in $H^1((0,T),\mathbb{R}^{M\times N})$ and the weak lower semicontinuity of the control cost $\|\partial_t \cdot\|_{L^p((0,T),\mathbb{R}^{M\times N})}$. Therefore, we conclude that

$$\begin{aligned}
J(u^*, y^*) + \frac{\alpha_d}{2} \|\partial_t u^*\|^2_{L^p((0,T),\mathbb{R}^{M\times N})} &\leq \liminf_{n\to\infty} \left(J(u_n, y_n) + \frac{\alpha_d}{2} \|\partial_t u_n\|^2_{L^p((0,T),\mathbb{R}^{M\times N})}\right) \\
&= \inf_u \left(J(u, y(u)) + \frac{\alpha_d}{2} \|\partial_t u\|^2_{L^p((0,T),\mathbb{R}^{M\times N})}\right).
\end{aligned}$$

Thus, the proof is complete. □

5.2 Remark. It is natural and numerically preferable to consider more general controls, specifically those belonging to $L^2((0,T),\mathbb{R}^{M\times N})$. In this case, the boundedness of the minimizing sequence follows either from the box constraints in eq. (5.6) or the presence of a strictly positive α in eq. (5.5). As mentioned before, the primary challenge lies in the lack of compactness for the state equation, making it unclear whether the weak limit of the minimizing control subsequence corresponds to the weak limit of the associated state subsequence.

In Theorem 5.1, we worked around this issue by exploiting additional regularity for the control and leveraging a compact embedding, which led to strong convergence. However, for controls in $L^2((0,T),\mathbb{R}^{M\times N})$, the lack of compactness becomes an issue for both the control and state, complicating the attainment of strong convergence.

Nevertheless, if the control enters the state equation linearly (rather than bilinearly) and appropriate structural conditions are imposed on the nonlinearity, the existence of an optimal control can still be established using only weak convergence. More precisely, we assume that

- The state equation has the form

  $$\delta y + L(a,x)y + \Lambda(a,x,y)y = \sigma(a)\Delta y + K(u),$$

  i.e. the control enters the state equation linearly.

- We can extend the nonlocal aspect of Λ backwards in time, i.e. we let

  $$\begin{aligned}\Lambda(t,a,x,y)^{hi} &= \int_0^t \int_0^{a_{\max}} \int_\Omega k^{hij}(t,\vartheta,a,\alpha,x,\xi) y_j(\vartheta,\alpha,\xi)\,\mathrm{d}\xi\,\mathrm{d}\alpha\,\mathrm{d}\vartheta \\ &= \left\langle k^{hi}(t,\cdot,a,\cdot,x,\cdot), I\right\rangle_{L^2((0,t)\times[0,a_{\max}]\times\Omega)}.\end{aligned}$$

  Moreover, we assume that the kernel k factorizes as follows (for brevity, let $G := [0,T]\times\mathcal{I}\times\Omega$, $z := (t,a,x)$, and $\zeta := (\vartheta,\alpha,\xi)$):

  $$k^{hij}(z,\zeta) = \sum_{\ell=1}^{N} k^{hij}_{1\ell}(z)\cdot k^{hij}_{2\ell}(\zeta),$$

  with element-wise multiplication of the 3-tensors k_{1j}, k_{2j}. Functions of this structure are dense in $L^2(G\times G)$, and if we allow $N=\infty$, they are even dense in $L^\infty(G, L^2(G))$, which is the natural domain for k (see e.g. [HvVW16, Lems. 1.2.19 and 2.1.4]). The temporal nonlocality can be interpreted as infections from germs present in the environment and thus from individuals that have been infectious in the past.

Under these assumptions, the above proof can be carried out without requiring strong convergence: Let u_n, u^*, and y_n as before. The boundedness of the y_n now merely shows that $y_n \rightharpoonup y^*$ in $L^2(G)$. The linear terms in the differential equation are weakly continuous by standard arguments (for example, for and $y' \in L^2(G)$ we have $\langle Ly_n, y'\rangle_{L^2(G)} = \langle y_n, L'y'\rangle_{L^2(G)}$ with the adjoint operator L', and this expression converges to $\langle y^*, L'y'\rangle_{L^2(G)} = \langle Ly^*, y'\rangle_{L^2(G)}$), and the nonlinearity also becomes weakly continuous due to the kernel structure. Specifically, for any sequence $y_n \rightharpoonup y$ in $L^2(G)$ and any $y' \in L^2(G)$, we have

$$\begin{aligned}\left\langle \Lambda(y_n)y_n, y'\right\rangle_{L^2(G)} &= \left\langle \langle k(z,\cdot), y_n\rangle_{L^2(G)} y_n, y'\right\rangle_{L^2(G)} = \left\langle \left\langle \sum_{\ell=1}^N k_{1\ell}(z)k_{2\ell}, y_n\right\rangle_{L^2(G)} y_n, y'\right\rangle_{L^2(G)} \\ &= \left\langle \sum_{\ell=1}^N k_{1\ell}\langle k_{2\ell}, y_n\rangle_{L^2(G)} y_n, y'\right\rangle_{L^2(G)} = \sum_{\ell=1}^N \langle k_{2\ell}, y_n\rangle_{L^2(G)} \langle k_{1\ell} y_n, y'\rangle_{L^2(G)}.\end{aligned}$$

This shows that $\langle \Lambda(y_n)y_n, v\rangle_{L^2((0,T),\mathcal{H})}$ converges to $\langle \Lambda(y^*)y^*, v\rangle_{L^2((0,T),\mathcal{H})}$. From the concept of weak solution from eq. (3.33) we deduce that for any $v \in L^2((0,T),\mathcal{V})$ we have

$$\begin{aligned}0 &= \langle \delta y_n, v\rangle_{L^2((0,T),\mathcal{V}')\times L^2((0,T),\mathcal{V})} + \langle L y_n, v\rangle_{L^2((0,T),\mathcal{H})} \\ &\quad + \langle \Lambda(y_n)y_n, v\rangle_{L^2((0,T),\mathcal{H})} + \langle K(u_n), v\rangle_{L^2((0,T),\mathcal{H})} + \langle \sigma(a)\nabla y_n, \nabla v\rangle L^2((0,T),\mathcal{H})^d \\ &\xrightarrow{n\to\infty} \langle \delta y^*, v\rangle_{L^2((0,T),\mathcal{V}')\times L^2((0,T),\mathcal{V})} + \langle L y^*, v\rangle_{L^2((0,T),\mathcal{H})} \\ &\quad + \langle \Lambda(y^*)y^*, v\rangle_{L^2((0,T),\mathcal{H})} + \langle K(u^*), v\rangle_{L^2((0,T),\mathcal{H})} + \langle \sigma(a)\nabla y^*, \nabla v\rangle L^2((0,T),\mathcal{H})^d,\end{aligned}$$

which shows that y^* is in fact the state corresponding to the control u^*.

5.2 First order optimality conditions

In this section, we derive first-order optimality conditions for the optimal control problem (OC), following Section 2.5. For this purpose, we define

$$\begin{aligned} U &:= L^2((0,T),\mathbb{R}^{M\times N}),\\ U^{\text{ad}} &:= \{u \in U \mid u \text{ satisfies eq. (5.6)}\},\\ Y &:= \left\{ y \in L^2((0,T),\mathcal{V}) \cap C([0,T],\mathcal{H}) \cap C(\bar{\mathcal{I}}, L^2((0,T),H)) \;\middle|\; \delta y \in L^2((0,T),\mathcal{V}') \right\},\\ Z &:= L^2((0,T),\mathcal{V}') \times \mathcal{H} \times L^2((0,T),H) \end{aligned}$$

and consider the mappings $J : U \times Y \to \mathbb{R}$ and $e : Y \times U \to Z$ defined by

$$\begin{aligned} J(u,y) &:= \frac{1}{2}\iiint \left| g \cdot y(t,a,x) \right|^2 \mathrm{d}(t,a,x) + \frac{\alpha}{2}\iiint |u(t,a,x)|^2 \,\mathrm{d}(t,a,x),\\ e(y,u) &:= \left(\delta y + Ly + \Lambda(y)y + \tilde{K}(u)y - \sigma\Delta y,\, y(t=0) - y_0,\, y(a=0) - \int_0^{a_{\max}} \beta y \,\mathrm{d}a \right). \end{aligned}$$

Here, for $u \in U$ we let

$$\tilde{K}(u) := K\left(\sum_{i=1}^{M}\sum_{j=1}^{N} u_{ij}(t)\mathbb{1}_{\Omega_i}(x)\mathbb{1}_{[a_{j-1},a_j]}(a) \right).$$

Then, the optimal control problem (OC) can be rewritten as

$$\inf_{u\in U} J(u,y) \quad \text{subject to } e(y,u) = 0, u \in U^{\text{ad}}.$$

We know that Y, Z are Banach spaces and U^{ad} is nonempty, convex and closed in $L^2((0,T),\mathbb{R}^{M\times N})$. In Theorem 3.20, we established the existence of a unique solution $y = y(u)$ to the equation $e(y(u),u) = 0$ for all $u \in U^{\text{ad}}$.

It is straightforward to verify that J is continuously Fréchet differentiable. Moreover, apart from the nonlinear terms $\Lambda(y)y$ and $\tilde{K}(u)y$, the mapping e consists of continuous linear terms, which are continuously Fréchet differentiable. The nonlinear terms themselves are continuous bilinear forms, which are also continuously Fréchet differentiable. In particular, for the nonlinear term $\Lambda(y)y$ and for any given $y,h \in Y$ we can write

$$\frac{1}{t}\left(\Lambda(y+th)(y+th) - \Lambda(y)y\right) = \frac{1}{t}\left(\Lambda(th)y + \Lambda(y)th + \Lambda(th)th\right) = \Lambda(h)y + \Lambda(y)h + t\Lambda(h)h.$$

Letting $t \to 0$, we obtain the directional derivative at the point y in the direction h. Similar calculations can be carried out for the term $\tilde{K}(u)y$. Hence we can conclude the following Lemma.

5.3 Lemma. Let $y \in Y$, $u \in U^{\text{ad}}$ be given. Then for every $h \in Y$ and $k \in U$ with $u + \varepsilon k \in U^{\text{ad}}$ for some $\varepsilon > 0$, the directional derivatives

$$e_y(y,u)h = \lim_{t\to 0}\frac{1}{t}\left(e(y+th,u) - e(y,u)\right), \quad e_u(y,u)k = \lim_{t\to 0}\frac{1}{t}\left(e(y,u+tk) - e(y,u)\right)$$

exist and it holds

$$e_y(y,u)h = \left(\delta h + Lh + \Lambda(y)h + \Lambda(h)y + \tilde{K}(u)h - \sigma\Delta h,\ h(t=0),\ h(a=0) - \int_0^{a_{\max}} \beta h \,\mathrm{d}a \right),$$
$$e_u(y,u)k = (\tilde{K}(k)y, 0, 0).$$

5.4 Lemma. Suppose that Assumption 3.19 holds. Then for all $u \in U^{\mathrm{ad}}$ the linear map $e_y(y(u),u)$ has a bounded inverse.

Proof: The statement is equivalent to demonstrating that for every given tuple $(f, h_0, B_0) \in Z$, the equation

$$\begin{gathered} \delta h + (L + \Lambda(y) + \tilde{K}(u))h + \Lambda(h)y - \sigma\Delta h = f, \\ h(t=0) = h_0, \quad h(a=0) - \int_0^{a_{\max}} \beta h \,\mathrm{d}a = B_0, \quad \partial_\nu h(x \in \partial\Omega) = 0 \end{gathered} \tag{5.7}$$

admits a unique weak solution $h \in Y$. To prove this, similarly to the proof of Theorem 3.18, we proceed through several steps. First, by neglecting the terms involving Λ, K and β, we consider the linearized equation

$$\begin{gathered} \delta h + Lh - \sigma\Delta h = f, \\ h(t=0) = h_0, \quad h(a=0) = B_0, \quad \partial_\nu h(x \in \partial\Omega) = 0. \end{gathered} \tag{5.8}$$

By Corollary 3.14, there exists a unique solution h to eq. (5.8), which can be expressed as in eq. (3.17):

$$h(t,a,x) = \begin{cases} U(a,0)B_0(t-a) + S(a,0)\, f|_{\mathrm{char}(t-a)} & t > a, \\ U(a,a-t)h_0(a-t) + S(a,a-t)\, f|_{\mathrm{char}(t-a)} & t \le a. \end{cases} \tag{5.9}$$

In the next step, we replace B_0 by a function $b \in L^2((0,T),H)$ satisfying

$$b(t) = h(a=0) = B_0(t) + \int_0^{a_{\max}} \beta(\alpha)h(\alpha)\,\mathrm{d}\alpha. \tag{5.10}$$

Together with eq. (5.9), and in a similar manner to eq. (3.24), we obtain the Volterra equation

$$\begin{aligned} b(t) = B_0(t) + &\int_0^{\min(t,a_{\max})} \beta(\alpha,x)U(\alpha,0)b(t-\alpha)\,\mathrm{d}\alpha \\ &+ \int_{\min(t,a_{\max})}^{a_{\max}} \beta(\alpha)U(\alpha,\alpha-t)h_0(\alpha-t)\,\mathrm{d}\alpha + \int_{t-a_{\max}}^{t} \beta(t-t_0)S(t-t_0,t_0^-)\, f|_{\mathrm{char}(t_0)}\,\mathrm{d}t_0. \end{aligned}$$

A similar proof as for Theorem 3.16 shows that there exists a solution $b \in L^2((0,T),H)$ satisfying

$$\|b\|_{L^2((0,T),H)} \lesssim \|h_0\|_{\mathcal{H}} + \|B_0\|_{L^2((0,T),H)} + \|f\|_{L^2((0,T),\mathcal{V}')}.$$

Hence, the solution of eq. (5.8) with initial condition for ($a = 0$) given by eq. (5.10) can be expressed as

$$h(t,a,x) = \begin{cases} U(a,0)b(t-a) + S(a,0)\, f|_{\mathrm{char}(t-a)} & t > a, \\ U(a,a-t)h_0(a-t) + S(a,a-t)\, f|_{\mathrm{char}(t-a)} & t \leq a. \end{cases}$$

Similar to eq. (3.25), we obtain an estimate for this h of the form

$$\|h\|^2_{L^2((0,T),\mathcal{V})} + \|h\|^2_{C([0,T],\mathcal{H})} + \|\delta h\|^2_{L^2((0,T),\mathcal{V}')} \lesssim \|h_0\|^2_{\mathcal{H}} + \|B_0\|_{L^2((0,T),H)} + \|f\|^2_{L^2((0,T),\mathcal{V}')}. \tag{5.11}$$

To include the remaining terms in the equation, we will employ the Banach fixed-point argument, as demonstrated in the proof of Theorem 3.20. For a given $h \in Y$, let $\Phi(h)$ denote the solution $k \in Y$ to the equation

$$\delta k + Lk - \sigma\Delta k = f - (\Lambda(y) + \tilde{K}(u))h - \Lambda(h)y,$$
$$k(t=0) = h_0, \quad k(a=0) - \int_0^{a_{\max}} \beta k \,\mathrm{d}a = B_0, \quad \partial_\nu k(x \in \partial\Omega) = 0.$$

With similar arguments as in the proof of Theorem 3.20, Φ is well-defined. Further, using eq. (5.11) a similar calculation as in eq. (3.30) yields the estimate

$$\begin{aligned} \|\Phi(h) - \Phi(k)\|_{L^\infty((0,\infty),\mathcal{H})} &\lesssim \left\|\Lambda(y)(h-k)\right\|_{L^2((0,T),\mathcal{V}')} + \left\|\tilde{K}(u)(h-k)\right\|_{L^2((0,T),\mathcal{V}')} \\ &+ \left\|\Lambda(h-k)y\right\|_{L^2((0,T),\mathcal{V}')} \lesssim \|h-k\|_{L^2((0,T),\mathcal{H})} \lesssim \sqrt{T}\|h-k\|_{L^\infty((0,T),\mathcal{H}))} \end{aligned}$$

Hence, choosing T sufficiently small, we can conclude that Φ is a contraction on $C([0,T],\mathcal{H}))$ (which is a superset of Y). Therefore, we have a local existence (in time) of the solution to eq. (5.7). The existence of the global solution follows from the following energy estimate which holds globally. Let h be any fixed point of Φ, then eq. (5.11) yields for almost all $t \in (0,T)$ that

$$\begin{aligned} \|h(t)\|^2_{\mathcal{H}} &\lesssim \|f\|^2_{L^2((0,T),V')} + \|h_0\|^2_{\mathcal{H}} + \|B_0\|^2_{L^2((0,T),H)} + \left\|\Lambda(y)h + \Lambda(h)y + \tilde{K}(u)h\right\|^2_{L^2((0,t),\mathcal{H})} \\ &\lesssim \|f\|^2_{L^2((0,T),V')} + \|h_0\|^2_{\mathcal{H}} + \|B_0\|^2_{L^2((0,T),H)} + \left(\left\|y\right\|^2_{L^\infty((0,T),\mathcal{H})} + \|u\|^2_{L^\infty((0,T),\mathcal{H})}\right)\|h\|^2_{L^2((0,t),\mathcal{H})}, \end{aligned}$$

where in the last line, we have used eq. (3.27). Now, an application of Gronwall's lemma yields the following estimate

$$\|h\|^2_{L^\infty((0,T),\mathcal{H})} \lesssim e^{\left(\|y\|^2_{L^\infty((0,T),\mathcal{H})} + \|u\|^2_{L^\infty((0,T),\mathcal{H})}\right)T}\left(\|f\|^2_{L^2((0,T),V')} + \|h_0\|^2_{\mathcal{H}} + \|B_0\|^2_{L^2((0,T),H)}\right).$$

This uniform estimate, combined with standard continuation arguments (cf. [Smo83, Thm. 14.4]), shows that the solution obtained via Banach's Fixed Point Theorem can indeed be extended to the entire time interval $(0,T)$. in addition to that, it establishes the uniqueness of the solution, as it directly shows that the difference between two solutions to the linear equation with identical initial and boundary conditions must be zero at all times. This completes the proof. □

Next, we will derive the adjoint equation. In other words, we find $p \in Z'$, where

$$Z' = L^2((0,T),\mathcal{V}) \times \mathcal{H} \times L^2((0,T),H),$$

which satisfies eq. (2.3), i.e.

$$e_y(y(u),u)^* p = -J_y(u,y(u)).$$

This means that for all $h \in Y$ that $\left\langle -J_y(u,y(u)), h \right\rangle_{Y' \times Y} = \left\langle p, e_y(y(u),u)h \right\rangle_{Z' \times Z}$. Recalling the weak formulation of the Laplace operator used in eq. (3.12) and using integration by parts, this yields

$$\begin{aligned}
-\left\langle gy(u), gh \right\rangle_{L^2((0,T),\mathcal{H})} &= \left\langle p_1, \delta h + Lh + \Lambda(y)h + \Lambda(h)y + \tilde{K}(u)h - \sigma \Delta h \right\rangle_{L^2((0,T),\mathcal{V}) \times L^2((0,T),\mathcal{V}')} \\
&\quad + \left\langle p_2, h(t=0) \right\rangle_{\mathcal{H}} + \left\langle p_3, h(a=0) - \int_0^{a_{\max}} \beta(\alpha) h(\alpha) \,\mathrm{d}\alpha \right\rangle_{L^2((0,T),H)} \\
&= -\left\langle h, \delta p_1 \right\rangle_{L^2((0,T),\mathcal{H})} + \left\langle h(t=T), p_1(t=T) \right\rangle_{\mathcal{H}} - \left\langle h(t=0), p_1(t=0) \right\rangle_{\mathcal{H}} \\
&\quad + \left\langle h(a=a_{\max}), p_1(a=a_{\max}) \right\rangle_{L^2((0,T)\times\Omega)} - \left\langle h(a=0), p_1(a=0) \right\rangle_{L^2((0,T)\times\Omega)} \\
&\quad + \left\langle h, L^{\mathsf{T}} p_1 \right\rangle_{L^2((0,T),\mathcal{H})} + \left\langle \Lambda(a,x,h)y + \Lambda(a,x,y)h + \tilde{K}(u)h, p_1 \right\rangle_{L^2((0,T),\mathcal{H})} \\
&\quad - \left\langle h, \sigma(a) \Delta p_1 \right\rangle_{L^2((0,T),\mathcal{V}) \times L^2((0,T),\mathcal{V}')} \\
&\quad + \left\langle h(t=0), p_2 \right\rangle_{\mathcal{H}} + \left\langle h(a=0) - \int_0^{a_{\max}} \beta(\alpha,x) h(t,\alpha,x) \,\mathrm{d}\alpha, p_3 \right\rangle_{L^2((0,T)\times\Omega)} .
\end{aligned}$$

Recall $\Lambda(h)^{\ell i}(t,a,x) = \int_0^{a_{\max}} \int_\Omega k^{\ell i j}(a,x,\alpha,\xi) h_j(t,\alpha,\xi) \,\mathrm{d}\xi \,\mathrm{d}\alpha$, where $\ell, i = 1, \dots, n$ and we use Einstein's convention over the index j. Setting $z := (a,x)$, $\zeta := (\alpha,\xi)$, we can rewrite the term with $\Lambda(h)y$ as

$$\begin{aligned}
\left\langle \Lambda(h)y, p_1 \right\rangle_{L^2((0,T),\mathcal{H})} &= \int_0^T \int_{\mathcal{I}\times\Omega} \int_{\mathcal{I}\times\Omega} k^{\ell i j}(z,\zeta) h_j(t,\zeta) \,\mathrm{d}\zeta \, y_i(t,z) p_{1\ell}(t,z) \,\mathrm{d}z \,\mathrm{d}t \\
&= \int_0^T \int_{\mathcal{I}\times\Omega} h_j(t,\zeta) \int_{\mathcal{I}\times\Omega} k^{\ell i j}(z,\zeta) y_i(t,z) p_{1\ell}(t,z) \,\mathrm{d}z \,\mathrm{d}\zeta \,\mathrm{d}t \\
&=: \left\langle h, \tilde{\Lambda}_y(p_1) \right\rangle_{L^2((0,T),\mathcal{H})},
\end{aligned}$$

where $\tilde{\Lambda}_y$ is a nonlocal linear operator. Further, the term with the birth condition can be rewritten as

$$\begin{aligned}
\left\langle \int_0^{a_{\max}} \beta(\alpha) h(\alpha) \,\mathrm{d}\alpha, p_3 \right\rangle_{L^2((0,T)\times\Omega)} &= \iiint_{(0,T)\times\mathcal{I}\times\Omega} \beta(\alpha,\xi) h(\vartheta,\alpha,\xi) \cdot p_3(\vartheta,\xi) \,\mathrm{d}(\vartheta,\alpha,\xi) \\
&= \left\langle h, \beta^{\mathsf{T}} p_3 \right\rangle_{L^2((0,T),\mathcal{H})}.
\end{aligned}$$

Therefore, we can write

$$\begin{aligned}
-\left\langle gy(u), gh\right\rangle_{\mathcal{I}\times\Omega} &= -\left\langle \delta p_1, h\right\rangle_{\mathcal{I}\times\Omega} + \left\langle p_1(t=T), h(t=T)\right\rangle_{\mathcal{H}} - \left\langle p_1(t=0), h(t=0)\right\rangle_{\mathcal{H}} \\
&\quad + \left\langle p_1(a=a_{\max}), h(a=a_{\max})\right\rangle_{L^2((0,T)\times\Omega)} - \left\langle p_1(a=0), h(a=0)\right\rangle_{L^2((0,T)\times\Omega)} \\
&\quad + \langle (L+\Lambda(y(u))+\tilde{K}(u))^{\mathsf{T}} p_1 + \tilde{\Lambda}_{y(u)}(p_1) - \sigma\Delta p, h\rangle_{L^2((0,T),\mathcal{V}')\times L^2((0,T),\mathcal{V})} \\
&\quad + \left\langle p_2, h(t=0)\right\rangle_{\mathcal{H}} + \left\langle p_3, h(a=0)\right\rangle_{L^2((0,T),H)} - \left\langle \beta^{\mathsf{T}} p_3, h\right\rangle_{L^2((0,T),\mathcal{H})},
\end{aligned}$$

which represents the weak formulation of the adjoint equation. For brevity we write $p := p_1$. Letting $h(t,a,x) = \delta_{t=0}\varphi(a,x)$, where φ is a test function on $(0,T)\times\Omega$ and $\delta_{t=0}$ is the Dirac distribution (this is a formal calculation, which can be made precise by an approximation argument), all that remains is

$$0 = \left\langle -p_1(t=0), \varphi\right\rangle_{\mathcal{H}} + \left\langle p_2, \varphi\right\rangle_{\mathcal{H}},$$

which shows that $p_2 = p(t=0)$. Similarly, it can be shown that $p_3 = p(a=0)$. Then letting h an arbitrary test function gives the weak formulation of the *adjoint equation*

$$\begin{gathered}
-\delta p + (L+\Lambda(y(u))+\tilde{K}(u))^{\mathsf{T}} p + \tilde{\Lambda}_{y(u)}(p) - \sigma\Delta p - \beta(a,x)^{\mathsf{T}} p(a=0) = -g^{\mathsf{T}} g y(u), \\
p(t=T) = 0, \quad p(a=a_{\max}) = 0, \quad \partial_\nu p(x\in\partial\Omega) = 0.
\end{gathered} \tag{5.12}$$

5.5 Remark. An interesting feature in the adjoint equation is the term of the form $\beta(a,x)^{\mathsf{T}} p(a=0)$, which arises from to the implicit boundary condition. We remark that a similar term $\beta^{\mathsf{T}} p(a=0)$ appears in the adjoint equation for a class of optimal control problems governed by age-structured models without spatial variable x (ODEs), as shown in [AAC11, Thm. 4.11].

5.6 Theorem. Suppose that Assumption 3.19 holds. Then the adjoint equation (5.12) has a unique solution in Y.

Proof: Setting $h(t,a) := p(T-t, a_{\max}-a)$, the adjoint equation can be rewritten as

$$\begin{gathered}
\delta h + (L+\Lambda(y)+\tilde{K}(u))^{\mathsf{T}}(T-t, a_{\max}-a)h + \tilde{\Lambda}_{y(T-t,a_{\max}-a)}(h) - \sigma(a_{\max}-a)\Delta h \\
\qquad\qquad - \beta(a_{\max}-a)^{\mathsf{T}} h(a=a_{\max}) = -g^{\mathsf{T}} g y(T-t, a_{\max}-a), \\
h(t=0)=0, \quad h(a=0)=0, \quad \partial_\nu h(x\in\partial\Omega) = 0.
\end{gathered}$$

Comparing this equation with eq. (5.7), we observe that they share a similar form, except for the initial conditions at $a=0$ and $t=0$, which in this case are zero, and instead of a term $\int_0^{a_{\max}} \beta h \,\mathrm{d}a$ on the boundary, in eq. (5.12) we have a term $\beta(a_{\max}-a)^{\mathsf{T}} h(t, a_{\max}, x)$ in the interior. Therefore, to prove the theorem, we can apply similar arguments to those in the proof of Lemma 5.4, with only slight adaptations. More specifically, we can easily obtain a solution h of the linearized equation

$$\begin{gathered}
\delta h + L^{\mathsf{T}}(T-t, a_{\max}-a)h - \sigma(a_{\max}-a)\Delta h - f = -g^{\mathsf{T}} g y(T-t, a_{\max}-a), \\
h(t=0)=0, \quad h(a=0)=0, \quad \partial_\nu h(x\in\partial\Omega) = 0,
\end{gathered}$$

as described in the proof of Corollary 3.14. Similarly to Theorem 3.2 and Definition 3.3, we obtain evolution operators $\tilde{U}$ and $\tilde{S}$ for the equation along characteristics, which we can use to represent the solution as

$$h(t,a,x) = \begin{cases} \tilde{S}(a,0)\, f|_{\mathrm{char}(t-a)} & t > a, \\ \tilde{S}(a,a-t)\, f|_{\mathrm{char}(t-a)} & t \le a. \end{cases} \tag{5.13}$$

The next step is to replace f in this equation with $f + \beta^{\mathsf{T}} b$, where $b \in L^2((0,T),H)$ is supposed to satisfy $b = h(a = a_{\max})$. Substituting this into eq. (5.13) and applying Duhamel's principle from Remark 3.5, we obtain the Volterra equation

$$\begin{aligned} b(t) &= \tilde{S}(a_{\max}, (a_{\max} - t)^+)\, f|_{\mathrm{char}(t) - a_{\max}} + \int\limits_{(a_{\max}-t)^+}^{a_{\max}} \tilde{U}(a_{\max}, r)\beta(a_{\max} - r)^{\mathsf{T}} b(t - a_{\max} + r)\,\mathrm{d}r \\ &= \tilde{S}(a_{\max}, (a_{\max} - t)^+)\, f|_{\mathrm{char}(t) - a_{\max}} + \int\limits_{0}^{\min(t,a_{\max})} \tilde{U}(a_{\max}, a_{\max} - s)\beta(s)^{\mathsf{T}} b(t - s)\,\mathrm{d}s. \end{aligned}$$

A solution $b \in L^2((0,T),H)$ can then be found as in Theorem 3.16. The remainder of the proof, specifically the inclusion of the missing terms, follows similarly to the proof of Lemma 5.4. □

The first-order necessary optimality conditions are then given by the following result.

5.7 Corollary. Suppose that Assumption 3.19 holds, and let the pair $(y^*, u^*) \in Y \times U^{ad}$ be a solution to Equation (OC). Then this pair satisfies the following variational inequality

$$\begin{aligned} &\alpha\langle u^*, u - u^*\rangle_{L^2((0,T),\mathbb{R}^{M\times N})} + \left\langle p^*, \tilde{K}(u - u^*)y^*\right\rangle_{L^2((0,T),\mathcal{H})} \\ &= \left\langle \alpha u^* + \mathcal{M}(p, y^*), u - u^*\right\rangle_{L^2((0,T),\mathbb{R}^{M\times N})} \geq 0 \quad \text{for all } u \in U^{ad}, \end{aligned}$$

where $y^* = y(u^*)$ is the solution to eq. (5.1) corresponding to u^*, p^* denotes the solution of the adjoint equation (5.12) associated with u^* and y^*, and the matrix $\mathcal{M}(p, y^*) \in L^2((0,T),\mathbb{R}^{M\times N})$ is defined as

$$\mathcal{M}(p, y^*)_{ij}(t) = \iint\limits_{\mathcal{I}\times\Omega} p^*(t,a,x)^{\mathsf{T}} K\left(\mathbb{1}_{\Omega_i}(x)\mathbb{1}_{[a_{j-1},a_j]}(a)\right) y^*(t,a,x)\,\mathrm{d}(a,x).$$

Proof: The linearity of K from Assumption 3.19 yields

$$\begin{aligned} &\left\langle p^*, \tilde{K}(u - u^*)y^*\right\rangle_{L^2((0,T),\mathcal{H})} \\ &= \iiint\limits_{(0,T)\times\mathcal{I}\times\Omega} p^*(t,a,x)^{\mathsf{T}} K\left(\sum_{i=1}^{M}\sum_{j=1}^{N}(u - u^*)_{ij}(t)\mathbb{1}_{\Omega_i}(x)\mathbb{1}_{[a_{j-1},a_j]}(a)\right) y^*(t,a,x)\,\mathrm{d}(t,a,x) \\ &= \sum_{i=1}^{M}\sum_{j=1}^{N}\int_0^T (u - u^*)_{ij}(t) \iint\limits_{\mathcal{I}\times\Omega} p^*(t,a,x)^{\mathsf{T}} K\left(\mathbb{1}_{\Omega_i}(x)\mathbb{1}_{[a_{j-1},a_j]}(a)\right) y^*(t,a,x)\,\mathrm{d}(a,x)\,\mathrm{d}t \\ &= \left\langle u - u^*, \mathcal{M}(p, y^*)\right\rangle_{L^2((0,T),\mathbb{R}^{M\times N})}. \end{aligned}$$

Then the claim is a direct consequence of Theorem 2.22. □

5.3 Numerical experiments

In this section we present a numerical implementation of the optimal control problem (OC) governed by the model (SVIR). We follow the so-called discretize-then-optimize approach, which means we first devise a numerical scheme to discretize the state equation, and then use optimization techniques for the discretized problem.

The spatial domain was chosen as the interval $\Omega = [0,1]$. In order to solve the state equation

$$\begin{gathered}\delta y + L(a,x)y + \Lambda(a,x,y)y + K(u)y = \sigma(a)\Delta y,\\ y(t=0) = y_0, \quad \partial_\nu y(x \in \partial\Omega) = 0,\\ y(a=0) = \int_0^{a_{\max}} \beta(\alpha,x)y(t,\alpha,x)\,\mathrm{d}\alpha\end{gathered}$$

from eq. (5.1) numerically, first let $x_1 < \ldots < x_K$ be equidistant points in space, the distance being Δx. We further choose equidistant time steps $0 = t_0 < t_1 < \ldots < t_M = T$ and age steps $0 = a_0 < a_1 < \ldots < a_N = a_{\max}$ in such a way that $\Delta t = \Delta a$. It should be noted that this choice, while convenient, poses challenges for real-world simulations: typically, Δt is on the order of days or less, while the maximum age $a_{\max}$ in I can span several decades. As a result, the age variable requires very fine discretization, potentially leading to the curse of dimensionality.

We proceed in three steps:

(1) We choose a finite difference scheme to get rid of the Laplacian:

$$\delta y(x_i) + L(x_i)y(x_i) + \Lambda(x_i,y)y(x_i) + K(u(x_i))y(x_i) = \sigma(x_i)\,\frac{y(x_{i-1}) - 2y(x_i) + y(x_{i+1})}{\Delta x^2}.$$

Since we want to approximate the Neumann Laplacian, we let $x_0 = x_1$ and $x_{K+1} = x_K$.

(2) We approximate the nonlocal operator Λ with the trapezoidal rule for numerical integration. To this end, we introduce the weight w defined by

$$w(a_\alpha,x_i) := \begin{cases}\frac{1}{4}, & a_\alpha \in \{a_0,a_{\max}\} \text{ and } x_i \in \{x_1,x_K\},\\ \frac{1}{2}, & \text{either } a_\alpha \in \{a_0,a_{\max}\} \text{ or } x_i \in \{x_1,x_K\},\\ 1, & \text{else.}\end{cases}$$

We obtain

$$\begin{aligned}\Lambda(a_j,x_i,y) &= \int_0^{a_{\max}} \int_\Omega k(a_j,\alpha,x_i,\xi)y(\alpha,\xi)\,\mathrm{d}\xi\,\mathrm{d}\alpha\\ &\approx \sum_{\alpha,\beta} k(a_j,a_\alpha,x_i,x_\beta)y(a_\alpha,x_\beta)w(a_\alpha,x_\beta)\Delta x\Delta a.\end{aligned}$$

The complete equation now reads

$$\begin{gathered}\delta y(a_j,x_i) + L(a_j,x_i)y(a_j,x_i) + \left(\sum_{\alpha,\beta} k(a_j,a_\alpha,x_i,x_\beta)y(a_\alpha,x_\beta)w(a_\alpha,x_\beta)\Delta x\Delta a\right)y(a_j,x_i)\\ +K(u(a_j,x_i))y(a_j,x_i) = \sigma(a_j,x_i)\,\frac{y(a_j,x_{i-1}) - 2y(a_j,x_i) + y(a_j,x_{i+1})}{\Delta x^2}.\end{gathered}$$

(3) We approximate δy by an approximation from an ODE solver. We choose the theta method from Section 2.4, which for $\vartheta = 0$ yields the explicit Euler method, $\vartheta = 1$ the implicit Euler method, and $\vartheta = 0.5$ the Crank-Nicholson method. We have to account for the age derivative as well: The equation $\delta x(t,a) = F(t,a,x(t,a))$ is approximated by

$$\frac{x(t+\Delta t, a+\Delta t) - x(t,a)}{\Delta t} = (1-\vartheta)F(t,a,x(t,a)) + \vartheta F(t+\Delta t, a+\Delta t, x(t+\Delta t, a+\Delta t)).$$

In our case we obtain

$$\begin{aligned}
&\frac{y(t_{n+1},a_{j+1},x_i) - y(t_n,a_j,x_i)}{\Delta t}\\
&\quad = (1-\vartheta)\Bigg\{ -(L(a_j,x_i) + K(u(t_n,a_j,x_i)))y(t_n,a_j,x_i)\\
&\quad - \left(\sum_{\alpha,\beta} k(a_j,a_\alpha,x_i,x_\beta) y(t_n,a_\alpha,x_\beta) w(a_\alpha,x_\beta)\Delta x \Delta a\right) y(t_n,a_j,x_i)\\
&\quad - \sigma(a_j,x_i)\frac{y(t_n,a_j,x_{i-1}) - 2y(t_n,a_j,x_i) + y(t_n,a_j,x_{i+1})}{\Delta x^2}\Bigg\}\\
&\quad + \vartheta\Bigg\{ -(L(a_{j+1},x_i) + K(u(t_{n+1},a_{j+1},x_i)))y(t_{n+1},a_{j+1},x_i)\\
&\quad - \left(\sum_{\alpha,\beta} k(a_{j+1},a_\alpha,x_i,x_\beta) y(t_{n+1},a_\alpha,x_\beta) w(a_\alpha,x_\beta)\Delta x \Delta a\right) y(t_{n+1},a_{j+1},x_i)\\
&\quad - \sigma(a_{j+1},x_i)\frac{y(t_{n+1},a_{j+1},x_{i-1}) - 2y(t_{n+1},a_{j+1},x_i) + y(t_{n+1},a_{j+1},x_{i+1})}{\Delta x^2}\Bigg\}.
\end{aligned}$$

For brevity let $y^n := y(t_n)$. Further, we use an Adams–Bashforth type approximation $\Lambda(y^{n+1}) \approx \Lambda(2y^n - y^{n-1})$ (cf. Section 2.4) in order to obtain a linear equation in y^{n+1}. The convention $y^{-1} = y^n$ corresponds to a simple Euler method for the first step. Then our algorithm can be reformulated to:

$$\begin{aligned}
&y^{n+1}(a_{j+1},x_i) + \vartheta\Delta t\Bigg\{ (L(a_{j+1},x_i) + K(u(t_{n+1},a_{j+1},x_i)))y^{n+1}(a_{j+1},x_i)\\
&+ \left(\sum_{\alpha,\beta} k(a_{j+1},a_\alpha,x_i,x_\beta)(2y^n - y^{n-1})(a_\alpha,x_\beta) w(a_\alpha,x_\beta)\Delta x\Delta a\right) y^{n+1}(a_{j+1},x_i)\\
&+ \sigma(a_{j+1},x_i)\frac{y^{n+1}(a_{j+1},x_{i-1}) - 2y^{n+1}(a_{j+1},x_i) + y^{n+1}(a_{j+1},x_{i+1})}{\Delta x^2}\Bigg\}\\
&- y^n(a_j,x_i) + (1-\vartheta)\Delta t\Bigg\{ (L(a_j,x_i) + K(u(t_n,a_j,x_i)))y^n(a_j,x_i)\\
&+ \left(\sum_{\alpha,\beta} k(a_j,a_\alpha,x_i,x_\beta) y^n(a_\alpha,x_\beta) w(a_\alpha,x_\beta)\Delta x\Delta a\right) y^n(a_j,x_i)
\end{aligned}$$

$$+ \sigma(a_j, x_i) \frac{y^n(a_j, x_{i-1}) - 2y^n(a_j, x_i) + y^n(a_j, x_{i+1})}{\Delta x^2} \Bigg\} = 0.$$

The only missing entry is $y(t_{n+1}, a_0, x_i)$, which can be obtained by the birth rule:

$$y(t_{n+1}, a_0, x_i) = \int_0^{a_{\max}} \beta(\alpha, x_i) y(t_{n+1}, \alpha, x_i) \,\mathrm{d}\alpha \approx \sum_\alpha \beta(a_\alpha, x_i) y(t_{n+1}, a_\alpha, x_i) \tilde{w}(a_\alpha) \Delta a,$$

where $\tilde{w}(a_\alpha) = \begin{cases} \frac{1}{2}, & a_\alpha \in \{a_0, a_{\max}\}, \\ 1, & \text{else.} \end{cases}$

In total, we now have a scheme that for a given control u produces the state $y(u)$ associated to u.

In order to obtain the gradient of the target functional with respect to the control, we can proceed as in eq. (2.4). To this end we derive a formula for the adjoint equation (2.3) by defining the discretized Lagrangian and setting $\frac{\partial \mathcal{L}}{\partial y}$ to zero. We assume the control to be piecewise constant on each time-age square $(t_{n-1}, t_n) \times (a_{j-1}, a_j)$ $(1 \leq n \leq M, 1 \leq j \leq N)$, where it takes the value $u_{n-1,j-1}$ (which may depend on x). The target functional

$$\begin{aligned} J &= \frac{1}{2} \|g \cdot y\|^2_{L^2((0,T),\mathcal{H})} + \frac{\alpha}{2} \|u\|^2_{L^2((0,T),\mathcal{H})} \\ &= \frac{1}{2} \int_0^t \int_0^{a_{\max}} \int_\Omega (g \cdot y)(t, a, x)^2 + \alpha u(t, a, x)^2 \,\mathrm{d}x \,\mathrm{d}a \,\mathrm{d}t \end{aligned}$$

is then approximated by

$$\begin{aligned} J \approx{} & \frac{1}{2} \sum_{n=0}^{M-1} \sum_{j=0}^{N-1} \sum_{i=1}^{K} (1 - \vartheta)(g \cdot y^n(a_j, x_i))^2 + \vartheta (g \cdot y^{n+1}(a_{j+1}, x_i))^2 \Delta x \Delta a \Delta t \\ & + \frac{\alpha}{2} \sum_{n=0}^{M-1} \sum_{j=0}^{N-1} \sum_{i=1}^{K} u_{n,j}(x_i)^2 \Delta x \Delta a \Delta t. \end{aligned}$$

Now we can define the Lagrangian by adding the target functional and the discretized equation multiplied with some Lagrange multiplier p:

$$\begin{aligned} \mathcal{L} = {} & \frac{1}{2} \sum_{n=0}^{M-1} \sum_{j=0}^{N-1} \sum_{i=1}^{K} (1 - \vartheta)(g \cdot y^n(a_j, x_i))^2 + \vartheta (g \cdot y^{n+1}(a_{j+1}, x_i))^2 \Delta x \Delta a \Delta t \\ & + \frac{\alpha}{2} \sum_{n=0}^{M-1} \sum_{j=0}^{N-1} \sum_{i=1}^{K} u_{n,j}(x_i)^2 \Delta x \Delta a \Delta t \\ & + \sum_{n=0}^{M-1} \sum_{j=0}^{N-1} \sum_{i=1}^{K} p^{n+1,j+1,i} \cdot \Bigg[y^{n+1}(a_{j+1}, x_i) + \vartheta \Delta t \Bigg\{ (L(a_{j+1}, x_i) + K(u_{n,j}(x_i))) y^{n+1}(a_{j+1}, x_i) \\ & + \left(\sum_{\alpha=0}^{N} \sum_{\beta=1}^{K} k(a_{j+1}, a_\alpha, x_i, x_\beta)(2y^n - y^{n-1})(a_\alpha, x_\beta) w(a_\alpha, x_\beta) \Delta x \Delta a \right) y^{n+1}(a_{j+1}, x_i) \end{aligned}$$

$$
\begin{aligned}
&+\sigma(a_{j+1},x_i)\,\frac{y^{n+1}(a_{j+1},x_{i-1})-2y^{n+1}(a_{j+1},x_i)+y^{n+1}(a_{j+1},x_{i+1})}{\Delta x^2}\Biggr\}\\
&-y^n(a_j,x_i)+(1-\vartheta)\Delta t\Biggl\{(L(a_j,x_i)+K(u_{n,j}(x_i)))y^n(a_j,x_i)\\
&+\left(\sum_{\alpha=0}^{N}\sum_{\beta=1}^{K}k(a_j,a_\alpha,x_i,x_\beta)y^n(a_\alpha,x_\beta)w(a_\alpha,x_\beta)\Delta x\Delta a\right)y^n(a_j,x_i)\\
&+\sigma(a_j,x_i)\,\frac{y^n(a_j,x_{i-1})-2y^n(a_j,x_i)+y^n(a_j,x_{i+1})}{\Delta x^2}\Biggr\}\Biggr]\\
&+\sum_{n=0}^{M-1}\sum_{i=1}^{K}p^{n+1,0,i}\left[y^{n+1}(a_0,x_i)-\sum_{\alpha}\beta(a_\alpha,x_i)y^{n+1}(a_\alpha,x_i)\tilde{w}(a_\alpha)\Delta a\right].
\end{aligned}
$$

The expression for the $p^{n,j,i}$ we are looking for can now be found by setting the partial derivatives of $\mathcal{L}$ with respect to the $y^n(a_j,x_i)$ to zero. Note that y^0 is calculated by means of the initial values, so we do not form the derivative with respect to it. We obtain

$$
\begin{aligned}
&\frac{\partial\mathcal{L}}{\partial y^n(a_j,x_i)}\\
&=(g\cdot y^n(a_j,x_i))g\cdot\Delta x\Delta a\Delta t\cdot\begin{cases}0, & n=M, j=0\\ \vartheta, & (n=M, j\neq 0)\text{ or } j=N\\ 1-\vartheta, & n\neq M, j=0\\ 1, & \text{else}\end{cases}\\
&+\Biggl[p^{n,j,i}+\vartheta\Delta t(L(a_j,x_i)+K(u_{n-1,j-1}(x_i)))^{\mathsf{T}}p^{n,j,i}\\
&+\vartheta\Delta t\left(\sum_{\alpha=0}^{N}\sum_{\beta=1}^{K}k^{\omega\gamma\delta}(a_j,a_\alpha,x_i,x_\beta)(2y_\delta^{n-1}-y_\delta^{n-2})(a_\alpha,x_\beta)w(a_\alpha,x_\beta)\Delta x\Delta a\right)p_\omega^{n,j,i}\\
&+\vartheta\Delta t\sigma(a_j,x_i)\,\frac{p^{n,j,i+1}-2p^{n,j,i}+p^{n,j,i-1}}{\Delta x^2}\Biggr]\cdot\begin{cases}0, & j=0\\ 1 & \text{else}\end{cases}\\
&+2\vartheta\Delta t\sum_{\alpha=0}^{N-1}\sum_{\beta=1}^{K}k^{\omega\gamma\delta}(a_{\alpha+1},a_j,x_\beta,x_i)y_\gamma^{n+1}(a_\alpha,x_\beta)p_\omega^{n+1,\alpha+1,\beta}w(a_j,x_\beta)\Delta x\Delta a\cdot\begin{cases}0, & n=M\\ 1, & \text{else}\end{cases}\\
&-\vartheta\Delta t\sum_{\alpha=0}^{N-1}\sum_{\beta=1}^{K}k^{\omega\gamma\delta}(a_{\alpha+1},a_j,x_\beta,x_i)y_\gamma^{n+2}(a_\alpha,x_\beta)p_\omega^{n+2,\alpha+1,\beta}w(a_j,x_\beta)\Delta x\Delta a\cdot\begin{cases}0, & n\in\{M,M-1\}\\ 1, & \text{else}\end{cases}\\
&+\Biggl[-p^{n+1,j+1,i}+(1-\vartheta)\Delta t(L(a_j,x_i)+K(u_{n,j}(x_i)))^{\mathsf{T}}p^{n+1,j+1,i}\\
&+(1-\vartheta)\Delta t\left(\sum_{\alpha=0}^{N}\sum_{\beta=1}^{K}k^{\omega\gamma\delta}(a_j,a_\alpha,x_i,x_\beta)y_\delta^{n}(a_\alpha,x_\beta)w(a_\alpha,x_\beta)\Delta x\Delta a\right)p_\gamma^{n+1,j+1,i}\\
&+(1-\vartheta)\Delta t\sum_{\alpha=0}^{N-1}\sum_{\beta=1}^{K}k^{\omega\gamma\delta}(a_\alpha,a_j,x_\beta,x_i)y_\gamma^{n}(a_\alpha,x_\beta)p_\omega^{n+1,\alpha+1,\beta}w(a_j,x_\beta)\Delta x\Delta a
\end{aligned}
$$

$$\left. + (1-\vartheta)\Delta t \sigma(a_j, x_i) \frac{p^{n+1,j+1,i+1} - 2p^{n+1,j+1,i} + p^{n+1,j+1,i-1}}{\Delta x^2} \right] \cdot \begin{cases} 0, & j = N \text{ or } n = M \\ 1 & \text{else} \end{cases}$$

$$- \beta(a_j, x_i)^\mathsf{T} p^{n,0,i} \tilde{w}(a_j)\Delta a + \begin{cases} p^{n,0,i}, & j = 0 \\ 0, & \text{else} \end{cases}$$

$$\overset{!}{=} 0.$$

First of all, we start by calculating $p^{M,0,i}$, for which we simply get the equation

$$(1 - \beta(a_0, x_i)^\mathsf{T} w(a_0)\Delta a) p^{M,0,i} = 0,$$

from which we can deduce that $p^{M,0,i} = 0$ for all i (if Δa is sufficiently small). This can be used to calculate the other $p^{M,j,i}$s: From

$$\begin{aligned}
&\vartheta(g \cdot y^L(a_j, x_i))g \cdot \Delta x \Delta a \Delta t \\
&+ \Bigg[p^{M,j,i} + \vartheta \Delta t (L(a_j, x_i) + K(u_{M-1,j-1}(x_i)))^\mathsf{T} p^{M,j,i} \\
&+ \vartheta \Delta t \left(\sum_{\alpha=0}^{N} \sum_{\beta=1}^{K} k^{\omega\gamma\delta}(a_j, a_\alpha, x_i, x_\beta)(2y_\delta^{M-1} - y_\delta^{M-2})(a_\alpha, x_\beta) w(a_\alpha)\Delta x \Delta a \right) p_\omega^{M,j,i} \\
&+ \vartheta \Delta t \sigma(a_j, x_i) \frac{p^{M,j,i+1} - 2p^{M,j,i} + p^{M,j,i-1}}{\Delta x^2} \Bigg] \\
&- \beta(a_j, x_i)^\mathsf{T} \underbrace{p^{M,0,i}}_{=0} w(a_j)\Delta a = 0
\end{aligned}$$

we can calculate the values for each j separately. (Note that for $\Delta t \to 0$, we obtain $p^{M,j,i} = 0$, in accordance to the non-discretized equation.) For the other values of n we get after some restructuring

$$\begin{aligned}
&p^{n,j,i} + \vartheta \Delta t \Bigg[(L(a_j, x_i) + K(u_{n-1,j-1}(x_i)))^\mathsf{T} p^{n,j,i} \\
&+ \left(\sum_{\alpha=0}^{N} \sum_{\beta=1}^{K} k^{\omega\gamma\delta}(a_j, a_\alpha, x_i, x_\beta)(2y_\delta^{n-1} - y_\delta^{n-2})(a_\alpha, x_\beta) w(a_\alpha)\Delta x \Delta a \right) p_\omega^{n,j,i} \\
&+ \sigma(a_j, x_i) \frac{p^{n,j,i+1} - 2p^{n,j,i} + p^{n,j,i-1}}{\Delta x^2} \Bigg] \cdot \begin{cases} 0, & j = 0 \\ 1 & \text{else} \end{cases} \\
&- \beta(a_j, x_i)^\mathsf{T} p^{n,0,i} w(a_j)\Delta a \\
=\; &- 2\vartheta \Delta t \sum_{\alpha=0}^{N-1} \sum_{\beta=1}^{K} k^{\omega\gamma\delta}(a_{\alpha+1}, a_j, x_\beta, x_i) y_\gamma^{n+1}(a_\alpha, x_\beta) p_\omega^{n+1,\alpha+1,\beta} w(a_j)\Delta x \Delta a \\
&+ \vartheta \Delta t \sum_{\alpha=0}^{N-1} \sum_{\beta=1}^{K} k^{\omega\gamma\delta}(a_{\alpha+1}, a_j, x_\beta, x_i) y_\gamma^{n+2}(a_\alpha, x_\beta) p_\omega^{n+2,\alpha+1,\beta} w(a_j)\Delta x \Delta a \cdot \begin{cases} 0, & n = M-1 \\ 1, & \text{else} \end{cases} \\
&- \Big[- p^{n+1,j+1,i} + (1-\vartheta)\Delta t (L(a_j, x_i) + K(u_{n,j}(x_i)))^\mathsf{T} p^{n+1,j+1,i}
\end{aligned}$$

$$
\begin{aligned}
&+(1-\vartheta)\Delta t\left(\sum_{\alpha=0}^{N}\sum_{\beta=1}^{K}k^{\omega\gamma\delta}(a_j,a_\alpha,x_i,x_\beta)y_\delta^n(a_\alpha,x_\beta)w(a_\alpha)\Delta x\Delta a\right)p_\gamma^{n+1,j+1,i}\\
&+(1-\vartheta)\Delta t\sum_{\alpha=0}^{N-1}\sum_{\beta=1}^{K}k^{\omega\gamma\delta}(a_\alpha,a_j,x_\beta,x_i)y_\gamma^n(a_\alpha,x_\beta)p_\omega^{n+1,\alpha+1,\beta}w(a_j)\Delta x\Delta a\\
&+(1-\vartheta)\Delta t\sigma(a_j,x_i)\frac{p^{n+1,j+1,i+1}-2p^{n+1,j+1,i}+p^{n+1,j+1,i-1}}{\Delta x^2}\Bigg]\cdot\begin{cases}0, & j=N\\ 1 & \text{else}\end{cases}\\
&-(g\cdot y^n(a_j,x_i))g\cdot\Delta x\Delta a\Delta t\cdot\begin{cases}\vartheta, & j=N\\ 1-\vartheta, & j=0\\ 1, & \text{else}\end{cases}.
\end{aligned}
$$

Because the nonlocal terms only appear in p^{n+1}, the resulting equation is rather easy to solve numerically.

Now, as stated in Equation (2.5), we can calculate the gradient $\nabla_u\mathcal{J}$ of the reduced target functional $\mathcal{J}(u) := J(u,y(u))$ with respect to the control variables by $\frac{\partial\mathcal{L}}{\partial u}$, which in our case is given by

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial u_{n,j}(x_i)} &= \alpha u_{n,j}(x_i)\Delta x\Delta a\Delta t\\
&\quad + p^{n+1,j+1,i}\left(\vartheta\Delta t\frac{\partial K}{\partial u_{n,j}(x_i)}y^{n+1}(a_{j+1},x_i)+(1-\vartheta)\Delta t\frac{\partial K}{\partial u_{n,j}(x_i)}y^n(a_j,x_i)\right).
\end{aligned}
$$

This allows us to employ the projected gradient method for the associated reduced problem defined as

$$
\min_{u\in U^{\mathrm{ad}}}\mathcal{J}(u) := \min_{u\in U^{\mathrm{ad}}}J(u,y(u)).
$$

That is, we use the iterative update rule

$$
u^{k+1} = P_{U^{\mathrm{ad}}}(u^k-\alpha_k\nabla_u\mathcal{J}(u^k)) \qquad \text{for } k\geq 0
$$

where $P_{U^{\mathrm{ad}}}$ is the orthogonal projection into U^{ad}, which in our case of box constraints, as stated in eq. (5.6), is simply given by

$$
P_{U^{\mathrm{ad}}}(v) = \max\{0,\min\{\bar{u},v\}\}.
$$

The step size α_k can be determined using a non-monotone line search algorithm (cf. [AB23]) which uses the Barziali-Borwein step sizes corresponding to $\mathcal{J}$ as the initial trial step size, as explained in [AK21, BB88]. This works as follows: Given two control estimates u^{k-1}, u^k and corresponding gradients $g^k = \nabla_u\mathcal{J}(u^k)$, $g^{k+1} = \nabla_u\mathcal{J}(u^{k+1})$, the stepsizes alternate between the two values α_k^1 and α_k^2 given by

$$
\alpha_k^1 = \frac{\left\langle u^k-u^{k-1},g^k-g^{k-1}\right\rangle}{\left\langle g^k-g^{k-1},g^k-g^{k-1}\right\rangle},\quad \alpha_k^2 = \frac{\left\langle u^k-u^{k-1},u^k-u^{k-1}\right\rangle}{\left\langle u^k-u^{k-1},g^k-g^{k-1}\right\rangle}.
$$

Parameter	Description	Value
T	Maximal time	5
$a_{\max}$	Maximum age	1
α	Control cost parameter	500
c	Loss of vaccine immunity	0.18564
μ	Natural death rate	$e^{-a} \cdot a^5$
φ_1	Vaccine protection from infection	0.0052
φ_2	Recovery protection from infection	0.00062
δ	Infection death rate	0.0018
γ	Recovery rate	0.278574
λ	Infection rate kernel	$(0.1 - \lvert x - \xi \rvert)^+$
β	Birth rate	$\frac{6.78}{a_{\max}} a^2 (a_{\max} - a)\left(1 + \sin\left(\pi \frac{a}{a_{\max}}\right)\right)$
σ_S	Susceptible diffusion coefficient	$0.1e^{-0.1a}$
σ_V	Vaccinated diffusion coefficient	$0.1e^{-0.1a}$
σ_I	Infective diffusion coefficient	$0.05e^{-0.1a}$
σ_R	Recovered diffusion coefficient	$0.1e^{-0.1a}$

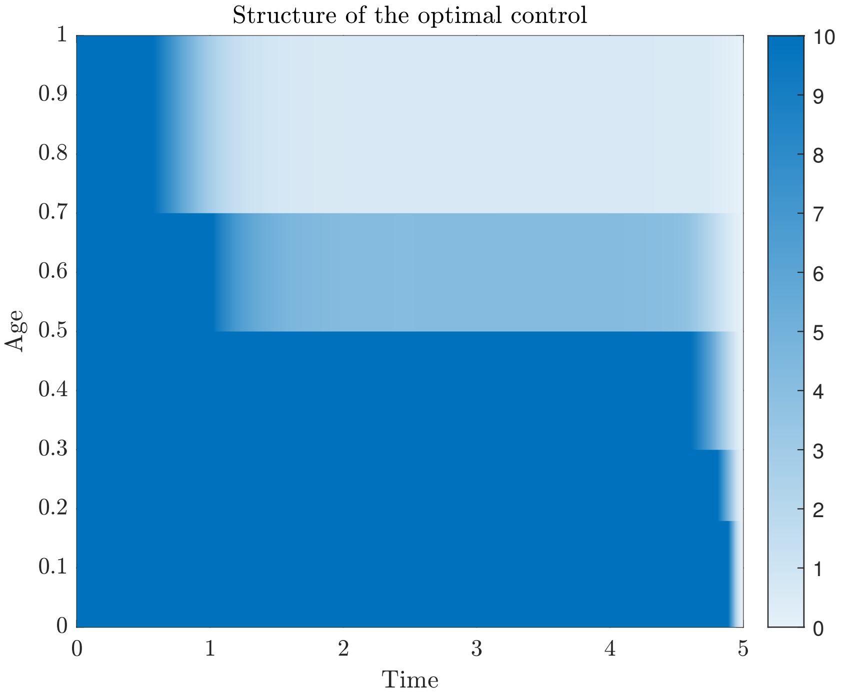
Table 5.1: Parameter Setting

The optimization algorithm was terminated when the norm of the difference between two successive iterations, divided by the norm of the previous iteration, was less than 10^{-8}.

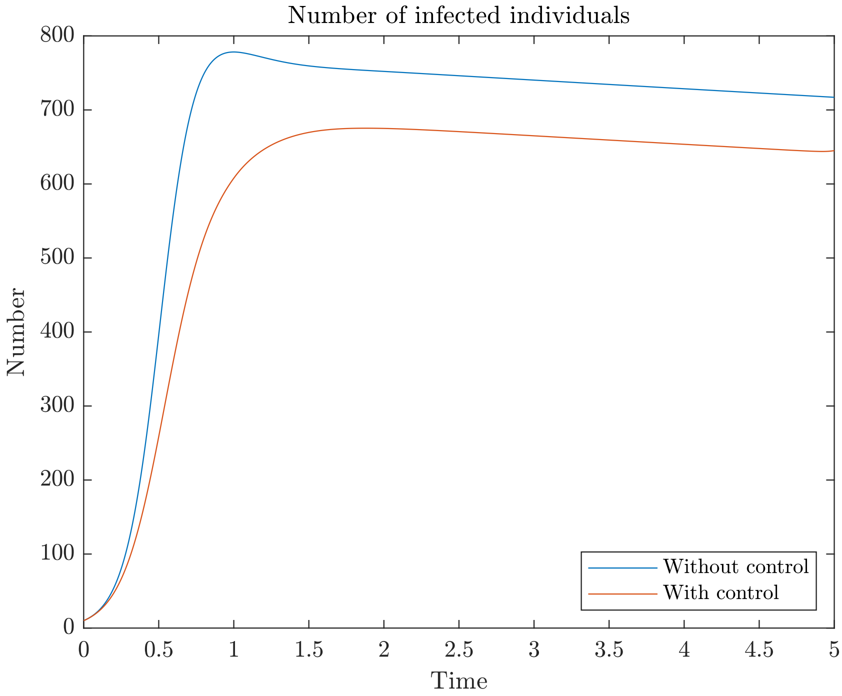
Throughout the numerical simulation, we used the parameters listed in Table 5.1. Most of these parameters are taken from [ARS25] and adjusted for our purposes. The birth and death rates are adopted from [AAC11, p. 155]. As initial conditions, we assume a population uniformly distributed across age and space, consisting of 1000 susceptible and 10 infectious individuals. The control is assumed to act only in the central region of the domain, $\Omega_1 = (0.45, 0.55)$, and uniformly across the age intervals defined by $a_0 = 0$, $a_1 = 0.18$, $a_2 = 0.3$, $a_3 = 0.5$, $a_4 = 0.7$, and $a_5 = 1$. Consequently, we set $N = 5$ and $M = 1$.

5.8 Example. In this example, we set $\Delta x = 0.01$, $\Delta t = 0.005$, and $\bar{u} = 10$. The structure of the optimal control for different age classes is illustrated in Figure 5a, where each strip corresponds to an age class. As shown, it is preferable to administer the vaccine during the early stages of the epidemic, and in later stages, prioritize younger age groups over older ones.

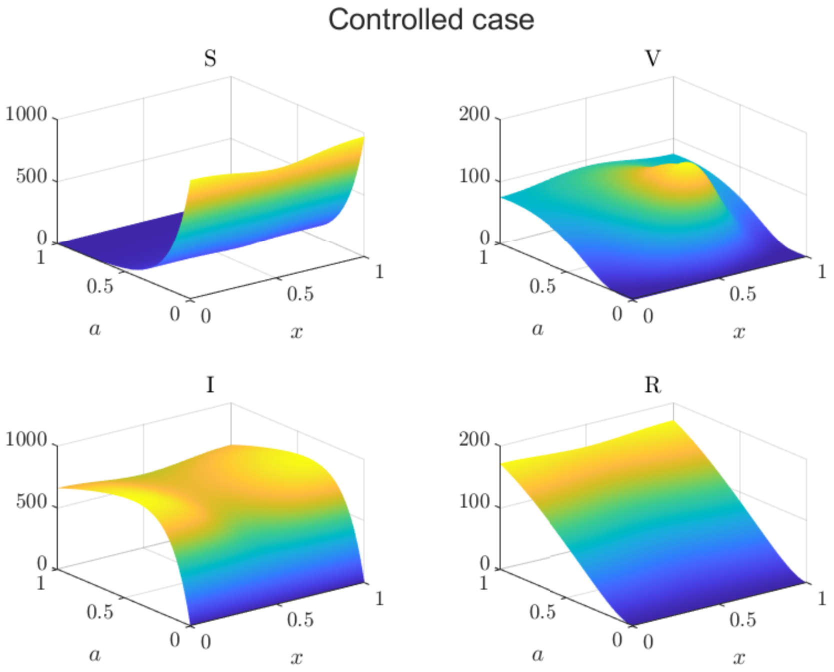
The evolution of the total number of infectious individuals for both the controlled and uncontrolled ($u = 0$) cases is depicted in Figure 5b. It can be observed that control measures or vaccination effectively reduce the number of infectious individuals over time, as desired. Figure 5c and Figure 5d present snapshots of the four compartments at the final time for the controlled and uncontrolled cases, respectively. Comparing these figures, a visible dent appears in the optimal state I. Additionally, the graph of the number of infected individuals shows a bump in the middle, reflecting the fact that the optimal control strategy favors vaccinating younger individuals.



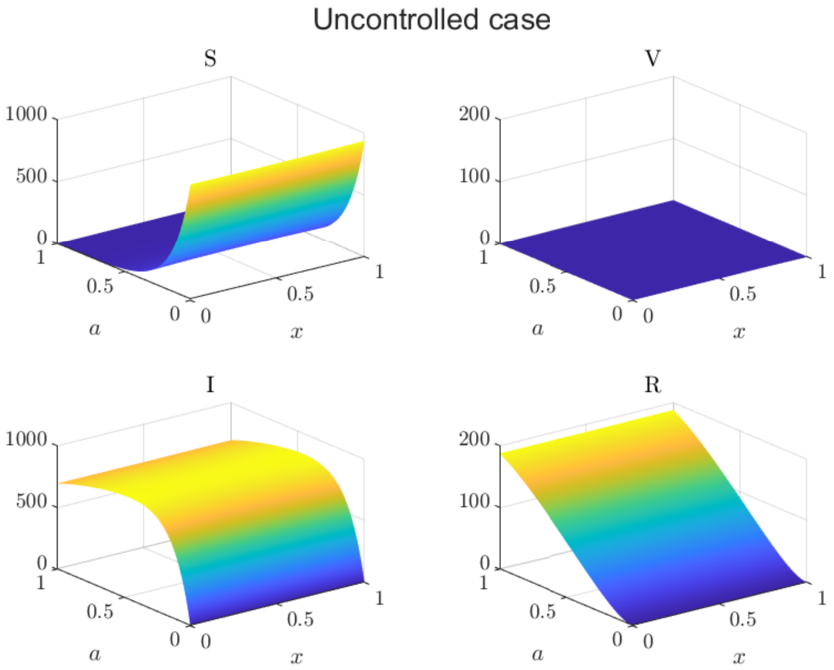
(a) Optimal control structure, with darker blue indicating higher values of u.



(b) Total number of infectious individuals: blue — controlled, red — uncontrolled.



(c) Controlled state at the final time



(d) Uncontrolled state at the final time

Figure 5: Numerical results of Example 5.8

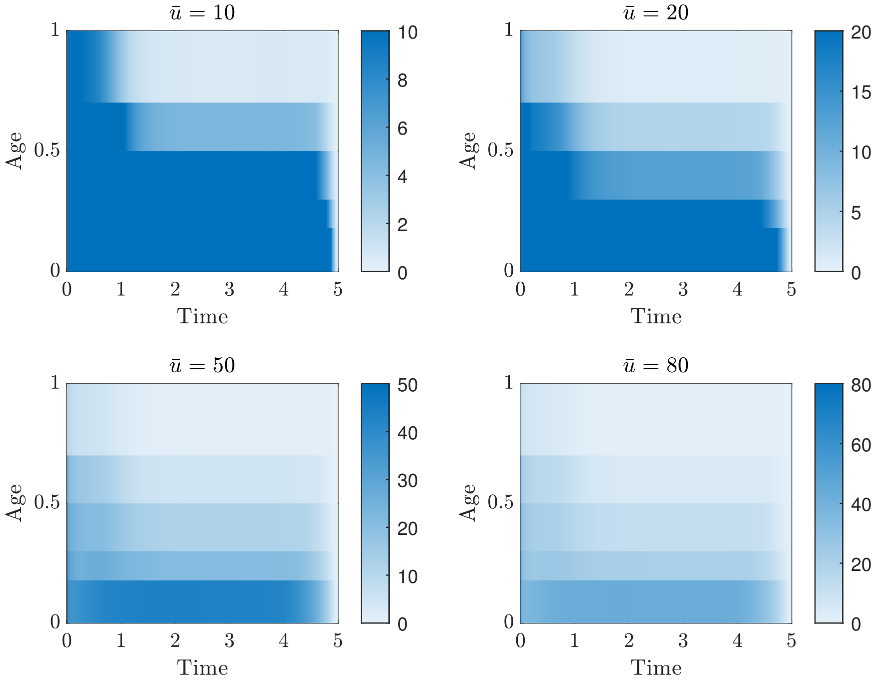
5.9 Example. It is insightful to vary $\bar{u}$ and observe how it impacts the optimal cost and state, as a higher control bound is more effective but also more expensive. In this example, we explore this trade-off. To speed up computations, we used a coarser grid with $\Delta x = 0.1$ and $\Delta t = 0.01$, and computed the optimal control for $\bar{u} \in \{10, 20, 50, 80\}$. The corresponding results are presented in Figure 6 and Table 5.2.

Comparing the top-left plot in Figure 6a with Figure 5a, we observe that the results are consistent, and different mesh sizes do not significantly influence the outcomes.

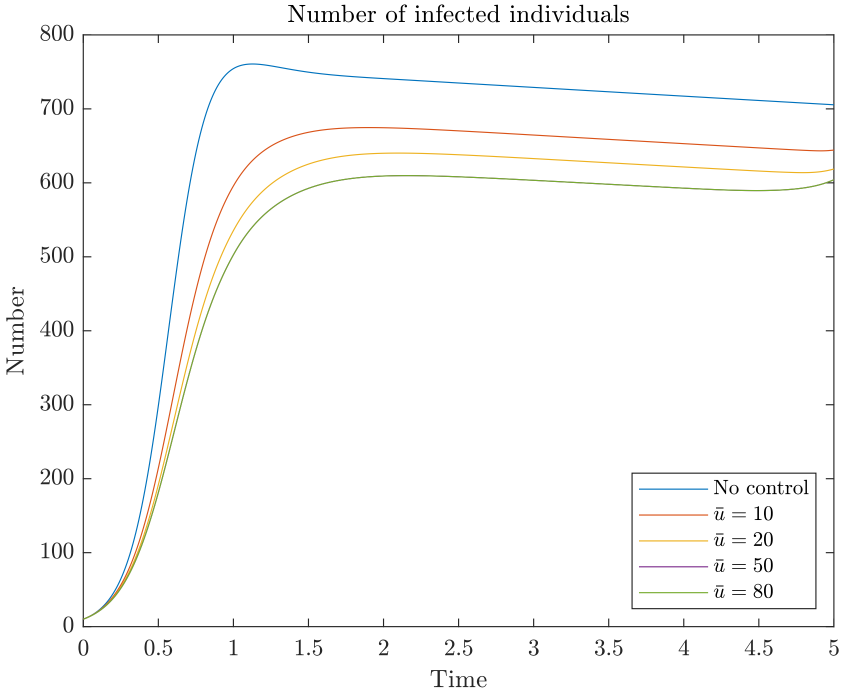
From Table 5.2, we see that as $\bar{u}$ increases, the optimal cost functional decreases, indicating that maintaining a high vaccination rate can be beneficial despite the associated costs. However, for $\bar{u} = 80$, the control reaches a maximum value of approximately 62.2, suggesting that beyond a certain point, increasing $\bar{u}$ provides no additional benefit. Most control values lie below 50, which explains why the optimal controls for $\bar{u} = 80$ and $\bar{u} = 50$ are nearly identical. In fact, the curves representing the total number of infected individuals for these two cases in Figure 6b completely overlap.

$\bar{u}$	$J(\bar{u}, y(\bar{u}))$
0	1111450
10	896491
20	822924
50	787393
80, ∞	787382

Table 5.2: Optimal value of the cost functional for different values of $\bar{u}$ from Example 5.9



(a) Optimal control structure for different values of $\bar{u}$



(b) Total number of infectious individuals for different values of $\bar{u}$

Figure 6: Comparison of different values for $\bar{u}$ of Example 5.9

Deutsche Zusammenfassung

Diese Arbeit beschäftigt sich auf zweierlei Weise mit Epidemiemodellen mit Alters- und Ortsvariablen, die als partielle Differentialgleichungen mit nichtlokalen Termen und impliziten Randbedingungen geschrieben werden können. Klassischerweise können diese als eine Art Reaktions-Diffusionsgleichung geschrieben werden. Anstelle der gewöhnlichen Zeitableitung steht hier jedoch die Summe aus Zeit- und Altersableitung, was die Behandlung deutlich erschwert.

Nach einigen vorbereitenden Bemerkungen gehen wir kurz auf die Frage nach der Existenz einer Lösung zu diesen Gleichungen ein. Danach modifizieren wir das Modell auf zwei unterschiedliche Arten und Weisen: Zum einen fügen wir einen sog. Relaxationsparameter ein, der die Gleichung in eine Art gedämpfte Wellengleichung transformiert. Nach einigen Bemerkungen zur Herleitung des neuen Modells, inklusive der Suche nach einer zweiten, notwendigen impliziten Randbedingung, folgt wie im vorigen Kapitel ein Existenzbeweis einer Lösung des neuen Modells. Ebenfalls vergleichen wir altes und neues Modell und geben ein quantitatives Konvergenzresultat an.

Im zweiten Teil fügen wir dem klassischen Modell einen Kontrollterm, der z. B. eine Impfung darstellt, hinzu und beschäftigen uns mit der Frage nach der optimalen Steuerung des Modells. Konkret versuchen wir dabei, die Anzahl der infizierten Individuen, gekoppelt mit den Kosten der Impfung, zu minimieren. Es stellt sich heraus, dass durch die Altersstruktur der Existenzbeweis einer optimalen Steuerung durch das Fehlen von geeigneten Kompaktheitssätzen deutlich erschwert wird, weswegen wir Kontrollen mit höherer Glattheit suchen müssen. Nachdem diese gefunden wurden, leiten wir Optimalitätsbedingungen her und führen numerische Versuche durch.

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1. Fachbereich Mathematik und Statistik

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