Control of a Lotka-Volterra System with Weak Competition

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Outline

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The Diffusive Lotka-Volterra Model

The Diffusive Lotka-Volterra Model

The Lotka-Volterra model reflects real ecological interactions where species compete for limited resources, potentially leading to coexistence, dominance of one species, or total extinction.

Comprehending the mechanisms governing these systems can yield critical insights for developing strategies in ecological management and biodiversity conservation.

The Diffusive Lotka-Volterra Model

Our problem is described below

$$\begin{cases} u_t = d_1 u_{xx} + u(1 - u - k_1 v), & (x,t) \in (0,L) \times \mathbb{R}^+ \\ v_t = d_2 v_{xx} + v(a - v - k_2 u), & (x,t) \in (0,L) \times \mathbb{R}^+ \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in (0,L) \end{cases} \tag{1}$$

where

- u and v are the population densities of the two species competing in $(0, L)$ and $(u(x,t), v(x,t))$ is the state to be controlled;
- $u_0 \in L^\infty((0,L); [0,1])$ and $v_0 \in L^\infty((0,L); [0,a])$ are the initial conditions;
- $d_1, d_2 > 0$ are constants representing the diffusion rates;
- $a > 0$ is a constant representing the intrinsic growth rate of v;
- $k_1, k_2 > 0$ are constants representing the inter-specific competition between u and v.

Note that 1 and a are the carrying capacities of u and v, respectively, and therefore it is natural to constrain the solutions by these values, i.e.

$$0 \leq u(x,t) \leq 1 \text{ and } 0 \leq v(x,t) \leq a \text{ for all } (x,t) \in (0,L) \times \mathbb{R}^+. \tag{2}$$

Moreover, we assume

$$k_1, k_2 < 1$$

and this condition results in a weak competition system.

We suppose boundary controls constraints

$$c_u(x,t), c_v(x,t) \in L^\infty(\{0,L\} \times \mathbb{R}^+),$$

$$\begin{cases} u(x,t) = c_u(x,t) & (x,t) \in \{0,L\} \times \mathbb{R}^+ \\ v(x,t) = c_v(x,t) & (x,t) \in \{0,L\} \times \mathbb{R}^+ \end{cases} \tag{3}$$

satisfying

$$0 \leq c_u \leq 1 \text{ and } 0 \leq c_v \leq a. \tag{4}$$

Definition 1
We say that (1) is controllable in infinite time towards $(\overline{u}(x), \overline{v}(x))$ if for any initial condition $(u_0(x), v_0(x))$ $(0 \leq u_0 \leq 1,\ 0 \leq v_0 \leq a)$, there exist controls $c_u \in L^\infty(\{0, L\} \times \mathbb{R}^+; [0, 1])$, $c_v \in L^\infty(\{0, L\} \times \mathbb{R}^+; [0, a])$ such that

$$(u(x,t), v(x,t)) \to (\overline{u}(x), \overline{v}(x))$$

uniformly in $[0, L]$ as $t \to \infty$.

Definition 2
We say that (1) is controllable in finite time towards $(\overline{u}(x), \overline{v}(x))$ if for any initial condition $(u_0(x), v_0(x))$ $(0 \leq u_0 \leq 1,\ 0 \leq v_0 \leq a)$, there exist $T > 0$ and controls $c_u \in L^\infty(\{0, L\} \times \mathbb{R}^+; [0, 1])$, $c_v \in L^\infty(\{0, L\} \times \mathbb{R}^+; [0, a])$ such that

$$(u(x,T), v(x,T)) = (\overline{u}(x), \overline{v}(x)).$$

The targets to be considered in this work are described below.

- a homogeneous state of species coexistence

$$(u^*, v^*) = \left(\frac{1 - k_1 a}{1 - k_1 k_2}, \frac{a - k_2}{1 - k_1 k_2} \right), \tag{5}$$

  which only makes sense to us when $k_2 < a < 1/k_1$;
- the extinction of the species $(0, 0)$;
- the survival of one of the species $(1, 0)$ and $(0, a)$;
- a heterogeneous state of species coexistence for the case $d_1 = d_2 = d$ and $a = 1$,

$$(u^{**}(x), v^{**}(x)) = \left(\left(\frac{1 - k_1}{1 - k_1 k_2} \right) \theta(x), \left(\frac{1 - k_2}{1 - k_1 k_2} \right) \theta(x) \right) \tag{6}$$

  where $\theta(x)$ is a smooth function that satisfies

$$\begin{cases} d\theta''(x) + \theta(x)(1 - \theta(x)) = 0 \ \ x \in (0, L) \\ \theta(0) = \theta(L) = 0, \\ 0 < \theta(x) < 1, \ \ x \in (0, L). \end{cases} \tag{7}$$

Main Results

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Theorem 1

If $k_2 < a < \frac{1}{k_1}$ then (1) is controllable in infinite time towards (u^*, v^*) defined in (5).

- The condition on the parameters a, k_1, k_2 is necessary for the existence of a coexistence state (u^*, v^*) and sufficient for the controllability towards it in the weak competition regime.

- Controllability towards the target (u^*, v^*) is independent of the domain size L and of the parameters d_1, d_2.

Theorem 2

(*i*) If $L \leq \sqrt{\frac{d_2}{a}}\pi$ or $k_2 > a$ then (1) is controllable in infinite time towards $(1, 0)$.

(*ii*) If $L \leq \sqrt{d_1}\pi$ or $k_1 > \frac{1}{a}$ then (1) is controllable in infinite time towards $(0, a)$.

- This theorem illustrates the complex balance between diffusion, competition, and domain size in determining species dominance.

- A natural question arises regarding controllability towards $(1, 0)$ or $(0, a)$ when none of these conditions hold. The answer is provided in the following theorem.

Theorem 3

If $k_2 < a < 1/k_1$ and

$$L > \max\left\{\sqrt{\frac{d_1}{1-ak_1}}\pi, \sqrt{\frac{d_2}{a-k_2}}\pi\right\} \tag{8}$$

then (1) is not controllable in infinite time towards either $(1,0)$ or $(0,a)$.

- Note that (8) contradicts all conditions present in Theorem 2. Non-controllability is demonstrated through the construction of barrier functions that prevent certain initial states from approaching $(1,0)$ or $(0,a)$.

Theorem 4

(i) If $L \leq \min\left\{\sqrt{d_1}\pi, \sqrt{\frac{d_2}{a}}\pi\right\}$, then the system (1) is controllable in infinite time towards $(0,0)$;

(ii) if $a < \frac{1}{k_1}$ and $L > \sqrt{\frac{d_1}{1-ak_1}}\pi$ or $a > k_2$ and $L > \sqrt{\frac{d_2}{a-k_2}}\pi$, then the system (1) is not controllable in infinite time towards $(0,0)$.

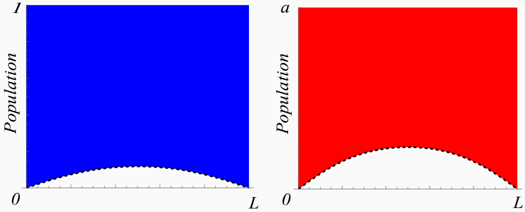

Figure 1: Barrier functions that prevent u and/or v from extinction.

Theorem 5

If $a = 1$, $d_1 = d_2 = d > 0$ and $L > \sqrt{d}\pi$, then (1) is controllable in infinite time towards a specific heterogeneous coexistence state (u^{**}, v^{**}) defined in (6).

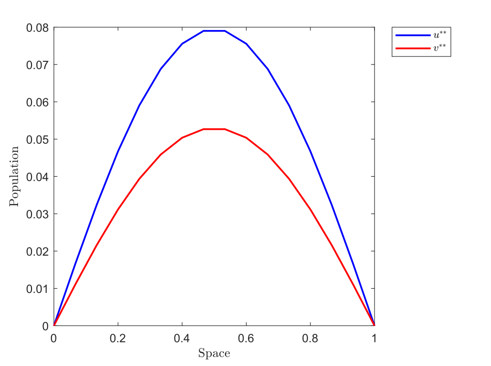

Figure 2: The target (u^{**}, v^{**}).

Numerical Simulations

Homogeneous Coexistence

In Figure 3, we have a simulation regarding the target

$$(u^*, v^*) = \left(\frac{1 - k_1 a}{1 - k_1 k_2}, \frac{a - k_2}{1 - k_1 k_2} \right) \approx (0.45, 0.68).$$

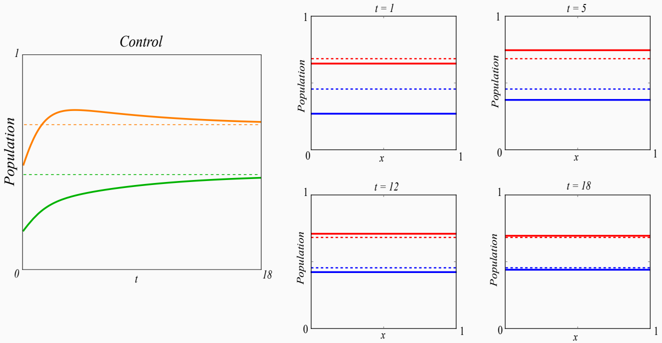

Figure 3: Controls c_u (green line) and c_v (orange line) approach u^* (dashed green line) and v^* (dashed orange line). Solutions u (blue line) and v (red line) approach u^* (dashed blue line) and v^* (dashed red line).

The initial condition assumed was $(u_0, v_0) = (0.2, 0.5)$, and we can observe that, for each fixed t, the solutions arising from this strategy are constant functions of x. This is not the case when we simulate the optimal control of the problem with the same parameters.

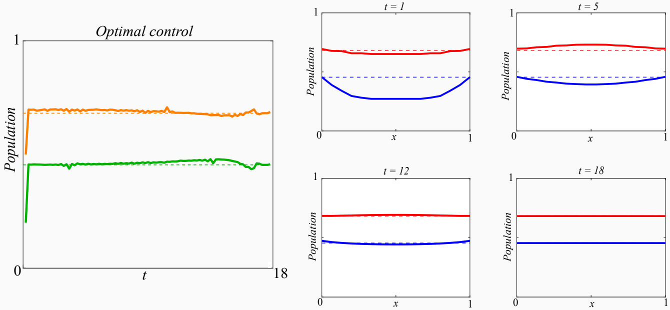

Figure 4: Optimal controls c_u (green line) and c_v (orange line) approach u^* (dashed green line) and v^* (dashed orange line). Solutions u (blue line) and v (red line) approach u^* (dashed blue line) and v^* (dashed red line).

Barrier Functions

Here, we assume $a < \dfrac{1}{k_1}$ and $L > \sqrt{\dfrac{d_1}{1 - ak_1}}\pi$, and then we will have the formation of a barrier solution that prevents u from approaching 0. Figure 5 was generated with the parameters: $k_1 = 0.8$, $k_2 = 0.7$, $a = 1$, $d_1 = 0.01$, $d_2 = 4$, and $L = 1$.

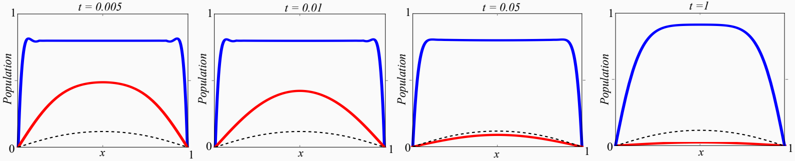

Figure 5: Solutions u (blue line), v (red line) and the barrier solution (dashed black line) (case $\sqrt{d_1/(1 - ak_1)}\pi < L < \sqrt{d_2/(a - k_2)}\pi$).

Note that, in this simulated case, the barrier solution only prevents the extinction of species u.

Assuming

$$L > \max\{\sqrt{d_1/(1-ak_1)}\pi, \sqrt{d_2/(a-k_2)}\pi\} \tag{9}$$

we have the existence of barrier functions for u and v.

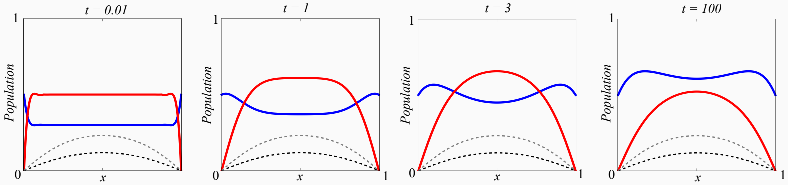

Figure 6: Solutions u (blue line), v (red line) and the barrier solutions to u and v (dashed black line and dashed gray line, respectively).

Finite-Time Controllability

Theorem 6

If $k_2 < a < \frac{1}{k_1}$ then (1) is controllable in finite time towards (u^*, v^*) defined in (5).

- For this theorem, our result on asymptotic controllability towards (u^*, v^*) is essential. Indeed, once the trajectory is sufficiently close to target (u^*, v^*) (Theorem 1), local finite-time controllability results can be employed to reach the target in finite time.
- This is possible because, locally, the control can oscillate above and below the target, allowing the trajectory to reach it in finite time. For this reason, controllability in finite time is not expected towards the targets $(1, 0)$, $(0, a)$, $(0, 0)$ and (u^{**}, v^{**}), in these cases, the local oscillation of the control is restricted by the constraints.

Concluding Remarks

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- All results can be generalized to n-dimensional domains ($n > 1$). Our choice of $n = 1$ was made to facilitate understanding.

The issues addressed here naturally lead us to envision new possibilities. Small changes in problem (1) significantly alter the model as well as its dynamics and results.

- Strong competition between the species ($k_1, k_2 > 1$).
- We can consider only one species with diffusion capacity, assuming, for example, $d_1 > 0$ and $d_2 = 0$.
- other relationships between the species can be studied, such as predator-prey, by assuming, for instance, that $k_1 > 0$ and $k_2 < 0$.
- It is certainly a great challenge to consider the case with 3 or more interacting species.

This presentation is based on the following work:

- Sonego, M.; Zuazua, E. **Control of a Lotka-Volterra system with weak competition**. 2024. *Submitted*.

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