Geometric Singular Perturbation Theory for Fast-Slow PDEs

(presented at FAU DCN Seminar, February 4th 2022)

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VolkswagenStiftung
DFG Deutsche Forschungsgemeinschaft
IAS TUM Institute for Advanced Study
Mathematisches Forschungsinstitut Oberwolfach
Seventh Framework Programme
ÖAW Österreichische Akademie der Wissenschaften
Alexander von Humboldt Stiftung/Foundation
FWF Der Wissenschaftsfonds.

Overview

Background: Fast-Slow ODEs

- a concrete example: van der Pol
- some general theory & techniques

New Results: Fast-Slow PDEs

- invariant slow manifolds
- geometric desingularization

Joint work with: Maximilian Engel, Felix Hummel.
Several further joint works in progress!

The Classical Example: Van der Pol Oscillator

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = y - \frac{x^3}{3} + x, \\ \frac{\mathrm{d}y}{\mathrm{d}t} = -\varepsilon x. \end{cases} \qquad 0 < \varepsilon \ll 1$$

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global unstable equilibrium

$q = (0, 0)$

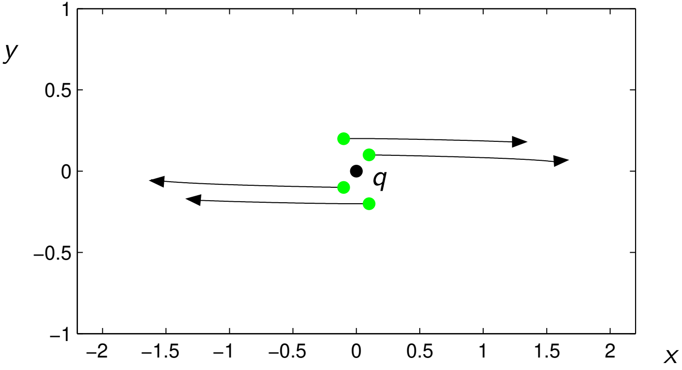

Figure: four green dots = initial conditions; $\varepsilon = 0.02$.

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$\varepsilon \to 0$, fast subsystem

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = y - \frac{x^3}{3} + x, \\ \frac{\mathrm{d}y}{\mathrm{d}t} = 0. \end{cases}$$

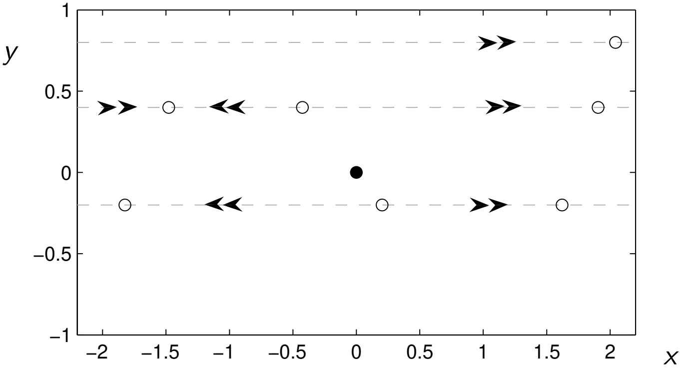

Figure: fast subsystems for $y \in \{-0.2, 0.4, 0.8\}$; $\varepsilon = 0$.

The Classical Example: Van der Pol Oscillator

$$\begin{cases} \varepsilon \frac{dx}{ds} = y - \frac{x^3}{3} + x, \\ \frac{dy}{ds} = -x. \end{cases}$$

$\varepsilon t =: s$, $\varepsilon \to 0$, slow subsystem

$$\begin{cases} 0 = y - \frac{x^3}{3} + x, \\ \frac{dy}{dt} = -x. \end{cases}$$

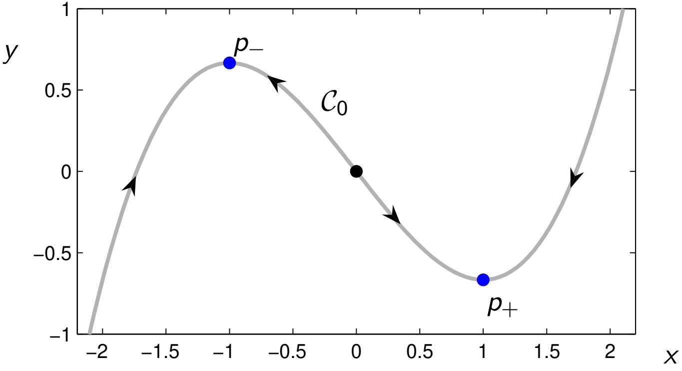

Figure: slow subsystem; $\varepsilon = 0$; fold points $p_\pm$.

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Candidate orbit γ_0:

- 2 slow and 2 fast segments
- jumps at fold points

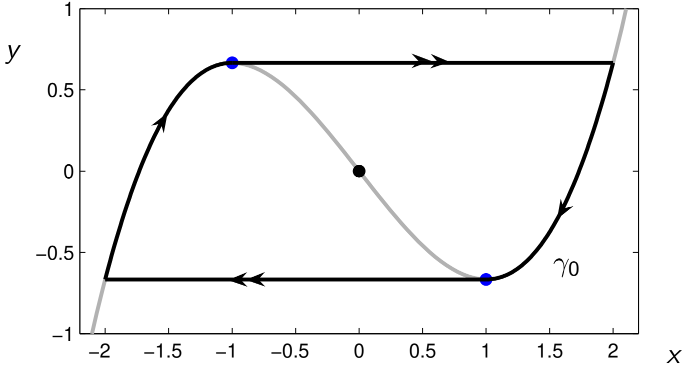

Figure: candidate orbit γ_0; $\varepsilon = 0$.

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Full orbit γ_ε:

- close to γ_0
- deviation near fold points

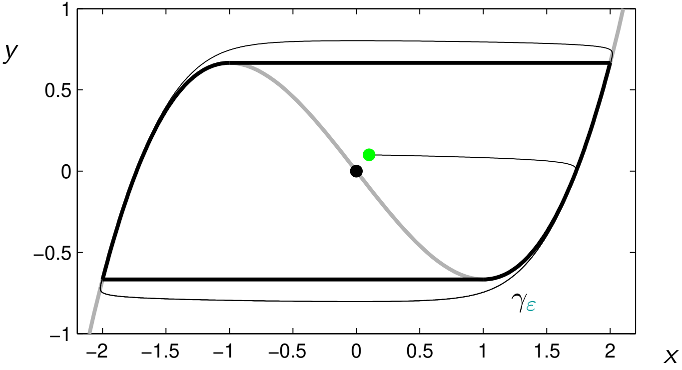

Figure: relaxation oscillation γ_ε for $\varepsilon = 0.02$.

Fast-Slow Systems - Standard Form

Fast variables $x \in \mathbb{R}^m$, slow variables $y \in \mathbb{R}^n$, time scale separation $0 < \varepsilon \ll 1$.

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = x' = f(x,y,\varepsilon) \\ \frac{\mathrm{d}y}{\mathrm{d}t} = y' = \varepsilon g(x,y,\varepsilon) \end{cases} \quad \overset{\varepsilon t = s}{\longleftrightarrow} \quad \begin{cases} \varepsilon \frac{\mathrm{d}x}{\mathrm{d}s} = \varepsilon \dot{x} = f(x,y,\varepsilon) \\ \frac{\mathrm{d}y}{\mathrm{d}s} = \dot{y} = g(x,y,\varepsilon) \end{cases}$$

$$\downarrow\ \varepsilon = 0 \qquad\qquad \downarrow\ \varepsilon = 0$$

$$\begin{cases} x' = f(x,y,0) \\ y' = 0 \end{cases} \qquad\qquad \begin{cases} 0 = f(x,y,0) \\ \dot{y} = g(x,y,0) \end{cases}$$

fast subsystem — slow subsystem

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fast subsystem — slow subsystem

- $\mathcal{C}_0 := \{f = 0\}$ = critical manifold = equil. of fast subsystem.
- $\mathcal{C}_0$ is called normally hyperbolic if the $m \times m$ matrix $(\mathrm{D}_x f)(p)$ has no eigenvalues λ_j with zero real part for all $p \in \mathcal{C}_0$.
- Attracting if $\mathrm{Re}(\lambda_j) < 0$ for all j; repelling if $\mathrm{Re}(\lambda_j) > 0$ for all j.

Multiple Time Scale Techniques / GSPT

Normal Hyperbolicity ($\mathrm{D}_x f(p)$ hyperbolic):

- asymptotics: regular expansions
- analysis: Fenichel(-Tikhonov) theorem
- algorithms: ILDM/CSP/ZDP/...

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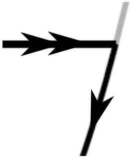
Normal Hyperbolicity & Transitions:

- asymptotics: matching
- analysis: exchange lemma
- algorithms: stiff IVPs, non-stiff BVPs



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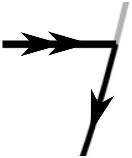

Loss of Normal Hyperbolicity:

- asymptotics: matching
- analysis: blow-up
- algorithms: bifurcation detection

Multiple Time Scale Techniques / GSPT

Links to many areas:

- applications (biology, chemistry, engineering, geoscience, ...)
- DAEs, non-smooth dynamics, control, hybrid systems, ...
- stochastic problems: SODEs, MCs, LDPs, ...
- algebraic geometry, polytopes, nonstandard analysis, ...
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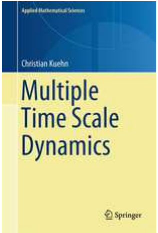

More on fast-slow systems
(aka "commercial break"):

$\rightarrow$ **CK**, Multiple Time Scale Dynamics
Springer, 814 pages, 2015.

Theorem (Fenichel, 1979; Tikhonov, 1952)

Suppose $S = S_0$ is a compact normally hyperbolic submanifold (with boundary) of the critical manifold $\mathcal{C}$. Then for $0 < \varepsilon \ll 1$:

(F1) *$\exists$ a locally invariant slow manifold S_ε diffeomorphic to S_0.*

(F2) *S_ε has a distance $\mathcal{O}(\varepsilon)$ from S_0.*

(F3) *The flow on S_ε converges to the slow flow as $\varepsilon \to 0$.*

(F4) *Similar conclusions hold for stable/unstable manifolds of S_0.*

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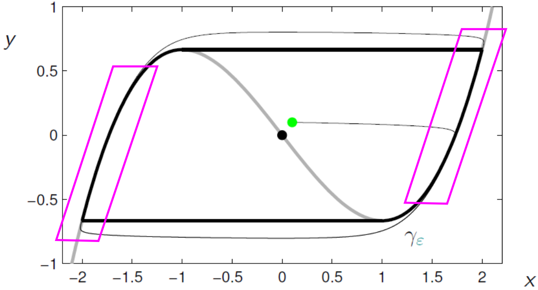

Meta-Theorem

Theorem (Dumortier/Roussarie, Krupa/Szmolyan, ... 1990s)

Let $(f, g) =: V$ *then* $\exists \Psi : \mathbb{S}^{m+n} \times [0, r_0] \to \mathbb{R}^{m+n+1}$ *and* $\exists k \in \mathbb{N}$ *s.t.*

$$V_{\text{blown-up}} = \frac{1}{r^k} (\mathrm{D}\Psi)^{-1} \circ V \circ \Psi$$

has only partially hyperbolic equilibria.

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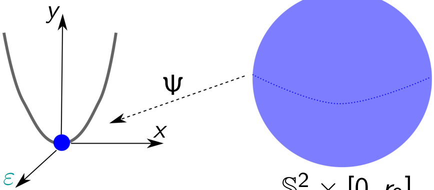
$$\begin{aligned} x' &= y - \tfrac{1}{3}x^3 + x, \\ y' &= -\varepsilon x, \\ \varepsilon' &= 0. \end{aligned} \qquad \to \begin{cases} x' = y - x^2, \\ y' = -\varepsilon, \\ \varepsilon' = 0. \end{cases}$$



$\mathbb{S}^2 \times [0, r_0]$

Fast-Slow PDEs

One motivating example:

$$\begin{aligned} \partial_t u &= \quad \partial_x^2 u + f(u,v), \\ \partial_t v &= \varepsilon(\partial_x^2 v + g(u,v)), \end{aligned}$$

for $u = u(t,x)$, $v = v(t,x)$, $0 < \varepsilon \ll 1$, $x \in [-a,a]$, with

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad \partial_x u(t,\pm a) = 0 = \partial_x v(t,\pm a).$$

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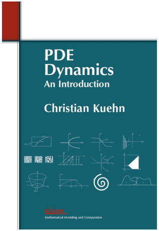

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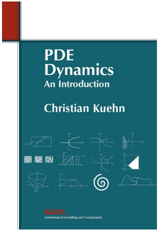

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Today's goals: first steps towards...

1. Fenichel theory for general fast-slow PDEs
2. blow-up desingularization for specific fast-slow PDEs

Overview & References

(I) *"Slow Manifolds for Infinite-Dimensional Evolution Equations"*, F. Hummel and **CK**, Commentarii Mathematici Helvetici, accepted / to appear, see also arXiv:2008.10700, pp. 1–53.

- identifies key issues with infinite-dimensional slow manifold
- general setting for multiple time scale evolution equations
- approximation result near the (attracting) critical manifold
- existence of (attracting) slow manifolds via subsplitting

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(III) *"Connecting a direct and a Galerkin Approach to Slow Manifolds in Infinite Dimensions"*, M. Engel, F. Hummel and **CK**, Proc. AMS, Vol. 8, pp. 252–266, 2021.

- comparison for existence of (attracting) slow manifolds
- Galerkin techniques close to direct Lyapunov-Perron

(I) Slow Manifolds in Infinite Dimensions

Problem 1: unbounded operators for slow dynamics

Example: abstract linear evolution equation

$$\partial_t u = Au + L_1 v,$$
$$\partial_t v = \varepsilon Bu + \varepsilon L_2 u,$$

with solution semigroup

$$T_\varepsilon(t) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} u_\varepsilon \\ v_\varepsilon \end{pmatrix}$$

$\mathcal{C}_0 := \{(u, v) \in \mathcal{D}(A) \times \mathcal{D}(B) : Au + L_1 v = 0\}$. Perturbed $\mathcal{C}_\varepsilon$?

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Best available theory (Bates et al., early 2000s) requires

$$\lim_{\varepsilon \to 0} \| T_\varepsilon(t) - T_0(t) \|_{C^1} = 0.$$

$\Rightarrow$ B is a bounded operator! Main example fails:

$$B = \partial_x^2 \qquad \Rightarrow \qquad \boxed{\lim_{\varepsilon \to 0} \varepsilon \partial_x^2 \simeq 0 \cdot \infty}.$$

(I) Slow Manifolds in Infinite Dimensions

Problem 2: fast-slow splitting

Example: two heat/diffusion equations

$$\begin{aligned}\partial_t u &= \partial_x^2 u,\\ \partial_t v &= \varepsilon\partial_x^2 v,\end{aligned}$$

with explicit solution

$$u(t,x)=\sum_{j=1}^{\infty} u_{0j}e_j(x)\mathrm{e}^{\lambda_k t},\quad v(t,x)=\sum_{j=1}^{\infty} v_{0j}e_j(x)\mathrm{e}^{\varepsilon\lambda_k t}$$

(λ_k, e_k) eigenpairs of ∂_x^2 on $[-a,a]$. Fast-slow dynamics?

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(λ_k, e_k) eigenpairs of ∂_x^2 on $[-a, a]$. Fast-slow dynamics?

Fix $\varepsilon > 0 \Rightarrow$ modes in v become fast

$$|\varepsilon \lambda_k| > |\lambda_1| \qquad \text{for all } k > k_0$$

$\Rightarrow$ direct fast-slow splitting fails! Spectral gaps help:

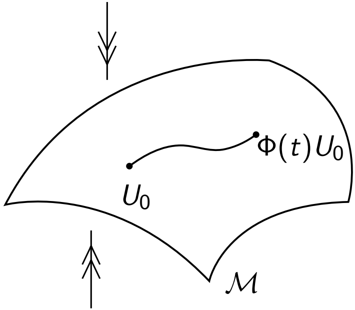
$$\lambda_k - \lambda_{k-1}.$$

(I) Slow Manifolds in Infinite Dimensions

Problem 3: localization in space-time

Classical: finite-dimensional attractors & inertial manifolds:

- e.g. books Temam, Springer (1997); Robinson, CUP (2001).
- **Defn:** inertial manifold $\mathcal{M}$ = finite-dimensional, invariant, and globally attracting for a given semiflow $\Phi(t)U_0 = U(t)$
- Fenichel: locally invariant and finite-time



(I) Slow Manifolds in Infinite Dimensions

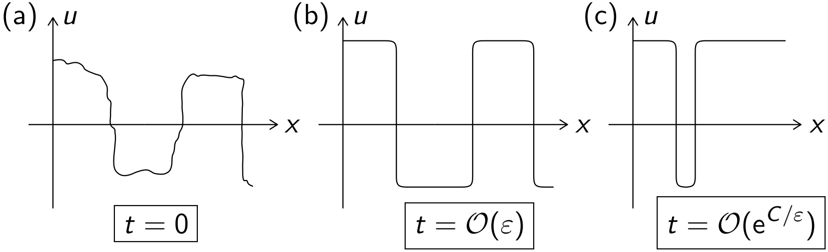
Problem 4: general setup

Classical: explicit approximately invariant manifolds:

- see e.g. analysis of Allen-Cahn metastability (Carr-Pego)

$$\partial_t u = \varepsilon^2 \partial_x^2 u + \frac{1}{2} u(1 - u^2)$$

- super-slow interface motion on manifolds, fast annihilation
- ad-hoc parametrization and construction



(I) Slow Manifolds in Infinite Dimensions

Abstract general setup:

$$\begin{aligned} \partial_t u &= \quad Au + f(u,v) \\ \partial_t v &= \varepsilon\,(Bv + g(u,v)) \end{aligned}$$

- Banach spaces X, Y with closed linear operators A, B
- continuous semigroups e^{tA}, e^{tB} with growth bounds
- interpolation-extrapolation scales X_α, Y_β (e.g. H^α, H^β)
- (local) Lipschitz conditions for f, g and $f(0,0) = 0 = g(0,0)$
- (local) attraction towards $S_0 := \{0 = Au + f(u,v)\}$
- abstract v-equation sub-splitting

$$\boxed{Y = Y^{\zeta}_{\text{fast}} \oplus Y^{\zeta}_{\text{slow}}}$$

- spectral gap-type condition

(I) Slow Manifolds in Infinite Dimensions

Theorem (F. Hummel & CK; 2020)

For all $\varepsilon, \zeta > 0$ sufficiently small and $\varepsilon < K\zeta$, there exists a family of slow invariant differentiable manifolds

$$S_{\varepsilon,\zeta} = \{(h^{\varepsilon,\zeta}(v_0), v_0) : v_0 \in Y_{\text{slow}} \cap Y_1\}$$

and for all $v_0 \in Y_{\text{slow}} \cap Y_1$

$$\left\| \begin{pmatrix} h^{\varepsilon,\zeta}_{X_1}(v_0) - h^0(v_0) \\ h^{\varepsilon,\zeta}_{Y_F^\zeta}(v_0) \end{pmatrix} \right\|_{X_1 \times Y_1} \leq C \left(\varepsilon + \frac{1}{N_S^\zeta - N_F^\zeta} \right) \|v_0\|_{Y_1},$$

(e.g. 1d RDE we have $N_S^\zeta - N_F^\zeta = \mathcal{O}(\zeta^{-1/2})$).

Method of proof: Lyapunov-Perron & suitable a-priori estimates.

(I) Slow Manifolds in Infinite Dimensions

Examples:

- *geoscience:* spatial Stommel model

$$\begin{aligned}\varepsilon\partial_t u &= \Delta u - u + 1 - \varepsilon u[1 + p_1(u^2 - v^2)],\\ \partial_t v &= \Delta v + p_2 - v[1 + p_1(u^2 - v^2)].\end{aligned}$$

- *neuroscience:* doubly-diffusive FitzHugh-Nagumo equation

$$\begin{aligned}\varepsilon\partial_t u &= \Delta u + u(1-u)(u-p_3) - w,\\ \partial_t v &= \Delta v + u - p_4 v.\end{aligned}$$

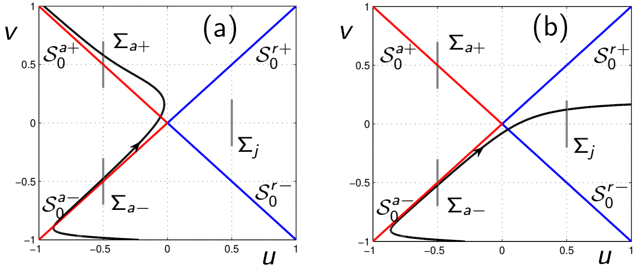
- *quantum mechanics:* Maxwell-Bloch eq. ($\rightarrow$ Menon/Haller)

$$\begin{aligned}\varepsilon\partial_t u_1 &= p_5 w u_2 - (1 + \mathrm{i}p_6)u_1,\\ \varepsilon\partial_t u_2 &= p_7(p_8 + 1 - u_2) - \tfrac{p_9}{2}\left(\overline{w}u_1 + w\overline{u_1}\right),\\ \partial_t w &= -\partial_x w + p_{10}\left(\tfrac{1}{p_9}u_1 - w\right).\end{aligned}$$

(II) Blow-Up in Infinite Dimensions

Example: classical ODE blow-up problem

$$\begin{aligned} \partial_t u &= u^2 - v^2 + \mu\varepsilon, \\ \partial_t v &= \varepsilon, \\ \partial_t \varepsilon &= 0. \end{aligned}$$

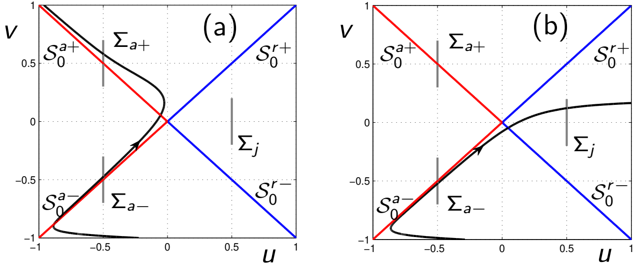


(II) Blow-Up in Infinite Dimensions

Example: natural PDE blow-up problem

$$\begin{aligned} \partial_t u &= \partial_x^2 u + u^2 - v^2 + \mu\varepsilon, \\ \partial_t v &= \varepsilon \partial_x^2 v + \varepsilon, \\ \partial_t \varepsilon &= 0. \end{aligned}$$

$x \in [-a, a]$ and Neumann boundary conditions.



- How to retain geometric interpretation?
- What object(s) to blow-up?
- How to treat the spatial domain?
- What happens in charts dynamically?

(II) Blow-Up in Infinite Dimensions

Start with concrete example

$$\begin{cases} \partial_t u = \partial_x^2 u + u^2 - v^2 + \mu\varepsilon, \\ \partial_t v = \varepsilon\partial_x^2 v + \varepsilon, \\ \partial_t \varepsilon = 0. \end{cases} \tag{tc}$$

$x \in [-a, a]$ and Neumann boundary conditions.

Q: Why start with (tc)?

- classical Crandall-Rabinowitz branch point

$$\mathcal{F}(u; v) = \partial_x^2 u + u^2 - v^2 = 0$$

- near-constant solutions *should* have ODE-type dynamics
- nonlinearity sufficiently simple for calculations
- non-trivial fast-slow dynamics via parameter μ
- local-in-time problem well-posed / parabolic regularity

(II) Blow-Up in Infinite Dimensions

$$\begin{cases} \partial_t u &= \partial_x^2 u + u^2 - v^2 + \mu\varepsilon, \\ \partial_t v &= \varepsilon \partial_x^2 v + \varepsilon, \\ \partial_t \varepsilon &= 0, \\ \partial_t a &= 0. \end{cases} \tag{tc}$$

Our approach:

- augment dynamics with $\partial_t a = 0$ (recall $x \in [-a, a]$)
- geometric space $L^2 = L^2([-a, a], \mathbb{R})$
- initial conditions $\|u_0 + K\|_{C^1} + \|v_0 + K\|_{C^1} < \delta$
- spectral Galerkin via $\partial_x^2 e_k = \lambda_k e_k$

$$u^{k_0}(t, x) = \sum_{k=1}^{k_0} u_k(t) e_k(x), \quad u_k = \langle u, e_k \rangle_{L^2}$$

$$v^{k_0}(t, x) = \sum_{k=1}^{k_0} v_k(t) e_k(x), \quad v_k = \langle v, e_k \rangle_{L^2}$$

- apply blow-up for each level $k_0 \in \mathbb{N}$

(II) Blow-Up in Infinite Dimensions

Theorem (M. Engel & CK; 2020)

$\exists$ local $\mathcal{S}_\varepsilon^{\text{a}\pm,k_0}$ for each k_0. For $\mu \neq 1$, $0 < \varepsilon \ll 1$, trajectories starting near $\mathcal{S}_\varepsilon^{\text{a}-,k_0}$ satisfy:

- *$\mu < 1$: exchange-of-stability to $\mathcal{S}_\varepsilon^{\text{a}+,k_0}$;*
- *$\mu > 1$: jump case near $\{v > 0, u = 0\}$ fast fiber.*

Methods of proof: novel blow-up for the Galerkin system, invariant manifold theory, calculation of transition map, shrinking/expanding domain through charts.

Interesting observation: free boundary PDEs in entry/exit charts!

(III) Connecting Galerkin & Abstract View

Theorem (M. Engel, F. Hummel & CK; 2021)

Fix $j, k \in \mathbb{N}$, $j \leq k$. Under suitable, yet generic, assumptions for $0 < \varepsilon, \zeta \ll 1$ with $\varepsilon = o(\zeta)$ and $\forall v_{0,S} \in Y_S^\zeta \cap Y_k$

$$\|h_X^{\varepsilon,\zeta}(v_{0,S}) - h_G^{\varepsilon,\zeta}(v_{0,S})\|_{X_j} + \|h_{Y_F^\zeta}^{\varepsilon,\zeta}(v_{0,S})\|_{Y_j}$$
$$\lesssim \left(\frac{\zeta^{k-j}}{(N_S^\zeta - N_F^\zeta)^{\delta_Y}} + \zeta^{k-j+\gamma_X} \right) \|v_{0,S}\|_{Y_k}.$$

$\Leftrightarrow$ "Galerkin slow manifold is close to abstract slow manifold"

The last slide...

Outlook:

- Question 1: Other singularities? $\rightarrow$ in progress.
- Question 2: Blow-up variants? $\rightarrow$ in progress.
- Question 3: "Non-standard form" systems? $\rightarrow$ starts 2022.
- Question 4: ...

There are many many further open questions in the area!

For further information and links to my papers/books:

- http://www.multiscale.systems

The last slide...

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Thank you very much for your attention!!