Conservation laws on a star-shaped network

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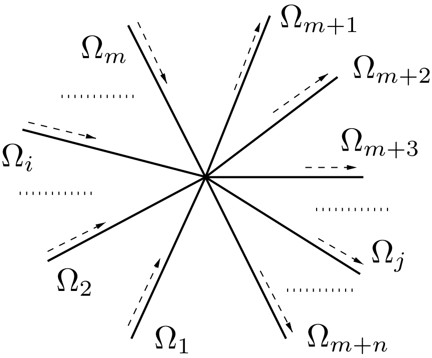
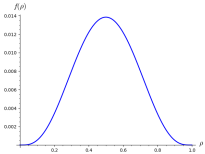
We consider a junction consisting of *m* incoming and *n* outgoing edges.



- Incoming edges: $x \in \Omega_i = \mathbb{R}_-$, $i = 1, \ldots, m$;
- Outgoing edges: $x \in \Omega_j = \mathbb{R}_+$, $j = m+1, \ldots, m+n$;
- The junction is located at $x = 0$.

On each edge we consider the evolution problem



$$\partial_t \rho_h + \partial_x f_h(\rho_h) = 0, \qquad h = 1, \ldots, m+n,$$

- ρ_h conserved quantity,
- f_h flux : possibly different, non degenerate nonlinear and bell-shaped
  - $f_h : [0, R] \to \mathbb{R}_+$, Lipschitz continuous,
  - $f_h(0) = 0 = f_h(R)$,
  - $\exists \bar{\rho} \in [0, R]$, such that $f_h'(\rho)(\bar{\rho} - \rho) > 0$, for a.e. $\rho \in [0, R]$.

We postulate conservation at the junction

$$\frac{d}{dt} \sum_{h=1}^{m+n} \int_{\Omega_h} \rho_h(t, x) dx = 0,$$

which we rewrite as

$$\sum_{i=1}^{m} f_i(\rho_i(t, 0^-)) = \sum_{j=m+1}^{m+n} f_j(\rho_j(t, 0^+)).$$

Weak solutions, edge-wise entropy admissible

We call weak solution on the star-shaped network $\vec{\rho} = (\rho_1, \dots, \rho_{m+n})$

- $\rho_h \in L^\infty(\mathbb{R}_+ \times \Omega_h; [0, R])$;
- ρ_h is a Kruzhkov entropy solution in $\mathbb{R}_+ \times \{\Omega_h \setminus \partial\Omega_h\}$.
  Namely $\forall k \in [0, R]$ and $\forall \varphi \in \mathcal{C}_c^1(\mathbb{R}_+ \times \Omega_h)$, $\varphi \geq 0$

$$\int_{\mathbb{R}_+} \int_{\Omega_h} |\rho_h - k|\varphi_t + \mathrm{sign}(\rho_h - k)\,(f_h(\rho_h) - f_h(k))\,\varphi_x \, dx\, dt + \int_{\Omega_h} |u_0^h(x) - k|\varphi(0, x)\, dx \geq 0 \,;$$

- conservation at the junction holds.

➽ Weak solutions are not unique in general.

Different approaches to solve the Riemann problem at the junction
(edge-wise constant initial conditions)

- Holden-Risebro 1995
  maximize a concave "entropy" function at the junction ;
- Coclite-Piccoli 2002, Coclite-Garavello-Piccoli 2005
  traffic distribution matrix + optimization ;
- Lebacque 1996, Lebacque-Khoshyaran 2002
  Supply-Demand model.
  Link between junctions and boundary value problems ;
- Coclite-Garavello 2010, Andreianov-Coclite-D. 2017, Coclite-D. 2020
  Vanishing Viscosity approximation, Germ of admissible solutions
- Fjordholm-Musch-Risebro 2022 (2 papers)
  Finite volumes approximation, Germ of admissible solutions ;
- Holle 2021, (Holle-Herty-Westdickenberg 2020, p-systems)
  Kinetic BGK approximation with maximal entropy dissipation .

The junction as a family of IBVPs

Fix $\vec{u}_0 = (u_0^1, \ldots, u_0^{m+n})$

We look for $\vec{\rho} = (\rho_1, \ldots, \rho_{m+n})$ s.t. $\forall h$, $\rho_h \in L^\infty(\mathbb{R}_+ \times \Omega_h, [0, R])$ solves

$$\begin{cases} \partial_t \rho_h + \partial_x f_h(\rho_h) = 0, & \text{on }]0, T[\times\Omega_h, \\ \rho_h(t, 0) = v_h(t), & \text{on }]0, T[, \\ \rho_h(0, x) = u_0^h(x), & \text{on } \Omega_h, \end{cases}$$

where $\vec{v} : \mathbb{R}_+ \to [0, R]^{m+n}$ is to be fixed at each $t > 0$

- to ensure conservation,
- depending on the state of the system,
- depending on the model.

. . . Solves? Weak entropy solution for the IBVP

u is a weak entropy solution for the IBVP

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, & \text{for } (t,x) \text{ in } \mathbb{R}_+ \times \mathbb{R}_- \\ u(t,0^-) = u_b(t), & \\ u(0,x) = u_0(x), & \end{cases}$$

if

- u is a Kruzhkov entropy solution in the interior of $\mathbb{R}_+ \times \mathbb{R}_-$,
- u satisfies the boundary condition in the sense of Bardos-LeRoux-Nédélec

  $$\operatorname{sign}(u(t,0^-) - u_b(t))\big(f(u(t,0^-)) - f(k)\big) \geq 0, \qquad \forall k \in \mathcal{I}(u(t,0^-), u_b(t)),$$

  which also write as

  $$f(u(t,0^-)) = \operatorname{God}(u(t,0^-), u_b(t)).$$

Geometrical meaning of the (BLN) condition

The solution of

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, & \text{for } (t,x) \text{ in } \mathbb{R}_+ \times \mathbb{R}_- \\ u(t,0^-) = u_b, \\ u(0,x) = u_0, \end{cases}$$

is the restriction to $\mathbb{R}_+ \times \mathbb{R}_-$ of the Riemann problem

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, & \text{for } (t,x) \text{ in } \mathbb{R}_+ \times \mathbb{R} \\ u(0,x) = \begin{cases} u_0, & x \leq 0, \\ u_b, & x > 0. \end{cases} \end{cases}$$

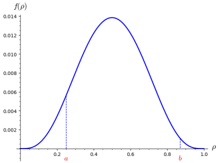
Then we say that the boundary condition is satisfied as soon as

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, & \text{for } (t,x) \text{ in } \mathbb{R}_+ \times \mathbb{R} \\ u(0,x) = \begin{cases} u(t,0^-), & x \leq 0, \\ u_b, & x > 0. \end{cases} \end{cases}$$

only contains waves of positive speed.

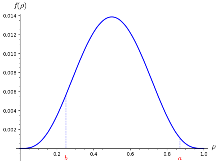
Godunov's flux

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u_0(x) = \begin{cases} a, & \text{if } x < 0, \\ b, & \text{if } x > 0. \end{cases} \end{cases}$$



The Godunov flux is the function $(a,b) \mapsto f(u(t,0^-)) = f(u(t,0^+))$. Analytically

$$\text{God}(a,b) = \begin{cases} \min_{s\in[a,b]} f(s) & \text{if } a \le b, \\ \max_{s\in[b,a]} f(s) & \text{if } a \ge b. \end{cases}$$



Basic properties:

- Consistency: for all $a \in [0,R]$, $\text{God}(a,a) = f(a)$;
- Monotonicity and Lipschitz continuity: $\exists L > 0$ s.t. $\forall (a,b) \in [0,R]^2$

$$0 \le \partial_a \text{God}(a,b) \le L, \qquad -L \le \partial_b \text{God}(a,b) \le 0.$$

To describe the Riemann Solver at the junction we define

- for $i = 1, \dots, m$
  Demand function : $\Delta_i(\rho_i(t, 0^-)) = \max_s \text{God}_{f_i}(\rho_i(t, 0^-), s)$;
- for $j = m+1, \dots, m+n$
  Supply function : $\Sigma_j(\rho_j(t, 0^+)) = \max_s \text{God}_{f_j}(s, \rho_j(t, 0^+))$;

and we use them to determine the passing flow at the junction from each of the incoming roads

$$\Gamma_i : [0, R]^3 \to [0, f_i^{\max}], \qquad i = 1, \dots, m.$$

At a 1-2 divide

Fix a distribution factor $\beta \in (0,1)$.

The passing flow at the junction is $\Gamma_1 : [0,R]^3 \to [0, f_1^{\max}]$ such that :

- If $\beta\Delta_1(\rho_{1,0}) \leq \Sigma_2(\rho_{2,0})$, and $(1-\beta)\Delta_1(\rho_{1,0}) \leq \Sigma_3(\rho_{3,0})$ then $\Gamma_1(\vec{\rho}_0) = \Delta_1(\rho_{1,0})$,
- otherwise, $\Gamma_1(\vec{\rho}_0) = \min\{\beta^{-1}\Sigma_2(\rho_{2,0}), (1-\beta)^{-1}\Sigma_3(\rho_{3,0})\}$.

In both cases

$$\begin{cases} v_1 = \left(f_{1|_{[\bar{\rho}_1, R]}}\right)^{-1}(\Gamma_1), \\ v_2 = \left(f_{2|_{[0,\bar{\rho}_2]}}\right)^{-1}(\beta\Gamma_1), \\ v_3 = \left(f_{3|_{[0,\bar{\rho}_3]}}\right)^{-1}((1-\beta)\Gamma_1). \end{cases}$$

Remark

The application $\vec{\rho}_0 \mapsto \vec{v}$ is not monotone : $\rho_{h,0} \geq \tilde{\rho}_{h,0} \not\Rightarrow v_h \geq \tilde{v}_h$.

At a $2-2$ junction

We introduce a distribution matrix of the form

$$A = \begin{pmatrix} \beta & \gamma \\ 1-\beta & 1-\gamma \end{pmatrix}$$

with β and γ in $]0,1[\setminus\{1/2\}$.
Then

- $(\Gamma_1, \Gamma_2) \in [0, \Delta_1] \times [0, \Delta_2]$;
- $A \cdot (\Gamma_1, \Gamma_2)^T$ must be in $[0, \Sigma_3] \times [0, \Sigma_4]$;
- $\Gamma_1 + \Gamma_2$ should be as large as possible, under the constraints above.

Remark

Counterexemples show that this solver lacks L^1-Lipschitz continuity with respect to the initial conditions.

See [Coclite-Garavello-Piccoli, 2005] and the book by Garavello and Piccoli *Traffic Flow on Networks*.

Vanishing viscosity approximations

[Coclite-Garavello, 2010]

Fix $\varepsilon > 0$ and consider

$$\begin{cases} \partial_t \rho_h^\varepsilon + \partial_x f_h(\rho_h^\varepsilon) = \varepsilon \partial_{xx}^2 \rho_h^\varepsilon, \\ \sum_{i=1}^m \left(f_i(\rho_i^\varepsilon(t,0)) - \varepsilon \partial_x \rho_i^\varepsilon(t,0)\right) = \sum_{j=m+1}^{m+n} \left(f_j(\rho_j^\varepsilon(t,0)) - \varepsilon \partial_x \rho_j^\varepsilon(t,0)\right), \\ \rho_h^\varepsilon(t,0) = \rho_{h'}^\varepsilon(t,0), \\ \rho_h^\varepsilon(0,x) = u_{h,\varepsilon}^0(x), \end{cases}$$

where the approximated initial conditions $\vec{u}_{0,\varepsilon}$ satisfy

$$u_{h,\varepsilon}^0 \in W^{2,1} \cap C^\infty(\Omega_h; [0,R]),$$

$$u_{h,\varepsilon}^0 \longrightarrow \rho_{0,h}, \text{ a.e. and in } L^p(\Omega_h),\ 1 \le p < \infty, \text{ as } \varepsilon \to 0,$$

$$\left\|u_{h,\varepsilon}^0\right\|_{L^1(\Omega_h)} \le \left\|\rho_{0,h}\right\|_{L^1(\Omega_h)}, \quad \left\|\partial_x u_{h,\varepsilon}^0\right\|_{L^1(\Omega_h)} \le TV(\rho_{0,h}), \quad \varepsilon\left\|\partial_{xx}^2 u_{h,\varepsilon}^0\right\|_{L^1(\Omega_h)} \le C_0,$$

with $C_0 > 0$ independent from ε, h.

Theory of semigroups $\Rightarrow$ for any fixed $\varepsilon > 0$ there exists a unique $\vec{\rho^{\varepsilon}}$ s.t.

$$\rho_h^{\varepsilon} \in C([0,\infty); L^2(\Omega_h)) \cap L^1_{loc}((0,\infty); W^{2,1}(\Omega_h)), \quad \forall h,$$

$$0 \le \rho_h^{\varepsilon} \le R, \qquad \sum_{h=1}^{m+n} \|\rho_h^{\varepsilon}(t,\cdot)\|_{L^1(\Omega_h)} \le \sum_{h=1}^{m+n} \|\rho_{0,h}\|_{L^1(\Omega_h)}, \quad \forall t \ge 0,$$

+ additional a priori estimates.

Compensated compactness $\Rightarrow$ existence of a sequence $\{\varepsilon_\ell\}_{\ell\in\mathbb{N}}$, $\varepsilon_\ell \to 0$ and a weak solution $\vec{\rho}$ of the inviscid Cauchy problem at the junction s.t.

$$\rho_h^{\varepsilon_\ell} \longrightarrow \rho_h, \text{ a.e. and in } L^p_{loc}(\mathbb{R}_+ \times \Omega_h),\ 1 \le p < \infty,$$

for every $h \in \{1, \ldots, m+n\}$.

In [Andreianov-D.-Coclite, 2017] we further characterize the limit solution and prove its uniqueness. More details in the following...

Analogously to [Diehl, 2009], [Andreianov-Mitrović, 2015] for $m = n = 1$

The condition $\rho_h^\varepsilon(t,0) = \rho_{h'}^\varepsilon(t,0)$, $\forall h,\ h' \in \{1,\dots,m+n\}$, translates into

$$v_h(t) = v_{h'}(t),$$

for the family of hyperbolic IBVPs at the junction.

$\vec{\rho} = (\rho_1,\dots,\rho_{m+n})$ is an admissible solution if there exists v in $L^\infty(\mathbb{R}_+, [0, R])$ s.t.

- $\vec{\rho}$ is a weak solution,
- each component ρ_h is weak entropy solution for the IBVP

$$\begin{cases} \rho_{h,t} + f_h(\rho_h)_x = 0, & \text{on }]0, T[\times\Omega_h, \\ \rho_h(t,0) = v(t), & \text{on }]0, T[, \\ \rho_h(0,x) = u_0^h(x), & \text{on } \Omega_h. \end{cases}$$

We call germ of vanishing viscosity the set

$$\mathcal{G}_{VV} = \left\{ \vec{k} \in [0, R]^{m+n}, \text{ stationary edge-wise constant admissible solution} \right\}$$

Lemma

If ρ_h is a Kruzhkov entropy solution in the interior of $\mathbb{R}_+ \times \Omega_h$, $\forall h \in \{1, \ldots, m+n\}$, TFAE

- $\vec{\rho}$ is an admissible solution;
- for a.e. $t \in \mathbb{R}_+$, the vector of traces $\gamma\vec{\rho}(t) = (\rho_1(t, 0^-), \ldots, \rho_{m+n}(t, 0^+))$ is in $\mathcal{G}_{VV}$;
- $\forall \vec{k} \in \mathcal{G}_{VV}$, $\vec{\rho}$ satisfies adapted entropy inequality on the network: $\forall \xi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R})$, $\xi \geq 0$,

$$\sum_{h=1}^{m+n} \left(\int_{\mathbb{R}_+} \int_{\Omega_h} \{ |\rho_h - k_h| \xi_t + \operatorname{sign}(\rho_h - k_h)(f_h(\rho_h) - f_h(k_h))\xi_x \} \, dx \, dt \right) \geq 0.$$

Well-posedness for admissible solutions

Theorem

- For any $\vec{u}_0$ there exists at most one admissible solution $\vec{\rho}$.
- If $\vec{\rho}$ and $\vec{\rho}^*$ are admissible solutions corresponding to $\vec{u}_0$ and $\vec{v}_0$, then

$$\sum_{h=1}^{m+n} \|\rho_h(t) - \rho_h^*(t)\|_{L^1(\Omega_h;\mathbb{R})} \leq \sum_{h=1}^{m+n} \left\| u_h^0 - v_h^0 \right\|_{L^1(\Omega_h;\mathbb{R})}$$

Fundamental properties of $\mathcal{G}_{VV}$

- completeness : we can associate an admissible solution to any Riemann datum.
- dissipativity : for any $\vec{k}_1$, $\vec{k}_2$ in $\mathcal{G}_{VV}$ with $\vec{k}_\ell = (k_1^\ell, \ldots, k_{m+n}^\ell)$, $\ell = 1, 2$,

$$\sum_{i=1}^{m} \operatorname{sign}(k_i^1 - k_i^2) \left(f_i(k_i^1) - f_i(k_i^2) \right) - \sum_{j=m+1}^{m+n} \operatorname{sign}(k_j^1 - k_j^2) \left(f_j(k_j^1) - f_j(k_j^2) \right) \geq 0.$$

- maximality : if $\vec{k}_1$ satisfies $\Uparrow$ for all $\vec{k}_2$ in $\mathcal{G}_{VV}$, then $\vec{k}_1 \in \mathcal{G}_{VV}$.

Vanishing viscosity with different coupling conditions?

See [Guarguaglini-Natalini, 2015 & 2021] for the linear case

We consider coupling conditions inspired by the Kedem-Katchalsky conditions for membrane permeability

$$
\begin{cases}
\partial_t \rho_h^\varepsilon + \partial_x f_h(\rho_h^\varepsilon) = \varepsilon \partial_{xx}^2 \rho_h^\varepsilon, & t > 0,\ x \in \Omega_h, \\
\rho_h^\varepsilon(0,x) = \rho_{h,0}^\varepsilon(x), & h = 1, \ldots, m+n, \\
f_i(\rho_i^\varepsilon(t,0)) - \varepsilon \partial_x \rho_i^\varepsilon(t,0) = \sum_j \mathfrak{c}_{ij}(\rho_i^\varepsilon(t,0) - \rho_j^\varepsilon(t,0)), & i = 1, \ldots, m, \\
f_j(\rho_j^\varepsilon(t,0)) - \varepsilon \partial_x \rho_j^\varepsilon(t,0) = \sum_i \mathfrak{c}_{ij}(\rho_i^\varepsilon(t,0) - \rho_j^\varepsilon(t,0)), & j = m+1, \ldots, m+n,
\end{cases}
$$

where $\mathfrak{c}_{ij} > 0$. We do not impose continuity at $x = 0$.

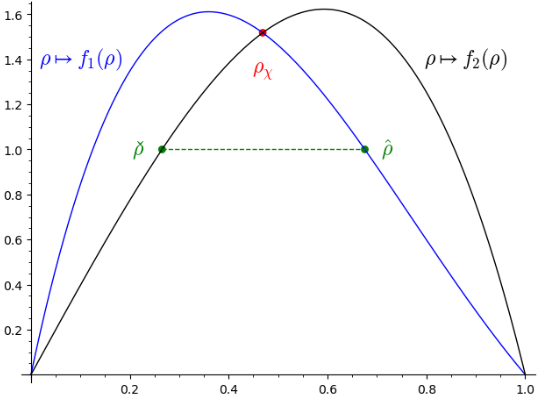
We can prove [Coclite-D. 2020] :

- Existence of parabolic approximations for any ε;
- Convergence (up to a subsequence) to a weak solution.

Characterization of the hyperbolic limit?

The 1-1 case

Assume $f_1(\rho_\chi) = f_2(\rho_\chi)$ and $f_1(\hat\rho) = f_2(\check\rho) = \mathfrak{c}(\hat\rho - \check\rho)$



$$\begin{cases} \partial_t \rho_1^\varepsilon + \partial_x f_1(\rho_1^\varepsilon) = \varepsilon \partial_{xx}^2 \rho_1^\varepsilon, & t > 0,\ x < 0, \\ \partial_t \rho_2^\varepsilon + \partial_x f_2(\rho_2^\varepsilon) = \varepsilon \partial_{xx}^2 \rho_2^\varepsilon, & t > 0,\ x > 0, \\ f_1(\rho_1^\varepsilon(t,0)) - \varepsilon \partial_x \rho_1^\varepsilon(t,0) = \mathfrak{c}(\rho_1^\varepsilon(t,0) - \rho_2^\varepsilon(t,0)), & t > 0, \\ f_2(\rho_2^\varepsilon(t,0)) - \varepsilon \partial_x \rho_2^\varepsilon(t,0) = \mathfrak{c}(\rho_1^\varepsilon(t,0) - \rho_2^\varepsilon(t,0)), & t > 0, \\ \rho_1^\varepsilon(0,x) = \hat{\rho}, & x < 0, \\ \rho_2^\varepsilon(0,x) = \check{\rho}, & x > 0. \end{cases}$$

As $\varepsilon \to 0$ the limit is $\rho_1(t,x) \equiv \hat{\rho}, \qquad \rho_2(t,x) \equiv \check{\rho}$.

- The couple $(\hat{\rho}, \check{\rho})$ is a connection as introduced by [Adimurthi-Mishra-Gowda, 2005].
- Already obtained by adapted vanishing viscosity regularization

$$\begin{cases} \partial_t \rho_1^\varepsilon + \partial_x f_1(\rho_1^\varepsilon) = \varepsilon \partial_{xx}^2 a_1\left(\rho_1^\varepsilon\right), & t > 0,\ x < 0, \\ \partial_t \rho_2^\varepsilon + \partial_x f_2(\rho_2^\varepsilon) = \varepsilon \partial_{xx}^2 a_2\left(\rho_2^\varepsilon\right), & t > 0,\ x > 0, \end{cases}$$

where a_1 and $a_2 : [0,1] \to [0,1]$ are strictly monotone increasing bijections and $a_1(\hat{\rho}) = a_2(\check{\rho})$.

Can be find a complete, maximal, L^1 dissipative germ for the limit problem?

$$\mathcal{G}_{KK} = \left\{(u_L, u_R) \text{ s.t. } u(t,x) = u_L \mathbb{1}_{\mathbb{R}_-} + u_R \mathbb{1}_{\mathbb{R}_+} \text{ is a stationary admissible solution}\right\}$$

$A = (\hat{\rho}, \check{\rho})$ defined by $f_1(\hat{\rho}) = f_2(\check{\rho}) = \mathfrak{c}(\hat{\rho} - \check{\rho})$ must be in, together with $u(t,x) \equiv 0$ and $u(t,x) \equiv 1$.

Of course, the germ contains all the couples (a, b) which

- are traces of IBVPs with boundary conditions $\vec{v} = (\hat{\rho}, \check{\rho})$, $\vec{v} = (0,0)$ or $\vec{v} = (1,1)$;
- satisfy the Rankine-Hugoniot condition.

$$\mathcal{H} = \left\{ \begin{aligned} (a,b)\colon\ & a \in [0, \check{u}_L] \cup \{\hat{\rho}\}, \\ & b \in [\hat{u}_R, 1] \cup \{\check{\rho}\}, \\ & f_1(a) = f_2(b). \end{aligned} \right\} \cup \{(0,0)\} \cup \{(1,1)\},$$

where $\check{u}_L$ and $\hat{u}_R$ satisfy $f_1(\check{u}_L) = f_1(\hat{\rho})$ and $f_2(\hat{u}_R) = f_2(\check{\rho})$

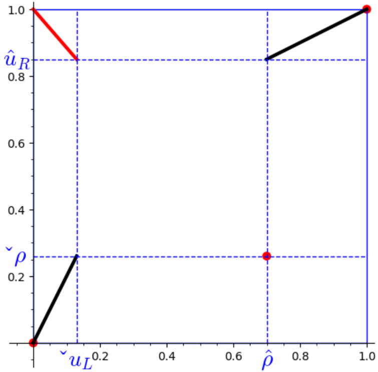
Some Riemann problems do not have solutions if we impose that the traces at $x = 0$ of the solution are in $\mathcal{H}$.

We call $\mathcal{G}_A^*$ the set of all couples (a, b) such that $f_1(a) = f_2(b)$ and

$$\operatorname{sign}(a - \hat{\rho})\,(f_1(a) - f_1(\hat{\rho})) - \operatorname{sign}(b - \check{\rho})\,(f_2(b) - f_2(\check{\rho})) \geq 0,$$

This set need to be in the germ because we need maximality and L^1-dissipativity.

A case by case study show that $\mathcal{G}_A^*$ is complete and maximal.
Therefore, we can use it to prove well-posedness of limit solutions to the hyperbolic problem.



Thank you for your attention!