Limits of the stabilization of a networked hyperbolic system with a circle

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Outline

Bastin, G.& Coron, J-M

In [1, Page 197], Bastin and Coron mention that for some systems of balance laws, there is an intrinsic limit of stabilization under local boundary control. It is proved that the following system

$$\begin{cases} \partial_t y_1 + \partial_x y_1 + y_2 = 0, & (t,x) \in (0,+\infty) \times (0,L), \\ \partial_t y_2 - \partial_x y_2 + y_1 = 0, & (t,x) \in (0,+\infty) \times (0,L), \\ y_2(t,L) = y_1(t,L), & t \in (0,+\infty), \\ y_1(t,0) = k y_2(t,0), & t \in (0,+\infty). \end{cases} \tag{1}$$

cannot be stabilized for any $k \in \mathbb{R}$ if $L \in (\frac{\pi}{c}, +\infty)$. On the other hand, the system (1) is stabilizable if $L \in (0, \frac{\pi}{2c})$ from [2].

Martin, G. & Stephan, G.[3]

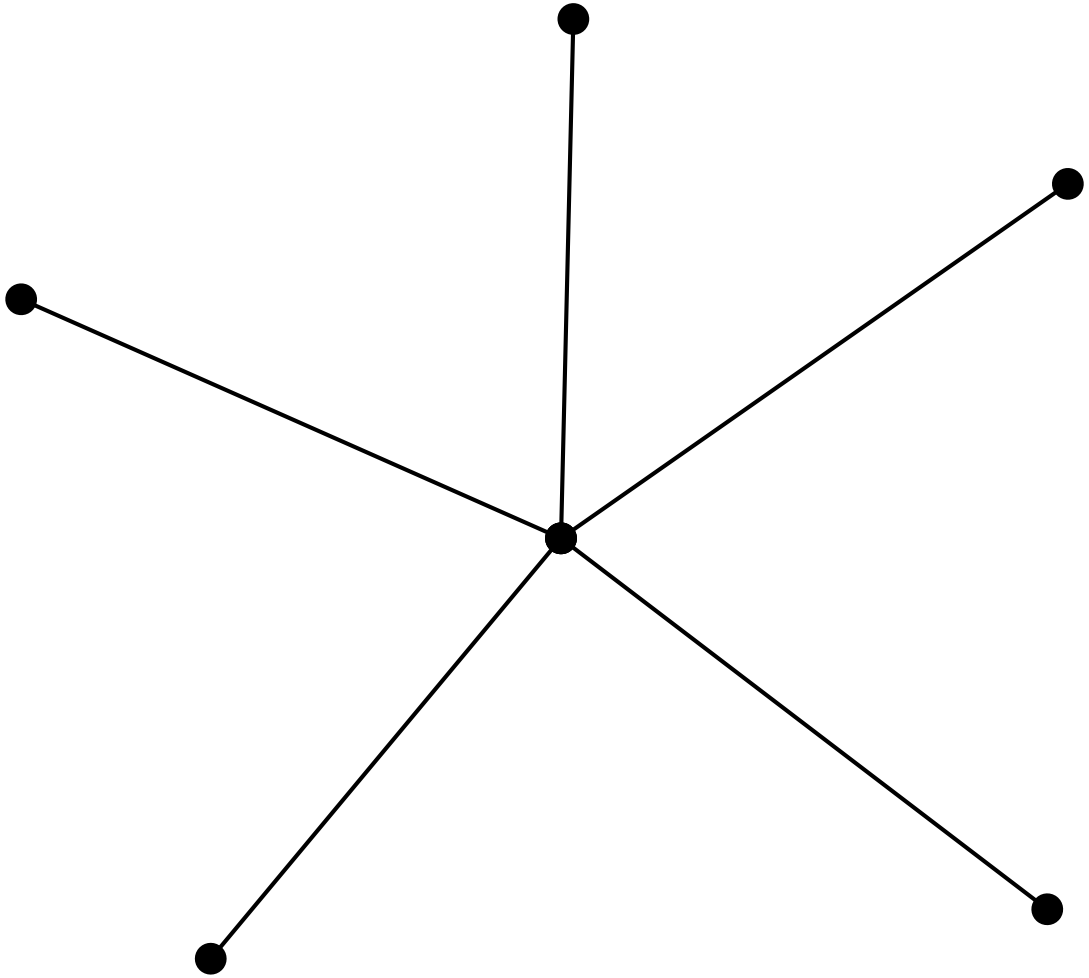


Figure 2: A star-shaped network.

A network with a circle

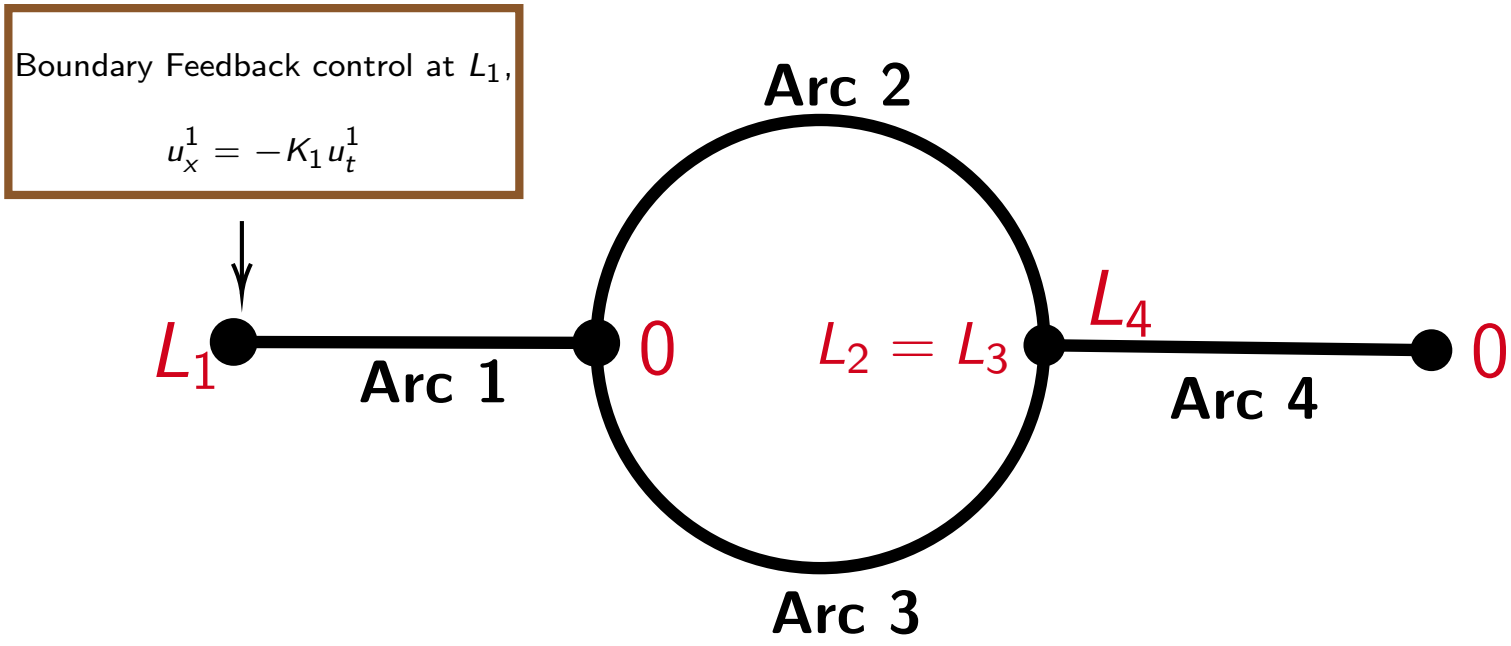


Figure 1: A network with a circle in 4 edges.

Model

For $k \in \{1,2,3,4\}$, let real numbers $c_k > 0, \varepsilon_k \geq 0$ be given. We consider the following system:

$$\begin{cases} u_{tt}^k = u_{xx}^k - 2\varepsilon_k u_t^k - (\varepsilon_k^2 - c_k^2)u^k, \\ u^1(t,0) = u^2(t,0) = u^3(t,0), \\ u^2(t,L_2) = u^3(t,L_3) = u^4(t,L_4), \\ \Sigma_{k=1,2,3} u_x^k(t,0) = 0, \\ \Sigma_{k=2,3,4} u_x^k(t,L_k) = 0, \\ u^4(t,0) = 0, \\ u_x^1(t,L_1) = -K_1 u_t^1(t,L_1). \end{cases} \tag{2}$$

$t \in (0,+\infty),\ x \in [0,L_k], k \in \{1,2,3,4\}.$

Model

The real number K_1 is the control gain. Besides, the initial state is given:

$$\begin{cases} U(0,x) = (u_0^1(x), ..., u_0^4(x)) = (u^1(0,x), ..., u^4(0,x)), \\ V(0,x) = (v_0^1(x), ..., v_0^4(x)) = (u_t^1(0,x), ..., u_t^4(0,x)). \end{cases}$$

Stabilizability

Definition

System (2) is stabilizable if there exists a control gain $K_1 \in \mathbb{R}$, such that for all $U(0,\cdot) \in \prod_{k=1}^{4} H^1(0, L_k)$ and all $V(0,\cdot) \in \prod_{k=1}^{4} L^2(0, L_k)$, we have

$$\lim_{t \to +\infty} \|U(t,\cdot)\|_{H^1} = 0.$$

Preliminaries

Preliminaries

- Well Posedness: By employing classical methods such as the method of characteristics or the theory of strongly continuous one-parameter semigroups of linear operators (see [4]), we can establish the well-posedness of the solution to the system.
- Sturm Liouville: Building upon the findings in [5] and [6], we identify that the eigenvalue problem of the system (2) corresponds to a Sturm-Liouville eigenvalue problem on the network.

Main Theorem

Theorem

Assume that $c_k = c_1 = c, \varepsilon_k = \varepsilon_1 = \varepsilon, L_k = L_1 = L$, *that is the length of the arcs in the network and the parameters are the same for all arcs.*

- *If* $L < L_{min} = \frac{\arctan\sqrt{\frac{2}{7}}}{\sqrt{c^2-\varepsilon^2}}$, *the system (2) is stabilizable (with* $|K_1|$ *sufficiently small).*
- *If* $L > L_{max} = \frac{\pi}{2\sqrt{c^2-\varepsilon^2}}$, *the system (2) is not stabilizable.*

Sketch of Proof- Spectral Analysis

For the stability analysis, we employ spectral analysis. For $L < L_{min} = \frac{\arctan\sqrt{\frac{2}{7}}}{\sqrt{c^2-\varepsilon^2}}$ with $K_1 = 0$ and $L_k = L$ $(k = 1, 2, 3, 4)$, we first establish the characteristic equation. All the eigenvalues of the system (2) lie in the left half-plane, indicating that the system (2) is L^2-exponentially stable. With the perturbation on the control parameter and the length, the eigenvalue still lies in the left half-plane.

Perturbation of the arcs L_k

Theorem

If $K_1 = 0, L < L_{min}$, we consider a small perturbation according to L_k $(k = 1, 2, 3, 4)$, that is:

$$\begin{cases} \widetilde{L}_1 = L + d_1 r, \\ \widetilde{L}_2 = \widetilde{L}_3 = L + d_2 r, \\ \widetilde{L}_4 = L + d_4 r. \end{cases} \tag{3}$$

Here d_1, d_2, d_4, r are real constants. The system 2 with $c_k = c_1 = c, \varepsilon_k = \varepsilon_1 = \varepsilon, L_k = \widetilde{L}_k$ is exponentially stable if $|r|$ is sufficiently small.

Perturbation of control parameter K_1

Theorem

The following system is exponentially stable if $L < L_{min}$ and $|K_1|$ is sufficiently small.

$$\begin{cases} u_{tt}^k = u_{xx}^k - 2\varepsilon u_t^k - (\varepsilon^2 - c^2)u^k, \\ u^1(t,0) = u^2(t,0) = u^3(t,0), \\ u^2(t,L) = u^3(t,L) = u^4(t,L), \\ \Sigma_{k=1,2,3} u_x^k(t,0) = 0, \\ \Sigma_{k=2,3,4} u_x^k(t,L) = 0, \\ u^4(t,0) = 0, \\ u_x^1(t,L) = K_1 u_t^1(t,L). \end{cases} \tag{4}$$

$t \in (0, +\infty),\ x \in [0, L], k \in \{1, 2, 3, 4\},$

Example 1

For the given initial value (5), we can calculate the characteristic function of the system (2). The numerical result generated with MATLAB using the upwind implicit scheme for the system (2), along with the numerical result obtained from the eigenfunctions calculated before, will be shown in the following Figure 1. We take the initial value:

$$\begin{cases} u^1(0,x) = -4\sin\left(\frac{\pi}{2}x\right), \\ u^i(0,x) = 2\sin\left(\frac{\pi}{2}x\right), \quad i \in \{2,3,4\}, \\ u_t^k(0,x) = 0, \quad k \in \{1,2,3,4\}. \end{cases} \tag{5}$$

The time evolution of the network

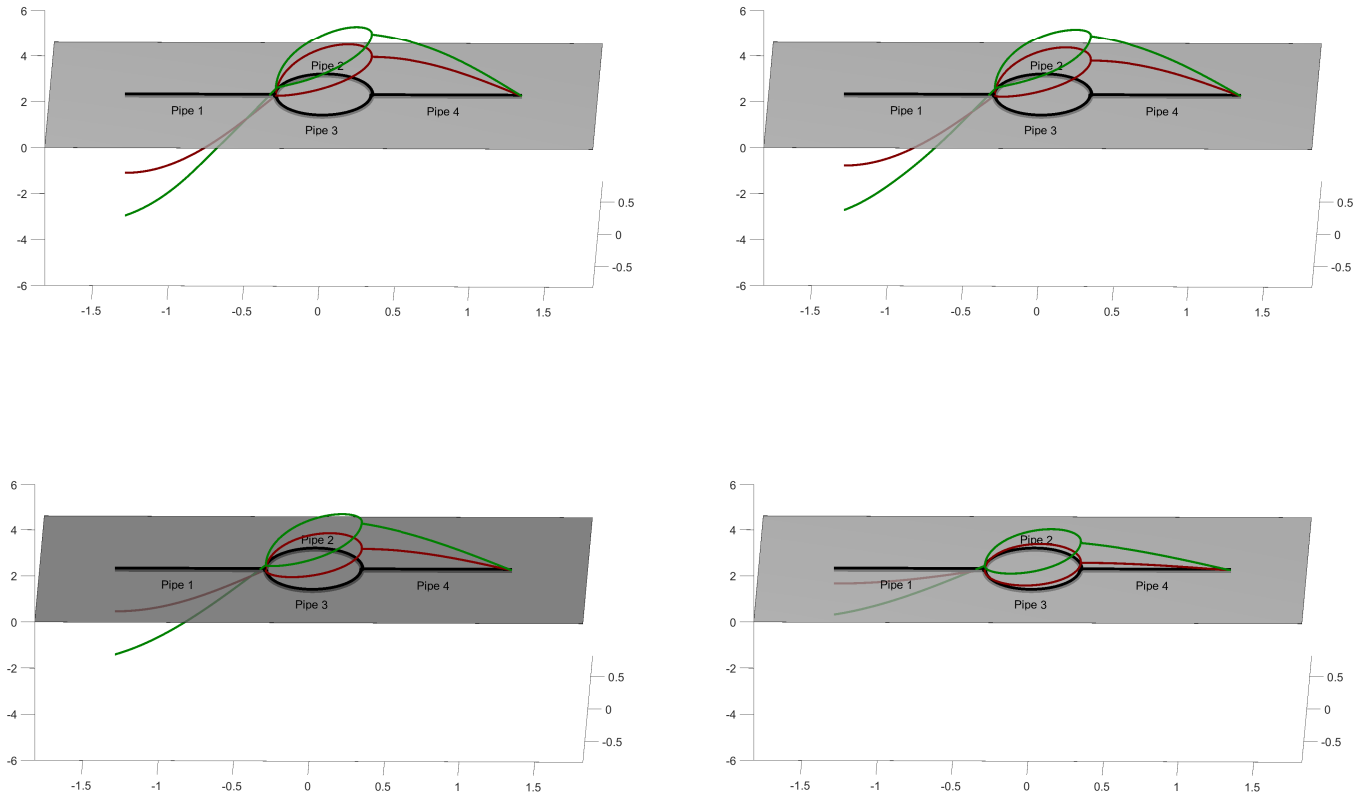
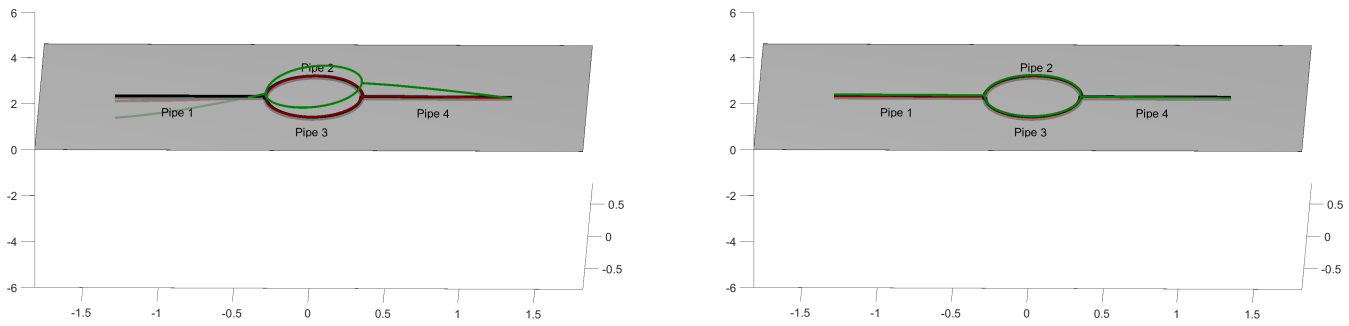


Figure 1 : The time evolution of the network with the initial value (5)

Example 1



$$\varepsilon = \pi, \quad c = \sqrt{1.01}\pi, \quad L = 1$$

Green Line: Numerical simulation Result Red Line: Exact Solution

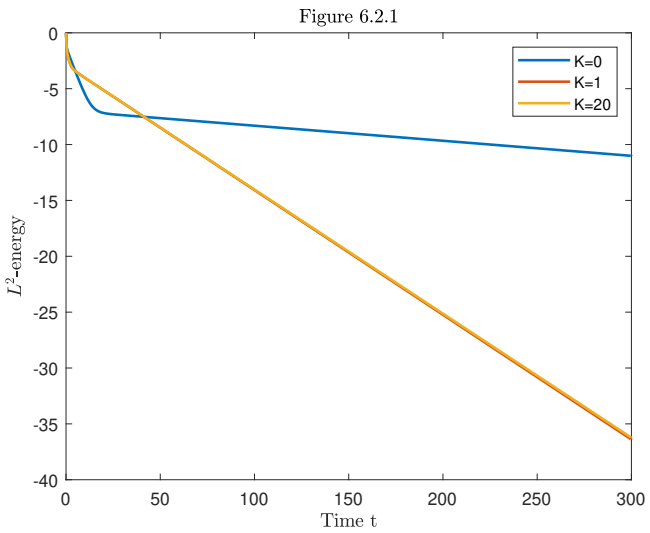
From the figure we can observe that the simulation result of the scheme is quite good.

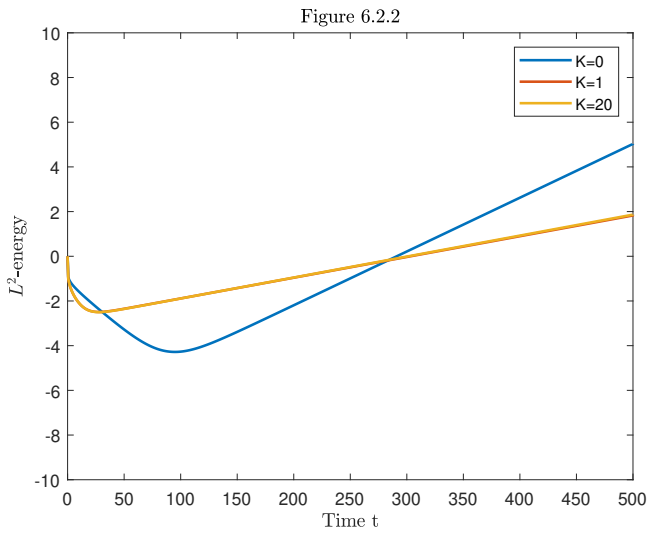
Example 2

We normalize the initial L^2 energy as 1. Taking $\varepsilon = \pi, c = \sqrt{1.01}\pi$, so Theorem 2 gives us $L_{min} = 1.5625,\ L_{max} = 5$. The time evolution of the log of L^2-energy of the networks with different length of the arcs can be shown in Figure 6.2.1, Figure 6.2.2, and Figure 6.2.3 for $K_1 = 0, 1, 20$. We take the initial value:

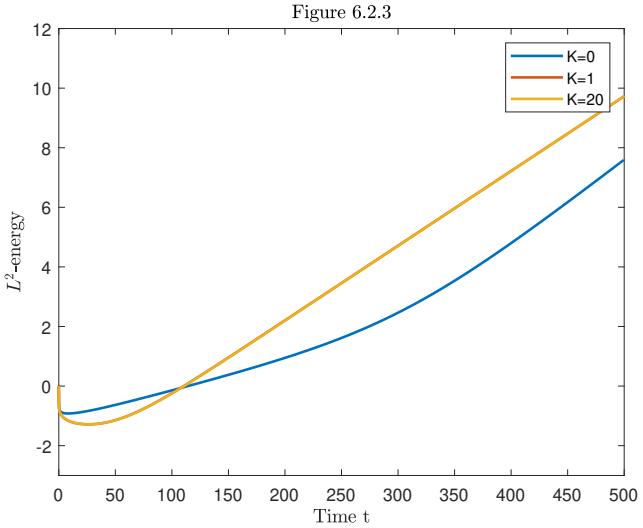
$$\begin{cases} u^1(0,x) = \sin\left(\frac{\pi x}{L}\right) + \frac{\pi}{L}x, \\ u^2(0,x) = u^3(0,x) = -\sin\left(\frac{\pi x}{L}\right), \\ u^4(0,x) = -\frac{2\pi}{L}x, \\ u_t^k(0,x) = 0, \quad k \in \{1,2,3,4\}. \end{cases} \tag{6}$$

Example 2





$L_1 = 1 < L_{min}$ $\qquad$ $L_2 = 3 \in (L_{min}, L_{max})$



$L_3 = 6 > L_{max}$

Example 2

From the numerical results, we conclude that if there exists a critical length L_c of the arc that divides the stabilization of the system, L_c has a higher probability to be equal to L_{min}.

Example 3

We take $\varepsilon = 4$ and $c = 5$, while $L = \frac{1}{2} > L_{min}$, only if $K_1 \in (0.8, 5.0)$ can the system possibly be stable. We try to simulate the values of $K_1 \in 0.9, 1, 2, 3, 4, 4.5$ using the initial value (6). However, because the five lines of the time evolution of L^2-energy of the networks are very close and increasing, we are only presenting the log of energy for $K_1 = 3$ in Figure 6.3.1.

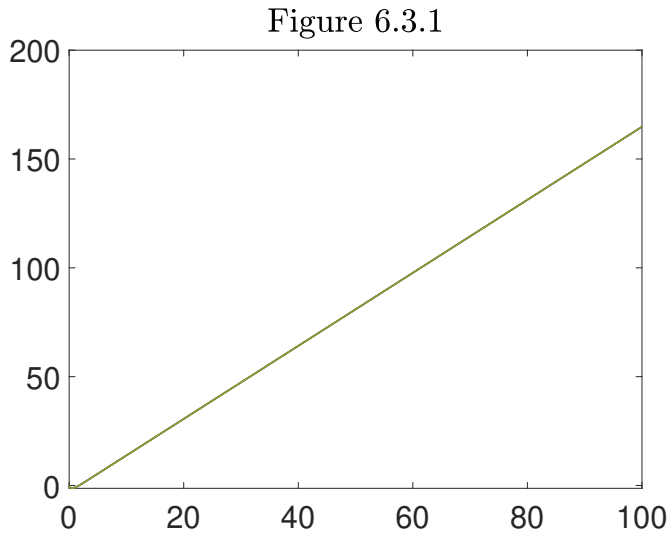


Figure 6.3.1: $c = 5,\ \varepsilon = 4,\ L = \frac{1}{2},\ K_1 = 3$

Example 3

The numerical results show that the system is not exponentially stable even if we cannot theoretically prove there exists an eigenvalue in the right part of the plane.

Perspectives

- The existence of the critical length to precisely separate the domains of stability and instability.
- More complicated system.

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Thank you for listening!

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