An inverse problem in Gravity Water Waves

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This talk is contained in works in collaboration with:

- M. A. Fontelos, J. López-Ríos, J. H. Ortega, & S. Zamorano
- Published in SICON (2017) and in Inverse Problems (2020)

Motivation

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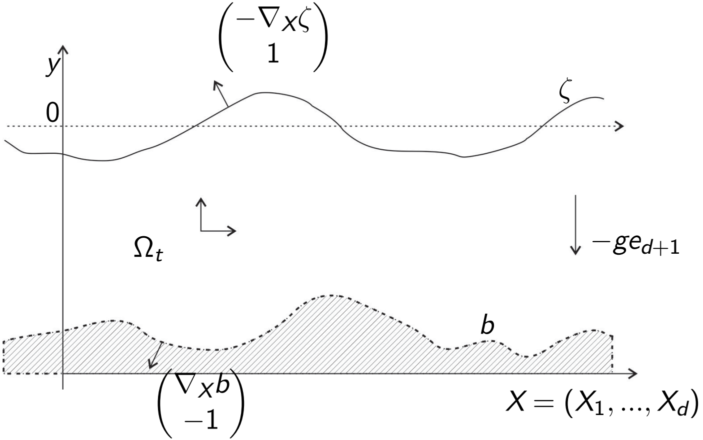

$\begin{pmatrix} -\nabla_X\zeta \\ 1 \end{pmatrix}$
y
0
ζ
Ω_t
$-ge_{d+1}$
b
$\begin{pmatrix} \nabla_X b \\ -1 \end{pmatrix}$
$X = (X_1, \ldots, X_d)$

Equations involved

The fluid is governed by the Euler equation, with restrictions on the divergence and the rotational of V:

- $V_t + (V \cdot \nabla)V = -\frac{1}{\rho}\nabla P - g e_{d+1}$
- $div\, V = 0$
- $\nabla \times V = 0$
- $V = \nabla \phi$ then $\Delta \phi = 0$
- $\phi_t + \frac{1}{2}|\nabla \phi|^2 = -\frac{1}{\rho}(P - P_{atm}) - gy + C$

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Boundary conditions

- Free boundary

$$\frac{d}{dt}(y - \zeta(t, X)) = 0$$

$$\phi_y - \zeta_t - \nabla_X \zeta \cdot \nabla_X \phi = 0$$

- bottom : wall condition

$$\left.\frac{\partial \phi}{\partial n}\right|_b = 0$$

General formulation of the Gravity Water Wave equation, with $(X, y) \in \mathbb{R}^d \times \mathbb{R}$ and $d = 1, 2$,

$$\begin{cases} \Delta_{X,y}\phi = 0, & b \leq y \leq \zeta, \\ \zeta_t = -\nabla\zeta \cdot \nabla\phi + \phi_y, & y = \zeta, \\ \phi_t + \frac{1}{2}|\nabla_{X,y}\phi|^2 + g\zeta = 0, & y = \zeta, \\ \frac{\partial\phi}{\partial n} = 0, & y = b. \end{cases} \tag{1}$$

Zakharov ('67); Craig, Sulem ('92)

- $\psi(t, X) := \phi(t, X, \zeta(t, X))$
- If in time t, ψ is known, it is possible to solve, in Ω_t

$$\begin{cases} \Delta_{X,y}\phi = 0, & \Omega_t \\ \phi|_{y=\zeta} = \psi, & \partial_n \phi|_{y=b} = 0 \end{cases}$$

$$G(\zeta, b)\psi = \sqrt{1 + |\nabla\zeta|^2}\partial_n \phi\big|_{y=\zeta}$$

System (1) becomes

$$\begin{cases} \zeta_t - G(\zeta, b)\psi = 0, & \mathbb{R}^d \times (0, T) \\ \psi_t + g\zeta + \frac{1}{2}|\nabla\psi|^2 - \frac{1}{2(1+|\nabla\zeta|^2)}(G(\zeta, b)\psi + \nabla\zeta \cdot \nabla\psi)^2 = 0, & \mathbb{R}^d \times (0, T). \end{cases} \tag{2}$$

Zakharov ('67); Craig, Sulem ('92)

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Some physical phenomena involving the Gravity Water Waves

- Wave generation through bottom disturbances (Zuazua '14; Peregrine '67; Wu '84)
- Tsunamis (generated by displacement of the plates), (Iguchi '11; Wu '84)
- Bottom detection through surface measurementse (Nicholls-Taber '08; Vasan-Deconinck '13; FLLO '15)
- Hydraulic jump (Fontelos-Friedman '04; FLLO '15)

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Local existence(Lannes, '13)

Theorem

Let $d_0 > \frac{d}{2}$ and $N \geq d_0 + max\{d_0, 2\} + 3/2$. Consider $U_0 = (\zeta_0, \psi_0) \in H^{d_0+2} \times H^2$, $\mathcal{E}^N \in L^\infty$, $b \in H^{N+max\{d_0,1\}+1}(\mathbb{R}^d)$ and further, suppose that

$$\exists h_{min} > 0,\ \exists a_0 > 0, \quad \zeta_0(X) - b(X) \geq h_{min} \text{ and } \mathfrak{a}(U_0) \geq a_0.$$

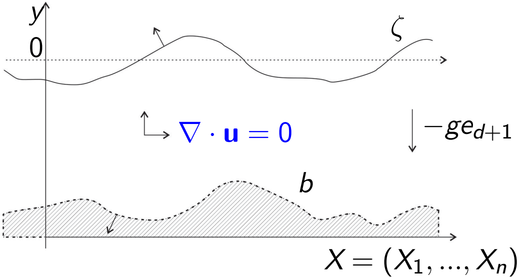
Then exists $T > 0$ and a unique solution $U \in C([0, T]; H^{d_0+2} \times H^2)$, $\mathcal{E}^N \in L^\infty$, of (2) with initial condition U_0.

$$\mathfrak{a} = g + w_t + V \cdot \nabla w, \quad \mathcal{E}^N = |B\psi|^2_{H^{d_0+3/2}} + \sum_{\alpha \in \mathbb{N}^d, |\alpha| \leq N} |\zeta_{(\alpha)}|_2^2 + |B\psi_{(\alpha)}|_2^2,$$

$$\zeta_{(\alpha)} = \partial^\alpha \zeta, \quad \psi_{(\alpha)} = \partial^\alpha \psi - \underline{\omega} \partial^\alpha \zeta, \quad B = \frac{|D|}{(1 + |D|)^{1/2}}, \quad \underline{\omega} = \frac{G(\zeta)\psi + \nabla_X \zeta \cdot \nabla_X \psi}{1 + |\nabla_X \zeta|^2}.$$

Bathymetry detection (Water)

Let us consider an incompressible, invised fluid, bounded below by an impermeable bottom (it can vary in time), and above by a free surface (waves).
Knowing the bottom, given an initial velocity field, $\mathbf{u}_0$, and an initial free surface, ζ_0, we want to determine $\mathbf{u}, \zeta$.



We consider the inverse geometric problem, of identifying the bottom, through (which) measurements on the surface (Lecaros, L-R, Ortega, SIAM '17)

Bottom detection through free surface measurements

- Teniou, Ait-Yahia, Hernane '08: stationary problem, profile measurement.
- Deconinck, Vasan '13: numerical recovery of bathymetry from $\zeta(x, t_0)$, $\zeta_t(x, t_0)$, $\zeta_{tt}(x, t_0)$, They assumed ϕ periodic and small disturbances of the flat band.
- Fontelos, Lecaros, López, Ortega '17: identifiability, from $\zeta(x, t_0)$, $\psi(x, t_0)$, $\zeta_t(x, t_0)$.

Theorem (M. Fontelos, **L.R.**, J. López-Ríos & J.H. Ortega, SICON '17)

Let $T > 0$ and $d_0 > \frac{d}{2}$. Suppose that $j = 1, 2$, $(\zeta_j, \psi_j) \in C^1([0, T]; H^{d_0+1}(\mathbb{R}^d) \times H^2(\mathbb{R}^d))$ are solutions of (2) with $b_1, b_2 \in H^{N+max\{d_0,1\}+1}(\mathbb{R}^d)$, such that exists $h_{min} > 0$, such that, for all $X \in \mathbb{R}^d$ and $t \in (0, T)$,

$$\zeta_j(t, X) - b_j(X) \geq h_{min}, \quad j = 1, 2.$$

Let S an open sub set of $\mathbb{R}^d$ and $t_0 \in (0, T)$ fixed. If $\forall X \in S$,

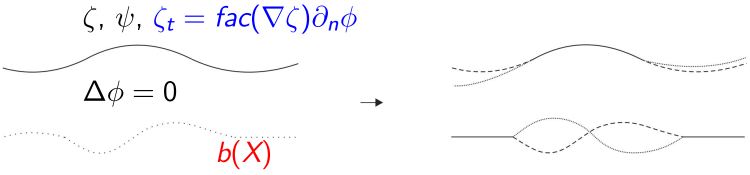
$$\zeta_1(t_0, X) = \zeta_2(t_0, X), \quad \psi_1(t_0, X) = \psi_2(t_0, X) \quad \text{and} \quad \partial_t\zeta_1(t_0, X) = \partial_t\zeta_2(t_0, X),$$

then

$$b_1(X) = b_2(X) \quad \forall X \in \mathbb{R}^d.$$

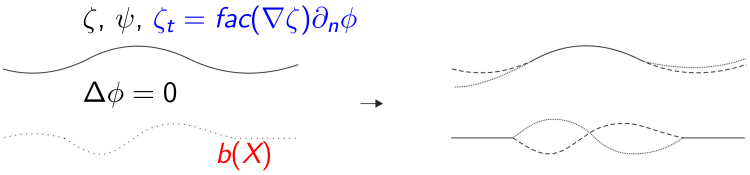
Objective: measure ϕ, $\partial_n\phi$ at the surface and estimate the bottom

We consider the inverse geometric problem, of identifying the bottom, through (which) measurements on the surface, where $fac(X) = \sqrt{1 + |X|^2}$



- Measurements are taken on an open $S \subset \mathbb{R}^d$
- We measure in a single time $t = t_0$
- Bathymetry is identified in all space: $b = b(X) \in \mathbb{R}^d$
- We ask for too many measurements: $\phi, \partial_n\phi, \zeta$.

Sketch of the proof



Using the Unique Continuation Property for the Laplacian, we obtain that if ζ_1, ψ_1 are equal to ζ_2, ψ_2 in a open subset of $(0, T) \times \mathbb{R}^d$, then $\zeta_1 = \zeta_2$ and $\psi_1 = \psi_2$ $\forall (t, x) \in (0, T) \times \mathbb{R}^d$. This U.C.P. are presented in other non-local models, for example: Benjamin-Ono equation

$$u_t - \mathcal{H}\partial_x^2 u + u\partial_x u = 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}.$$

See C. E. Kenig, G. Ponce, and L. Vega. Uniqueness properties of solutions to the benjamin-ono equation and related models. Journal of Functional Analysis, 2020.

Optimal control problem

Given a function $\tau(X) \in L^2(\mathbb{R}^d)$, we consider $F : H^{N+max\{d_0,1\}+2}(\mathbb{R}^d) \to \mathbb{R}$

$$F(b) = \frac{1}{2} \int_{\mathbb{R}^d} \left| \partial_t \zeta(X)|_{t=t_0} - \tau(X) \right|^2 dX$$

Problem

Find $b^{min} \in \mathcal{B}_{ad}$, such that,

$$F(b^{min}) = \min_{b \in \mathcal{B}_{ad}} F(b).$$

- $\mathcal{B}_{ad} = \{ b \in H^{N+max\{d_0,1\}+2}(\mathbb{R}^d) : \mathrm{supp}(b) \subset K, |b|_{H^{N+max\{d_0,1\}+2}(\mathbb{R}^d)} \leq C \}$

Theorem: Existence of minimizers (M. Fontelos, **L.R.**, J. López-Ríos & J.H. Ortega, SICON '17)

Let $\zeta \in H^{d_0+2}(\mathbb{R}^d)$ and $\psi \in H^2(\mathbb{R}^d)$. we assume $\tau(X) \in L^2(\mathbb{R}^d)$. Then the minimization problem,

$$\min_{b \in \mathcal{B}_{ad}} F(b),$$

has a minimizer $b^m \in \mathcal{B}_{ad}$.

- Let $\{b_n\} \subset \mathcal{B}_{ad}$ a minimizing sequence of F
- $\exists$ sub-succession, such that $b_n \rightharpoonup \bar{b}$ in $H^{N+max\{d_0,1\}+2}(\mathbb{R}^d)$, with $\bar{b} \in \mathcal{B}_{ad}$.
- $b_n \to \bar{b}$ strongly in $H^{N+max\{d_0,1\}+1}(\mathbb{R}^d)$
- Let ϕ_n and $\bar{\phi}$ the corresponding solutions with bathymetries b_n and $\bar{b}$
- $G_n(\zeta, b)\psi = G(\zeta, b_n)\psi \to G(\zeta, \bar{b})\psi, \quad \text{strongly in } L^2(\mathbb{R}^d)$
- $\inf_{b \in \mathcal{B}_{ad}} F(b) = \lim_{n\to\infty} F(b_n) = F(\bar{b})$
- The uniqueness of this minimum follows from the identifiability.

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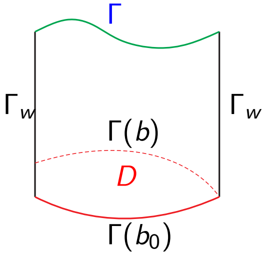
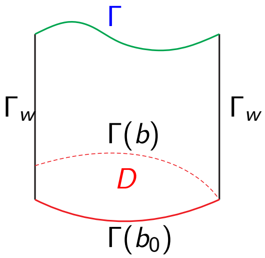
- The uniqueness of this minimum follows from the identifiability.

The identifiability result can be improved, with the following "stability", where the size of the difference of the backgrounds is estimated, through Dirichlet and / or Neumann measurements on the surface (Lecaros, LR, Ortega, Zamorano, IP '20)





Theorem: Stability with partial measurements (L.R., J. López-Ríos, J.H. Ortega & S. Zamorano, IP '20)

$$\frac{\left(\int_\Gamma(\partial_n\phi-\partial_n\phi_0)\psi-\int_\Gamma\partial_n\phi(\psi-\psi_0)\right)^2}{\int_\Gamma\psi_0\partial_n\phi_0\int_\Gamma\psi\partial_n\phi}\leq|D|\leq\frac{\int_\Gamma\partial_n\phi(\psi-\psi_0)+\int_\Gamma(\partial_n\phi_0-\partial_n\phi)\psi_0}{\int_\Gamma\psi_0\partial_n\phi_0}$$

Sketch of the proof

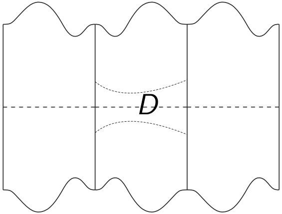
$$\begin{cases} \Delta\phi_0 = 0, & \Omega, \\ \phi_0 = \psi_0, & \Gamma, \\ \partial_n\phi_0 = 0, & \Gamma(b_0) \cup \Gamma_w(b_0, \zeta_0), \end{cases} \qquad \begin{cases} \Delta\phi = 0, & \Omega \setminus D, \\ \phi = \psi, & \Gamma, \\ \partial_n\phi = 0, & \Gamma(b) \cup \Gamma_w(b, \zeta_0). \end{cases}$$

For the lower bound

$$\int_{\Gamma(b)} \partial_n\phi_0\phi = \int_\Gamma (\partial_n\phi - \partial_n\phi_0)\psi_0 + \int_\Gamma \partial_n\phi_0(\psi_0 - \psi).$$

For the upper bound

$$\int_{\Omega\setminus D} |\nabla(\phi - \phi_0)|^2 + \int_D |\nabla\phi_0|^2 = \int_\Gamma \partial_n\phi(\psi - \psi_0) + \int_\Gamma (\partial_n\phi_0 - \partial_n\phi)\psi_0.$$



Theorem: Stability with partial measurements (**L.R.**, J. López-Ríos, J.H. Ortega & S. Zamorano, IP '20)

Let $\phi_0 \in H^2(\Omega)$, $\phi \in H^2(\Omega \setminus D)$. Given $k \in (0,1)$, exists $C_5, \delta_0 > 0$ such that, for all $\delta \in (0, \delta_0)$ and

$$\|\phi_0, \phi\|_{H^2} \leq M, \quad \|\phi_0, \phi\|_{H^1(\Gamma_{up})} + \|\partial_n \phi_0, \phi\|_{L^2(\Gamma_{up})} \leq \delta,$$

with $M > 0$. We have

$$|D| \leq C^* \left(\|\nabla \phi\|_{H^1(\Omega \setminus D)} + \|\phi_0\|_{H^1(\Omega)} \right) \log \left(\frac{M}{\|\psi - \psi_0\|_{H^1(\Gamma^*)} + \|\partial_n(\phi - \phi_0)\|_{L^2(\Gamma^*)}} \right)^{-k}.$$

The constant C^* depends on $|\Omega|$, r_0, M_0, M, $\frac{\|\partial_n \phi_0\|_{L^2(\Gamma_{up})}}{\|\partial_n \phi_0\|_{H^{-1/2}(\Gamma_{up})}}$.

Some open problems

- The estimation in an infinite band, of the bottom of the channel
- The estimates of the cavities, for general elliptical operators, $\nabla \cdot (A \nabla u) = 0$, and with any type of embedding.
- Estimation of cavities in Stokes-type fluids, $\nabla(\sigma(\mathbf{u}, p)) = 0$, $\sigma(\mathbf{u}, p) = 2\mu e(\mu) - pI$.
- Estimation of the size of a moving object, if the angular velocity of its center of mass is known

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R. Lecaros, J. López-Ríos, J. Ortega, S. Zamorano. *T*he stability for an inverse problem of bottom recovering in water–waves. Inverse Problems, Vol.–, 2020.

Thank you very much for your attention