VARIATIONAL PRINCIPLES IN STOCHASTIC SETTING

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MOTIVATION AND BASIC IDEA

Functional derivative notation

V (possibly infinite dimensional) vector space

V^* vector space in weak duality $\langle \cdot , \cdot \rangle : V^* \times V \to \mathbb{R}$ with V; in finite dimensions, V^* is the usual dual vector space, but in infinite dimensions it rarely is the topological dual.

If $f : V^* \to \mathbb{R}$ is smooth, the *functional derivative* $\frac{\delta f}{\delta a} \in V$, if it exists, is defined by

$$\lim_{\epsilon \to 0} \frac{f(a + \epsilon b) - f(a)}{\epsilon} = \left\langle \frac{\delta f}{\delta a}, b \right\rangle, \quad a, b \in V^*.$$

Geometric mechanics setup

Given are a left (right) invariant Lagrangian $L : TG \to \mathbb{R}$ and a left (right) invariant Hamiltonian $H : T^*G \to \mathbb{R}$. Below $+$ is for left, $-$ is for right.

(i) **Hamilton's Principle:** $g(t) \in G$ is a critical point of the action

$$\int_{t_0}^{t_1} L\left(g(t), \dot{g}(t)\right) dt$$

for variations $\delta g(t) \in T_{g(t)}G$ such that $\delta g(t_i) = 0$ for $i = 0, 1$.

(ii) **The Euler-Poincaré Variational Principle:** $v(t) \in \mathfrak{g}$ is a critical point of the reduced action

$$\int_{t_0}^{t_1} l\left(v(t)\right) dt$$

for variations of the form $\delta v(t) = \dot{\eta}(t) \pm [v(t), \eta(t)] \in \mathfrak{g}$, where $\eta(t)$ is a curve in $\mathfrak{g}$ such that $\eta(t_i) = 0$, for $i = 1, 2$.

(iii) The **Euler-Lagrange equations** on G hold:

$$\frac{\partial L}{\partial g} - \frac{d}{dt}\frac{\partial L}{\partial \dot{g}} = 0 \quad (\text{intrinsic } \mathcal{EL} : T^{(2)}G \to T^*G).$$

(iv) The **Euler-Poincaré equations** on $\mathfrak{g}^*$ hold:

$$\frac{d}{dt}\frac{\delta l}{\delta v} = \pm \,\mathrm{ad}_v^* \frac{\delta l}{\delta v}, \qquad v(t) = g^{-1}(t)\dot{g}(t) \quad (v(t) = \dot{g}(t)g^{-1}(t)).$$

(v) **Hamilton's Phase Space Principle:** $(g(t), p(t)) \in T^*G$ is a critical point of the action

$$\int_{t_0}^{t_1} (p \cdot \dot{g} - H(g,p))\, dt;$$

variations $(\delta g, \delta p) \in T_{(g,p)}(T^*G)$, $\delta g(t_i) = 0$, $i = 0, 1$, $\delta p(t)$ arbitrary.

(vi) **The Lie-Poisson Variational Principle:** $(v(t), \mu(t)) \in \mathfrak{g} \times \mathfrak{g}^*$ is a critical point of the action

$$\int_{t_0}^{t_1} (\langle \mu(t), v(t) \rangle - h(\mu(t)))\, dt$$

for variations of the form $\delta v(t) = \dot{\eta}(t) \pm [v(t), \eta(t)] \in \mathfrak{g}$, $\eta(t) \in \mathfrak{g}$ such that $\eta(t_i) = 0$, for $i = 0, 1$, and where the variations $\delta\mu$ are arbitrary.

(vii) The **Hamilton equations** on T^*G hold:

$$(\dot{g}(t), \dot{p}(t)) = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial g} \right) \quad \text{(intrinsic } \mathbf{i}_{X_H}\Omega_{\text{can}} = \mathbf{d}H\text{)}.$$

(viii) The **Lie-Poisson equations** on $\mathfrak{g}^*$ hold:

$$\dot{\mu} = \pm \operatorname{ad}^*_{\delta h/\delta\mu}\, \mu.$$

(**i**) $\Leftrightarrow$ (**ii**) $\Leftrightarrow$ (**iii**) $\Leftrightarrow$ (**iv**) for general Lagrangians. (Poincaré)
(**v**) $\Leftrightarrow$ (**vi**) $\Leftrightarrow$ (**vii**) $\Leftrightarrow$ (**viii**) for general Hamiltonians. (reduction)
If L or H are hyperregular, all statements are equivalent.

Hyperregular: $\mathbb{F}L : TQ \to T^*Q$ is a diffeomorphism, $\langle \mathbb{F}L(v_q), w_q \rangle := \frac{d}{dt}\Big|_{t=0} L(v_q + tw_q)$. $H(\mathbb{F}L(v_q)) := \langle \mathbb{F}L(v_q), v_q \rangle - L(v_q)$. Then **reduced Legendre transformation** $\xi \ni \mathfrak{g} \stackrel{\sim}{\longleftrightarrow} \mu := \delta l / \delta \xi \in \mathfrak{g}^*$ is also a diffeomorphism. $h(\mu) := \langle \mu, \xi \rangle - l(\xi)$.

Motion is such that $\mu := \delta l / \delta \xi$ lies on coadjoint orbits in $\mathfrak{g}^*$.

Hamiltonian side: geometry, Lie theory.

Lagrangian side: analysis, variational principles.

I am not aware of any general results for such constrained variational principles.

Reconstruction

Solve the motion equations for a left invariant $L : TG \to \mathbb{R}$, $H : T^*G \to \mathbb{R}$. Define $l := L|_{\mathfrak{g}} : \mathfrak{g} \to \mathbb{R}$, $h := H|_{\mathfrak{g}^*} : \mathfrak{g}^* \to \mathbb{R}$.

- Solve the Euler-Poincaré equations: $\frac{d}{dt}\frac{\delta l}{\delta v} = \mathrm{ad}^*_\xi \frac{\delta l}{\delta v}$ with $\xi(0) = \xi_0$.

- Solve $\dot{g}(t) = g(t)\xi(t)$, $g(0) = e$; linear, time dependent coefficients.

- For any initial condition $V(0) = g_0\xi_0 \in TG$, $g_0 \in G$, $\xi_0 \in \mathfrak{g}$, the solution of the Euler-Lagrange equations on TG is $V(t) = g_0 g(t)\xi(t)$.

- Solve the Lie-Poisson equations $\dot{\mu} = \mathrm{ad}^*_{\delta h/\delta\mu}\,\mu$ with $\mu(0) = \mu_0$.

- Solve $\dot{g}(t) = g(t)\frac{\delta h}{\delta\mu(t)}, g(0) = e$; linear, time-dependent coefficients.

- For any initial condition $\alpha(0) = g_0\mu_0 \in T^*G$, $g_0 \in G$, $\mu_0 \in \mathfrak{g}^*$, the solution of Hamilton's equations on T^*G is $\alpha(t) = g_0 g(t)\mu(t)$.

There is a right invariant version of this theorem. There are relative sign changes, so must be very careful.

SEMIDIRECT PRODUCT THEORY

$\rho : G \to \mathrm{Aut}(V)$ denote a *right* Lie group representation. Form the semidirect product $S = G \circledS V$ whose group multiplication is

$$(g_1, v_1)(g_2, v_2) := (g_1 g_2, v_2 + \rho_{g_2}(v_1)) := (g_1 g_2, v_1 g_2 + v_2).$$

The Lie algebra $\mathfrak{s} = \mathfrak{g} \circledS V$ of S has bracket

$$\mathrm{ad}_{(\xi_1, v_1)}(\xi_2, v_2) = [(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, \xi_2], v_1\xi_2 - v_2\xi_1),$$

where $v\xi$ denotes the induced action of $\mathfrak{g}$ on V, that is,

$$v\xi := \frac{d}{dt}\Big|_{t=0} \rho_{\exp(t\xi)}(v) = \frac{d}{dt}\Big|_{t=0} v \exp(t\xi) \in V.$$

If $(\xi, v) \in \mathfrak{s}$ and $(\mu, a) \in \mathfrak{s}^*$ we have

$$\mathrm{ad}^*_{(\xi, v)}(\mu, a) = (\mathrm{ad}^*_\xi \mu + v \diamond a, a\xi), \quad a\xi \in V^*, \quad v \diamond a \in \mathfrak{g}^*,$$

$$a\xi := \frac{d}{dt}\Big|_{t=0} \rho^*_{\exp(-t\xi)}(a) \quad \text{and} \quad \langle v \diamond a, \xi \rangle_{\mathfrak{g}} := -\langle a\xi, v \rangle_V,$$

$\langle \cdot, \cdot \rangle_{\mathfrak{g}} : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ and $\langle \cdot, \cdot \rangle_V : V^* \times V \to \mathbb{R}$ are the duality parings.

$v \diamond a$ is the momentum map of cotangent lifted G-action to $V \times V^*$.

Lagrangian semidirect product theory

- Given is $L : TG \times V^* \to \mathbb{R}$ which is right G-invariant.

- So, if $a_0 \in V^*$, define the Lagrangian $L_{a_0} : TG \to \mathbb{R}$ by $L_{a_0}(v_g) := L(v_g, a_0)$. Then L_{a_0} is right invariant under the lift to TG of the right action of G_{a_0} on G, where $G_{a_0} := \{g \in G \mid \rho_g^* a_0 = a_0\}$.

- Right G-invariance of L permits us to define $l : \mathfrak{g} \times V^* \to \mathbb{R}$ by

$$l(T_g R_{g^{-1}}(v_g), \rho_g^*(a_0)) = L(v_g, a_0).$$

- For curve $g(t) \in G$, let $\xi(t) := TR_{g(t)^{-1}}(\dot{g}(t))$ and $a(t) = \rho_{g(t)}^*(a_0)$; $a(t)$ is the the unique solution of the following linear differential equation with time dependent coefficients

$$\dot{a}(t) = -a(t)\xi(t), \quad a(0) = a_0.$$

i *With a_0 held fixed, Hamilton's variational principle*

$$\delta \int_{t_1}^{t_2} L_{a_0}(g(t), \dot{g}(t))dt = 0,$$

holds, for variations $\delta g(t)$ of $g(t)$ vanishing at the endpoints.

ii *$g(t)$ satisfies the Euler-Lagrange equations for L_{a_0} on G.*

iii *The constrained variational principle*

$$\delta \int_{t_1}^{t_2} l(\xi(t), a(t))dt = 0,$$

holds on $\mathfrak{g} \times V^*$, upon using variations $(\delta\xi, \delta a)$ of the form

$$\delta\xi = \frac{\partial \eta}{\partial t} - [\xi, \eta], \quad \delta a = -a\eta,$$

where $\eta(t) \in \mathfrak{g}$ vanishes at the endpoints.

iv *The Euler-Poincaré equations hold on $\mathfrak{g} \times V^*$:*

$$\frac{\partial}{\partial t}\frac{\delta l}{\delta \xi} = -\operatorname{ad}^*_\xi \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a} \diamond a.$$

Hamiltonian semidirect product theory

- $H : T^*G \times V^* \to \mathbb{R}$ which is right G-invariant.

- So, if $a_0 \in V^*$, define the Hamiltonian $H_{a_0} : TG \to \mathbb{R}$ by $H_{a_0}(\alpha_g) := H(\alpha_g, a_0)$. Then H_{a_0} is right invariant under the lift to TG of the right action of G_{a_0} on G.

- Right G-invariance of H permits us to define $h : \mathfrak{g}^* \times V^* \to \mathbb{R}$ by

$$h(T_e^* R_g(\alpha_g), \rho_g^*(a_0)) = H(\alpha_g, a_0).$$

For $\alpha(t) \in T^*_{g(t)}G$ and $\mu(t) := T^* R_{g(t)}(\alpha(t)) \in \mathfrak{g}^*$, the following are equivalent:

i *$\alpha(t)$ satisfies Hamilton's equations for H_{a_0} on T^*G.*

ii *The Lie-Poisson equation holds on* $\mathfrak{s}^*$*:*

$$\frac{\partial}{\partial t}(\mu,a) = -\,\mathrm{ad}^*_{\left(\frac{\delta h}{\delta \mu},\frac{\delta h}{\delta a}\right)}(\mu,a) = -\left(\mathrm{ad}^*_{\frac{\delta h}{\delta \mu}}\mu + \frac{\delta h}{\delta a}\diamond a, a\frac{\delta h}{\delta \mu}\right), \quad a(0)=a_0$$

where $\mathfrak{s}$ *is the semidirect product Lie algebra* $\mathfrak{s}=\mathfrak{g}Ⓢ V$*. The associated Poisson bracket is the Lie-Poisson bracket on the semidirect product Lie algebra* $\mathfrak{s}^*$*, that is,*

$$\{f,g\}(\mu,a) = \left\langle \mu, \left[\frac{\delta f}{\delta \mu},\frac{\delta g}{\delta \mu}\right]\right\rangle + \left\langle a, \frac{\delta f}{\delta a}\frac{\delta g}{\delta \mu} - \frac{\delta g}{\delta a}\frac{\delta f}{\delta \mu}\right\rangle.$$

As on the Lagrangian side, the evolution of the advected quantities is given by $a(t)=\rho^*_{g(t)}(a_0)$*.*

Reduced Legendre transformation: $h(\mu,a) := \langle \mu,\xi\rangle - l(\xi,a)$, where $\mu=\frac{\delta l}{\delta \xi}$. If it is invertible, since

$$\frac{\delta h}{\delta \mu}=\xi \quad \text{and} \quad \frac{\delta h}{\delta a} = -\frac{\delta l}{\delta a},$$

the Lie-Poisson equations for h are equivalent to the Euler-Poincaré equations for l together with the advection equation $\dot{a}+a\xi=0$.

NOTATIONS AND CONVENTIONS

$\mathbb{R}^+ := [0, \infty[$, $(\Omega, \mathcal{P}, P)$ probability space.

Non-decreasing filtration $(\mathcal{P}_t)_{t \in \mathbb{R}^+}$ on the probability space:

- $(\mathcal{P}_t)_{t \in \mathbb{R}^+}$ given family of sub-σ-algebras of $\mathcal{P}$
- non-decreasing: $\mathcal{P}_s \subseteq \mathcal{P}_t$ if $0 \leq s \leq t$
- right-continuous: $\cap_{\epsilon > 0} \mathcal{P}_{t+\epsilon} = \mathcal{P}_t$, $\forall t \in \mathbb{R}^+$.

A stochastic process $X : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ is $(\mathcal{P}_t)$-*adapted* if $X(t)$ is $\mathcal{P}_t$-measurable for every t. Typically, filtrations describe the past history of a process: one starts with a process X and defines

$$\mathcal{P}_t := \cap_{\epsilon > 0} \sigma\{X(s), 0 \leq s \leq t\}.$$

Then the process X will be automatically $(\mathcal{P}_t)$-adapted.

$\mathbb{E}$ denotes the *expectation* of a random variable

$\mathbb{E}_s(M(t,\omega)) := \mathbb{E}(M(t,\omega)|\mathcal{P}_s)$, for each $s \geq 0$, is the *conditional expectation* of the random variable $M_\omega(t)$, $t > s$, relative to the σ-algebra $(\mathcal{P}_s)$, i.e., $\Omega \ni \omega \mapsto \mathbb{E}_s(M_\omega(t)) \in \mathbb{R}$ is a $\mathcal{P}_s$-measurable function satisfying

$$\int_A \mathbb{E}_s(M(t,\omega))dP(\omega) = \int_A M(t,\omega)dP(\omega), \quad \forall A \in \mathcal{P}_s.$$

A stochastic process $M: \mathbb{R}^+ \times \Omega \to \mathbb{R}$ is a *martingale* if
(i) $\mathbb{E}|M(t,\omega)| < \infty$ for all $t \geq 0$;
(ii) $M(t,\omega)$ is $(\mathcal{P}_t)$-adapted;
(iii) $\mathbb{E}_s(M(t,\omega))) = M(s,\omega)$ a.s. for all $0 \leq s < t$.
Condition **(iii)** is equivalent to $\mathbb{E}((M_\omega(t) - M_\omega(s))\chi_A) = 0$, $\forall A \in \mathcal{P}_s$, $\forall t, s \in \mathbb{R}$ satisfying $t > s \geq 0$; χ_A characteristic function of set A.

Work only with processes defined on compact time intervals $[0,T]$, continuous in t for almost all $\omega \in \Omega$, i.e., *continuous processes*.

If a martingale M is continuous and $\mathbb{E}(M_\omega(t)^2) < \infty$, $\forall t \geq 0$, then M has a *quadratic variation* $\{[\![M, M]\!]_t,\ t \in [0, T]\}$ if $M^2(t) - [\![M, M]\!]_t$ is a martingale, and $[\![M, M]\!]_t$ is a continuous, $\mathcal{P}_t$-adapted, a.s. non-decreasing process with $[\![M, M]\!]_0 = 0$. Such a process is unique and coincides with the following limit (convergence in probability),

$$\lim_{n\to\infty} \sum_{t_i, t_{i+1} \in \sigma_n} (M(t_{i+1}) - M(t_i))^2;$$

σ_n is a partition of $[0, t]$ and the mesh converges to zero as $n \to \infty$. Def. of the quadratic variation requires only right-continuity of M.

M, N martingales, same assumptions; their *covariation* is

$$[\![M, N]\!]_t := \lim_{n\to\infty} \sum_{t_i, t_{i+1} \in \sigma_n} (M(t_{i+1}) - M(t_i))(N(t_{i+1}) - N(t_i)),$$

which extends the notion of quadratic variation. Clearly,

$$2[\![M, N]\!]_t = [\![M + N, M + N]\!]_t - [\![M, M]\!]_t - [\![N, N]\!]_t.$$

Stopping time: random variable $\tau : \Omega \to \mathbb{R}^+$ such that

$$\{\omega \in \Omega \mid \tau(\omega) \leq t\} \in \mathcal{P}_t,\ \forall t \geq 0.$$

Local martingale: stochastic process M for which $\exists$ sequence of stopping times $\{\tau_n, n \geq 1\}$, such that $\lim_{n\to\infty} \tau_n = \infty$ a.s., and $M^n(t) := M(t \wedge \tau_n)$ is a square integrable martingale for all $n \geq 1$, where $t \wedge \tau_n := \min(t, \tau_n)$. Define $[\![M, M]\!]_t := [\![M^n, M^n]\!]_t$ if $t \leq \tau_n$.

Real-valued Brownian motion: continuous martingale $W(t)$, $t \in [0, T]$, such that $W^2(t) - t$ is a martingale $\iff [\![W, W]\!]_t = t$.

Semimartingale: stochastic process $X : \Omega \times [0, T] \to \mathbb{R}$ such that

$$X(t) = X(0) + M(t) + A(t), \quad \forall t \geq 0,$$

where M is a local martingale with $M(0) = 0$ and A is a càdlàg (A is right-continuous with left limits at each $t \geq 0$) adapted process of locally bounded variation with $A(0) = 0$ a.s. We consider only processes that are continuous in time.

Local semimartingale: same as above with M a local martingale and A a locally bounded variation process. Define $[\![X, X]\!]_t := [\![M, M]\!]_t$.

Martingales (hence Brownian motion) are not a.s. t-differentiable (unless they are constant), so cannot integrate with respect to martingales as one does with respect to functions of bounded variation. Needed stochastic integrals: the Itô and the Stratonovich integrals.

If X and Y are continuous real-valued semimartingales such that

$$\mathbb{E}\left(\int_0^T |X(t)|^2 dt + \int_0^T |Y(t)|^2 dt\right) < \infty,$$

the *Itô stochastic integral* of $X(t)$ on $[0,t]$, $0 < t \leq T$, with respect to Y is defined as the limit in probability (if limit exists) of the sums

$$\int_0^t X(s)dY(s) := \lim_{n\to\infty} \sum_{t_i,t_{i+1}\in\sigma_n} X(t_i)(Y(t_{i+1}) - Y(t_i));$$

σ_n is a partition of $[0,t]$ with mesh converging to zero as $n \to \infty$.

If Y is a martingale such that $\mathbb{E}\left(\int_0^T |X(t)|^2 d[\![Y,Y]\!]_t\right) < \infty$, then $\int_0^t X(s)dY(s)$, $t \in [0,T]$, is also a martingale.

The *Stratonovich stochastic integral* is defined by

$$\int_0^t X(s) \circ dY(s) := \lim_{n\to\infty} \sum_{t_i, t_{i+1}\in\sigma_n} \frac{(X(t_i) + X(t_{i+1}))}{2}(Y(t_{i+1}) - Y(t_i))$$

whenever this limit exists.

These integrals do not coincide in general, even though X is a continuous process, due to the lack of differentiability of the paths of Y. The Itô and the Stratonovich integrals are related by

$$\int_0^t X_s \circ dY_s = \int_0^t X_s dY_s + \frac{1}{2}\int_0^t d[\![X, Y]\!]_s$$

Itô's formula: for any $f \in C^2(\mathbb{R})$,

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s))dX(s) + \frac{1}{2}\int_0^t f''(X(s))d[\![X, X]\!]_s$$

For Stratonovich integrals, this formula is:

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s)) \circ dX(s)$$

Stratonovich integral: standard differential calculus rules apply, works on manifolds.

Itô integral with respect to a martingale M is again a martingale, a very important property. For example, we have, as an immediate consequence, that $\mathbb{E}_s \int_s^t X(r) dM(r) = 0$ for all $0 \leq s < t$.

Itô's formula: X a $\mathbb{R}^d$-valued semimartingale; for any $f \in C^2(\mathbb{R}^d)$,

$$f(X(t)) = f(X(0)) + \sum_{i=1}^{d} \int_0^t \partial_i f(X(s)) dX^i(s) + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \partial^2_{i,j}(X(s)) d[\![X^i, X^j]\!]_s$$

$$= f(X(0)) + \sum_{i=1}^{d} \int_0^t \partial_i f(X(s)) \circ dX^i(s)$$

The difference between the two integrals is given by the Hessian.

Rules for Brownian motions $W^1, \ldots W^k$, where $\iota(t) := t$:

$$d[\![W^i, W^j]\!]_t = \delta_{ij} dt, \quad d[\![W^i, \iota]\!]_t = 0, \quad d[\![\iota, \iota]\!]_t = 0, \quad \forall i, j = 1, \ldots, d$$

(covariation of semimartingales determined by their martingale parts)

GENERALIZED DERIVATIVE FOR LIE GROUP VALUED SEMIMARTINGALES

G Lie group. L_g, R_g left and right translation by $g \in G$. If $v \in T_eG$, $v^L(g) := T_eL_g v$ and $v^R(g) := T_eR_g v$ are the left and right invariant vector fields on G such that $v^L(e) = v^R(e) = v$. $[v_1, v_2] := \left[v_1^L, v_2^L\right](e)$, for $v_1, v_2 \in T_eG$, defines a (left) Lie bracket on T_eG. Denote $\mathrm{ad}_u v := [u, v]$ and $\mathrm{ad}_u^* : T_e^*G \to T_e^*G$ its dual map.

Let ∇ be a right invariant linear connection on G, i.e., $\nabla_{v_1^R} v_2^R$ is a right invariant vector field, for any $v_1, v_2 \in T_eG$. Define

$$\nabla_{v_1} v_2 := \nabla_{v_1^R} v_2^R(e), \quad \forall v_1, v_2 \in T_eG.$$

Right invariant ∇ is torsion free $\Longleftrightarrow$

$$\nabla_{v_1} v_2 - \nabla_{v_2} v_1 = -[v_1, v_2], \quad \text{for all} \quad v_1, v_2 \in T_eG.$$

There is a sign change in front of the Lie bracket because we work with right invariant vector fields: $[v_1^R, v_2^R] = -[v_1, v_2]^R$.

In general: on (M, ∇), the *Hessian* of $f \in C^2(M)$ is defined by

$$\mathrm{Hess} f(X,Y) := (\nabla\nabla f)(X,Y) := \langle \nabla_X(\nabla f), Y\rangle = \langle \nabla_X(\mathbf{d}f), Y\rangle\,,$$

$X, Y \in \mathfrak{X}(M)$, since $\nabla f = \mathbf{d}f$ (definition). This is $\mathbb{R}$-bilinear. Hence

$$X[Y[f]] = \nabla_X\nabla_Y f = \nabla_X(\langle \mathbf{d}f, Y\rangle) = \langle \nabla_X \mathbf{d}f, Y\rangle + \langle \mathbf{d}f, \nabla_X Y\rangle \;\Rightarrow$$
$$\mathrm{Hess} f(X,Y) = X[Y[f]] - (\nabla_X Y)[f].$$

If $0 = (\mathrm{Tor}\,\nabla)(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$, $\mathrm{Hess} f$ is symmetric.

∇ right invariant linear torsion free connection, $f \in C^2(G)$, the *Hessian* $\mathrm{Hess} f(g) : T_gG \times T_gG \to \mathbb{R}$ at $g \in G$ is defined by

$$\mathrm{Hess} f(g)(v_1, v_2) := \tilde{v}_1[\tilde{v}_2[f]](g) - (\nabla_{\tilde{v}_1}\tilde{v}_2)[f](g), \quad v_1, v_2 \in T_gG,$$

for $\tilde{v}_i$, $i = 1, 2$, arbitrary vector fields on G such that $\tilde{v}_i(g) = v_i$. $\mathrm{Hess}\, f(g)$ is a symmetric $\mathbb{R}$-bilinear form on each T_gG.

Probability space $(\Omega, \mathcal{P}, P)$ endowed with a non-decreasing filtration $(\mathcal{P}_t)_{t\geq 0}$ A *semimartingale with values in G* (with respect to $(\mathcal{P}_t)_{t\geq 0}$) *is a $\mathcal{P}_t$-adapted stochastic process $g : \Omega\times\mathbb{R}^+ \to G$ such that $f\circ g : \Omega\times \mathbb{R}^+ \to \mathbb{R}$ is a real-valued semimartingale* (on $(\Omega, \mathcal{P}, P)$) $\forall f \in C^2(G)$.

A G-valued semimartingale is a ∇-(*local*) *martingale* if $\forall f \in C^2(G)$

$$t \longmapsto f(g_\omega(t)) - f(g_\omega(0)) - \frac{1}{2}\int_0^t \mathrm{Hess} f(g_\omega(s)) d[\![g_\omega, g_\omega]\!]_s ds$$

is a real-valued (local) martingale, where $[\![g_\omega, g_\omega]\!]_t$ is the quadratic variation of g_ω.

Quadratic variation is well defined for finite dimensional G and some some infinite dimensional groups (e.g., $Diff^s(\mathbb{T}^3)$).

G is finite dimensional, then (Emery [1989], Ikeda-Watanabe [1981])

$$d[\![g_\omega, g_\omega]\!]_t := d\left[\!\!\left[\int_0^\cdot \mathbf{P}_s^{-1}\circ dg_\omega(s), \int_0^\cdot \mathbf{P}_s^{-1}\circ dg_\omega(s) \right]\!\!\right]_t,$$

$\mathbf{P}_t : T_{g_\omega(0)}G \to T_{g_\omega(t)}G$ is the (stochastic) ∇-parallel translation along the (stochastic) curve $t \mapsto g_\omega(t)$.

Modeling

For a G-valued semimartingale $g_\omega(\cdot)$, suppose there exist an integer $m > 0$ and $\mathcal{P}_t$-adapted processes $\mathbf{v}, \mathbf{w}^i, M^i : \Omega \times \mathbb{R}^+ \to T_eG$, $1 \leq i \leq m$, s.t. M^i is a $\mathbb{R}$-valued martingale with continuous sample paths and

$$dg_\omega(t) = T_eR_{g_\omega(t)}\Big(\sum_{i=1}^{m} \mathbf{w}_\omega^i(t) \circ dM_\omega^i(t) + \mathbf{v}_\omega(t)dt\Big). \qquad (1)$$

The choice of $\{(\mathbf{w}_\omega^i, M_\omega^i) \mid 1 \leq i \leq m\}$ in (1) may not be unique, but the decomposition into the martingale part $(\sum_{i=1}^m \mathbf{w}_\omega^i(t)dM_\omega^i(t))$ and the drift part without contraction $(T_eR_{g_\omega(t)}\mathbf{v}_\omega(s)dt)$ in (1) is unique.

Define the velocity derivative of $g_\omega(\cdot)$ (which is independent of ∇)

$$\frac{\mathscr{D}g_\omega(t)}{dt} := T_eR_{g_\omega(t)}\mathbf{v}_\omega(t) \in T_{g_\omega(t)}G.$$

From now on fix:

$$\Big\{(\mathbf{w}^i_\omega, M^i_\omega)_{i=1}^m \mid \mathbf{w}^i, M^i : \Omega \times \mathbb{R}^+ \to T_eG \text{ are } \mathcal{P}_t\text{-adapted processes,}$$
$$M^i \text{ real valued martingales with continuous sample paths}\Big\}.$$

$\mathscr{S}(G) := \Big\{G\text{-valued semimartingales on } [0,T] \text{ with smooth coeff.}\Big\}$,

$\widetilde{\mathscr{S}(G)} := \Big\{(g_\omega, \mathbf{w}^i_\omega, M^i_\omega)_{i=1}^m \mid g_\omega \in \mathscr{S}(G)\Big\}$.

Given $\{(\mathbf{w}^i_\omega, M^i_\omega) \mid 1 \le i \le m\}$, define the ($T_{g_\omega(t)}G$-valued) $m \times m$ **contraction matrix** $\frac{\mathbf{D}^{\nabla,(\mathbf{w}^i_\omega, M^i_\omega)_{i=1}^m} g_\omega(t)}{dt}$ by its (i,j)-entries

$$T_eR_{g_\omega(t)}\left(\nabla_{\mathbf{w}^i_\omega(t)}\mathbf{w}^j_\omega(t)\frac{d[\![M^i_\omega, M^j_\omega]\!]_t}{dt} + \frac{d[\![\mathbf{w}^i_\omega, M^i_\omega]\!]_t}{dt}\delta_{ij}\right), \quad 1 \le i,j \le m.$$

(i,j)-entry: contraction between the noises in vectors $\mathbf{w}^i_\omega$ and $\mathbf{w}^j_\omega$.

$\frac{\mathscr{D}}{dt}$, $\frac{\mathbf{D}^{\nabla,(\mathbf{w}^i_\omega, M^i_\omega)_{i=1}^m}}{dt}$ well defined for semimartingales with values in a finite dimensional Lie group and some infinite dimensional groups (e.g., $Diff(\mathbb{T}^3)$, Arnaudon-Chen-Cruzeiro [2014]).

For each fixed t define the $T_{g_\omega(t)}G$-valued random variable

$$\mathbf{Sum}\left(\frac{\mathbf{D}^{\nabla,(\mathsf{w}^i_\omega,M^i_\omega)^m_{i=1}}g_\omega(t)}{dt}\right) := \sum_{i,j=1}^m \left(\frac{\mathbf{D}^{\nabla,(\mathsf{w}^i_\omega,M^i_\omega)^m_{i=1}}g_\omega(t)}{dt}\right)_{ij} \in T_{g_\omega(t)}G.$$

For a G-valued semimartingale $g_\omega(\cdot)$ of the form (1), define the *∇-generalized derivative* of $g_\omega(t)$ by (Emery [1989])

$$\frac{D^\nabla g_\omega(t)}{dt} := \mathbf{P}_t\left(\lim_{\epsilon\to 0}\mathbb{E}_t\left[\frac{\eta_\omega(t+\epsilon)-\eta_\omega(t)}{\epsilon}\right]\right)$$
$$= \frac{1}{2}\mathbf{Sum}\left(\frac{\mathbf{D}^{\nabla,(\mathsf{w}^i_\omega,M^i_\omega)^m_{i=1}}g_\omega(t)}{dt}\right) + \frac{\mathscr{D}g_\omega(t)}{dt},$$

where $\mathbf{P}_t : T_eG \to T_{g_\omega(t)}G$ is the stochastic parallel translation defined by ∇, $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot|\mathcal{P}_t]$ denotes the conditional expectation, and

$$\eta_\omega(t) = \int_0^t \mathbf{P}_s^{-1}\circ dg_\omega(s) \in T_eG.$$

Therefore, according to the definition, if a G-valued semimartingale $g_\omega(t)$ satisfies $\frac{D^\nabla g_\omega(t)}{dt} = 0$, then $g_\omega(t)$ is a ∇-martingale.

In $\mathbb{R}^n$, $\frac{D^\nabla g_\omega(t)}{dt}$ = usual generalized derivative for $\mathbb{R}^n$-valued semimartingales (Cipriano-Cruzeiro [2007], Yasue [1981], Zambrini [2015]).

The conditional expectation $\mathbb{E}_t$ in the definition of the ∇-generalized derivative eliminates the martingale part of the semimartingale. So: velocities are given by the drift (the bounded variation part), diffusion part (the martingale) can be seen as a stochastic perturbation; in other words, the drift determines the directions where the particles flow, the martingale part describes their random fluctuations.

Generalized derivative coincides with the drift of a diffusion process. It was first associated with a dynamical interpretation, as a mean velocity, in Nelson's Stochastic Mechanics [1967].

The generalized derivative is sufficient to generate the viscosity terms (second order differential terms) in some PDEs such as incompressible Navier-Stokes; work of Cruzeiro. However, for a large class of hydrodynamics equations, e.g., compressible Navier-Stokes, the viscous terms depend on more than the contraction terms. This is why we need the decomposition of the ∇-derivative $\frac{D^\nabla}{dt}$.

Important special case

We make special choices in (1). Recall this equation:

$$dg_\omega(t) = T_eR_{g_\omega(t)}\Big(\sum_{i=1}^{m} \mathbf{w}^i_\omega(t) \circ dM^i_\omega(t) + \mathbf{v}_\omega(t)dt\Big), \quad g_\omega(0) = e.$$

Given are a $\mathbb{R}^m$-valued martingale $M_\omega(t) = (M^1_\omega(t), ..., M^m_\omega(t))$, $t \in [0, T]$, which has a continuous sample paths, (non-random) vectors $H_i \in T_eG$, $1 \le i \le m$, and a $\mathcal{P}_t$-adapted, T_eG-valued semi-martingale $u : \Omega \times [0, T] \to T_eG$. Consider the following SDE on G: $g_\omega(0) = e$,

$$\begin{aligned} dg_\omega(t) &= T_eR_{g_\omega(t)}\left(\sum_{i=1}^{m} H_i \circ dM^i_\omega(t) + u_\omega(t)dt\right) \\ &= T_eR_{g_\omega(t)}\left(\sum_{i=1}^{m} H_i dM^i_\omega(t) + \frac{1}{2}\sum_{i,j=1}^{m} \nabla_{H_i}H_j d[\![M^i_\omega, M^j_\omega]\!]_t + u_\omega(t)dt\right). \end{aligned}$$

If G is a finite dimensional Lie group, there exists a unique strong solution (Ikeda-Watanabe [1981], Emery [1989]).

If G is the diffeomorphism group of a torus, u is less regular, under suitable conditions on H_i, a weak solution still exists (Arnaudon-Chen-Cruzeiro [2014], Cipriano-Cruzeiro [2007]). In our application to compressible Navier-Stokes these hypotheses hold.

In this case, the formulas we need are:

$$dg_\omega(t) = T_eR_{g_\omega(t)}\left(\sum_{i=1}^m H_i \circ dM_\omega^i(t) + u_\omega(t)dt\right) \qquad (1)$$

$$\frac{\mathscr{D}g_\omega(t)}{dt} = T_eR_{g_\omega(t)}u_\omega(t),$$

$$\left(\frac{\mathbf{D}^{\nabla,(H_i,M_\omega^i)_{i=1}^m}g_\omega(t)}{dt}\right)_{ij} = T_eR_{g_\omega(t)}(\nabla_{H_i}H_j)\frac{d[\![M_\omega^i, M_\omega^j]\!]_t}{dt}.$$

STOCHASTIC SEMIDIRECT PRODUCT EULER-POINCARÉ EQUATIONS

U vector (Banach) space, U^* dual, $\langle \cdot, \cdot \rangle_U : U^* \times U \to \mathbb{R}$ duality pairing.

G Lie group (enough: topological group, manifold, smooth right translation). T_eG its Lie algebra (ILB). U right representation space for G. So there are naturally induced right representations of G and T_eG on U and U^*. All representations are denoted by concatenation. $\langle \cdot, \cdot \rangle_{T_eG} : T_e^*G \times T_eG \to \mathbb{R}$, $\langle \cdot, \cdot \rangle_U : U^* \times U \to \mathbb{R}$ are the duality pairings.

Recall the operator $\diamond : U \times U^* :\to T_e^*G$ defined by

$$\langle a \diamond \alpha, v \rangle_{T_eG} := - \langle \alpha v, a \rangle_U = \langle \alpha, av \rangle_U \,, \quad v \in T_eG, \quad a \in U, \quad \alpha \in U^*.$$

$a \diamond \alpha$ is the value at (a, α) of the momentum map $U \times U^* \to T_e^*G$ of the cotangent lifted action induced by the right representation of G on U.

Recall: $\mathscr{S}(G) := \{G$-valued semimartingales on $[0,T]$ with smooth coefficients$\}$, $\widetilde{\mathscr{S}(G)} := \left\{(g_\omega, \mathbf{w}^i_\omega, M^i_\omega)_{i=1}^m \mid g_\omega \in \mathscr{S}(G)\right\}$; fix

$$\left\{(\mathbf{w}^i_\omega, M^i_\omega)_{i=1}^m \mid \mathbf{w}^i, M^i : \Omega \times \mathbb{R}^+ \to T_eG \text{ are } \mathcal{P}_t\text{-adapted processes,} \right.$$
$$\left. M^i \text{ real valued martingales with continuous sample paths}\right\}.$$

If $\mathscr{M}_m := \left\{A = (a_{ij})_{i,j=1}^m \mid a_{ij} \in T_eG\right\}$ then $\mathscr{M}_m^* := \left\{\Xi = (\xi_{ij})_{i,j=1}^m \mid \xi_{ij} \in T_e^*G\right\}$ relative to the pairing $\langle \Xi, A\rangle_{\mathscr{M}_m} := \mathbf{Tr}\left(\left\langle \Xi^T, A\right\rangle\right) := \sum_{i,j=1}^m \left\langle \xi_{ij}, a_{ij}\right\rangle_{T_eG}$. Define $\mathscr{M} := \cup_{m=1}^\infty \mathscr{M}_m$, $\mathscr{M}^* := \cup_{m=1}^\infty \mathscr{M}_m^*$.

Given are:

- a right invariant torsion free linear connection ∇ on G
- $\alpha_0 \in U^*$
- a non-random (Lagrangian) function $l : [0,T] \times T_eG \times U^* \to \mathbb{R}$
- a (viscosity force) function $p : \mathscr{M} \times \mathscr{M} \times T_eG \to \mathbb{R}$.

For the G-valued $\widetilde{\text{semimartingales}}$ g^j_ω, $j = 1, 2, 3$, define the **action functional** $\tilde{J}^\nabla : \widetilde{\mathscr{S}(G)} \times \widetilde{\mathscr{S}(G)} \times \widetilde{\mathscr{S}(G)} \to \mathbb{R}$ by

$$\tilde{J}^{\nabla}\left(\left(g^1_\omega, \mathbf{w}^{1,i}_\omega, M^{1,i}_\omega\right)_{i=1}^{m_1}, \left(g^2_\omega, \mathbf{w}^{2,i}_\omega, M^{2,i}_\omega\right)_{i=1}^{m_2}, g^3_\omega\right)$$

$$:= \int_0^T l\left(t, T_{g^1_\omega(t)} R_{g^1_\omega(t)^{-1}} \frac{\mathscr{D} g^1_\omega(t)}{dt}, \tilde{\alpha}(t)\right) dt$$

$$+ \int_0^T p\left(T_{g^1_\omega(t)} R_{g^1_\omega(t)^{-1}} \frac{\mathbf{D}^{\nabla,(\mathbf{w}^{1,i}_\omega, M^{1,i}_\omega)_{i=1}^{m_1}} g^1_\omega(t)}{dt},\right.$$

$$\left. T_{g^2_\omega(t)} R_{g^2_\omega(t)^{-1}} \frac{\mathbf{D}^{\nabla,(\mathbf{w}^{2,i}_\omega, M^{2,i}_\omega)_{i=1}^{m_2}} g^2_\omega(t)}{dt}, T_{g^1_\omega(t)} R_{g^1_\omega(t)^{-1}} \frac{\mathscr{D} g^1_\omega(t)}{dt}\right) dt,$$

where $\tilde{\alpha}(t) := \mathbb{E}\left[\alpha_\omega(t)\right] \in U^*$ and $\alpha_\omega(t) := \alpha_0 g^3_\omega(t)^{-1} \in U^*$.

MODELING DECISION: **What kind of deformations, and hence variations, should one take?** Results depend on choices. Consider now only deformations leading to deterministic equations.

Other types variations, for more complicated random Lagrangians that include stochastic forces, yield stochastic equations. This is completely worked out.

For every $\varepsilon \in [0,1)$ and $\mathfrak{g} \in C^1([0,1]; T_eG)$, satisfying $\mathfrak{g}(0) = \mathfrak{g}(T) = 0$, let $e_{\varepsilon,\mathfrak{g}} \in C^1([0,T]; G)$ be the unique solution of the time-dependent ordinary differential equation on G

$$\frac{d}{dt} e_{\varepsilon,\mathfrak{g}}(t) = \varepsilon T_e R_{e_{\varepsilon,\mathfrak{g}}(t)} \dot{\mathfrak{g}}(t) = \varepsilon \dot{\mathfrak{g}}(t) e_{\varepsilon,\mathfrak{g}}(t), \qquad e_{\varepsilon,\mathfrak{g}}(0) = e.$$

This choice is inspired by the proof of the deterministic Euler-Poincaré variational principle. Note: $e_{0,\mathfrak{g}}(t) = e$ for all $t \in [0,T]$.

Remark. In general, $\mathfrak{g} : \Omega \times [0,T] \to T_eG$ is a $\mathcal{P}_t$-adapted process satisfying $\mathfrak{g}_\omega(0) = \mathfrak{g}_\omega(T) = 0$ and $\mathfrak{g}_\omega(\cdot) \in C^1([0,1]; T_eG)$ a.s. Let $e_{\omega,\varepsilon,\mathfrak{g}}(\cdot) \in C^1([0,T]; G)$ be the unique solution of the (random) time-dependent ordinary differential equation on G

$$\frac{d}{dt} e_{\omega,\varepsilon,\mathfrak{g}}(t) = \varepsilon T_e R_{e_{\omega,\varepsilon,\mathfrak{g}}(t)} \dot{\mathfrak{g}}_\omega(t), \qquad e_{\omega,\varepsilon,\mathfrak{g}}(0) = e.$$

We have $e_{\omega,0,\mathfrak{g}}(t) = e$ a.s. for all $t \in [0,T]$. $\Diamond$

Suppose $g_\omega \in \mathscr{S}(G)$ has the form (1) and define the deformations

$$g_{\omega,\varepsilon,\mathfrak{g}}(t) := e_{\varepsilon,\mathfrak{g}}(t) g_\omega(t), \quad t \in [0,T], \quad \varepsilon \in [0,1).$$

Then we have:

$$dg_{\omega,\varepsilon,\mathscr{g}}(t) = T_eR_{g_{\omega,\varepsilon,\mathscr{g}}(t)}\Big(\sum_{i=1}^{m} \mathsf{Ad}_{e_{\varepsilon,\mathscr{g}}(t)} \mathbf{w}_\omega^i(t) \circ dM_\omega^i(t) + \mathsf{Ad}_{e_{\varepsilon,\mathscr{g}}(t)} \mathbf{v}_\omega(t)dt + \varepsilon\dot{\mathscr{g}}(t)dt\Big). \tag{2}$$

Based on (2), natural to consider $\left(g_{\omega,\varepsilon,\mathscr{g}}, \mathsf{Ad}_{e_{\varepsilon,\mathscr{g}}(t)} \mathbf{w}_\omega^i, M_\omega^i\right)_{i=1}^m$ as a deformation for $\left(g_\omega, \mathbf{w}_\omega^i, M_\omega^i\right)_{i=1}^m$ with $g_\omega \in \mathscr{S}(G)$ given by (1).

VARIATIONAL PRINCIPLE

$\left(\left(g_\omega^1, \mathbf{w}_\omega^{1,i}, M_\omega^{1,i}\right)_{i=1}^{m_1}, \left(g_\omega^2, \mathbf{w}_\omega^{2,i}, M_\omega^{2,i}\right)_{i=1}^{m_2}, g_\omega^3\right) \in \widetilde{\mathscr{S}(G)} \times \widetilde{\mathscr{S}(G)} \times \mathscr{S}(G)$ is a **critical point** of $\tilde{J}^\nabla$ if for every $\mathscr{g}(\cdot) \in C^1([0,T];T_eG)$, $\mathscr{g}(0) = \mathscr{g}(T) = 0$, we have

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \tilde{J}^\nabla \left(\left(g_{\omega,\varepsilon,\mathscr{g}}^1, \mathsf{Ad}_{e_{\varepsilon,\mathscr{g}}^{-1}} \mathbf{w}_\omega^{1,i}, M_\omega^{1,i}\right)_{i=1}^{m_1}, \left(g_{\omega,\varepsilon,\mathscr{g}}^2, \mathsf{Ad}_{e_{\varepsilon,\mathscr{g}}^{-1}} \mathbf{w}_\omega^{2,i}, M_\omega^{2,i}\right)_{i=1}^{m_2}, g_{\omega,\varepsilon,\mathscr{g}}^3\right) = 0, \tag{3}$$

$$g_{\omega,\varepsilon,\mathscr{g}}^i(t) := e_{\varepsilon,\mathscr{g}}(t)g_\omega^i(t), \quad t \in [0,T], \quad i = 1,2,3, \quad \varepsilon \in [0,1). \tag{4}$$

Note: deformations in G determined by the directions $\mathscr{g}(t) \in T_eG$.

Go back to important special case and make non-random choices.

G finite dimensional Lie group, ∇ right invariant linear connection on G, U finite dimensional right representation space for G.

Fix non-random $\{H_i^j\}_{i=1}^{m_j} \in T_eG$ and $\mathbb{R}^{m_j}$-valued martingales $M_\omega^j(t) = (M_\omega^{j,1}(t), ..., M_\omega^{j,m_j}(t))$ such that $[\![M^{j,i}, M^{j,k}]\!]_t := [\![M_\omega^{j,i}, M_\omega^{j,k}]\!]_t$, $1 \leq j \leq 3$, $1 \leq i, k \leq m_j$ is non-random (eliminate the index ω to emphasize non-randomness). Suppose that the semimartingales $g_\omega^j(\cdot) \in \mathscr{S}(G)$, $j = 1, 2, 3$, have the form (1), i.e.,

$$dg_\omega^j(t) = T_eR_{g_\omega^j(t)}\left(\sum_{i=1}^{m_j} H_i^j \circ dM_\omega^{j,i}(t) + u(t)dt \right), \quad g_\omega^j(0) = e,$$

with $u \in C^1([0,T]; T_eG)$ non-random and the same for $j = 1, 2, 3$. Consider deformations (4), i.e., $g_{\omega,\varepsilon,\mathscr{g}}^j(t) := e_{\varepsilon,\mathscr{g}}(t) g_\omega^j(t)$.

(i) Then $\left(\left(g_\omega^1, H_i^1, M_\omega^{1,i} \right)_{i=1}^{m_1}, \left(g_\omega^2, H_i^2, M_\omega^{2,i} \right)_{i=1}^{m_2}, g_\omega^3 \right)$ is a critical point of $\tilde{J}^\nabla$ if and only if $u(t)$ coupled with $\tilde{\alpha}(t) := \mathbb{E}[\alpha_\omega(t)] \in U^*$, where $\alpha_\omega(t) := \alpha_0 g_\omega^3(t)^{-1} \in U^*$, satisfies the following system of ODEs:

$$\begin{cases} d\left(\frac{\delta l}{\delta u}(t,u(t),\tilde{\alpha}(t)) + \frac{\delta p}{\delta u}\left(\tilde{H}_1(t),\tilde{H}_2(t),u(t)\right)\right) \\ = -\mathrm{ad}^*_{u(t)}\left(\frac{\delta l}{\delta u}(t,u(t),\tilde{\alpha}(t))\right)dt - \mathrm{ad}^*_{u(t)}\left(\frac{\delta p}{\delta u}\left(\tilde{H}_1(t),\tilde{H}_2(t),u(t)\right)\right)dt \\ \quad + \left(\frac{\delta l}{\delta \alpha}(t,u(t),\tilde{\alpha}(t))\right)\diamond \tilde{\alpha}(t)dt - K\left(t,\tilde{H}_1(t),\tilde{H}_2(t),u(t)\right)dt, \\ d\tilde{\alpha}(t) = \frac{1}{2}\sum_{i,k=1}^{m_3}\left(\tilde{\alpha}(t)H_i^3\right)H_k^3 d[\![M^{3,i},M^{3,k}]\!]_t - \tilde{\alpha}(t)u(t)dt; \end{cases}$$

$\tilde{H}_j(t) \in \mathscr{M}_{m_j}$ is the $m_j \times m_j$ matrix with entries

$$(\tilde{H}_j(t))_{ik} := \left(\nabla_{H_i^j} H_k^j\right)\frac{d[\![M^{j,i},M^{j,k}]\!]_t}{dt}, \quad 1 \le i,k \le m_j, \quad j = 1,2;$$

the operator $K : [0,T] \times \mathscr{M} \times \mathscr{M} \times T_eG \to T_e^*G$ is defined by

$$\left\langle K(t,A_1,A_2,u),v\right\rangle_{T_eG} := -\sum_{j=1}^{2}\left\langle \frac{\delta p}{\delta A_j}(A_1,A_2,u), B_j(t,v)\right\rangle_{\mathscr{M}_{m_j}},$$

where $A_j \in \mathscr{M}_{m_j}$ for $j = 1,2$, $u,v \in T_eG$, and $B_j(t,v) \in \mathscr{M}_{m_j}$ is the $m_j \times m_j$ matrix whose entries for $1 \le i,k \le m_j$ are

$$\left(B_j(t,v)\right)_{ik} := \left(\nabla_{H_i^j}(\mathrm{ad}_v H_k^j) + \nabla_{\mathrm{ad}_v H_i^j} H_k^j\right)\frac{d[\![M^{j,i},M^{j,k}]\!]_t}{dt}.$$

(ii) The first equation is equivalent to the following constrained stochastic variational principle

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left(\int_0^T l(t,u_\varepsilon(t),\tilde{\alpha}_\varepsilon(t))dt+\int_0^T p(\tilde{H}_{1,\varepsilon}(t),\tilde{H}_{2,\varepsilon}(t),u_\varepsilon(t))dt\right)=0$$

on $T_eG\times U^*$ for variations of the form

$$\begin{cases}\dfrac{du_\varepsilon(t)}{d\varepsilon}\Big|_{\varepsilon=0}=\dot{v}(t)-\mathrm{ad}_{u(t)}v(t),\\ \dfrac{d\tilde{\alpha}_\varepsilon(t)}{d\varepsilon}\Big|_{\varepsilon=0}=-v(t)\tilde{\alpha}(t),\\ \dfrac{d\tilde{H}_{j,\varepsilon}(t)}{d\varepsilon}\Big|_{\varepsilon=0}=B_j(t,v(t)),\ \ j=1,2,\\ u_0(t)=u(t),\quad \tilde{\alpha}_0(t)=\tilde{\alpha}(t),\quad \tilde{H}_{j,0}(t)=\tilde{H}_j(t),\end{cases}$$

where $v\in C^1([0,T];T_eG)$ with $v(0)=0$, $v(T)=0$ is non-random.

It is a *stochastic variational principle*, even though the action functional is deterministic, because in the variational principle the action functional is taken for stochastic Lagrangian paths (the contraction matrix for stochastic Lagrangian paths is used).

COMPRESSIBLE NAVIER-STOKES

Applying the main theorem to $G^s := Diff(\mathbb{T}^3)$, $s > \frac{3}{2} + 1$ is not possible; G^s is not a Lie group. But the idea and outline of the proof(s) is still valid. Need to work out explicitly the Itô differential of the advected quantity (e.g., function, one-form, density, on $\mathbb{T}^3$); this used to be $\alpha_\omega(t) \in U^*$ previously. Then one computes the induced variations using some results in Arnaudon-Chen-Cruzeiro [2014]. The upshot is that all goes through (even in the stochastic case). A key ingredient is the right invariant connection ∇^0 on G^s:

$$\left(\nabla^0_{\mathcal{X}}\mathcal{Y}\right)(\eta) = \Bigg[\frac{\partial}{\partial t}\left(\mathcal{Y}(\eta_t)\circ\eta_t^{-1}\right)\Big|_{t=0} + \nabla_{\mathcal{X}(\eta)\circ\eta^{-1}}\left(\mathcal{Y}(\eta)\circ\eta^{-1}\right)$$
$$+ \nabla\left(\mathcal{X}(\eta)\circ\eta^{-1}\right)^T\cdot\left(\mathcal{Y}(\eta)\circ\eta^{-1}\right) + \nabla\left(\mathcal{Y}(\eta)\circ\eta^{-1}\right)^T\cdot\left(\mathcal{X}(\eta)\circ\eta^{-1}\right)$$
$$+ \left(\mathcal{X}(\eta)\circ\eta^{-1}\right)\mathrm{div}\left(\mathcal{Y}(\eta)\circ\eta^{-1}\right) + \left(\mathcal{Y}(\eta)\circ\eta^{-1}\right)\mathrm{div}\left(\mathcal{X}(\eta)\circ\eta^{-1}\right)\Bigg]\circ\eta,$$

where ∇ is the Levi-Civita connection on M, $\mathcal{X},\mathcal{Y}\in\mathfrak{X}(G^s)$ and hence $\mathcal{X}(\eta),\mathcal{Y}(\eta)\in T_\eta G^s$, and $t\mapsto\eta_t$ is a C^1 curve in G^s such that $\eta_0=\eta$ and $\frac{d}{dt}\Big|_{t=0}\eta_t = \mathcal{X}(\eta)$.

Warning: Right invariant metric is not the hydrodynamic metric! *Hydrodynamic metric* $\langle\langle\cdot,\cdot\rangle\rangle_\eta$ on G^s is the L^2-weak metric given by

$$\langle\langle U_\eta, V_\eta\rangle\rangle_\eta := \int_M \langle U_\eta(m), V_\eta(m)\rangle_{\eta(m)}\, d\mu_g(m),$$

$\eta \in G^s$, $U_\eta, V_\eta \in T_\eta G^s = \left\{W : M \to TM \text{ of class } H^s \mid W(m) \in T_{\eta(m)}M\right\}$. The Euler equations are the spatial representation of the geodesic spray of this metric; Arnold [1966] formal, Ebin-Marsden [1970] rigorous: *the spray is a smooth vector field on TG^s; induces smooth spray on TG^s_{vol}, hence well-posedness for Euler.*

The spray of the weak right-invariant metric is not smooth. However, metrics involving at least one derivative will give a smooth spray. This leads to the α-Euler and n-dimensional CH equations.

We take $U^* = \{\text{vector space of all densities on } \mathbb{T}^3\}$ and define $\alpha_0 := D_0(\theta)d^3\theta \in U^*$, $\theta \in \mathbb{T}^3$. As before, let $\mathscr{M}_m(G^s)$ be the collection of all $m \times m$ matrices whose entries are in $\mathfrak{X}^s(\mathbb{T}^3)$. Define $\mathscr{M}(G^s) := \bigcup_{m=1}^\infty \mathscr{M}_m(G^s)$.

Declare $(\mathfrak{X}^s(\mathbb{T}^3))^*$ to be $\Omega^1(\mathbb{T}^3)$ (use Lebesgue measure on $\mathbb{T}^3$).

Define the **contraction force** $\tilde{p} : \mathscr{M}(G^s) \times \mathscr{M}(G^s) \times \mathfrak{X}^s(\mathbb{T}^3) \to \mathbb{R}$ by

$$\tilde{p}(A, B, u) := \frac{1}{2} \int_{\mathbb{T}^3} u(\theta) \cdot \mathbf{Tr}(A)(\theta) d^3\theta + \frac{1}{2} \sum_{i,j=1}^{m} \int_{\mathbb{T}^3} \mathbf{P}_i(u(\theta)) \mathbf{P}_j\Big((B)_{ij}(\theta)\Big) d^3\theta,$$

$$\forall A \in \mathscr{M}_n(G^s),\ B \in \mathscr{M}_m(G^s),\ \forall u \in \mathfrak{X}^s(\mathbb{T}^3);\ \ u(\theta), (B)_{ij}(\theta) \in \mathbb{R}^3,\ \forall \theta \in \mathbb{R}^3,$$

where $\mathbf{Tr} : \mathscr{M}(G^s) \to \mathfrak{X}^s(\mathbb{T}^3)$ is the trace and $\mathbf{P}_i : \mathbb{R}^3 \to \mathbb{R}$ is the projection operator defined by

$$\mathbf{P}_i(x_1, x_2, x_3) := \begin{cases} x_i, & \text{if } 1 \leq i \leq 3, \\ 0, & \text{if } i > 3. \end{cases}$$

View $H_{1,\nu} := \sqrt{2\nu}(1, 0, 0)$, $H_{2,\nu} := \sqrt{2\nu}(0, 1, 0)$, $H_{3,\nu} := \sqrt{2\nu}(0, 0, 1)$, $H_{1,\mu} := \sqrt{2\mu}(1, 0, 0)$, $H_{2,\mu} := \sqrt{2\mu}(0, 1, 0)$, $H_{3,\mu} := \sqrt{2\mu}(0, 0, 1) : \mathbb{T}^3 \to \mathbb{R}^3$ as constant maps $\mu, \nu \geq 0$. They define, by translation, vector fields on $\mathbb{T}^3$, i.e., in the trivialization $T\mathbb{T}^3 = \mathbb{T}^3 \times \mathbb{R}^3$, these vector fields are constant.

Suppose that g_ω^ν and $\tilde{g}_\omega^\mu$ are the solutions of the following SDEs:

$$\begin{cases} dg_\omega^\nu(t,\theta) = \sum_{i=1}^3 H_{i,\nu} dW_\omega^i(t) + u_\omega(t, g_\omega^\nu(t,\theta))dt, & g_\omega^\nu(0,\theta) = \theta, \\ d\tilde{g}_\omega^\mu(t,\theta) = \sum_{i=1}^3 H_{i,\mu} d\tilde{W}_\omega^i(t) + u_\omega(t, \tilde{g}_\omega^\mu(t,\theta))dt, & \tilde{g}_\omega^\mu(0,\theta) = \theta, \end{cases} \tag{5}$$

$W_\omega^1, W_\omega^2, W_\omega^3$ independent $\mathbb{R}$-valued Brownian motions, $\tilde{W}_\omega^1 = \tilde{W}_\omega^2 = \tilde{W}_\omega^3$, and $u : \Omega \times [0,T] \to \mathfrak{X}^s(\mathbb{T}^3)$ is such that $u_\omega(t,\theta)$ is a $\mathcal{P}_t$-adapted semimartingale for every $\theta \in \mathbb{T}^3$.

Let $v \in C^1([0,1]; \mathfrak{X}^s(\mathbb{T}^3))$ (s large) with $v(0,\theta) = v(T,\theta) = 0$. Solve

$$\frac{de_{\varepsilon,v}(t,\theta)}{dt} = \varepsilon \dot{v}(t, e_{\varepsilon,v}(t,\theta)), \quad e_{\varepsilon,v}(0,\theta) = \theta; \quad e_{\varepsilon,v}(t,\cdot) \in G^s \tag{6}$$

Define $g_{\omega,\varepsilon,v}^\nu(t,\theta) := e_{\varepsilon,v}(t, g_\omega^\nu(t,\theta))$, where g_ω^ν is the solution to (5).

The equation for the induced deformation is

$$\begin{aligned} dg_{\omega,\varepsilon,v}^\nu(t) = T_e R_{g_{\omega,\varepsilon,v}^\nu(t)} \Big(\sum_{i=1}^3 \mathrm{Ad}_{e_{\varepsilon,v}(t)} H_{i,\nu} \circ dM_\omega^i(t) \\ + \mathrm{Ad}_{e_{\varepsilon,v}(t)} u_\omega(t) dt + \varepsilon \dot{v}(t) dt \Big). \end{aligned} \tag{7}$$

The action integral is:

$$\mathbf{J}\left(\left(g^1_\omega, \mathbf{w}^{1,i}_\omega, M^{1,i}_\omega\right)_{i=1}^{m_1}, \left(g^2_\omega, \mathbf{w}^{2,i}_\omega, M^{2,i}_\omega\right)_{i=1}^{m_2}, g^3_\omega\right)$$

$$:= \int_0^T \int_{\mathbb{T}^3} \left(\frac{1}{2}\left|w_\omega(t,\theta)\right|^2 \tilde{D}(t,\theta) - \tilde{D}(t,\theta) e\left(t, \tilde{D}(t,\theta)\right)\right) d^3\theta dt$$

$$+ \int_0^T \tilde{p}\left(\frac{\mathbf{D}^{\nabla^0,(\mathbf{w}^{1,i}_\omega, M^{1,i}_\omega)_{i=1}^{m_1}} g^1_\omega(t)}{dt}, \frac{\mathbf{D}^{\nabla^0,(\mathbf{w}^{2,i}_\omega, M^{2,i}_\omega)_{i=1}^{m_2}} g^2_\omega(t)}{dt}, w_\omega(t)\right) dt,$$

$$\forall \left(\left(g^1_\omega, \mathbf{w}^{1,i}_\omega, M^{1,i}_\omega\right)_{i=1}^{m_1}, \left(g^2_\omega, \mathbf{w}^{2,i}_\omega, M^{2,i}_\omega\right)_{i=1}^{m_2}, g^3_\omega\right) \in \widetilde{\mathscr{S}(G^s)} \times \widetilde{\mathscr{S}(G^s)} \times \mathscr{S}(G^s),$$

where the specific internal energy $e(t, \tilde{D})$ is non-random, and

$$w_\omega(t,\cdot) := T_{g^1_\omega(t)} R_{g^1_\omega(t)^{-1}}\left(\frac{\mathscr{D} g^1_\omega(t)}{dt}\right), \quad \tilde{D}(t,\cdot) d^3\theta = \mathbb{E}\left[\left(g^3_\omega(t,\cdot)^{-1}\right)^* D_0 d^3\theta\right].$$

Suppose that $u_\omega = u$ in (5) is non-random. Then

$$\left(\left(g_\omega^\nu, H_{i,\nu}, W_\omega^i\right)_{i=1}^3, \left(\tilde{g}_\omega^\mu, H_{i,\mu}, \tilde{W}_\omega^i\right)_{i=1}^3, g_\omega^0\right), \quad g_\omega^0 := g_\omega^\nu|_{\nu=0},$$

is a critical point of $\mathbf{J}$, using deformations (7) induced by (6), if and only if (the non-random variables) $\left(u, \tilde{D}\right)$ satisfy the following (deterministic) classical Navier-Stokes equations for compressible fluid flow

$$\begin{cases} du(t) = -\left(u(t)\cdot\nabla u(t)\right)dt + \frac{1}{\tilde{D}(t)}\Big(\nu\Delta u(t) + \mu\nabla \text{div} u(t) - \nabla p(t)\Big)dt, \\ d\tilde{D}(t) = -\text{div}\left(u(t)\tilde{D}(t)\right)dt, \quad p := \tilde{D}^2\frac{de}{d\tilde{D}} \text{ non-random pressure,} \end{cases}$$

e is the internal energy density of the fluid.

• Similar result for compressible MHD. Variables: velocity, magnetic field, mass density, entropy. Dissipation (two viscosity coefficients like in Navier-Stokes) in velocity, magnetic field, entropy equations.

• Kelvin-Noether theorem gives the derivative of circulation, not zero, depends on the advected quantities and on the bulk viscosity.

THANK YOU FOR

YOUR ATTENTION