On the constrained exact boundary controllability of a quasilinear model for pipeline gas flow

Xu HUANG
Chair for Dynamics, Control, Machine Learning and Numerics
Department of Mathematics Friedrich-Alexander-Universität
Erlangen-Nürnberg (FAU)

Friedrich-Alexander-Universität
DYNAMICS, CONTROL,
MACHINE LEARNING
AND NUMERICS

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Gas Pipeline Network

Figure: Gas Pipeline Network

Outline

The isothermal Euler equations

$$\begin{cases} \rho_t + q_x = 0, \\ q_t + (p + \frac{q^2}{\rho})_x = -\frac{1}{2}\theta \frac{q|q|}{\rho} - \rho g s_{lope}. \end{cases} \tag{1}$$

$s_{lope} := \sin(\varphi)$, $\theta := \frac{\lambda_{fric}}{D}$, g the gravitational constant

$\rho > 0$ the gas density

$p > 0$ the pressure

q the cross-sectional mass flow rate.

$$p = Z(p) R_s^e T^e \rho, \tag{2}$$

where

$$Z(p) = 1 + \alpha p. \tag{3}$$

$$\begin{pmatrix} R_+ \\ R_- \end{pmatrix}_t + diag(\lambda_+, \lambda_-) \begin{pmatrix} R_+ \\ R_- \end{pmatrix}_x = \widetilde{F}(R_+, R_-), \qquad (4)$$

with

$$\widetilde{F}(R_+, R_-) = \left[\frac{1}{8}\theta\sqrt{R_s^e T^e}|R_+ - R_-|(R_+ - R_-) + \frac{1}{\sqrt{R_s^e T^e}} g s_{lope}\right] \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

R_s^e is the gas constant and T^e is the temperature.

$$\begin{cases} \lambda_- = \sqrt{R_s^e T^e}\left[\frac{R_+ - R_-}{2} - 1 - \alpha \exp\left(\frac{R_+ + R_-}{2}\right)\right], \\ \lambda_+ = \sqrt{R_s^e T^e}\left[\frac{R_+ - R_-}{2} + 1 + \alpha \exp\left(\frac{R_+ + R_-}{2}\right)\right]. \end{cases} \qquad (5)$$

2021, Gugat, M. etc.[1]

$$\begin{pmatrix} R_+ \\ R_- \end{pmatrix}_t + diag(c, -c) \begin{pmatrix} R_+ \\ R_- \end{pmatrix}_x = \widetilde{F}(R_+, R_-), \qquad (6)$$

with

$$\widetilde{F}(R_+, R_-) = \left[-\frac{1}{4}\theta c \left| \frac{R_+ + R_-}{R_+ - R_-} \right| (R_+ + R_-) - \frac{1}{c} g s_{lope} \frac{R_+ - R_-}{2} \right] \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This paper studies the existence of continuous solutions of the semilinear model subject to

- bounds on the pressure,
- bounds on the Mach number,

and also the **constrained exact boundary controllability** of the system with the same constraints on the pressure and Mach number.

2018, Gugat, M.& Ulbrich, S [2]

This paper investigates the existence of semi-global Lipschitz continuous solutions of the initial boundary value problem of the quasilinear Euler equation on **networks** but with **horizontal pipes**. Its basic method is the fixed point iteration. It studies the exact controllability **without constraint**.

Preliminaries

Evaluation on the eigenvalues

Assumptions:

$$(R_+(t,x), R_-(t,x)) \in M_1(u_{max}),$$

$$M_1(u_{max}) := \{(R_+, R_-) : |R_+| \leq u_{max}, |R_-| \leq u_{max}\}$$

Result:

$$\max\left\{|\lambda_+(R_+, R_-)|, |\lambda_-(R_+, R_-)|\right\} \leq \Lambda(u_{max}),$$

$$\lambda_+(R_+, R_-) \geq \underline{\Lambda}(u_{max}),\ \ \lambda_-(R_+, R_-) \leq -\underline{\Lambda}(u_{max}),$$

Assumptions:

- $(R_+(t,x), R_-(t,x)) \in M_1(u_{max})$.
- $|R_+(t,x) - R_-(t,x)|$ are bounded by 2κ.
- $R_\pm(t,x)$ are Lipschitz continuous with respect to x with the Lipschitz constant K_M.

Result:

$$|\sigma(R_+, R_-) - \sigma(S_+, S_-)| \leq K_\sigma(\kappa)(|R_+ - R_-| + |S_+ - S_-|),$$

$$|\sigma(R_+, R_-)| \leq \sigma_{max}(\kappa).$$

Assumptions:

- $(R_+, R_-) \in C([0, T] \times [0, L])^2$ is Lipschitz continuous with respect to x with the Lipschitz-constant L_r.
- $(R_+, R_-) \in M_1(u_{max})$.

Characteristic Curves:

$$\xi_{\pm}^{R}(s,x,t) = x + \int_{t}^{s} \lambda_{\pm}(R_{+},R_{-})(\tau,\xi_{\pm}^{R}(\tau,x,t))d\tau. \tag{7}$$

Important time variable $t_{\pm}^{R}(x,t)$:(Example:$t_{+}^{R}(x,t)$)

- If $\xi_{+}^{R}(0,x,t) \in [0,L]$, $t_{+}^{R}(x,t) = 0$.
- If $\xi_{+}^{R}(0,x,t) < 0$, let $t_{+}^{R}(x,t) \in [0,T]$ be defined as the uniquely determined time with

$$\xi_{+}^{R}(t_{+}^{R}(x,t),x,t) = 0.$$

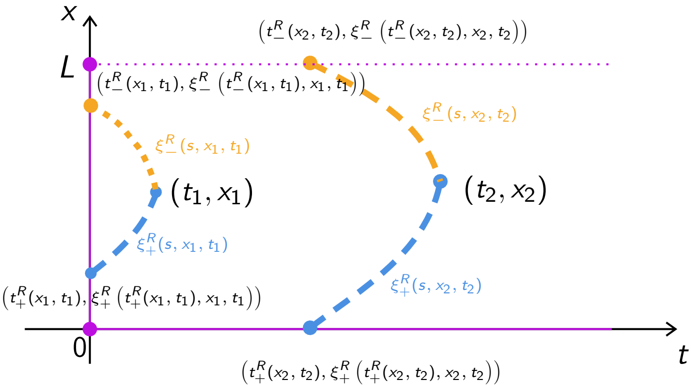


Figure 1: $\xi_\pm^R(s,x,t)$ and $t_\pm^R(x,t)$

Approach: Picard-Lindelöf Theorem

Result:

- $\xi_{\pm}^{R}(s, x, t)$
  - With regard to s: L_s
  - With regard to x: L_x
  - With regard to $R = (R_+, R_-)$: H_ξ
- $t_{\pm}^{R}(t, x)$
  - With regard to x: K_B
  - With regard to $R = (R_+, R_-)$: H_B

Constants

$$
\begin{cases}
L_s = \Lambda(u_{max}), \\
L_x = \exp\left(2K_M T\Lambda(u_{max})\right), \\
K_B = \frac{1}{\underline{\Lambda}(u_{max})} \exp\left(2K_M T\Lambda(u_{max})\right), \\
H_\xi = T\Lambda(u_{max}) \exp(K_M T\Lambda(u_{max})), \\
H_B = \frac{1}{\underline{\Lambda}(u_{max})} T\Lambda(u_{max}) \exp(K_M T\Lambda(u_{max})), \\
\Lambda_1 = \Lambda(u_{max}), \\
\Lambda_2 = \frac{1}{\underline{\Lambda}(u_{max})}.
\end{cases}
\tag{8}
$$

Theorem

Consider a pipe with physical parameters $L > 0$, $\theta \geq 0$ and s_{lope}, α, g, R_s^e, T^e. Define the number

$$u_{max} \in \big(0, \min\{\frac{1}{2}, -\ln(2|\alpha|)\}\big).$$

Let $T > 0$ be given, define the sets

$$\Gamma_+ = \{0\} \times [0, L] \cup [0, T] \times \{0\},$$

$$\Gamma_- = \{0\} \times [0, L] \cup [0, T] \times \{L\}.$$

With the given values of R_+ on Γ_+ and R_- on Γ_-that prescribe initial conditions at $t = 0$ and boundary conditions at $x = 0$ and $x = L$, and assume that the C^0- compatibility conditions are satisfied for $R_+(t,x)$ on Γ_+ and for $R_-(t,x)$ on Γ_-. Let Lipschitz continuous states R_+ on Γ_+ and R_- on Γ_- be given. Let K_R denote a common Lipschitz constant for R_+ on Γ_+ and R_- on Γ_-, which means that for $(t_1,x_1),(t_2,x_2) \in \Gamma_\pm$, we have

$$|R_\pm(t_1,x_1) - R_\pm(t_2,x_2)| \leq K_R\big(|t_2 - t_1| + |x_2 - x_1|\big). \tag{9}$$

Define the numbers:

$$B_\pm = \sup_{(t,x)\in\Gamma_\pm} \{|R_\pm(t,x)|\}, B = \max\{B_+, B_-\}.$$

Assume that there exists a number $\kappa > 0$ satisfies:

$$B + \frac{1}{2} T\theta\sqrt{R_s^e T^e}\kappa^2 + \frac{1}{\sqrt{R_s^e T^e}} g|s_{lope}|T \leq \min\{u_{max}, \kappa\}, \quad (10)$$

and that $K_M \geq K_R$ satisfied these two following inequalities:

$$L_x\Big(K_R\big(1 + \Lambda_1\Lambda_2 + \Lambda_2\big) + K_\sigma(\kappa)\big(2K_M T + \Lambda_2\big)\Big) \leq K_M, \quad (11)$$

$$\begin{aligned} &L_x K_R T\Big(\Lambda_1 + \Lambda_1\Lambda_2 + \Lambda_1^2\Lambda_2\Big) \\ &+K_\sigma(\kappa) T\Big(2 + L_x\Lambda_1\Lambda_2 + 2L_x K_M T\Lambda_1\Big) < 1 \end{aligned} \quad (12)$$

with the notation before.

Then the system (4) has a unique solution on $[0, T]$ that solves the initial boundary value problem in the sense of characteristics. Moreover we have $(R_+(t,x), R_-(t,x)) \in M_2(K_M)$ for $(t,x) \in [0, T] \times [0, L]$ with the set $M_2(K_M)$ defined as follows

$$M_2(K_M) = \{(R_+, R_-) \in M_1(u_{max}) : |R_+(t,x) - R_-(t,x)| \leq 2\kappa, \text{ and } R_+ \text{ and } R_- \text{ are continuous on } [0, T] \times [0, L] \text{ and Lipschitz continuous with respect to } x \text{ with the Lipschitz constant } K_M\}.$$

Sketch of Proof- Banach's fixed point theorem

Define the operator $P(t,x) = (P_+, P_-)$, with

$$
\begin{aligned}
&P_\pm(R_+, R_-)(t,x) \\
=&R_\pm(t_\pm(x,t), \xi_\pm^R(t_\pm(x,t), x, t)) \mp \int_{t_\pm(x,t)}^{t} \sigma(R_+, R_-)(\tau, \xi_\pm^R(\tau, x, t)) d\tau.
\end{aligned}
$$

- The fixed point iteration is well-defined.
  (From equation (10))
- Uniform boundedness of the Lipschitz constants.
  (From equation (11))
  $(R_+, R_-) \in M_2(K_M)$, then
  $$(P_+(R_+, R_-), P_-(R_+, R_-)) \in M_2(K_M).$$
- Contractivity. (From equation (12))
  $$|P_\pm(R_+, R_-) - P_\pm(S_+, S_-)| < K_{ct} \|R - S\|_{C^0} \ (K_{ct} < 1).$$

Some Details

For $(S_+, S_-), (R_+, R_-) \in M_2(K_M)$, $S_+ = R_+$ on Γ_+ and $S_- = R_-$ on Γ_-, we set

$$|P_\pm(R_+, R_-)(t,x) - P_\pm(S_+, S_-)(t,x)| \le I_1 + I_2,$$

with

$$I_1 = |R_\pm(t_\pm^R(x,t), \xi_\pm^R(t_\pm^R(x,t), x, t)) - S_\pm(t_\pm^S(x,t), \xi_\pm^S(t_\pm^S(x,t), x, t))|,$$

$$I_2 = |\int_{t_\pm^R(x,t)}^{t} \sigma(R_+, R_-)(\tau, \xi_\pm^R(\tau, x, t)) d\tau - \int_{t_\pm^S(x,t)}^{t} \sigma(S_+, S_-)(\tau, \xi_\pm^S(\tau, x, t)) d\tau|.$$

Some Details

$$
\begin{aligned}
I_1 =& \Big| R_\pm\big(t_\pm^R(x,t), \xi_\pm^R(t_\pm^R(x,t),x,t)\big) - S_\pm\big(t_\pm^S(x,t), \xi_\pm^S(t_\pm^S(x,t),x,t)\big)\Big| \\
\leq& \Big| R_\pm\big(t_\pm^R(x,t), \xi_\pm^R(t_\pm^R(x,t),x,t)\big) - R_\pm\big(t_\pm^S(x,t), \xi_\pm^S(t_\pm^S(x,t),x,t)\big)\Big| \\
+& \Big| R_\pm\big(t_\pm^S(x,t), \xi_\pm^S(t_\pm^S(x,t),x,t)\big) - S_\pm\big(t_\pm^S(x,t), \xi_\pm^S(t_\pm^S(x,t),x,t)\big)\Big| \\
\leq& K_R\Big(\Big|\xi_\pm^R(t_\pm^R(x,t),x,t) - \xi_\pm^S(t_\pm^S(x,t),x,t)\Big| + \Big| t_\pm^R(x,t) - t_\pm^S(x,t)\Big|\Big) \\
\leq& K_R\Big|\xi_\pm^R(t_\pm^R(x,t),x,t) - \xi_\pm^S(t_\pm^R(x,t),x,t)\Big| \\
&+ K_R\Big|\xi_\pm^S(t_\pm^R(x,t),x,t) - \xi_\pm^S(t_\pm^S(x,t),x,t)\Big| \\
&+ K_R\Big| t_\pm^R(x,t) - t_\pm^S(x,t)\Big| \\
\leq& K_R\|\xi_\pm^R - \xi_\pm^S\|_{C^0} + K_R(L_s+1)|t_\pm^R(x,t) - t_\pm^S(x,t)|.
\end{aligned}
$$

Some Details

$$
\begin{aligned}
I_2 =& \left| \int_{t_\pm^S(x,t)}^{t} \sigma(R_+, R_-)(\tau, \xi_\pm^R(\tau,x,t)) - \sigma(S_+,S_-)(\tau, \xi_\pm^S(\tau,x,t)) d\tau \right| \\
& + \left| \int_{t_\pm^R(x,t)}^{t_\pm^S(x,t)} \sigma(S_+,S_-)(\tau, \xi_\pm^S(\tau,x,t) d\tau \right| \\
\leq & \int_{t_\pm^S(x,t)}^{t} \Big| \sigma(R_+,R_-)\big(\tau, \xi_\pm^R(\tau,x,t)\big) - \sigma(S_+,S_-)\big(\tau, \xi_\pm^R(\tau,x,t)\big) \Big| d\tau + \\
& \int_{t_\pm^S(x,t)}^{t} \Big| \sigma(S_+,S_-)\big(\tau, \xi_\pm^R(\tau,x,t)\big) - \sigma(S_+,S_-)\big(\tau, \xi_\pm^S(\tau,x,t)\big) \Big| d\tau \\
& + K_\sigma(\kappa) | t_\pm^R(x,t) - t_\pm^S(x,t) | \\
\leq & 2TK_\sigma(\kappa) \| R - S \|_{C^0} + 2TK_\sigma(\kappa) K_M \| \xi_\pm^R - \xi_\pm^S \|_{C^0} \\
& + K_\sigma(\kappa) | t_\pm^R(x,t) - t_\pm^S(x,t) |.
\end{aligned}
$$

Semi-global Solution

If we assume that $T > \frac{L}{\underline{\Lambda}(u_{max})}$, we can find a value $T_{in} \in [\frac{L}{\underline{\Lambda}(u_{max})}, T]$ where the conditions (10)-(12) hold for $T = T_{in}$. The reason for being able to substitute T_{in} for T in all the estimates provided in the proof is that for any $(t, x) \in [0, T] \times [0, L]$, we have:

$$|t - t_{\pm}(x, t)| < \frac{L}{\underline{\Lambda}(u_{max})} < T_{in}.$$

This implies that the estimates still hold for T_{in}. Therefore, based on the assumption, we establish the sufficient condition (10)-(12) with $T = T_{in}$, ensuring the existence of a solution to the system. Consequently, for $T > \frac{L}{\underline{\Lambda}(u_{max})}$, it is sufficient to demonstrate the conditions (10)-(12) for some $T = T_{in} \in [\frac{L}{\underline{\Lambda}(u_{max})}, T]$.

Constrained Exact Controllability

Box Constraint

- Pressure constraint

$$\underline{p} \leq p \leq \overline{p},$$

which can be expressed in terms of the Riemann invariants as:

$$2\ln\left(\underline{p}\right) \leq R_+ + R_- \leq 2\ln(\overline{p}). \tag{13}$$

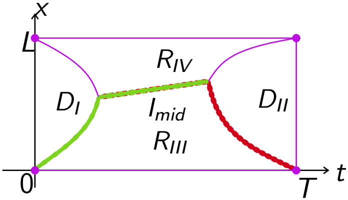
- Velocity constraint

$$|v| \leq \lambda_0. \tag{14}$$

Expressing v as $v = \frac{\sqrt{R_s^e T^e}(R_+ - R_-)}{2\lambda_0}$, we can rephrase condition (14) in terms of the Riemann invariants as:

$$-\frac{2\lambda_0}{\sqrt{R_s^e T^e}} \leq R_+ - R_- \leq \frac{2\lambda_0}{\sqrt{R_s^e T^e}}. \tag{15}$$

Range



$$D_I := \{(t,x)|t \in (0,t_I), x \in [\xi_+(t,0,0), \xi_-(t,L,0)]\},$$

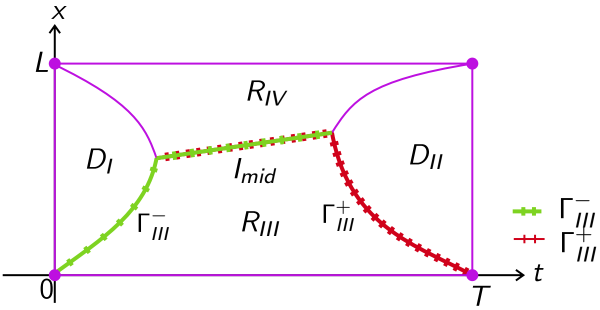
$$D_{II} := \{(t,x)|t \in (t_{II},T), x \in [\xi_+(t,0,0), \xi_-(t,L,0)]\}.$$

$$k_I = \frac{\xi_+(t_{II},0,T) - \xi_+(t_I,0,0)}{t_{II} - t_I}.$$

Then the segment is defined as

$$I_{mid} = \left\{ \left(t, \xi_+(t_I,0,0) + k_I(t - t_I) \right) \middle| t \in [t_I, t_{II}] \right\}.$$

Constrained Exact Controllability



we define the following sets:

$$\Gamma = \{0\} \times [0, L] \cup I_{mid} \cup \{T_0\} \times [0, L], \tag{16}$$

$\Gamma_{+}^{III} := I_{mid} \cup \{(t, \xi_{-}(t, 0, T)) | t \in [t_{II}, T_0]\}$, $\Gamma_{-}^{III} := I_{mid} \cup \{(t, \xi_{+}(t, 0, 0)) | t \in [0, t_I]\}$,

$\Gamma_{+}^{IV} := I_{mid} \cup \{(t, \xi_{-}(t, L, 0)) | t \in [0, t_I]\}$, $\Gamma_{-}^{IV} := I_{mid} \cup \{(t, \xi_{+}(t, L, T)) | t \in [t_{II}, T_0]\}$,

Boundary Control

The control input, denoted as $u_{\pm}(t)$, is derived from the boundary trace of the constructed state $\mathcal{R} = (R_+, R_-)$.

$$\begin{cases} u_+(t) = R_+(0, t), \\ u_-(t) = R_-(L, t), \quad t \in [0, T_0]. \end{cases} \tag{17}$$

Lipschitz Constants

$$
\begin{cases}
\widetilde{L}_s := \Lambda(u_{max}), \\
L_x := \exp\left(2K_M T\Lambda(u_{mid})\right), \\
\widetilde{L}_x := \exp\left(2K_M T\Lambda(u_{max})\right), \\
K_A := 2TK_\sigma(\kappa)K_M L_x + \sigma_{max}(\kappa) + K_R L_x, \\
K_B := \frac{1}{\underline{\Lambda}(u_{max})} \exp\left(2K_M T\Lambda(u_{max})\right), \\
K_\sigma(s) := \frac{1}{2}\theta\sqrt{R_s^e T^e}s + \frac{1}{\sqrt{R_s^e T^e}} g s_{lope}, \\
\widetilde{K}_B := \frac{1}{\underline{\Lambda}(u_{max}) - k_l} \exp\left(L_r T\Lambda(u_{max})\right), \\
H_\xi(s) = T\Lambda(u_{max})\exp(sT\Lambda(u_{max})), \\
\widetilde{H}_B(s) := \frac{T\Lambda(u_{max})\exp(sT\Lambda(u_{max}))}{\underline{\Lambda}(u_{max}) - k_l}.
\end{cases}
\tag{18}
$$

The exact constrained controllability

Consider a pipe with physical parameters $L > 0$, $\theta \geq 0$ and s_{lope}, α, g, R_s^e, T^e. Define the number $u_{max} \in \left(0, \min\left\{\frac{1}{2}, -\ln(2|\alpha|)\right\}\right)$. Choose the control time $T_0 > \frac{L}{\underline{\Lambda}(u_{max})}$ and an intermediate time

$$T_1 \in [\frac{L}{\underline{\Lambda}(u_{max})}, T_0].$$

Let $k_I \in (-\infty, \underline{\Lambda}(u_{max}))$. Let the stationary initial state $R_\pm(0)$ and the desired stationary terminal state $R_\pm(T_0)$ be given. Assume that $R_\pm(0)$ and $R_\pm(T_0)$ are Lipschitz continuous with the Lipschitz constant K_R and satisfy the conditions included in the following estimate (19). We construct the boundary controls $u_\pm(t)$ defined in (17) that steer the system from the initial state to the terminal state under the following assumption (19)-(24).

Let K_R denote a common Lipschitz constant for $R_\pm$ on Γ defined in (16), which means for $(t_1, x_1), (t_2, x_2) \in \Gamma$,

$$|R_\pm(t_1, x_1) - R_\pm(t_2, x_2)| \leq K_R\big(|t_2 - t_1| + |x_2 - x_1|\big). \qquad (19)$$

Define the numbers:

$$B = \sup_{(t,x)\in\Gamma} \{R_+(t,x), R_-(t,x)\}, \quad A = \inf_{(t,x)\in\Gamma} \{R_+(t,x), R_-(t,x)\}.$$

Assume that there exists numbers $\kappa > 0, u_{mid} \in (0, u_{max})$ that satisfy:

$$
\begin{cases}
B + T_1 K_\sigma(\kappa) \in (-u_{mid}, u_{mid}), \\
A - T_1 K_\sigma(\kappa) \in (-u_{mid}, u_{mid}), \\
B - A + 2T_1 K_\sigma(\kappa) \leq \kappa,
\end{cases} \tag{20}
$$

and that there exists a number $\widetilde{\kappa} > \kappa$ that satisfies:

$$
\begin{cases}
B + T_1 K_\sigma(\kappa) + T_1 K_\sigma(\widetilde{\kappa}) \in (-u_{max}, u_{max}), \\
A - T_1 K_\sigma(\kappa) - T_1 K_\sigma(\widetilde{\kappa}) \in (-u_{max}, u_{max}), \\
B - A + 2T_1 K_\sigma(\kappa) + 2T_1 K_\sigma(\widetilde{\kappa}) \leq \widetilde{\kappa}.
\end{cases} \tag{21}
$$

Moreover, $\kappa, \widetilde{\kappa}$ also satisfy

$$
\begin{cases}
B + T_1 K_\sigma(\kappa) + T_1 K_\sigma(\widetilde{\kappa}) \in (-u_{max}, u_{max}), \\
A - T_1 K_\sigma(\kappa) - T_1 K_\sigma(\widetilde{\kappa}) \in (-u_{max}, u_{max}), \\
B - A + 2T_1 K_\sigma(\kappa) + 2T_1 K_\sigma(\widetilde{\kappa}) \leq \widetilde{\kappa}.
\end{cases} \tag{22}
$$

with the notation before.

Assume that $K_M \geq K_R$ satisfies the two following inequalities:

$$\begin{aligned} & L_x \Big(K_R \big((\Lambda_3 + \Lambda_3 k_I) + \widetilde{L}_x (1 + 2\Lambda_1\Lambda_3 + \Lambda_3 k_I) \big) \\ & + K_\sigma(\kappa) \big(\Lambda_3 + 2\widetilde{L}_x T_1 K_M (1 + \Lambda_3 + \Lambda_1\Lambda_3 + \Lambda_3 k_I) \big) \\ & + K_\sigma(\widetilde{\kappa}) (2 T_1 K_M + \Lambda_3) \Big) \leq K_M. \end{aligned} \tag{23}$$

and

$$\begin{aligned} & L_x K_R T_1 \Lambda_1 \Lambda_3 \big((\Lambda_3 + \Lambda_3 k_I) + \widetilde{L}_x (1 + 2\Lambda_1\Lambda_3 + \Lambda_3 k_I) \big) \\ & + K_\sigma(\widetilde{\kappa}) T_1 \big(2 + 2 L_x K_M T_1 \Lambda_1 + L_x \Lambda_1 \Lambda_3 \big) \\ & + L_x K_\sigma(\kappa) T_1 \Lambda_1 \Lambda_3 \big(\Lambda_3 + 2\widetilde{L}_x T_1 K_M (1 + \Lambda_3 + \Lambda_1\Lambda_3 + \Lambda_3 k_I) \big) < 1. \end{aligned} \tag{24}$$

Define the set

$$M_3(K_M) = \{(R_+, R_-) \in M_1(u_{max}) : |R_+(t,x) - R_-(t,x)| \leq 2\widetilde{\kappa},$$
$$R_+ \text{ and } R_- \text{ are continuous on}[0, T_0] \times [0, L] \text{ and Lipschitz}$$
$$\text{continuous with respect to } x \text{ with the Lipschitz constant } K_M\}.$$

Then the system (4) has a unique solution on $[0, T_0]$ under the corresponding controls $u_\pm(t)$ from (17). Moreover we have that $(R_+, R_-) \in M_3(K_M)$ satisfies the box constraints (13) and (15) on $[0, T_0] \times [0, L]$.

Idea of Proof

By imposing suitable assumptions on the value of I_{mid}, we establish the unique existence of the solution in each part and ensure its satisfaction of the constraint. Consequently, we obtain a solution on $[0, T] \times [0, L]$ that satisfies the initial state and the desired terminal state. The uniqueness of the solution implies the unique determination of the control input, leading to the achievement of constrained exact controllability.

Constrained Exact Controllability

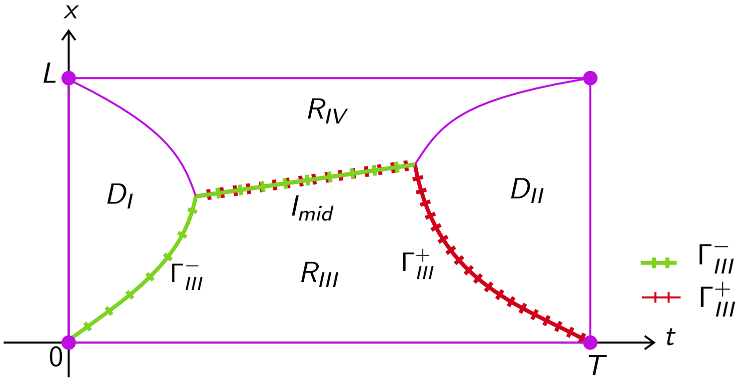


Figure 2. Domain and boundaries for the exact controllability boundary analysis

Constrained Exact Controllability

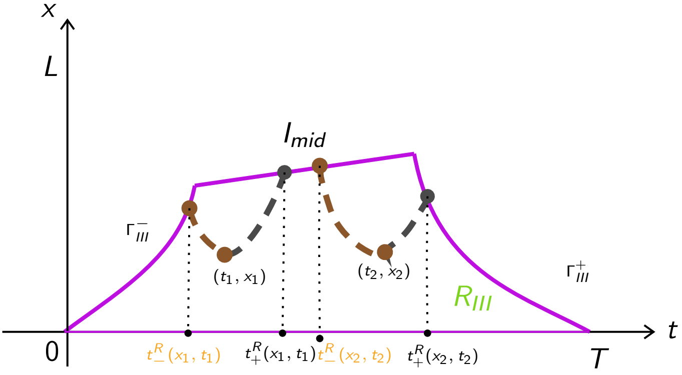


Figure 3. $t_{\pm}(x, t)$ in R_{III}

Perspectives

- Streamline and simplify the existing conditions in order to facilitate the verification process.
- Extend our research to investigate the constrained controllability between arbitrary stationary states for the Saint-Venant system, building upon the controllability established by Gugat and Leugering in their work [3].

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Thank you for listening!

Xu Huang

xuxu.huang@fau.de