Boundary and Interior Control in a Diffusive Lotka-Volterra Model

João Carlos Barreira, Maicon Sonego and Enrique Zuazua

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CAPES

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Mathematical Biology and Lotka-Volterra Model (LVM)

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Introduction and main results

Mathematical Biology and Lotka-Volterra Model (LVM)

- The LVM, developed by Alfred J. Lotka and Vito Volterra in the early 20th century, models population interactions.
- Population dynamics studies the size and age composition of populations.
- LVM is widely used to analyze ecological systems such as competition and mutualism.

- [6], [7], [11]

Control Problem and Lotka-Volterra Model

We are interested in the following control problem applied to the general diffusive Lotka-Volterra competition model:

$$\begin{cases} u_t = d_1 \Delta u + u(a_1 - b_1 u - c_1 v) + h 1_\omega u, & (x,t) \in \Omega \times \mathbb{R}^+ \\ v_t = d_2 \Delta v + v(a_2 - b_2 u - c_2 v), & (x,t) \in \Omega \times \mathbb{R}^+ \\ (u(x,0), v(x,0)) = (u_0, v_0), & x \in \Omega \\ u(x,t) = c_u(x,t), \quad v(x,t) = c_v(x,t), & (x,t) \in \partial\Omega \times \mathbb{R}^+ \end{cases} \tag{1}$$

where

- a_i, b_i, c_i, d_i $(i = 1, 2)$ are positive parameters, and $\Omega \subset \mathbb{R}^N$ $(N = 1, 2, 3)$ is a smooth domain.
- $\Omega \subset \mathbb{R}^N$ $(N = 1, 2, 3)$ is a smooth domain with a regular boundary $\partial\Omega$.
- $h 1_\omega u$ represents an interior multiplicative control, where $\omega \subset \Omega$ is a non-empty open set.
- 1_ω is the characteristic function of ω, and c_u, c_v are boundary controls.

Constraints on the Solutions

For this model we assume constraints based on carrying capacities:

$$0 \leq u \leq \frac{a_1}{b_1}, \quad 0 \leq v \leq \frac{a_2}{c_2}. \tag{2}$$

Similarly, boundary controls c_u and c_v are constrained by:

$$0 \leq c_u \leq \frac{a_1}{b_1}, \quad 0 \leq c_v \leq \frac{a_2}{c_2}. \tag{3}$$

On one hand, conditions (2) and (3) add complexity to the problem, but on the other, they bring it closer to the reality, which is one of the main motivations for this study.

Equilibrium State

The strategy of combining internal multiplicative control with boundary controls proved to be crucial for obtaining our results. The inclusion of multiplicative control prevents the emergence of barriers that could prevent the trajectories from asymptotically reaching the desired targets, as demonstrated by [10]. We consider the equilibrium state $(0, a_2/c_2)$, which represents the survival of species v and the extinction of species u. Our goal is to steer the system toward this target using boundary and interior controls.

Theorem 1 (Asymptotic Control to Equilibrium State)

There exist boundary controls $c_u, c_v \in L^\infty(\partial\Omega)$ and an interior control $h \in L^\infty(\Omega)$ such that for every $(u_0, v_0) \in L^\infty(\Omega) \times L^\infty(\Omega)$, the solution (u, v) of (1) satisfies

$$\lim_{t\to\infty} (u(\cdot, t), v(\cdot, t)) = (0, \frac{a_2}{c_2}) \quad \textit{uniformly in } \Omega.$$

Symmetry and Target for Other Equilibrium

By symmetry, an analogous result can be obtained for the target $(a_1/b_1, 0)$:

$$\begin{cases} u_t = d_1\Delta u + u(a_1 - b_1 u - c_1 v), & (x,t) \in \Omega \times \mathbb{R}^+ \\ v_t = d_2\Delta v + v(a_2 - b_2 u - c_2 v) + \bar{h}1_\omega v, & (x,t) \in \Omega \times \mathbb{R}^+ \\ (u(x,0), v(x,0)) = (u_0, v_0), & x \in \Omega \\ u(x,t) = c_u(x,t), \quad v(x,t) = c_v(x,t), & (x,t) \in \partial\Omega \times \mathbb{R}^+ \end{cases} \tag{4}$$

Heterogeneous Coexistence

In the case of heterogeneous coexistence, we assume $d_1 = d_2 = d$ and $a_1 = a_2 = a$, leading to the following equilibrium:

$$(u^{**}(x), v^{**}(x)) = \left(\frac{c_2 - c_1}{b_1 c_2 - c_1 b_2}\theta(x), \frac{b_1 - b_2}{b_1 c_2 - c_1 b_2}\theta(x) \right)$$

where $\theta(x)$ is a solution of:

$$\begin{cases} d\Delta\theta + \theta(a-\theta) = 0, \;\; x \in \Omega \\ \theta = 0, \;\; x \in \partial\Omega \\ 0 < \theta < 1, \;\; x \in \Omega \end{cases}$$

We assume $b_1 > b_2$ and $c_1 < c_2$ to ensure positivity of u^{**} and v^{**}.

Finite-Time Controllability

For the target (u^{**}, v^{**}), finite-time controllability can be achieved by applying internal control to either the first or the second equation.

Theorem 2 (Finite-Time Controllability)

Assume $d_1 = d_2 = d$, $a_1 = a_2 = a$, $b_1 > b_2$, $c_1 < c_2$. *If*

$$d < \frac{a}{\lambda_1} \tag{5}$$

then for every $(u_0, v_0) \in L^\infty(\Omega) \times L^\infty(\Omega)$, *there exist boundary controls* $c_u, c_v \in L^\infty(\partial\Omega)$ *and interior control* $h \in L^2(\hat{\rho}^2(0, T); L^2(\omega)) \cap L^\infty(\omega \times (0, T))$, *where* $L^2(\hat{\rho}^2(0, T); L^2(\omega))$ *is a weighted Banach space, such that the solution satisfies*

$$(u(\cdot, T), v(\cdot, T)) = (u^{**}, v^{**}). \tag{6}$$

Biological Understanding and Control

- **Interior Multiplicative Control**: $h1_\omega u$ acts as an external agent affecting the reproduction rate of u in a subregion ω.
- **Positive** h: Represents a resource supplement, genetic or environmental factor enhancing u's growth.
- **Negative** h: Reduces the growth rate of u, due to pesticide or disease affecting only u.
- **Bounded Control**: h must remain bounded due to biological interpretation. L^2 regularity for control is typical, but L^∞ estimates are essential here.
- **Indirect Influence**: The interior control affects not only u, but also indirectly affects v through the coupling of the equations. - Symbolically: $h \sim u \sim v$.

Boundary Controls

- **Boundary Controls c_u and c_v**: Directly influence population densities at the boundary of the habitat.
- **Possible Interpretations**: Forced relocation of individuals at the boundary, or biological control strategies (e.g., introduction of predators, removal of competitors).

Dirichlet and Neumann Controls

- **Dirichlet Controls**: Focus of this work, guiding the solution to the desired target.
- **Extension to Neumann Controls**: Results can be extended to Neumann controls, where boundary flux can be viewed as a Neumann control. This interplay is inherent in problems where controls are applied along the entire boundary. Hence, the results presented here can also be interpreted as strategies for regulating boundary flux, with practical implications in areas such as boundary regulation, filtration, and migration facilitation.

Motivation of the study

- **Applications of LVM**: Widely used in economics, technology, and marketing to model industry competition (e.g., [9], [12]).
- **Control Theory**: LVM is appealing due to the significant impact of minor changes (e.g., sign alterations) on model dynamics and outcomes.
- **Notable Studies**:
  - *Microbiota Control*: [1] used LVM to control microbiotas with probiotics, antibiotics, and bactericides.
  - *Predator-Prey Dynamics*: [5] formulated an optimal control problem with hunting as a control variable, analyzing long-term optimal strategies and the "Turnpike" property.
- **Expansive Literature**: Numerous studies in control theory and related fields (e.g., [3], [4], [8]).

So some of the objectives of this work are:

- Obtain mathematical results for the competitive LVM to understand the ecological dynamics of biological populations.
- Insights gained help in species conservation and ecosystem management.
- Competition between two species for limited resources, with mathematical and numerical analysis.

Analisys of results

Results about global asymptotic controllability (Theorem 1)

We consider the targets $(a_1/b_1, 0)$ or $(0, a_2/c_2)$. We establish a global asymptotic controllability result (formalized in Theorem 1). That is, given any initial condition satisfying the constraints (2), we construct boundary controls c_u and c_v, along with a bounded interior control h (assuming $\omega = \Omega$), that steer the trajectories toward the target as $t \to \infty$.

Barrier Effect

As discussed by [10], when considering only boundary controls in a particular weak competition model (i.e., when in our model (1) we consider $c_1, b_2 < 1, a_1 = 1, b_1 = 1, c_2 = 1$), if $b_2 < a_2 < 1/c_1$ and $\lambda_1 < \min\{(1 - a_2 c_1)/d_1, (a_2 - b_2)/d_2\}$, then a barrier effect arises. This means barriers can be designed to prevent trajectories from reaching the desired targets, regardless of the boundary controls, which may vary in space and time.

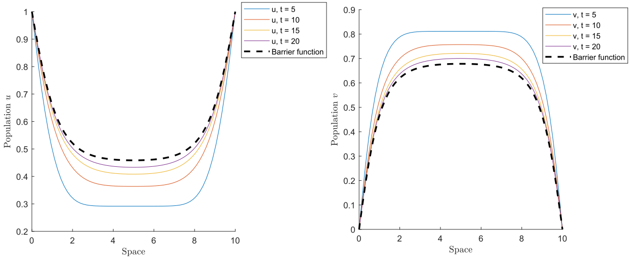


Figure: Barriers preventing the trajectories from approaching the target $(a_1/b_1, 0) = (1, 0)$.

Internal Control and Independence of Initial Conditions

In this context, internal control becomes crucial, as the action of this multiplicative control, acting as a potential, allows the system to be asymptotically directed toward the target.
Although our technique requires the internal control to act throughout the entire domain, we achieve asymptotic controllability regardless of the parameter values, using a multiplicative control in the interior.

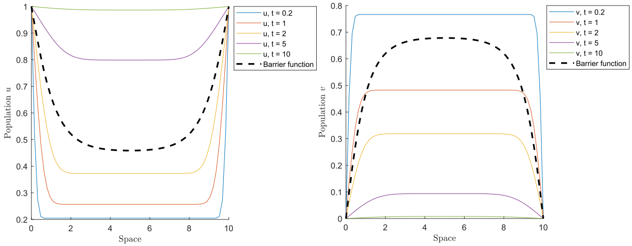


Figure: Trajectories crossing the barrier under the action of the internal control $\overline{h}$.

Challenges and Future Research

It is worth noting that these targets are located at the boundaries of the prescribed constraint intervals, which may complicate achieving a finite-time controllability result. In general, finite-time controllability would require oscillations of the trajectories around the target, leading to a violation of the constraints.

Undoubtedly, finite-time controllability for these targets is a significant challenge and should certainly be a topic for future research.

Results about Finite-Time Controllability (Theorem 2)

- The target (u^{**}, v^{**}) can be exactly reached in finite time when (5) holds. This inequality relates the geometry of the domain Ω (represented by λ_1), the diffusion capacity, and the reproduction rate of the species. It is necessary for the asymptotic approximation to the target, which corresponds to the first step of the control strategy.
- The finite-time controllability obtained in Theorem 2 is particularly interesting because, although $0 < u^{**}(x) < \frac{a_1}{b_1}$ and $0 < v^{**}(x) < \frac{a_2}{c_2}$ for $x \in \Omega$, we also have $u^{**}(x) = v^{**}(x) = 0$ for $x \in \partial\Omega$.
  As a consequence, the trajectory oscillations generally required to achieve finite-time controllability could potentially violate some of the constraints, since the target vanishes at the boundary. However, we establish a strategy that prevents this issue, primarily through the use of the multiplicative interior control.

Control Strategy

The strategy is as follows: given any initial condition satisfying (2), we keep the interior control $h \equiv 0$ as well as the boundary controls $c_u \equiv c_v \equiv 0$. Under these conditions, the system's dynamics drive the trajectories asymptotically toward the target.
Once the trajectories are sufficiently close to the target, we activate h and establish a finite-time local controllability result to ensure the target is reached exactly.

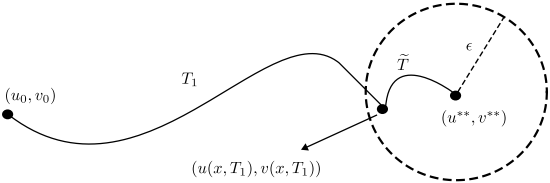


Figure: Controlled trajectory reaching exactly the target (u^{**}, v^{**}) at time $T = T_1 + \widetilde{T}$.

Key Aspects of the Strategy

Two crucial aspects of our strategy are:

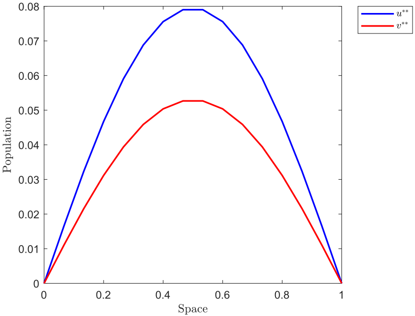
• Throughout the entire process, the values of u and v are zero at the boundary, thus not violating the constraint (3).

• The possibility of activating the interior control $h1_\omega u$ only when the trajectories are sufficiently close to the target. This allows us to choose h sufficiently small (thanks to the L^∞ estimates of h) and use comparison arguments to ensure the constraints in state (2).

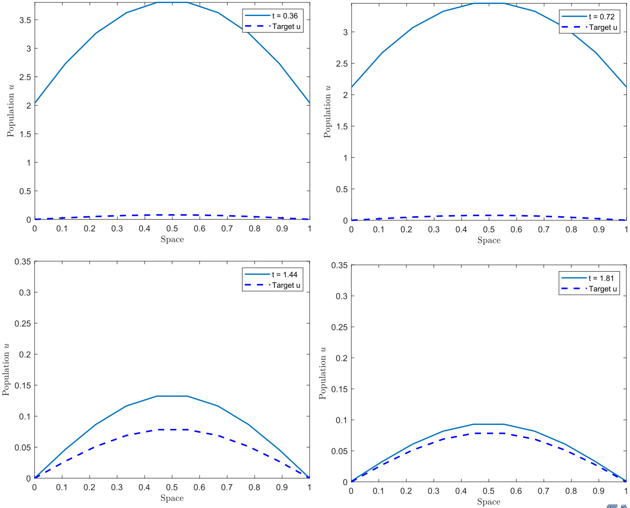
Numerical Simulations

Achieving finite-time controllability for this target using only boundary controls is a very delicate and challenging task, as its potential oscillations can violate the constraints. This highlights the essential role of multiplicative control within the domain in achieving finite-time controllability.

Under these assumptions, inequality (5) is satisfied as well as the other conditions of Theorem 2.



We recall that constraints must be assumed on the controls c_u and c_v (see (3)), and h must be bounded. In our example, we again assume $|h| \leq 1$. The minimum time obtained was $t = 1.8055$, and figures display the behaviors of u and v, respectively.



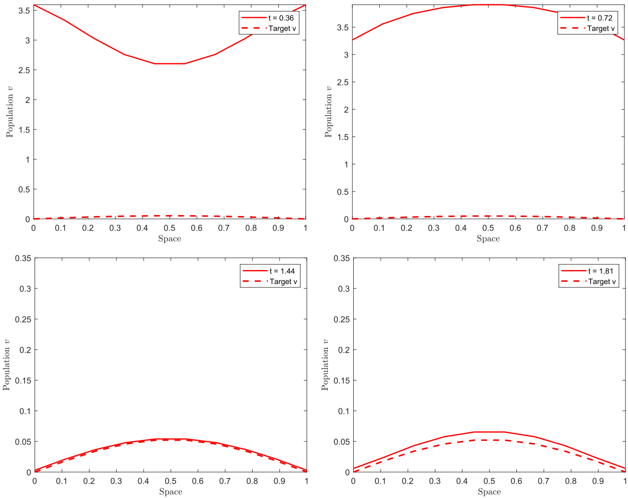


Figure: Trajectories approaching v^{**}.

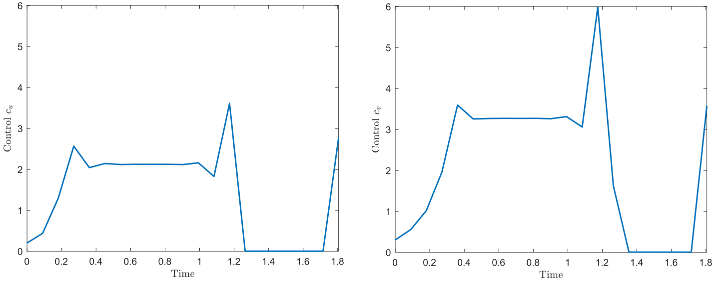


Figure: Boundary controls c_u and c_v.

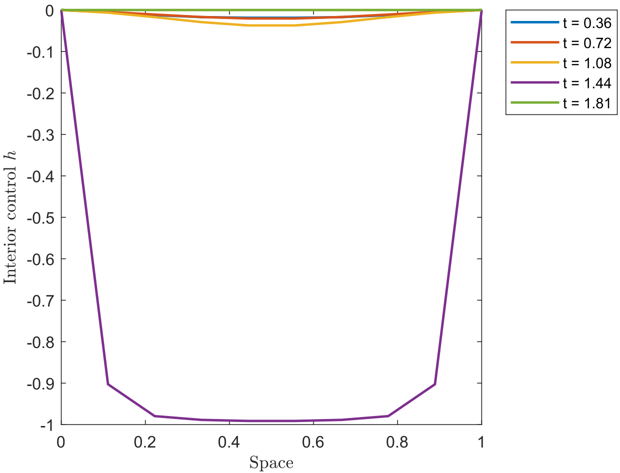


Figure: Interior control h.

Conclusions

- *Survival of one of the species:* $(a_1/b_1, 0)$ *and* $(0, a_2/c_2)$. The asymptotic controllability is achieved through the combination of controls acting on the boundary and in the interior. More specifically, we provide boundary controls c_u and c_v satisfying (3), along with a multiplicative interior control hu such that for any initial condition (u_0, v_0) $(0 \leq u_0 \leq a_1/b_1$ and $0 \leq v_0 \leq a_2/c_2)$, The trajectory converges to the desired target as $t \to \infty$. This result holds independently of the problem parameters and the domain Ω. As mentioned earlier, achieving finite-time controllability for these targets is particularly challenging due to the considered constraints on controls and trajectories.

- *Heterogeneous coexistence:* (u^{**}, v^{**}). In this case, provides finite-time global controllability; roughly speaking, the target is exactly reached at some time T from any initial condition satisfying the constraints. This case is particularly interesting since the target (u^{**}, v^{**}) is zero on the boundary, and thus finite-time controllability could require some oscillation of the trajectories, potentially leading to a violation of one of the constraints. However, the interior multiplicative control $h1_{\omega}u$, combined with an appropriate control strategy, allows us to consider h sufficiently small so that comparison arguments can be used to ensure the assumed constraints.

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Thanks