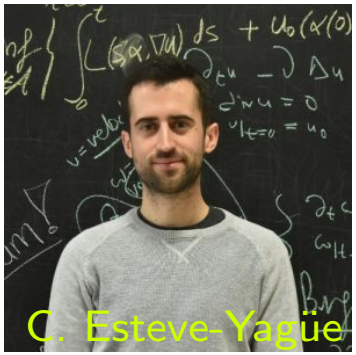
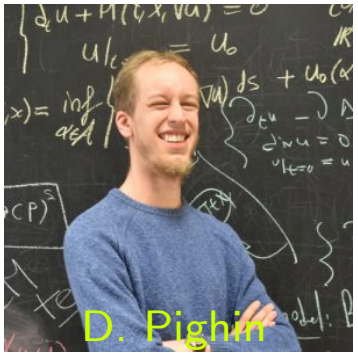
Observations on turnpike & optimal actuator design

Borjan Geshkovski
DCN Seminar

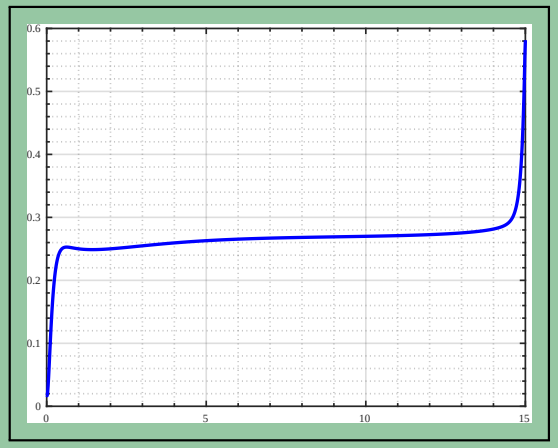
May 11 2022

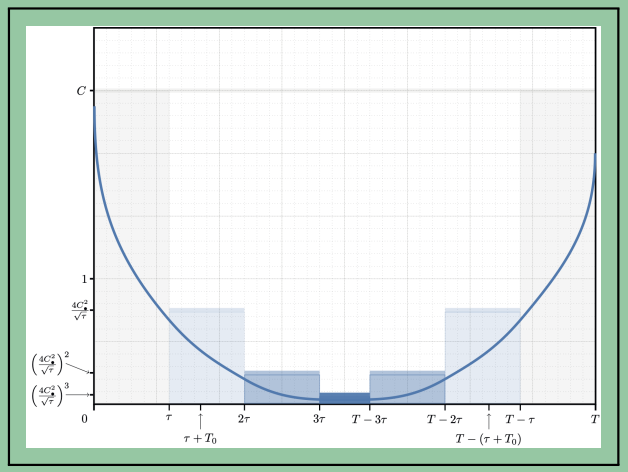




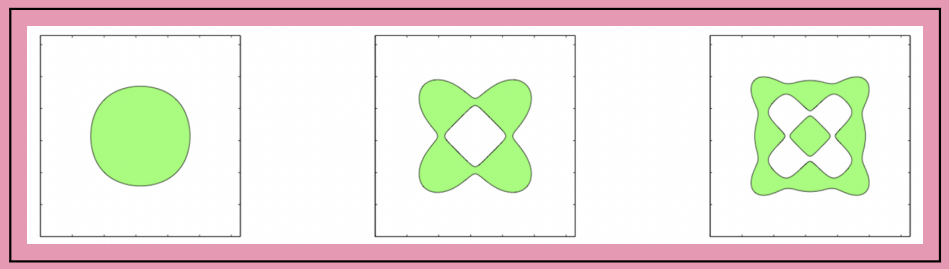
Massachusetts Institute of Technology

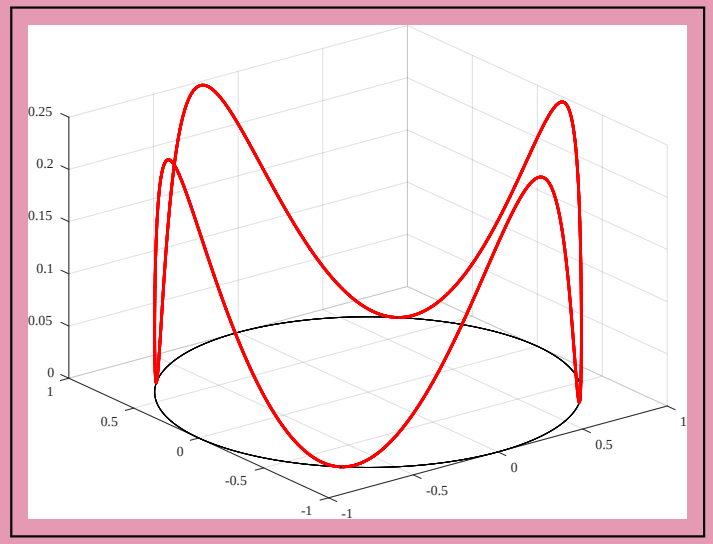
1. A method for proving nonlinear turnpike



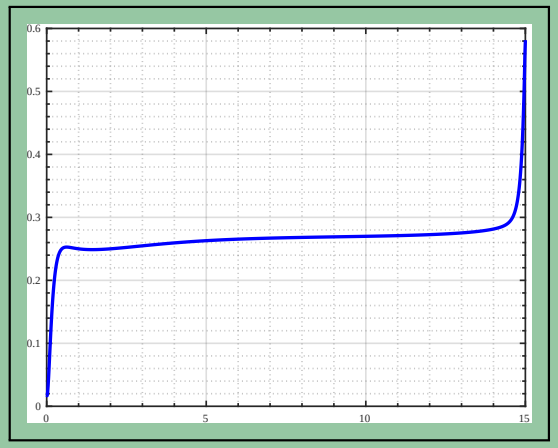


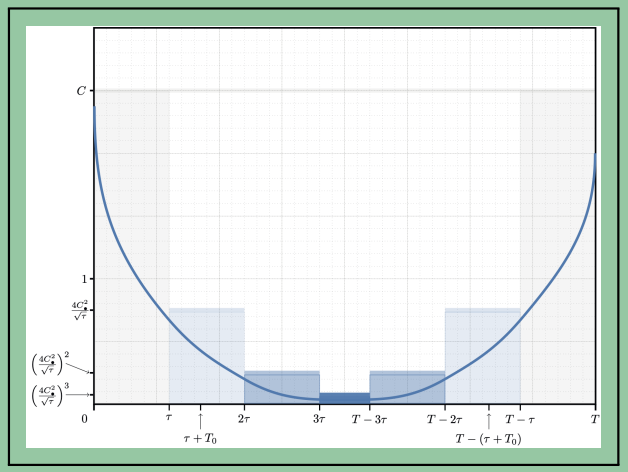
2. Hardships of optimal actuator design





1. A method for proving nonlinear turnpike





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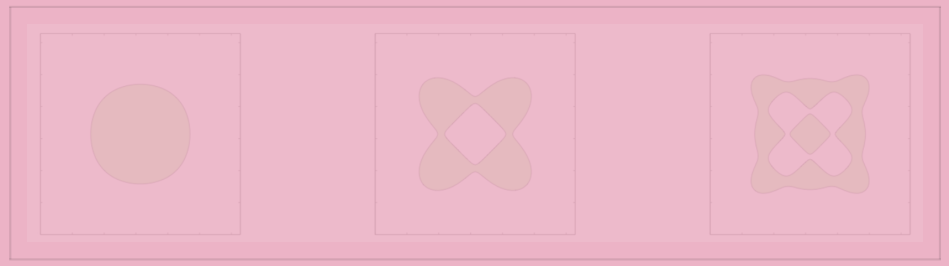

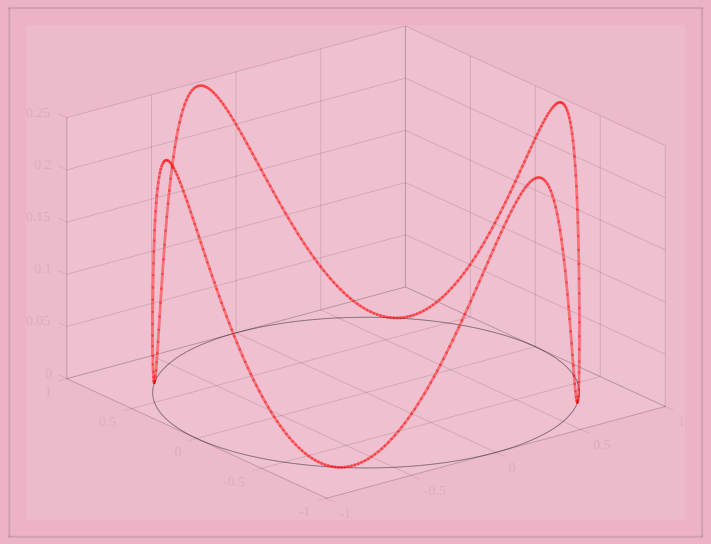

Why solve

$$\min_{u\in L^2((0,T)\times\omega)} \int_0^T \int_{\omega_\circ} (y(t,x)-y_d(x))^2 + \int_0^T \int_\omega u(t,x)^2 \qquad (1)$$

$$\text{s.t.} \qquad \begin{cases} \partial_t y - \Delta y = u 1_\omega & (0,T)\times\Omega, \\ y = 0 & (0,T)\times\partial\Omega, \\ y|_{t=0} = y^0 & \Omega, \end{cases}$$

when you can solve

$$\min_{u\in L^2(\omega)} \int_{\omega_\circ} (y-y_d)^2 + \int_\omega u^2 \quad \text{s.t.} \quad \begin{cases} -\Delta y = u 1_\omega & \text{in} \quad \Omega, \\ y = 0 & \text{on} \quad \partial\Omega \end{cases} \qquad (2)$$

?

The turnpike property

- Given $y_d = y_d(x)$, $\omega, \omega_\circ$, y^0, there exist $C, \lambda > 0$ independent of T such that for T large enough,

$$\|y_T(t) - \bar{y}\|_{L^2(\Omega)} + \|u(t) - \bar{u}\|_{L^2(\omega)} \leqslant C\left(e^{-\lambda t} + e^{-\lambda(T-t)}\right)$$

  for all $t \in [0, T]$.
- Here (u_T, y_T) solution to (1) and $(\bar{u}, \bar{y})$ to (2)

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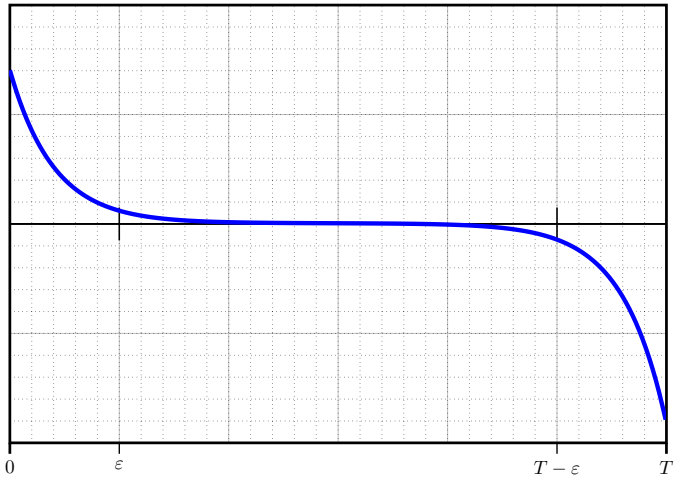

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Turnpike in optimal control of PDEs, ResNets, and beyond

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Dedicated to the memory of Roland Glowinski

Linear theory

Theorem[1]

Let $y^0 \in L^2(\Omega)$ and $y_d \in L^2(\omega_\circ)$ be fixed; $\omega, \omega_\circ \subset \Omega$ open, non-void. There exist $C, \lambda > 0$, independent of y^0 and y_d, such that for large enough $T > 0$,

$$\begin{aligned} &\|y_T(t) - \bar{y}\|_{L^2(\Omega)} + \|u_T(t) - \bar{u}\|_{L^2(\omega)} \\ &\qquad \leqslant C\left(\left\|y^0 - \bar{y}\right\|_{L^2(\Omega)} e^{-\lambda t} + \|\overline{p}\|_{L^2(\Omega)} e^{-\lambda(T-t)}\right) \end{aligned}$$

holds for $t \in [0, T]$.

How does one prove such a result?

[1] [Porretta, Zuazua, SICON '13]

Linear theory

Necessary and sufficient conditions for optimality: $(u_T = p_T 1_\omega, \bar{u} = \bar{p} 1_\omega)$

Transient

$$\begin{cases} \partial_t y_T - \Delta y_T = p_T 1_\omega \\ \partial_t p_T + \Delta p_T = (y_T - y_d) 1_{\omega_\circ} \\ y_{|t=0} = y^0 \\ p_{|t=T} = 0 \end{cases} \tag{3}$$

Steady

$$\begin{cases} -\Delta \bar{y} = \bar{p} 1_\omega \\ -\Delta \bar{p} = -(\bar{y} - y_d) 1_{\omega_\circ} \end{cases} \tag{4}$$

At least 2 (transparent) ways to proceed:

- Riccati[2]
- Diagonalization[3]

Both uncouple the optimality system by a feedback operator, and use stabilizability to get decay in phase space (y_T, p_T).

[2] [Porretta, Zuazua; 2013]
[3] [Trélat, Zuazua; 2015]

Wave equation and others

Theory applies more generally to

1. abstract first-order systems $\boxed{\dot{y} = Ay + Bu}$ (principal part of A symmetric on $\mathcal{H}$) and cost

$$\phi(y(T)) + \int_0^T \|My(t) - y_d\|_{\mathcal{H}}^2 + \int_0^T \|u(t)\|_{\mathcal{U}}^2,$$

   as long as (A,B) and (A^*,M^*) stabilizable;

2. second-order systems like the wave equation $\boxed{\partial_t^2 y - \Delta y = u 1_\omega}$ and cost like

$$\phi(y(T), \partial_t y(T)) + \int_0^T \|y(t) - y_d\|_{H_0^1(\Omega)}^2 + \int_0^T \int_\omega u(t)^2.$$

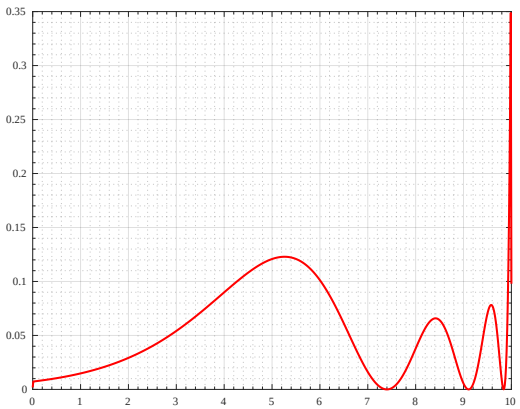
   Stabilizability translates to ω, Ω satisfying GCC.

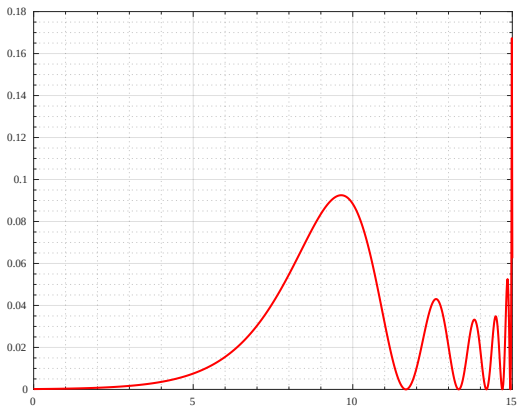
Caution

$$\min_{u\in L^2((0,T)\times\omega)} \int_{\omega_\circ} (y(T,x)-y_d(x))^2 + \int_0^T \int_\omega u(t,x)^2$$

$$\text{s.t.} \qquad \begin{cases} \partial_t y - \Delta y = u 1_\omega & (0,T)\times\Omega, \\ y = 0 & (0,T)\times\partial\Omega, \\ y|_{t=0} = y^0 & \Omega, \end{cases}$$

Turnpike doesn't hold!





Critical elements for turnpike: 1). stabilizability for (A,B) and detectability for (A^*,M^*), and 2). state-tracking term.

Nonlinear theory

$$\min_{u \in L^2((0,T)\times\omega)} \int_0^T \int_\Omega |y(t) - y_d|^2 + \int_0^T \int_\omega |u(t)|^2$$

$$\text{s.t.} \qquad \begin{cases} \partial_t y - \Delta y + y^3 = u 1_\omega & (0,T)\times\Omega, \\ y = 0 & (0,T)\times\partial\Omega, \\ y|_{t=0} = y^0 & \Omega \end{cases}$$

- Write and linearize optimality system for perturbation variables $(\delta y, \delta p)$:

$$\begin{cases} \partial_t \delta y - \Delta \delta y + f'(\bar{y})\delta y = \delta p 1_\omega, \\ \partial_t \delta p + \Delta p - f'(\bar{y})\delta p = \ \delta y + f''(\bar{y})\bar{p}\delta y. \end{cases} \tag{5}$$

Nonlinear theory

- Clue: (5) is the optimality system for

$$\min_{v} \int_0^T \int_\Omega \rho(x)\zeta^2(t,x) + \int_0^T \int_\omega v(t,x)^2$$

$$\text{s.t.} \qquad \begin{cases} \partial_t \zeta - \Delta\zeta + f'(\bar{y})\zeta = v 1_\omega & (0,T)\times\Omega, \\ y = 0 & (0,T)\times\partial\Omega, \\ \zeta|_{t=0} = \delta y^0 & \Omega, \end{cases}$$

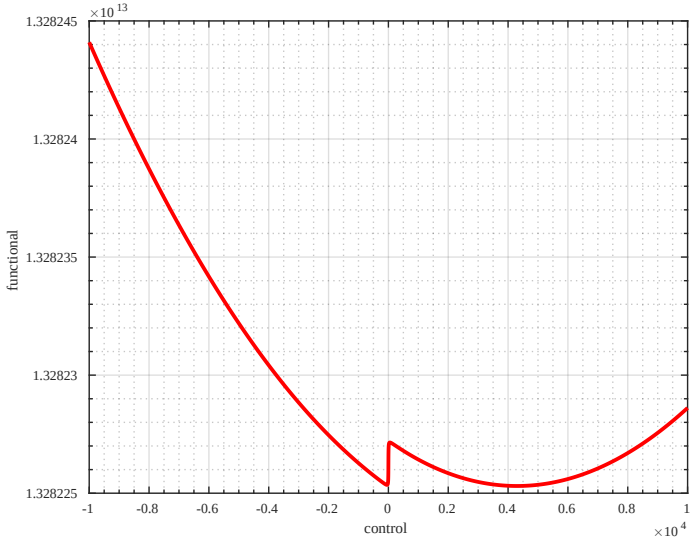
  where $\rho(x) = 1 - f''(\bar{y}(x))\bar{p}(x)$.
- Turnpike holds for this LQ whenever $\inf_\Omega \rho(x) > 0$. This can be ensured when $\|y_d\|_{L^2(\Omega)} \ll 1$!

Bottleneck

Question

Can we have y_d large?

A: Lack of uniqueness of minimizers for the steady problem![4]



Question

Can we avoid $f \in C^2$? The theory doesn't even apply for Lipschitz-only f!

[4] [Pighin; JEMS 2022]

Our setup

$$\min_{u\in L^2((0,T)\times\omega)} \phi\big((\zeta,\partial_t\zeta)(T)\big) + \int_0^T \big\|(\zeta(t),\partial_t\zeta(t)) - (\bar{\zeta},0)\big\|^2_{H_0^1\times L^2(\Omega)} + \int_0^T \int_\omega u(t,x)^2$$

$$\text{s.t.} \qquad \begin{cases} \partial_t^2\zeta - \Delta\zeta + f(\zeta) = u1_\omega & (0,T)\times\Omega, \\ (\zeta,\partial_t\zeta)|_{t=0} = (\zeta^0,\zeta^1). \end{cases}$$

Key assumptions:

- $f \in \mathrm{Lip}(\mathbb{R})$
- $\bar{\zeta}$ is a steady state: $-\Delta\bar{\zeta} + f(\bar{\zeta}) = 0$ in Ω.
- ω chosen so that exact-controllability holds (multiplier condition)

Set $y = (\zeta, \partial_t \zeta)$ and $\bar{y} = (\bar{\zeta}, 0)$:

$$\min_{u \in L^2((0,T)\times\omega)} \phi(y(T)) + \int_0^T |y(t) - \bar{y}|^2_{\mathcal{H}} + \int_0^T \int_\omega u(t,x)^2$$

$$\text{s.t.} \qquad \begin{cases} \partial_t y - Ay = \mathfrak{f}(y) + Bu & (0,T), \\ y|_{t=0} = y^0 \end{cases}$$

[5] [Esteve-Yagüe, Geshkovski, Pighin, Zuazua. Nonlinearity '22]

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$$\text{s.t.} \qquad \begin{cases} \partial_t y - Ay = \mathfrak{f}(y) + Bu & (0,T), \\ y|_{t=0} = y^0 \end{cases}$$

Theorem[5]

Suppose $f(0) = 0$. For any $y^0 \in \mathcal{H}$, there exists $T^* > 0$ and constants $C, \lambda > 0$ such that for all $T \geqslant T^*$, any minimizer (u_T, y_T) satisfies

$$\|u_T\|_{L^2((0,T)\times\omega)} \leqslant C$$

and

$$\|y_T(t) - \bar{y}\|_{\mathcal{H}} \leqslant C(e^{-\lambda t} + e^{-\lambda(T-t)})$$

holds for all $t \in [0,T]$.

[5] [Esteve-Yagüe, Geshkovski, Pighin, Zuazua. Nonlinearity '22]

Tool #1

Assumption (satisfied by wave eq):

(a) There exists a time $T_0 > 0$ such that (10.4) is exactly controllable at time T_0. Namely, for any data $(y^0, y^1) \in \mathscr{H} \times \mathscr{H}$, there exists a control $u \in L^2((0,T_0)\times\omega)$ such that the unique solution y to (10.4) set on $(0,T_0)$ satisfies $y(0) = y^0$ and $y(T_0) = y^1$.

(b) There exists some $r > 0$ and some constant $C(T_0) > 0$ such that

$$\inf_{\substack{u\in L^2((0,T_0)\times\omega)\\ y(0)=y^0\\ y(T_0)=\overline{y}}} \|u\|^2_{L^2((0,T_0)\times\omega)} \leq C(T_0)\|y^0-\overline{y}\|^2_{\mathscr{H}} \tag{10.9}$$

and

$$\inf_{\substack{u\in L^2((0,T_0)\times\omega)\\ y(0)=\overline{y}\\ y(T_0)=y^1}} \|u\|^2_{L^2((0,T_0)\times\omega)} \leq C(T_0)\|y^1-\overline{y}\|^2_{\mathscr{H}} \tag{10.10}$$

for every $y^0, y^1 \in \mathfrak{B}_r(\overline{y})$, where

$$\mathfrak{B}_r(\overline{y}) := \{z \in \mathscr{H} \mid \|z-\overline{y}\|_{\mathscr{H}} \leq r\}.$$

Tool #2

Lemma ("Lipschitz interpolation")

There exists $C_1 > 0$ such that for any(!) $u \in L^2((0,T) \times \omega)$, and any $T > 0$, for any u, it holds

$$\sup_{t \in [0,T]} \|y(t) - \bar{y}\|_{\mathcal{H}} \leqslant C_1 \left(\|y(0) - \bar{y}\|_{\mathcal{H}} + \|y - \bar{y}\|_{L^2(0,T;\mathcal{H})} + \|u\|_{L^2} \right)$$

[6] [Sontag 2008]

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Comment

- Not aware how to extend it to non-globally Lipschitz f.
- Possibly milder form of a quantitative "input-to-state" stability (ISS)[6]

[6] [Sontag 2008]

Step 1: Global estimate

Lemma

Exists $C_2 > 0$ such that $\mathcal{J}_T(u_T) \leqslant C_2$ for all $T \geqslant T_0$.

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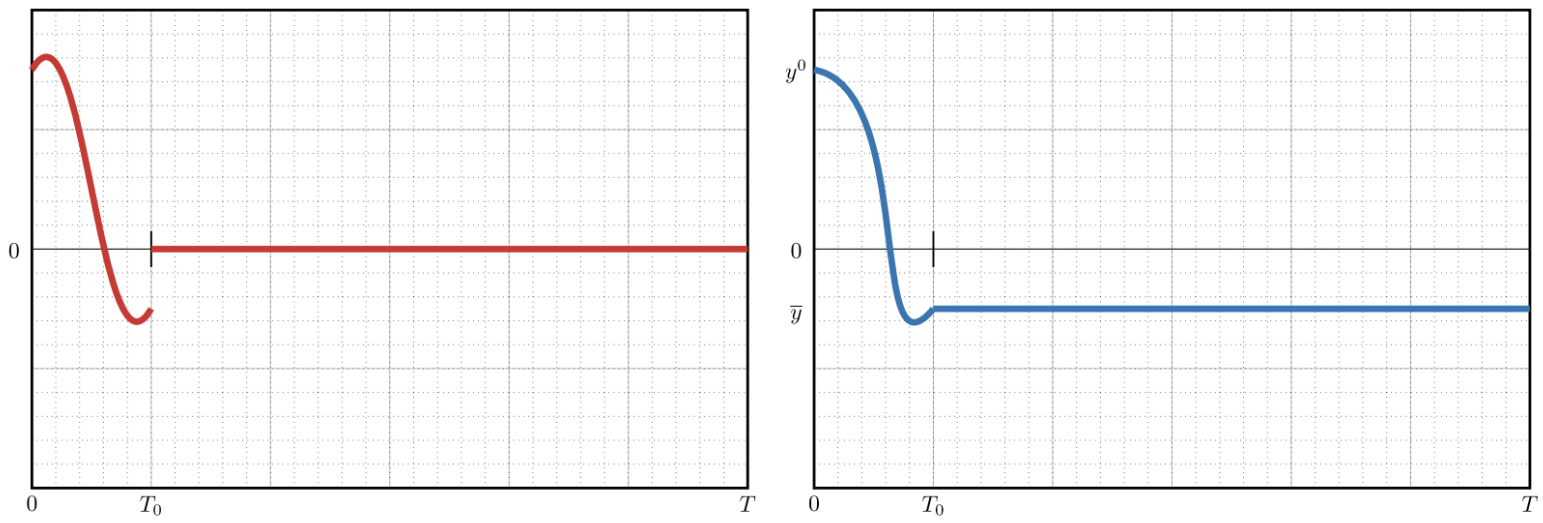
Proof. **Quasi-turnpike principle**:

1. Controllability yields u_1 such that $y_1(T_0) = \bar{y}$;
2. Consider $u_2 = u_1 1_{[0,T_0]}$ on $[0,T]$. Then $\mathcal{J}(u_T) \leqslant \mathcal{J}(u_2) =: C_2$. $\square$



Step 1: Global estimate

1. Used in conjunction with Lipschitz interpolation:

$$\sup_{t\in[0,T]} \|y(t)-\bar{y}\|_{\mathcal{H}}^2 + \mathcal{J}_T(u_T) \leqslant C_3^2 \tag{6}$$

for $C_3 > 0$ independent of T.

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2. Consequently, turnpike holds on intervals whose length is independent of T! For $t \in [0, \tau + T_0]$,

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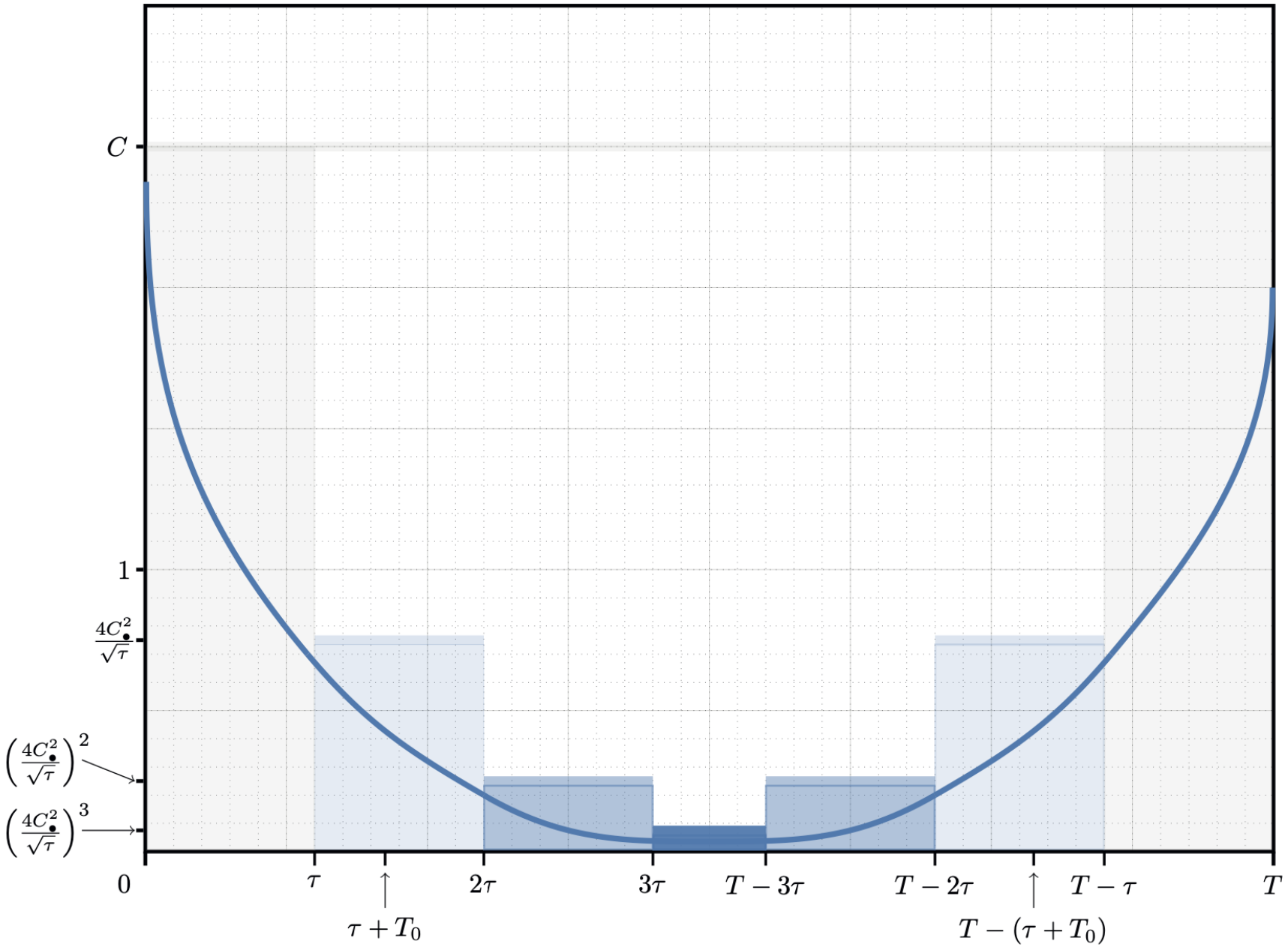
   Same for $t \in [T-(\tau+T_0), T]$.


C
1
$\frac{4C_\bullet^2}{\sqrt{\tau}}$
$\left(\frac{4C_\bullet^2}{\sqrt{\tau}}\right)^2$
$\left(\frac{4C_\bullet^2}{\sqrt{\tau}}\right)^3$
0
τ
$\tau + T_0$
2τ
3τ
$T - 3\tau$
$T - 2\tau$
$T - (\tau + T_0)$
$T - \tau$
T

Step 2: Localization

Localize study to $[\tau + T_0, T - (\tau + T_0)]$. $\tau > 0$ is a degree of freedom independent of T, and T will be chosen sufficiently large.

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$$\|y_T(\tau_j) - \bar{y}\|_{\mathcal{H}} \leqslant \frac{\|y_T - \bar{y}\|_{L^2(0,T;\mathcal{H})}}{\sqrt{\tau}}$$

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2. Restrict u_T to $[\tau_1, \tau_2]$: it solves

$$\min_{\substack{u \\ y_t = Ay + f(y) + Bu \text{ in } (\tau_1, \tau_2) \\ y(\tau_1) = y_T(\tau_1) \\ y(\tau_2) = y_T(\tau_2)}} \int_{\tau_1}^{\tau_2} \|y(t) - \bar{y}\|_{\mathcal{H}}^2 dt + \int_{\tau_1}^{\tau_2} \|u(t)\|_{L^2}^2$$

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3. Quasi-turnpike again! Controllability 2 times (from τ_1 to τ_1+T_0 and then τ_2-T_0 to τ_2) yields

$$\sup_{t\in[\tau_1,\tau_2]} \|y_T(t)-\bar{y}\|_H \leqslant C_\bullet\big(\|y_T(\tau_1)-\bar{y}\|_{\mathcal{H}} + \|y_T(\tau_2)-\bar{y}\|_{\mathcal{H}}\big) \tag{8}$$

for some $C_\bullet>0$ independent of T,τ,τ_1,τ_2.

Step 2: Localization

4. Combining (8) and (7):

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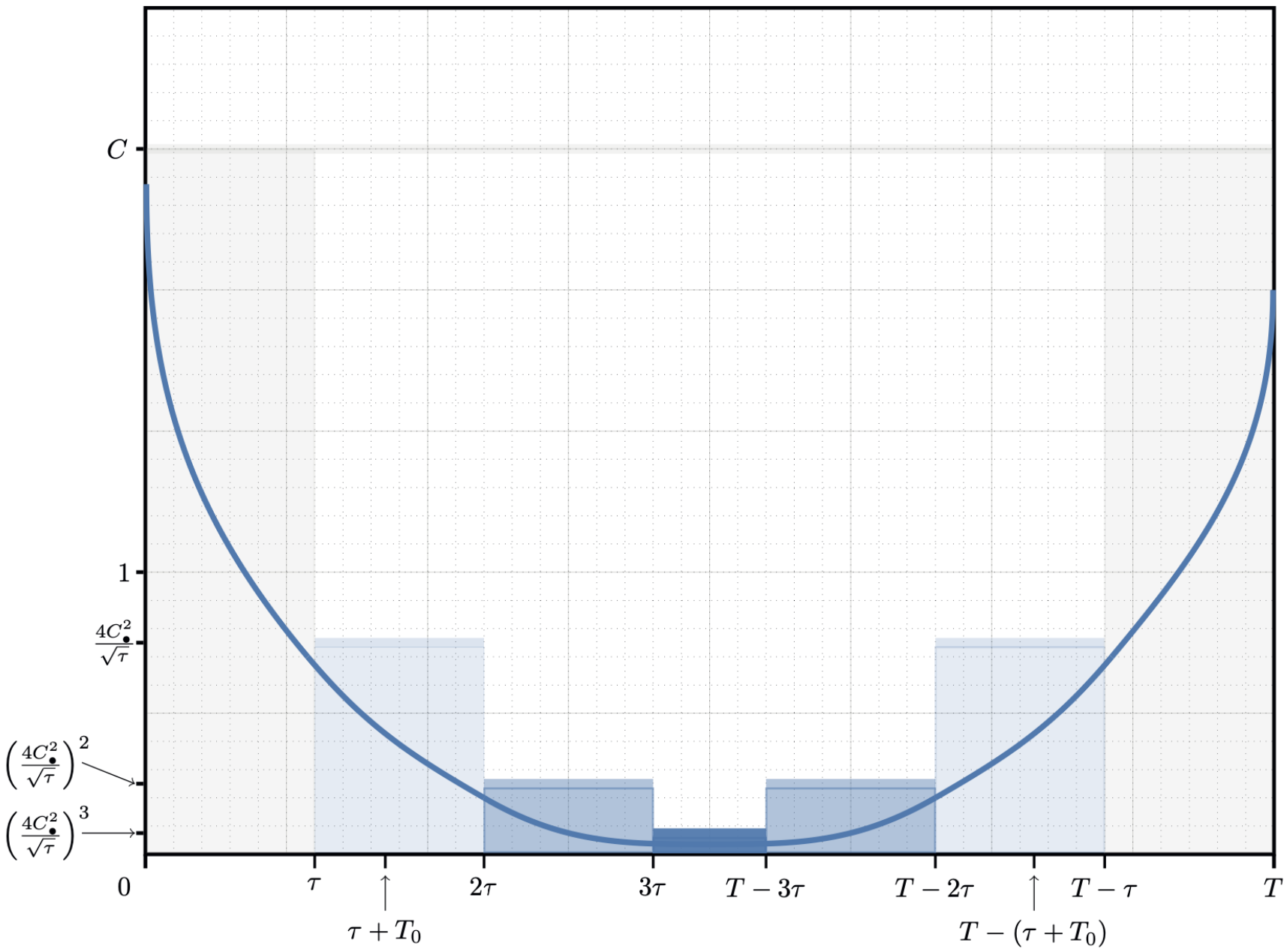
3. A judicious choice of n will yield the desired estimate.



Iterative quasi-turnpike.

Discussion and outlook

1. Proof is "modular": to generalize to other examples, need to improve individual steps. For instance, "Lipschitz interpolation" inequality to more general nonlinearities. What matters is that RHS in "Lipschitz interpolation" is roughly the cost functional!

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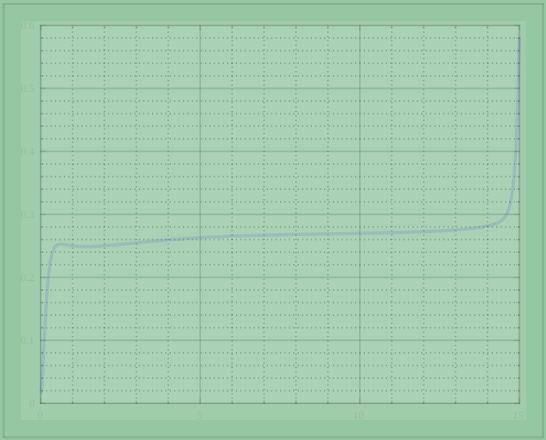
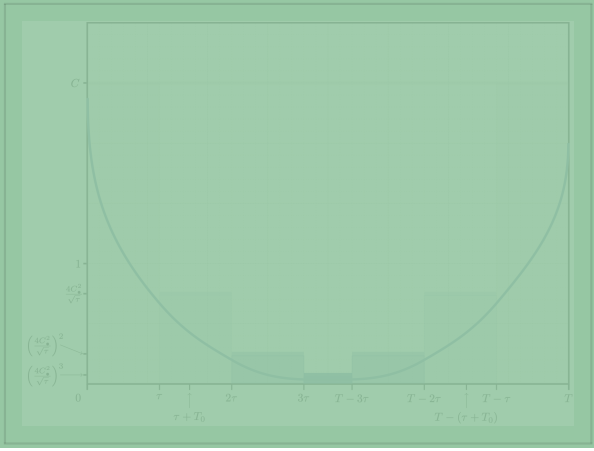
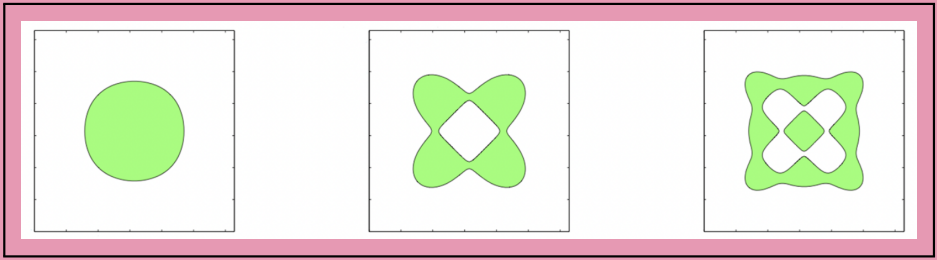
4. Finite-dimensional case with Lagrangian $\|P(y-\bar{y})\|^2 + \|u\|^2$; $y \in \mathbb{R}^d$ and $P : \mathbb{R}^d \to \mathbb{R}^m$. Can expect turnpike estimate for $\|P(y_T - \bar{y})\|$, but "Lipschitz interpolation" is difficult to get for this term! Loyasewicz's inequality can be useful here (en cours).

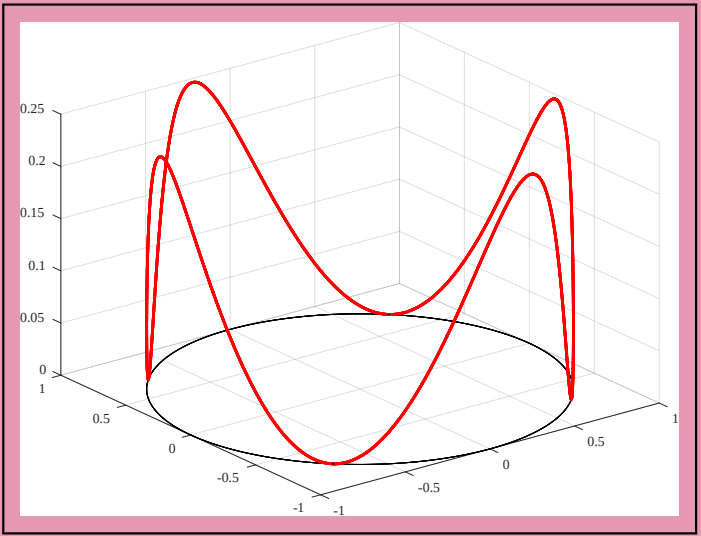
1. A method for proving nonlinear turnpike





2. Hardships of optimal actuator design





Optimal actuators, optimal sensors

- The heat equation $(\partial_t - \Delta)y = 0$ is **observable** from any open $\omega \subset \Omega$ in any time $T > 0$: there exists $C_T(\omega) > 0$ such that

$$C_T(\omega)\|y(T,\cdot)\|^2_{L^2(\Omega)} \leqslant \int_0^T \int_\omega |y(t,x)|^2$$

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Question[7]

Given $L \in (0,1)$, what is ω of fixed volume $\text{meas}(\omega) = L\,\text{meas}(\Omega)$, which maximizes sensing/observation per $C_T^*(\omega)$?

[7] [Privat, Trélat, Zuazua, 2013-2019]

Bottleneck

- Fourier is our friend: $b_j = a_j e^{-\lambda_j T}$,

$$C_T^*(\omega) = \inf_{\sum_{j=1}^{\infty} |b_j|^2 = 1} \int_0^T \int_\omega \left| \sum_{j=1}^{\infty} b_j e^{\lambda_j t} \phi_j(x) \right|^2$$

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$$= \inf \sigma \left(\frac{e^{\lambda_k + \lambda_j} - 1}{\lambda_k + \lambda_j} \int_\omega \phi_j(x) \phi_k(x) dx \right)_{j,k \geqslant 1}.$$

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  Quite opaque. Gramian is (still) not very well understood (at least, not by the speaker).

- So, "worst-case scenario" problem is too hard.

Randomization

- Randomize the Fourier coefficients of the initial data: $a_j^\nu = \beta_j^\nu a_j$ for every j, where β_j^ν are independent Bernoulli r.v.

$$\boxed{C^*_{T,\mathrm{rand}}(\omega) := \inf_{\sum_{j=1}^\infty |b_j|^2=1} \mathbb{E} \int_0^T \int_\omega \left| \sum_{j=1}^\infty \beta_j^\nu b_j e^{\lambda_j t} \phi_j(x) \right|^2 .}$$

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- We have $0 \leqslant C^*_T(\omega) \leqslant C^*_{T,\mathrm{rand}}(\omega)$; inequalities can be (and are often) strict; furthermore

$$C^*_{T,\mathrm{rand}}(\omega) = \inf_{j \geqslant 1} \frac{e^{2\lambda_j T} - 1}{2\lambda_j} \int_\omega |\phi_j(x)|^2 dx.$$

Similar ideas for wave, Schrödinger (different optimal shapes).

Heat ($N^*(T)$ ϕ_j suffice):

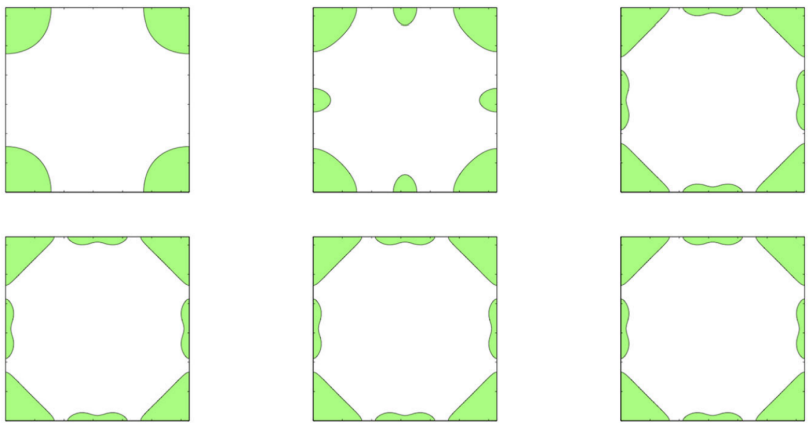

Fig. 2. On this figure, $\Omega = (0, \pi)^2$, $L = 0.2$, $T = 0.05$, and $A_0 = -\triangle$ is the Neumann–Laplacian defined on the domain $D(A_0) = \{y \in H^2(\Omega, \mathbb{C}) \mid \int_\Omega y = 0 \text{ and } \frac{\partial y}{\partial n} = 0 \text{ on } \partial\Omega\}$. Row 1, from left to right: optimal domain ω^N (in *green*) for $N = 1, 2, 3$. Row 2, from *left* to *right*: optimal domain ω^N (in *green*) for $N = 4, 5, 6$ (color figure online)

Wave (spillover phenomenon):

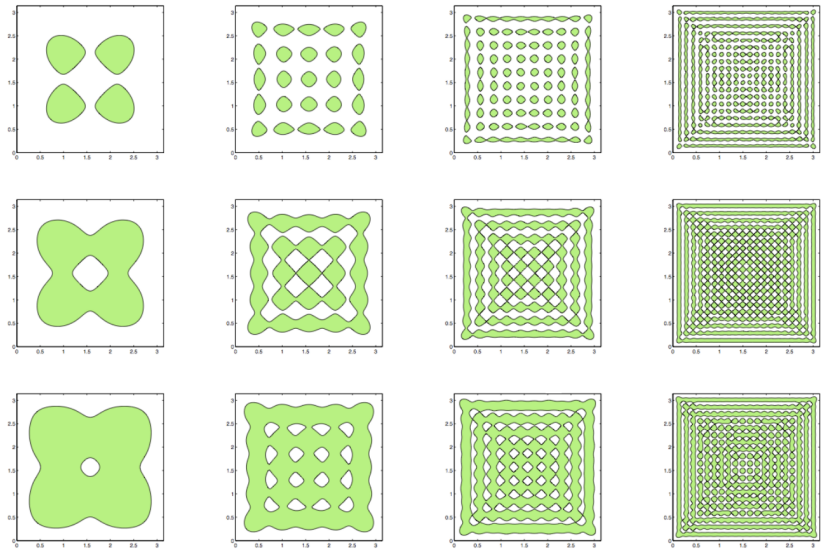

Fig. 3. $\Omega = (0, \pi)^2$, with Dirichlet boundary conditions. Row 1: $L = 0.2$; row 2: $L = 0.4$; row 3: $L = 0.6$. From left to right: $N = 2$ (4 eigenmodes), $N = 5$ (25 eigenmodes), $N = 10$ (100 eigenmodes), $N = 20$ (400 eigenmodes). The optimal domain is in green.

The picture for the randomized problem is relatively clear[8][9]. But we still know almost nothing about the deterministic problem.

[8] [Privat, Trélat, Zuazua; ARMA 2015]
[9] [Privat, Trélat, Zuazua; JEMS 2016]

The "simplest" case

Consider[10]

$$\begin{cases} \dot{y}(t) - Ay(t) = bu(t) \quad (0,T), \\ y_{|t=0} = y^0 \end{cases}$$

where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, so $u(t) \in \mathbb{R}$.

- Kalman rank condition: $\operatorname{rank}\begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix} = n$. Minimal L^2-norm control satisfies $\|u\|_{L^2(0,T)} \leqslant C_T(b)\|y^0\|$ for all y^0.
- Define $C_T^*(b)$ as the smallest such constant.

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Optimal actuator design

$$\min_{b\in\mathbb{S}^{n-1}} \; C_T^*(b) \tag{10}$$

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Linear algebra

Let $A \in \mathbb{R}^{n \times n}$.

1. The **companion matrix** $\mathfrak{A}$ to A is given by

$$\mathfrak{A} := \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 0 & 1 \\ -a_n & \dots & \dots & \dots & -a_1 \end{bmatrix}$$

   where $\{a_1, \dots, a_n\}$ are the coefficients of the characteristic polynomial of A.

2. We say that A is **similar** to $\mathfrak{A}$ if there exists an invertible $P \in \mathbb{R}^{n \times n}$ such that $A = P\mathfrak{A}P^{-1}$.

Brunovsky's normal form

Lemma (Brunovsky normal form)

Let $A \in \mathbb{R}^{n\times n}$, $n \geqslant 2$. For any $b \in \mathbb{R}^n$ such that (A,b) satisfies the Kalman rank condition, there exists an invertible matrix $P = P(b) \in \mathbb{R}^{n\times n}$ such that

$$A = P\mathfrak{A}P^{-1} \qquad \text{and} \qquad b = P\mathbf{e}_n. \tag{11}$$

Moreover, the matrix $P(b)$ ensuring (11) is unique, and its columns are given by

$$f_k = \begin{cases} b & k = n \\ \left(A^{n-k} + \sum_{j=1}^{n-k} a_j A^{n-k-j}\right) b & 1 \leqslant k \leqslant n-1 \end{cases} := p_k(A)b$$

Rewriting the cost

Lemma

We have

$$C_T(b) = \left\| P^{-1}(b) \right\| \kappa(T), \tag{12}$$

where $\kappa(T)$ is the controllability cost for $(\mathfrak{A}, \mathbf{e}_n)$.

- Proof: The system $\dot{z}(t) - \mathfrak{A}z(t) = \mathbf{e}_n u(t)$ is equivalent to $\dot{y}(t) - Ay(t) = bu(t)$ via $y = P(b)z \quad \square$.

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- Hence, (10) equivalently rewrites as

$$\min_{b \in \mathbb{S}^{n-1}} \left\| P^{-1}(b) \right\| = \boxed{\max_{b \in \mathbb{S}^{n-1}} \lambda_{\min} \left(P(b) P(b)^\top \right)} \tag{13}$$

Non-uniqueness

Proposition

Let $A \in \mathbb{R}^{n\times n}$ be similar to its companion matrix. Let $\mathbf{R} \in \mathbb{R}^{n\times n}$ be such that

1. $[A, \mathbf{R}] = 0$ ("A and $\mathbf{R}$ commute")
2. $\mathbf{R}\mathbf{R}^\top = \mathbf{R}^\top\mathbf{R} = \mathrm{I}_n$ ("$\mathbf{R}$ is orthogonal")

Then

$$\max_{b\in\mathbb{S}^{n-1}} \lambda_{\min}\left(P(b)P(b)^\top\right) = \max_{b\in\mathbb{S}^{n-1}} \lambda_{\min}\left(P(\mathbf{R}b)P(\mathbf{R}b)^\top\right).$$

Proof: Just note $\lambda_{\min}\left(P(b)P(b)^\top\right) = \lambda_{\min}\left(\sum_{k=1}^n p_k(A)bb^\top p_k(A)^\top\right)$, use Rayleigh quotient $\square$.

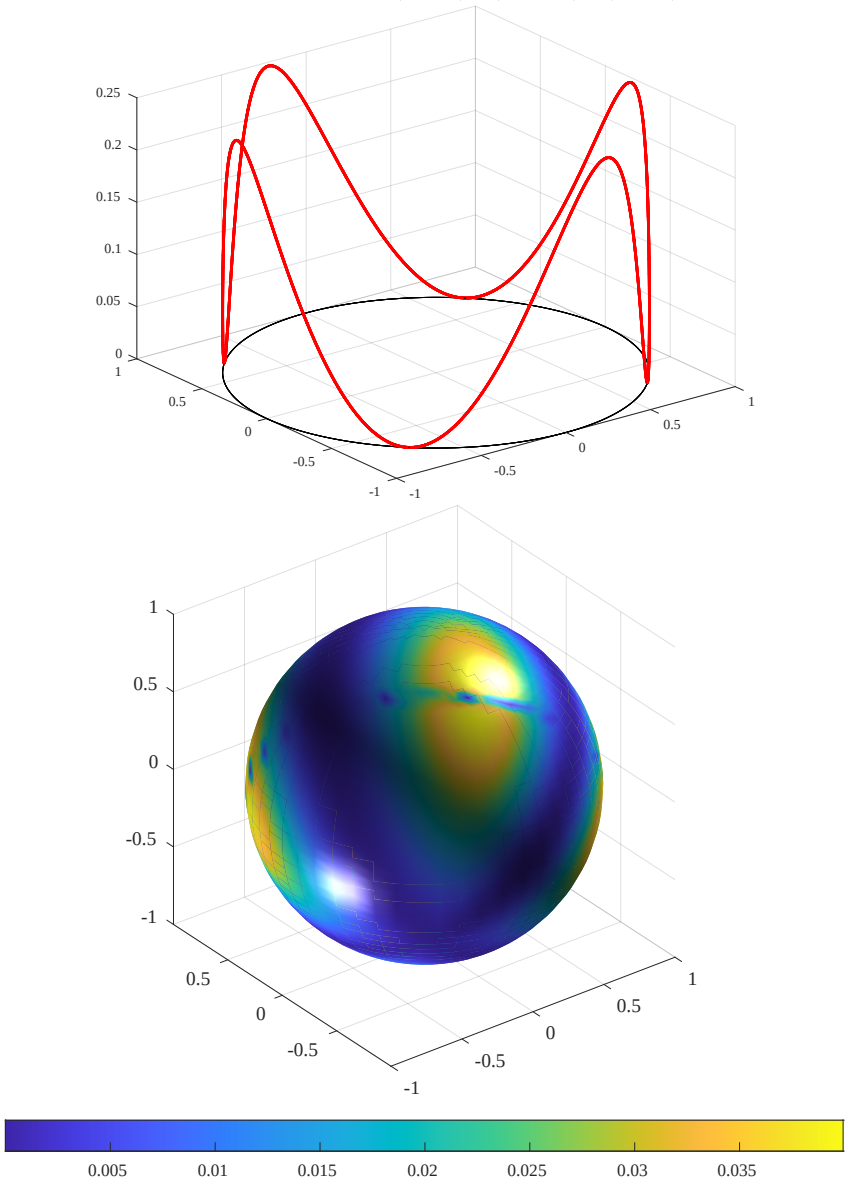
Computational hardships

- Also hard to find numerical solutions. Works up to $n = 9$ for typical Toeplitz matrices. Then $8h$ on a PC. Why?
- Problem is not concave, due to rank-1 structure:

$$\max_{\substack{B \in \mathbb{R}^{n \times n} \\ B \succeq 0 \\ \mathrm{trace}(B)=1 \\ \mathrm{rank}(B)=1}} \lambda_{\min}(L(B))$$

- NP hard combinatorial optimization problem. Blindly applying IPOPT doesn't work; genetic algorithms work slightly better.

$$b \mapsto \lambda_{\min}(P(b)P(b)^\top)$$



Discussion and outlook

1. Uniqueness modulo rotations? We need to compute gradients of $f : \mathbb{S}^{n-1} \to \mathbb{R}_+$

$$f(b) \simeq \min_{x \neq 0} \frac{\langle Mb, x \rangle^2}{\|x\|^2}$$

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- To compute $\nabla \tilde{f}$, we could appeal to Danskin's theorem.
- Given maximizer b^*, study curvature of f away from set $\{\mathbf{R} b^* : \ \mathbf{R} \text{ in } \ \mathit{Prop.}\}$

Discussion and outlook

2. What is a good relaxation to render problem tractable? If we remove the rank constraint, probably fine and concave. But who is to say that solution to relaxed problem will be of rank-1 (or even low rank)?

[11] [Waldspurger 2021, Cours Peccot]

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5. PDEs? Heat equation through flatness (en cours).

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