Concentration limit for non-local dissipative convection-diffusion kernels on the hyperbolic space

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Structure of the talk

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2. The Euclidean case
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3. The hyperbolic case
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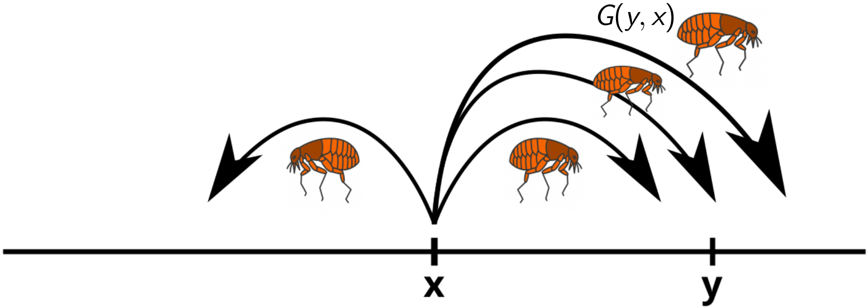
   - Non-local convection diffusion
   - Dissipative kernels on the hyperbolic space
   - Concentration limit on $\mathbb{H}^N$
   - Compactness result on Riemannian manifolds

Intuition – fleas jumping on a mattress I

Intuition – fleas jumping on a mattress II



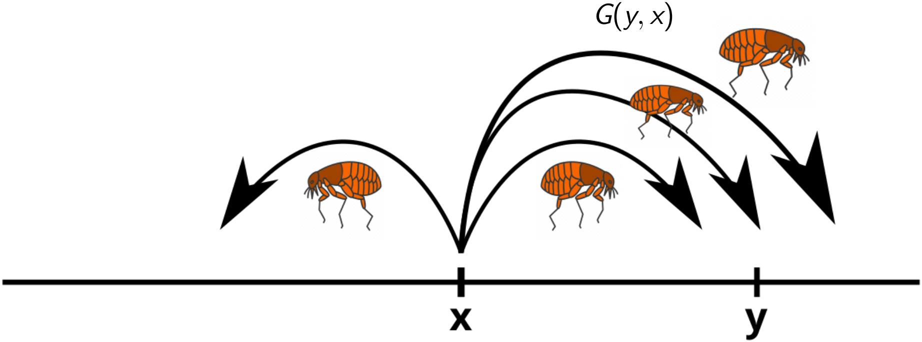
$G(y, x)$ accounts for the probability that the fleas will jump to y from x.

If $G(y, x) = J(d(x, y)) \Rightarrow$ equal movement in every direction

If, for example, $G(y, x) > G(x, y)$ for $x < y$, we obtain a drift to the right.

How many fleas run away from x?

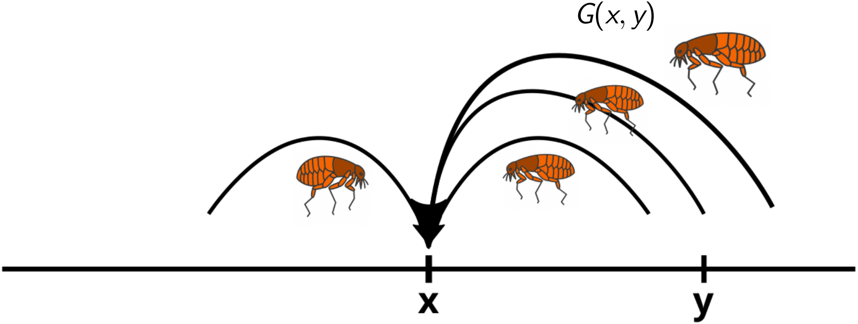
$u(x) =$ number of fleas at point x.



$$\int_{\mathbb{R}} G(y,x)u(x)\mathrm{d}y \quad \text{fleas running away from } x$$

How many fleas are coming to x?

$u(y) =$ number of fleas at point y.



$$\int_{\mathbb{R}} G(x,y)u(y)\mathrm{d}y \quad \text{fleas comming in } x$$

The evolution equation

$$\begin{cases} \partial_t u(t,x) = \int_{\mathbb{R}} G(x,y)u(t,y)\mathrm{d}y - \int_{\mathbb{R}} G(y,x)u(t,x)\mathrm{d}y, & x \in \mathbb{R}, t \geq 0 \\ u(0,x) = u_0(x), & x \in \mathbb{R} \end{cases} \tag{NLTR}$$

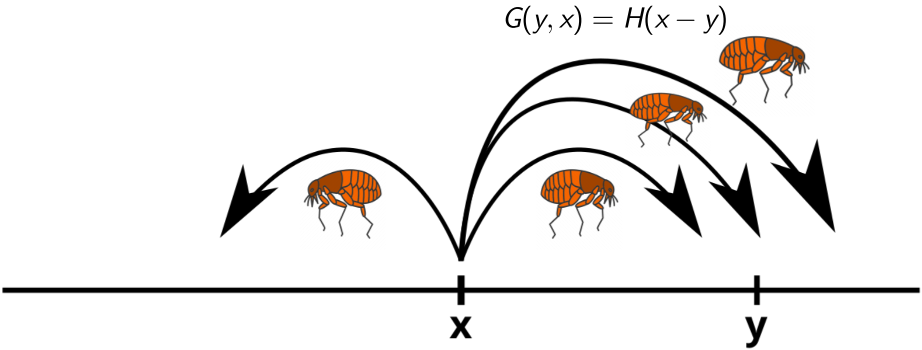
If we further have that G is a dissipative kernel, i.e.,

$$\int [G(x,y) - G(y,x)]\mathrm{d}y = 0,\ \forall x \in \mathbb{R},$$

the equation is more compact:

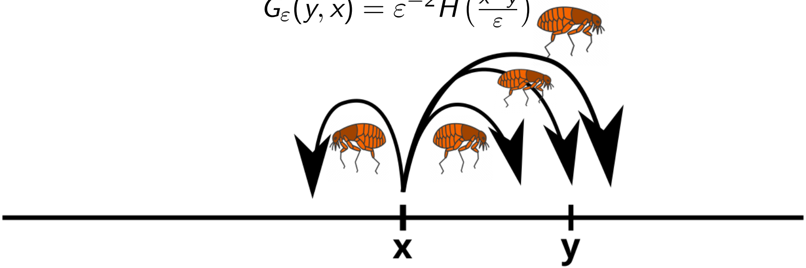
$$\begin{cases} \partial_t u(t,x) = \int_{\mathbb{R}} G(x,y)[u(t,y) - u(t,x)]\mathrm{d}y, & x \in \mathbb{R}, t \geq 0 \\ u(0,x) = u_0(x), & x \in \mathbb{R} \end{cases} \tag{NLTR2}$$

Concentrating the jumps I

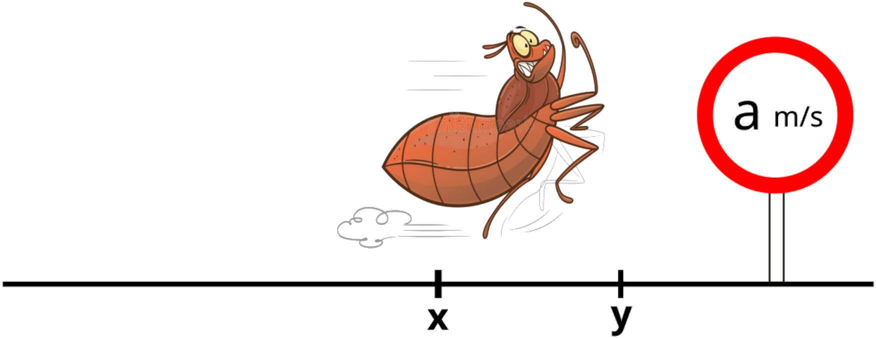


Concentrating the jumps II

$$G_\varepsilon(y,x) = \varepsilon^{-2} H\left(\frac{x-y}{\varepsilon}\right)$$



The transport equation



In the limit: drift effect = transport equation:

$$\partial_t u(t,x) = -a\,\partial_x u(t,x)$$

$$a = -\int_{\mathbb{R}} H(x)\,x\,\mathrm{d}x$$

Non-local convection-diffusion on $\mathbb{R}^N$

[Ignat & Rossi, 2007] – convection-diffusion equation on $\mathbb{R}^N$

$$\begin{cases} \partial_t u(t,x) = \displaystyle\int_{\mathbb{R}^N} J(y-x)(u(t,y)-u(t,x))\mathrm{d}y \\ \qquad\qquad + \displaystyle\int_{\mathbb{R}^N} \widetilde{G}(y-x)\left[f(u(t,y)) - f(u(t,x))\right]\mathrm{d}y, \quad x \in \mathbb{R}^N, t \geq 0; \\ u(0,x) = u_0(x), \qquad\qquad x \in \mathbb{R}^N. \end{cases} \tag{1}$$

$f(r) = |r|^{q-1}r, q \geq 1$

J is a radial function, $\widetilde{G}$ is not radial.

Concentrating the kernels on $\mathbb{R}^N$

$$
\begin{cases}
\partial_t u^\varepsilon(t,x) = \displaystyle\int_{\mathbb{R}^N} \varepsilon^{-N-2} J\left(\frac{y-x}{\varepsilon}\right)(u(t,y)-u(t,x))\mathrm{d}y \\
\qquad + \displaystyle\int_{\mathbb{R}^N} \varepsilon^{-N-1} \widetilde{G}\left(\frac{y-x}{\varepsilon}\right)\left[f(u(t,y)) - f(u(t,x))\right]\mathrm{d}y, \quad x \in \mathbb{R}^N, t \geq 0; \\
u^\varepsilon(0,x) = u_0(x), \qquad x \in \mathbb{R}^N.
\end{cases}
\tag{2}
$$

Convergence to a local problem on $\mathbb{R}^N$

$$\begin{cases} \partial_t u(t,x) = A_J \Delta u(t,x) + \nabla_x f(u)(t,x) \cdot X_G & , x \in \mathbb{R}^N, t \geq 0 \\ u(0,x) = u_0(x) & , x \in \mathbb{R}^N \end{cases} \tag{3}$$

$$A_J = \frac{1}{2N} \int_{\mathbb{R}^N} J(|x|)|x|^2 dx$$

$$X_G = -\int_{\mathbb{R}^N} \widetilde{G}(x)\, x\, dx \in \mathbb{R}^N \text{ constant vector.}$$

Essential property for the convergence:

$$\int_{\mathbb{R}^N} [\widetilde{G}(y-x) - \widetilde{G}(x-y)] dx = 0, \forall y \in \mathbb{R}^N$$

i.e. $G(x,y) = \widetilde{G}(y-x)$ is a *dissipative* kernel,

the L^2 norm of the solution decreases in time.

The Hyperbolic space I

The half-space model

$\mathbb{R}^N_+ = \left\{x = (x', x_N) \in \mathbb{R}^N : x_N > 0\right\}$

$$g_{i,j}(x) = \frac{1}{x_N^2}\delta_{i,j}.$$

$$\nabla f = x_N^2 \nabla_e f, \quad \operatorname{div}(Y) = x_N^N \operatorname{div}_e\left(\frac{1}{x_N^N} Y\right), \quad \Delta f = x_N^N \operatorname{div}_e\left(\frac{1}{x_N^{N-2}}\nabla_e f\right).$$

$$\int_{\mathbb{H}^N} f(x)\mathrm{d}\mu(x) = \int_{\mathbb{R}^N_+} f(x)\frac{1}{x_N^N}\mathrm{d}x$$

Geodesics:

- vertical lines
- half-circles centred and perpendicular on the ground

The Hyperbolic space II

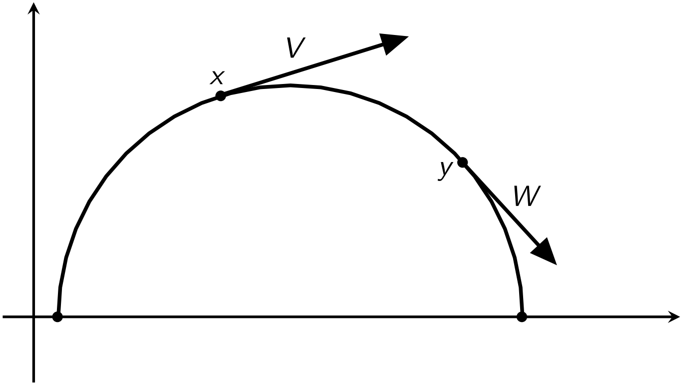


Figure: A geodesic through x in the half-space model, tangent to V

Non-local convection-diffusion problem on hyperbolic space

$$\begin{cases} \partial_t u(t,x) = \displaystyle\int_{\mathbb{H}^N} J(d(x,y))(u(t,y)-u(t,x))\mathrm{d}\mu(y) \\ \qquad\qquad + \displaystyle\int_{\mathbb{H}^N} G(x,y)\left[f(u(t,y))-f(u(t,x))\right]\mathrm{d}\mu(y), \quad x\in\mathbb{H}^N, t\geq 0; \\ u(0,x) = u_0(x), \qquad\qquad x\in\mathbb{H}^N. \end{cases} \tag{4}$$

Both $J \rightsquigarrow J_\varepsilon$ and $G \rightsquigarrow G_\varepsilon$ are concentrated as $\varepsilon \to 0$ to obtain:

$$\begin{cases} u_t(t,x) = A_J \Delta u(t,x) - \mathrm{div}(f(u(t))\, X_G)(x), \quad x\in\mathbb{H}^N, t\geq 0; \\ u(0,x) = u_0(x), \qquad\qquad x\in\mathbb{H}^N. \end{cases} \tag{5}$$

where $A_J > 0$ and X_G is a bounded C^1 vector field.

Construction of a dissipative kernel on $\mathbb{H}^N$ I

Aim:

$$\int_{\mathbb{H}^N} [G(x,y) - G(y,x)] \mathrm{d}\mu(x) = 0, \forall y \in \mathbb{H}^N$$

Definition (Geodesic flow)

$(x, V) \in T\mathbb{H}^N$

$\gamma_{x,V}$ be the unique geodesic s.t. $\gamma(0) = x$, $\gamma'(0) = V$

$y = \gamma_{x,V}(t)$, $W = \gamma'_{x,V}(t)$

$$\Phi_t(x, V) = (y, W)$$

If $t = 1$, $|V| = d(x, y)$. In this case, $V_{x,y} := V$ accounts for $(y - x)$.

$$exp_x(V_{x,y}) = y$$

Construction of a dissipative kernel on $\mathbb{H}^N$ II

Remark

$$\Phi_1(x, V_{x,y}) = (y, -V_{y,x})$$

This relation accounts for

$$(y - x) = -(x - y)$$

We define

$$G(x, y) = \widetilde{G}(x, V_{x,y})$$

$\widetilde{G} : T\mathbb{H}^N \to [0, \infty)$

$$\widetilde{G}(\Phi_t(x, V)) = \widetilde{G}(x, V), \forall t \in \mathbb{R}.$$

G is a dissipative kernel

$$\widetilde{G}(x, V_{x,y}) = \widetilde{G}(y, -V_{y,x})$$

Proposition

$$\int_{\mathbb{H}^N} [G(x,y) - G(y,x)] \mathrm{d}\mu(x) = 0, \forall y \in \mathbb{H}^N$$

Proof idea.

$$\int_{\mathbb{H}^N} [\widetilde{G}(y, -V_{y,x}) - G(y, V_{y,x})] \mathrm{d}\mu(x) = 0$$

equivalent to

$$\int_{T_y\mathbb{H}^N} [\widetilde{G}(y, -V) - G(y, V)] \left| J_{\exp_y}(V) \right| dV = 0$$

□

Concentrating the kernel G

$$G_\varepsilon(x,y) = \varepsilon^{-N-1}\, \widetilde{G}\left(x, \frac{1}{\varepsilon} V_{x,y}\right)$$

Still invariant to the geodesic flow.

Particular case: G compactly supported around the diagonal of $\mathbb{H}^N \times \mathbb{H}^N$:

$$d(x,y) > M \Rightarrow G(x,y) = 0$$

$$\widetilde{G}(x,V) = 0, \text{ if } |V| > M$$

$$d(x,y) > \varepsilon M \Rightarrow G_\varepsilon(x,y) = 0$$

Non-local non-linear convection-diffusion on $\mathbb{H}^N$

$u_0 \in L^1(\mathbb{H}^N) \cap L^\infty(\mathbb{H}^N)$, $\quad f(r) = |r|^{q-1}r$, $q \geq 1$.

$$\begin{cases} \partial_t u^\varepsilon(t,x) = \varepsilon^{-N-2} \displaystyle\int_{\mathbb{H}^N} J\left(\frac{d(x,y)}{\varepsilon}\right)(u^\varepsilon(t,y) - u^\varepsilon(t,x))\mathrm{d}\mu(y) \\ \qquad\qquad + \displaystyle\int_{\mathbb{H}^N} G_\varepsilon(x,y)\left[f(u^\varepsilon(t,y)) - f(u^\varepsilon(t,x))\right]\mathrm{d}\mu(y), \qquad t \geq 0, x \in \mathbb{H}^N; \\ u^\varepsilon(0,x) = u_0(x), \qquad\qquad x \in \mathbb{H}^N. \end{cases} \tag{NLCD}$$

Local non-linear convection-diffusion on $\mathbb{H}^N$

$$\begin{cases} u_t(t,x) = A_J \Delta u(t,x) - \mathrm{div}(f(u(t))\, X_G)(x), & x \in \mathbb{H}^N, t \geq 0; \\ u(0,x) = u_0(x), & x \in \mathbb{H}^N. \end{cases} \tag{LCD}$$

$$A_J = \frac{1}{2N} \int_{\mathbb{R}^N} J(|x|)|x|^2 dx \qquad X_G(x) = -\int_{T_x\mathbb{H}^N} \widetilde{G}(x,W)\, W \,\mathrm{d}\mu(W)$$

Concentration limit for convection-diffusion on $\mathbb{H}^N$

Theorem

Under some integrability constraints on J and G, the sequence $(u^\varepsilon)_{\varepsilon>0}$ of solutions of (NLCD) converges weakly in $L^2([0,T]\times\mathbb{H}^N)$ for every $T>0$ to the unique distributional solution of (LCD) with the same initial data u_0.

Compactness result on Riemannian manifolds I

Inspired from [Ignat & Ignat & Stancu-Dumitru, 2015] – Euclidean case.

M is a N-dimensional, complete, connected Riemannian Manifold.
$J : [0, \infty) \to [0, \infty)$ bounded away from zero in a neighbourhood of 0.
$(u^\varepsilon)_{\varepsilon>0}$ a bounded sequence in $L^2([0, T] \times M)$ satisfying:

$$\varepsilon^{-N-2} \int_0^T \int_M \int_M J\left(\frac{d(x,y)}{\varepsilon}\right) |u^\varepsilon(t,y) - u^\varepsilon(t,x)|^2 \mathrm{d}\mu_g(x) \mathrm{d}\mu_g(y) dt \leq K < \infty$$

Compactness result on Riemannian manifolds II

Then

1. If $u^\varepsilon \rightharpoonup u$ in $L^2([0,T], L^2(M))$, then:

$$u \in L^2([0,T], H^1(M))$$

$$\int_0^T \|\nabla u(t)\|^2_{L^2(M)} dt \leq CK.$$

2. If $D \subseteq M$ open, bounded and

$$\|\partial_t u^\varepsilon\|_{L^2([0,T],H^{-1}(D))} \text{ uniformly bounded in } \varepsilon > 0$$

then $(u^\varepsilon)_{\varepsilon>0}$ converges strongly in $L^2([0,T] \times D)$ on a subsequence.

Further directions of research

1. Study the convergence of $u^\varepsilon \to u$ in other L^p or Sobolev norms.
2. Study the long-time asymptotic behaviour of the difference

$$\|U(t) - u(t)\|_{L^p(\mathbb{H}^N)}$$

U is the solution of the non-local convection-diffusion equation with initial data u_0 and some fixed J and G.

u is the solution of the local convection-diffusion equation with the same initial data and the corresponding A_J and X_G.

References

[González & Ignat & Manea & Moroianu, 2023]
González, M. d. M., Ignat, L. I., Manea, D., and Moroianu, S. (2023).
Concentration limit for non-local dissipative convection-diffusion kernels on the hyperbolic space.
preprint, arXiv:2302.02624.

[Ignat & Ignat & Stancu-Dumitru, 2015]
Ignat, L. I., Ignat, T. I., and Stancu-Dumitru, D. (2015).
A compactness tool for the analysis of nonlocal evolution equations.
SIAM Journal on Mathematical Analysis, 47(2):1330–1354.

[Ignat & Rossi, 2007]
Ignat, L. I. and Rossi, J. D. (2007).
A nonlocal convection–diffusion equation.
Journal of Functional Analysis, 251(2):399–437.