Coagulation with Collisional Fragmentation and Applications in Wave Turbulence Kinetics

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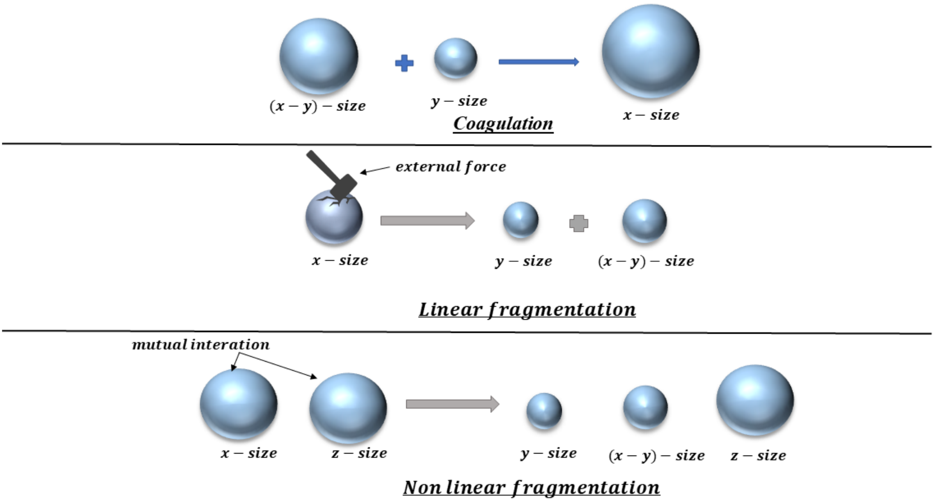
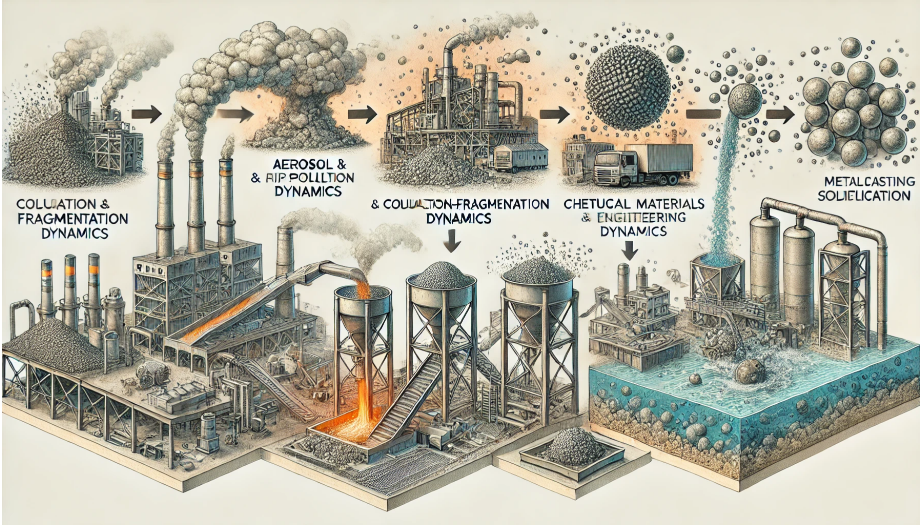
Outline of the presentation

- General introduction: The nonlinear population balance models
- Analytical results: Well-posedness, gelation and shattering phenomena
- Numerical aspects: Development of schemes and simulations
- Applications in wave turbulence kinetics: Numerical approaches

General introduction

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Coagulation with collisional fragmentation equation

The pure binary nonlinear collisional breakage equation is given by

$$
\begin{aligned}
&\partial_t f(x,t) = \mathcal{A}(f)(x,t) - \mathcal{F}(f)(x,t) := \mathcal{Q}(f)(x,t),\\
&\mathcal{A}(f) := \frac{1}{2}\int_0^x \mathcal{C}(x-y,y)f(x-y,t)f(y,t)\mathrm{d}y - f(x,t)\int_0^x \mathcal{C}(x,y)f(y,t)\mathrm{d}y\\
&\mathcal{F}(f)(x,t) := \int_0^\infty \int_x^\infty \mathcal{B}(x|y,z)\mathcal{K}(y,z)f(y,t)f(z,t)\mathrm{d}y\mathrm{d}z\\
&\qquad\qquad -f(x,t)\int_0^\infty \mathcal{K}(x,y)f(y,t)\mathrm{d}y
\end{aligned}
\tag{1}
$$

Supported with the initial data: $f(x,0) = f_0(x)(\geq 0)$ for all $x \in (0,\infty)$.
$\mathcal{C},\mathcal{K}$ both are nonnegative symmetric functions and $\mathcal{B}$ satisfies the following properties:

$$\mathcal{B}(x|y,z) = 0 \quad \text{for all} \quad x \geq y, \quad \text{and} \quad \int_0^y x\mathcal{B}(x|y,z)\mathrm{d}x = y; \tag{2}$$

$$\int_0^y \mathcal{B}(x|y,z)\mathrm{d}x = \nu(y,z) < \infty \quad \text{for all} \quad y>0, z>0. \tag{3}$$

Moment functions

- The $p-$th order moment: $\mathcal{M}^{(p)}(t) = \int_0^\infty x^p f(x,t)\mathrm{d}x, .$
- Total mass of the particles present in the system: $\mathcal{M}^{(1)} = \int_0^\infty x f(x,t)\mathrm{d}x.$
- Total number of the particles present in the system: $\mathcal{M}^{(0)} = \int_0^\infty f(x,t)\mathrm{d}x.$
- The system obeys the mass conservation law: $\mathcal{M}^{(1)}(t) = \mathcal{M}^{(1)}(0)$. If not

$$T_{gel}/T_{sht} := \inf\left\{t \geq 0 : \int_0^\infty x f(x,t)\mathrm{d}x < \int_0^\infty x f^{\mathrm{in}}(x)\mathrm{d}x\right\} \in [0,+\infty].$$

Assumptions

(A1) The initial data $f^{in} \in L^1\left[(0,\infty)\,;(1+x)\,\mathrm{d}x\right]$;

(A2) The coagulation kernel $\mathcal{C}(x,y) \leq k\left(x^{\alpha}y^{\beta} + x^{\beta}y^{\alpha}\right)$, for some constants $0 \leq \alpha \leq \beta \leq 1$ satisfying $\lambda := \alpha + \beta \in [0,1)$;

(A3) The collisional kernel satisfy the growth condition $\mathcal{K}(x,y) \leq a_0(x+y)^{\gamma}$, for some positive constant a_0.

(A4) The breakage distribution function $\mathcal{B}(x|y,z)$ have the 'Power-Law' growth rate

$$\mathcal{B}(x|y,z) = \frac{1}{y}\beta\left(\frac{x}{y},z\right) \text{ where } 0 < x < y,\ \ z > 0.$$

Here β is a nonnegative function and $\displaystyle\int_0^1 z^*\beta\left(z^*,z\right)\mathrm{d}z^* = 1$ for $z > 0$.

Well-posedness

- **Kernel truncation:**

$$\mathcal{C}_n(x, y) = \mathcal{C}\left(\min\{x, n\}, \min\{y, n\}\right), \quad (x, y) \in (0, \infty)^2,$$
$$\mathcal{K}_n(x, y) = \mathcal{K}\left(\min\{x, n\}, \min\{y, n\}\right), \quad (x, y) \in (0, \infty)^2$$

- **Relative compactness:** The sequence of solution $\{f_n\}_{n=1}^{\infty}$ is relatively compact over a compact rectangular subset $\Xi_1 = \{(x, t) : 0 < x < X, 0 \leq t \leq T\}$ of Ξ of $(0, \infty) \times [0, T]$.
  - $f_n(x, t) \leq G(k, a_0, X, T)$,
  - For a arbitrary $\epsilon > 0$ there exist some constants $\mathcal{X}$ depending on X and T such that $\sup\limits_{|x'-x|<\delta, |t'-t|<\delta} \left|f_n(x', t') - f_n(x, t)\right| < \mathcal{X}(X, T)\epsilon$.

Combining all these results along with the Arzelà-Ascoli theorem ensure that

$$\lim_{n \to \infty} f_n = f$$

uniformly on Ξ_1.

Mass loss or gelation in coagulation process ($\mathcal{K} = 0$)

For any $\omega \in L^{\infty}(0, \infty)$, the Smoluchowski coagulation equation can be written as:

$$\int_0^{\infty} \omega(x) \left[f(x,t) - f^{in}(x) \right] \mathrm{d}x = \frac{1}{2} \int_0^t \int_0^{\infty} \int_0^{\infty} \bar{\omega}(x) \mathcal{C}(x,y) f(x,t) f(y,s) \mathrm{d}y \mathrm{d}x \mathrm{d}s.$$

Where $\bar{\omega}(x) = \omega(x+y) - \omega(x) - \omega(y)$.
Now we choose $\omega(x) = x\chi_{(0,q)}(x) \in L^{\infty}(0,\infty)$ for $q \in (0, \infty)$

$$\int_0^q x f(x,t) \mathrm{d}x - \int_0^q x f^{in}(x) \mathrm{d}x = -\int_0^t \int_0^q \int_{q-x}^{\infty} x \mathcal{C}(x,y) f(x,s) f(y,s) \mathrm{d}y \mathrm{d}x \mathrm{d}s$$

$$\Longrightarrow \text{ for some } \mathcal{C} : \int_0^{\infty} x f(x,t) \mathrm{d}x \neq \int_0^{\infty} x f^{in}(x) \mathrm{d}x$$

(*for example:* $\mathcal{C}(x,y) = (xy)^{\alpha}$, $\alpha > 1$ *and* $f^{in} \in L^1 \left[(0,\infty) ; (1+x) \, \mathrm{d}x \right]$)

Divergence form and their non-conservative truncation

The mass conserving form of nonlinear population balance equation:

$$\frac{\partial\left(xf(x,t)\right)}{\partial t} = -\frac{\partial \mathcal{A}\left(f(x,t)\right)}{\partial x} + \frac{\partial \mathcal{F}\left(f(x,t)\right)}{\partial x}, \tag{4}$$

with the initial data $f(x,0) = f_0(x)$.

$$\mathcal{A}\left(f(x,t)\right) := \int_0^x \int_{x-u}^{\infty} u\mathcal{C}(u,v)f(u,t)f(v,t)\mathrm{d}v\mathrm{d}u$$

$$\mathcal{F}\left(f(x,t)\right) := \int_0^{\infty}\int_x^{\infty}\int_0^x u\mathcal{B}(u|v,w)\mathcal{K}(v,w)f(v,t)f(w,t)\mathrm{d}u\mathrm{d}v\mathrm{d}w$$

The non-conservative truncation of the coupled CF-equation is given by

$$x\frac{\partial f}{\partial t}(x,t) = -\frac{\partial \mathcal{A}_{\mathrm{nc}}^R\left(f\right)}{\partial x}(x,t) + \frac{\partial \mathcal{F}_{\mathrm{c}}^R\left(f\right)}{\partial x}(x,t), \quad \text{where} \quad (x,t)\in(0,R]\times[0,T].$$

$$\mathcal{A}_{\mathrm{nc}}^R\left(f\right)(x,t) = \int_0^x\int_{x-u}^R u\mathcal{C}(u,v)f(u,t)f(v,t)\mathrm{d}v\mathrm{d}u,$$

$$\mathcal{F}_c^R\left(f\right)(x,t) = \int_0^R\int_x^R\int_0^x u\mathcal{B}(u,v;w)\mathcal{K}(v,w)f(v,t)f(w,t)\mathrm{d}u\mathrm{d}v\mathrm{d}w.$$

Approximate number density

- Let $\Lambda := (0, x_{\max}]$ be the computational domain and $\Lambda_i := \left(x_{i-1/2}, x_{i+1/2}\right]$, $i = 0, 1, 2, \ldots, I^h$ with $x_{-1/2} = 0$, $x_{I^h+1/2} = x_{\max}$ and $\Delta x_i := x_{i+1/2} - x_{i-1/2} \leq h$.
- For any given integers i and j satisfying $x_{i+1/2} - x_j \geq 0$, define a integer $\gamma_{i,j} \in \{0, 1, ..., I^h\}$ such that $x_{i+1/2} - x_j \in \Lambda^h_{\gamma_{i,j}}$.
- To discretize the time variable t, we split the time interval $[0, T]$ into N subintervals $\tau_n := [t_n, t_{n+1})$ for $n \in \{0, 1, ..., N-1\}$, with $t_n = n\Delta t$.

The discrete number density function over the cell Λ_i is calculated as

$$f_i^n \approx \frac{1}{\Delta x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} f(x, t_n) \mathrm{d}x.$$

Discrete scheme

The mass conserving finite volume scheme of equation (4) is written as

$$x_i \Delta x_i \left(f_i^{n+1} - f_i^n\right) = -\Delta t \left(\mathcal{C}_{i+1/2}^n - \mathcal{C}_{i-1/2}^n\right) + \Delta t \left(\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n\right) \quad (5)$$

where $\mathcal{C}_{i+1/2}^n$ and $\mathcal{F}_{i+1/2}^n$ can be computed as

$$\mathcal{C}_{i+1/2}^n := \sum_{p=0}^{i} \sum_{q=\gamma_{i,p}}^{I^h} x_p C_{p,q} f_p^n f_q^n \Delta x_p \Delta x_q, \quad (6)$$

$$\mathcal{F}_{i+1/2}^n = \sum_{p=0}^{I^h} \sum_{q=i+1}^{I^h} \sum_{r=0}^{i} x_r B_{q,p}^r K_{q,p} f_p^n f_q^n \Delta x_p \Delta x_q \Delta x_r. \quad (7)$$

Since $x_{-1/2} = 0$, and $x_{I^h+1/2} = R$, the numerical fluxes at the boudaries are

$$\mathcal{C}_{-1/2} = \mathcal{F}_{-1/2} = \mathcal{F}_{I^h+1/2} = 0.$$

Kernel approximations

The finite volume approximation of the kinetic kernels is given as

$$C^h(u,v) = \sum_{p=0}^{I^h}\sum_{q=0}^{I^h} C_{p,q}\chi_{\Lambda_p^h}(u)\chi_{\Lambda_q^h}(v) \text{ where } C_{p,q} = \frac{1}{\Delta x_p \Delta x_q}\int_{\Lambda_p^h\times\Lambda_q^h} \mathcal{C}(u,v)\mathrm{d}u\mathrm{d}v,$$

$$K^h(u,v) = \sum_{p=0}^{I^h}\sum_{q=0}^{I^h} K_{p,q}\chi_{\Lambda_p^h}(u)\chi_{\Lambda_q^h}(v) \text{ where } K_{p,q} = \frac{1}{\Delta x_p \Delta x_q}\int_{\Lambda_p^h\times\Lambda_q^h} \mathcal{K}(u,v)\mathrm{d}u\mathrm{d}v,$$

$$B^h(u,v;w) = \sum_{r=0}^{I^h}\sum_{q=r+1}^{I^h}\sum_{p=0}^{I^h} B_{q,p}^r\chi_{\Lambda_r^h}(u)\chi_{\Lambda_q^h}(v)\chi_{\Lambda_p^h}(w)$$

$$\text{where} \quad B_{q,p}^r = \frac{1}{\Delta x_p \Delta x_q \Delta x_r}\int_{\Lambda_p^h\times\Lambda_q^h\times\Lambda_r^h} \mathcal{B}(u,v;w)\mathrm{d}u\mathrm{d}v\mathrm{d}w$$

With the discretize number density function f_i^n, we define a function f^h as follows

$$f^h(x,t) = \sum_{n=0}^{N-1}\sum_{i=0}^{I^h} \chi_{\tau_n}(t)\chi_{\Lambda_i^h}(x)f_i^n$$

Convergence to weak solution

Theorem

Let $\mathcal{C}(x,y) \in L^\infty_{loc}(\mathbb{R}_+ \times \mathbb{R}_+)$, $\mathcal{B}(x,y;z) \in L^\infty_{loc}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+)$ and $\mathcal{K}(x,y) = K_0(x+y)$ with the initial data $f^{in} \in L^1(\mathbb{R}^+, (1+x)\mathrm{d}x)$. Moreover, assume that the time step Δt satisfies that there exist a positive constant θ such that

$$C_{T,R}\Delta t \leq \theta < 1 \quad \textit{where},$$

where $C_{T,R} = \left(\|\mathcal{C}\|_{L^\infty} + 2K_0\left(\|\mathcal{B}\|_{L^\infty}R^2 + M_1^{in}\right)\right)\|f^{in}\|_{L^1} e^{2K_0 R\|\mathcal{B}\|_\infty M_1^{in} T}$. *Then there exist a subsequence such that*

$$f^h \longrightarrow f \quad \textit{in} \quad L^\infty\left(0,T;L^1(0,R)\right)$$

where f is a weak solution to the equation (4) on $[0,T]$ satisfying

$$\int_0^T\int_0^R xf(x,t)\frac{\partial\varphi}{\partial t}(x,t)\mathrm{d}x\mathrm{d}t + \int_0^R xf^{in}(x)\varphi(x,0)\mathrm{d}x$$
$$+ \int_0^T\int_0^R\left[\mathcal{C}^R_{nc}(x,t) - \mathcal{F}^R_c(x,t)\right]\frac{\partial\varphi}{\partial x}(x,t)\mathrm{d}x\mathrm{d}t = \int_0^T \mathcal{C}^R_{nc}(R,t)\varphi(R,t)\mathrm{d}t,$$

where φ be a test function $(0,R]\times[0,T)$.

Convergence analysis

- (Dunford-Pettis theorem) Let Ω be a subset of $\mathbb{R}$ such that $|\Omega| < \infty$ and $f^h : \Omega \to \mathbb{R}$ be a sequence in $L^1(\Omega)$ satisfies the following conditions
  (i) $\sup \|f^h\|_{L^1(\Omega)} < \infty$,
  (ii) there exist a increasing function $\Phi : [0, \infty) \to [0, \infty)$ satisfies

$$\lim_{r \to \infty} \frac{\Phi(r)}{r} \longrightarrow \infty \qquad \text{and} \qquad \int_\Omega \Phi(f^h)\mathrm{d}x < \infty.$$

  Then the sequence f^h is weakly sequentially compact in $L^1(\Omega)$.

- For $0 \leq s \leq t \leq T$, the approximate solution f^h is a non-negative function such that

$$\int_0^R X^h(x) f^h(x,t)\mathrm{d}x \leq \int_0^R X^h(x) f^h(x,s)\mathrm{d}x \leq \int_0^R X^h(x) f^h(x,0)\mathrm{d}x := M_1^{\text{in}}.$$

  and for all $t \in [0, T]$,

$$\int_0^R f^h(x,t)\mathrm{d}x \leq \|f^{\text{in}}\|_{L^1} e^{2K_0 R \|\mathcal{B}\|_\infty M_1^{\text{in}} t} \tag{8}$$

Continue...

- Since $f^{\text{in}} \in L^1(0,R)$, so, according to De la Vallée Poussin theorem, there exist a continuously differentiable non-negative convex function Φ on $\mathbb{R}^+$ with $\Phi(0)=0, \Phi'(0)=1$ and Φ' is a concave function satisfy the following conditions

$$\frac{\Phi(r)}{r} \to \infty \quad \text{as} \quad r \to \infty \qquad \text{and} \qquad \int_0^R \Phi(f^{\text{in}})(x)\mathrm{d}x < +\infty. \tag{9}$$

- The integral of $\Phi(f^h)$ can be expressed as

$$\int_0^T \int_0^R \Phi(f^h)\mathrm{d}x\mathrm{d}t = \sum_{n=0}^{N-1} \sum_{i=0}^{I^h} \Phi\left(f_i^n\right) \Delta x_i \Delta t$$

Using the convexity of Φ, we can get the following inequality

$$\begin{aligned}\sum_{i=0}^{I^h} \Delta x_i \Phi(f_i^n) &\le \mathcal{P}^n \sum_{i=0}^{I^h} \Delta x_i \Phi(f_i^{\text{in}}) + \mathcal{Q}\frac{\mathcal{P}^{n-1}-1}{\mathcal{P}-1} \\ &= \mathcal{P}^n \int_0^R \Phi(f^{\text{in}})(x)\mathrm{d}x + \mathcal{Q}\frac{\mathcal{P}^{n-1}-1}{\mathcal{P}-1} < \infty.\end{aligned}$$

Continue...

- By a diagonal technique, we can extract subsequences of $\left(f^h\right)_h, \left(C^h\right)_h, \left(K^h\right)_h$ and $\left(B^h\right)_h$ such that

$$
\begin{aligned}
&f^h \rightharpoonup f, \quad \text{weakly in} \quad L^1\left((0,R)\times(0,T)\right) \text{ as } h \to \infty,\\
&C^h(u,v) \to \mathcal{C}(u,v),\ K^h(u,v) \to \mathcal{K}(u,v) \text{ a.e. } (u,v) \in (0,R)^2 \text{ as } h \to 0, \text{and}\\
&B^h(u,v;w) \to B(u,v;w) \quad \text{a.e.} \quad (u,v,w) \in (0,R)^3 \text{ as } h \to 0.
\end{aligned}
$$

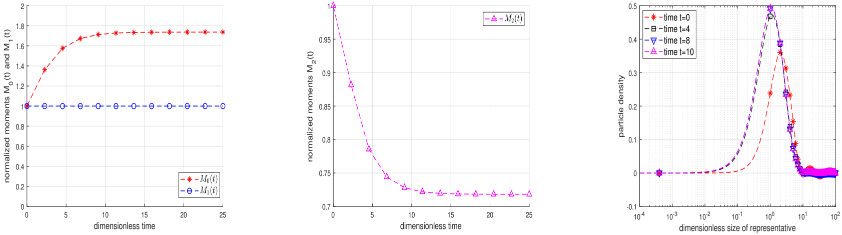
- Then for the approximate coagulation fragmentation fluxes $\mathcal{C}^h$, $\mathcal{F}^h$ there exist a subsequence of $(f^h)_h$, such that

$$\mathcal{C}^h \rightharpoonup \mathcal{C}^R_{\text{nc}} \quad \text{and} \quad \mathcal{F}^h \rightharpoonup \mathcal{F}^R_{\text{c}}$$

in $L^1\left((0,R]\times(0,T)\right)$ as $h \to 0$.

Numerical simulations

$$\begin{cases} \mathcal{C}(x,y) = A, & \text{with} \quad A \geq 0, \\ \mathcal{K}(x,y) = C\,(x+y)\,, & \text{with} \quad C \geq 0, \quad \text{with } f^{\text{in}}(x) = x\exp(-x). \\ \mathcal{B}(x,y;z) = 2\delta\left(x - \frac{y}{2}\right) & \end{cases}$$

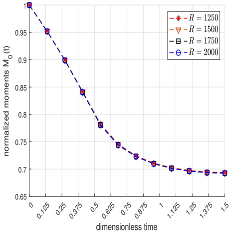


(a) (b) (c)

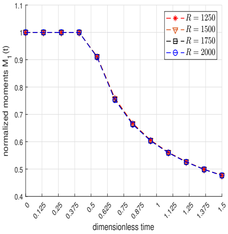
Figure: (a) Time evolution of the total mass of particles $M_1(t)$ and total number of particles $M_0(t)$, (b) Time evolution of second order moment $M_2(t)$, (c) Time evolution of number density $f(x,t)$ in log scale at time $t = 0, 4, 8$ and 10 for $A = 0.5$ and $C = 0.1$

Finite time shattering and blowup of moments

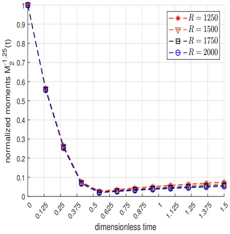
$$\begin{cases} \mathcal{C}(x,y) = Axy, & \text{with} \quad A \geq 0, \\ \mathcal{K}(x,y) = C\,(x+y)\,, & \text{with} \quad C \geq 0, \\ \mathcal{B}(x,y;z) = \frac{4x^2}{y^3} & \end{cases} \quad \text{with } f^{\text{in}}(x) = x\exp(-x).$$



(a)



(b)



(c)

Figure: (a) Time evolution of the total mass of particles $M_1(t)$ and total number of particles $M_0(t)$, (b) Time evolution of second order moment $M_2(t)$, (c) Time evolution of number density $f(x,t)$ in log scale at time $t = 0, 4, 8$ and 10 for $A = 0.5$ and $C = 0.1$

Improved formulation with weighted numerical flux

Scheme 2 (MCNP):

$$x_i \frac{\mathrm{d}N_i}{\mathrm{d}t} = \tilde{\mathcal{F}}_{i+1/2}(t) - \tilde{\mathcal{F}}_{i-1/2}(t). \tag{10}$$

Here, $\tilde{\mathcal{G}}_{i+1/2}$ is the revised weighted numerical flux at the i^{th} cell and is defined as

$$\tilde{\mathcal{F}}_{i+1/2} := \sum_{p=1}^{I} \sum_{q=i+1}^{I} \sum_{r=1}^{i} \beta_{r,q}^{p} \Theta_{q,p} \mathcal{K}_{q,p} N_q(t) N_p(t), \tag{11}$$

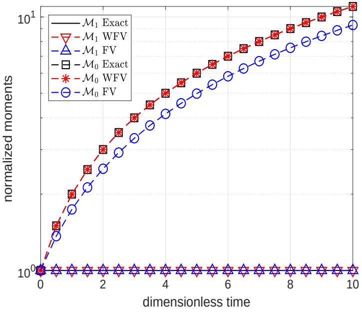
where, the weight function $\Theta_{q,p}$ is defined as,

$$\Theta_{q,p} := \frac{x_q \left(\nu(x_q, x_p) - 1\right)}{\sum_{j=1}^{q} (x_q - x_j) \int_{\Lambda_j} \mathcal{B}(x|x_q, x_p) \mathrm{d}x} \quad \text{with} \quad \Theta_{1,p} = 0., \tag{12}$$

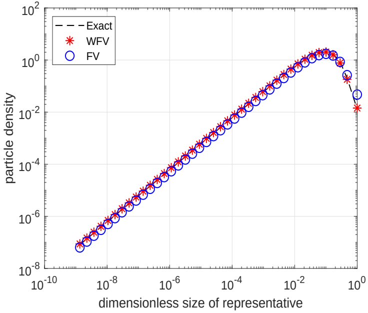
Test case for unbounded collision rate

Problem: $\mathcal{B}(x|y;z) = \dfrac{2}{y}$, $\mathcal{K}(x,y) = xy$ and $n_0(x) = \delta(x-1)$.

Exact solution: $n(t,x) = \exp(-tx)[2t + t^2(1-x)] + \delta(x-1)\exp(-t)$



(a) Dimensionless moments



(b) Number density

Figure: A comparison of numerical results of Test problem I

3−Wave turbulence kinetics

The general form of 3-wave kinetic equations reads

$$\partial_t f(t,p) = \iint_{\mathbb{R}^{2d}} \Big[R_{p,p_1,p_2}[f] - R_{p_1,p,p_2}[f] - R_{p_2,p,p_1}[f]\Big]\mathrm{d}p_1\mathrm{d}p_2, \ f(0,p) \ = \ f_0(p), \tag{13}$$

where $f(t,p)$ is the wave density at wavenumber $p \in \mathbb{R}^d$, $d \geq 2$ and $f_0(p)$ is the initial condition. Moreover,

$$R_{p,p_1,p_2}[f] := |V_{p,p_1,p_2}|^2 \delta(p - p_1 - p_2)\delta(\omega - \omega_1 - \omega_2)(f_1 f_2 - ff_1 - ff_2),$$

with the short-hand notations $f = f(t,p)$, $\omega = \omega(p)$ and $f_j = f(t,p_j)$, $\omega_j = \omega(p_j)$, for $p,\ p_j,\ j \in \{1,2\}$.

C-F formulation of 3− wave kinetics

$$\frac{\partial \mathbf{N}_\omega}{\partial \mathbf{t}} = \mathcal{Q}\left[\mathbf{N}_\omega\right](\mathbf{t}) := \mathbf{S_1}\left[\mathbf{N}_\omega\right] - \mathbf{S_2}\left[\mathbf{N}_\omega\right] - \mathbf{S_3}\left[\mathbf{N}_\omega\right], \quad \omega \in \mathbb{R}_+, \quad \mathbf{N}_\omega(\mathbf{0}) = \mathbf{N}^{\mathbf{in}}_\omega. \tag{14}$$

Here $\mathbf{N}_\omega$ be the **wave frequence spectrum** and the operators S_1, S_2 and S_3 can be expressed as

$$\begin{aligned}
S_1\left[N_\omega\right] &= \int_0^\omega K_1(\omega-\mu,\mu)N_{\omega-\mu}N_\mu \mathrm{d}\mu - 2\int_0^\infty K_1(\omega,\mu)N_\omega N_\mu \mathrm{d}\mu,\\
S_2\left[N_\omega\right] &= -\int_0^\omega K_2(\mu,\omega-\mu)N_\omega N_\mu \mathrm{d}\mu + \int_\omega^\infty K_2(\omega,\mu-\omega)N_{\mu-\omega}N_\mu \mathrm{d}\mu\\
&\quad + \int_0^\infty K_2(\omega,\mu)N_\omega N_{\omega+\mu}\mathrm{d}\mu,\\
S_3\left[N_\omega\right] &= -\int_0^\omega K_3(\mu,\omega-\mu)N_\omega N_{\omega-\mu}\mathrm{d}\mu + \int_\omega^\infty K_3(\omega,\mu-\omega)N_\omega N_\mu \mathrm{d}\mu\\
&\quad + \int_0^\infty K_3(\omega,\mu)N_\mu N_{\omega+\mu}\mathrm{d}\mu.
\end{aligned}$$

Energy cascading phenomena

The total wave action **N** and total wave energy **E** is given by:

$$N = \int_0^\infty N_\omega \mathrm{d}\omega, \qquad \text{and} \qquad E = \int_0^\infty \omega N_\omega \mathrm{d}\omega.$$

- For all $T_1 > 0$ we can always find a larger time $T_2 > T_1$ such that

$$\int_0^\infty \omega N_\omega(T_2)\mathrm{d}\omega < \int_0^\infty \omega N_\omega(T_1)\mathrm{d}\omega. \tag{15}$$

- Energy $g_\omega(t) = \omega N_\omega(t)$ at any time t as follows

$$g_\omega(t) = \bar{g}_\omega(t) + \tilde{g}(t)\delta_{\{\omega=\infty\}}. \tag{16}$$

  with $\bar{g}_\omega(0) = g_\omega(0)$ and $\tilde{g}(0) = 0$.
- For arbitrary truncation parameter R,

$$\int_0^R \omega N_\omega(t)\mathrm{d}\omega = \int_{\mathbb{R}_+} \chi_{[0,R]}(\omega)\omega N_\omega \mathrm{d}\omega \leq \mathcal{O}\left(\frac{1}{\sqrt{t}}\right), \quad \text{as} \quad t \longrightarrow \infty. \tag{17}$$

Numerical schemes

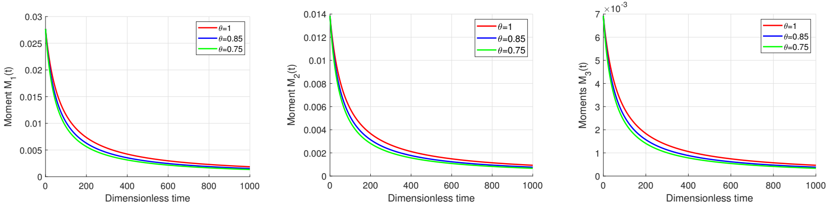
$$\mathcal{I}^i_{j,k} := \{(j,k) \in \mathbb{N} \times \mathbb{N} : \omega_{i-1/2} \leq \omega_j + \omega_k < \omega_{i+1/2}\},$$
$$\mathcal{J}^i_{j,k} := \{(j,k) \in \mathbb{N} \times \mathbb{N} : \omega_{i-1/2} \leq \omega_j - \omega_k < \omega_{i+1/2}\}.$$

The numerical *Finite Volume Scheme* (FVS) can be written as follows:

$$\begin{aligned} N_i^{n+1} = N_i^n + \Delta t^n \Bigg(& \sum_{(j,k)\in\mathcal{I}^i_{j,k}} K^1_{j,k} N_j^n N_k^n \frac{\Delta\omega_j \Delta\omega_k}{\Delta\omega_i} - 2\sum_{j=1}^{I} K^1_{i,j} N_i^n N_j^n \Delta\omega_j \\ & + \sum_{j=i+1}^{I} \left(K^2_{j-i,i} + K^3_{j-i,i}\right) N_i^n N_j^n \Delta\omega_j - \sum_{j=1}^{i-1} \left(K^2_{i-j,j} + K^3_{i-j,j}\right) N_i^n N_j^n \Delta\omega_j \\ & + \sum_{(j,k)\in\mathcal{J}^i_{j,k}} \left(K^2_{j-k,k} + K^3_{j-k,k}\right) N_j^n N_k^n \frac{\Delta\omega_j \Delta\omega_k}{\Delta\omega_i} \Bigg). \end{aligned} \tag{18}$$

Numerical Test

$$N_{\omega}^{in} = 1.25\omega e^{-100(\omega-0.25)^2},\ \omega \geq 0$$
$$\text{with } K_1(\omega,\mu) = K_2(\omega,\mu) = K_3(\omega,\mu) = (\omega\mu)^{\theta}$$



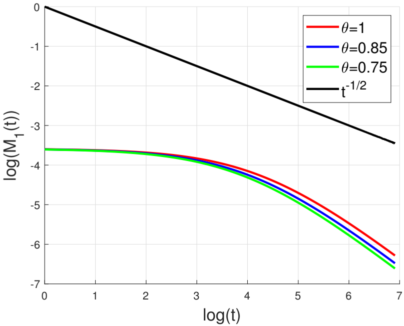
(a) (b) (c)

Figure: Time evolution of the (A) first, (B) second, and (C) third moments for different degrees of homogeneity θ.

Estimation of decay rate

For an arbitrary parameter R, we have the estimation

$$\int_{\mathbb{R}_+} \chi_{[0,R]}(\omega)\omega N_\omega \mathrm{d}\omega \leq \mathcal{O}\left(\frac{1}{\sqrt{t}}\right), \quad \text{as} \quad t \longrightarrow \infty.$$



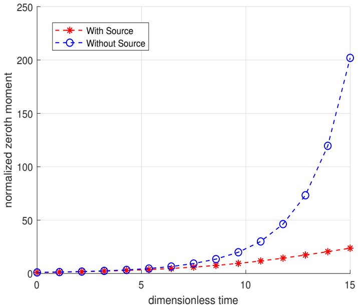
(a) In logarithmic scale

Figure: Decay of total energy for different values of θ.

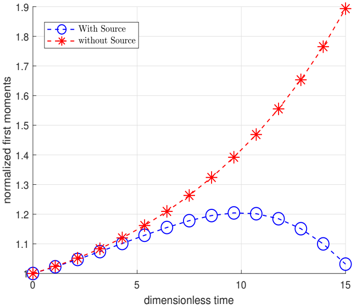
Observation & future scopes

- Collisional fragmentation with source term

$$\partial_t f(x,t) = \mathcal{F}(f)(x,t) - \underbrace{V(x,t)\, f(x,t)}_{source},$$



(a) Total Number of Particles



(b) Total Mass of the Syatem

Figure: Evolution of moment functions for $V = 100xe^{-100x}$

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Thank you