Controllability for the viscous van Wijngaarden–Eringen equation

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Plan of the Talk

1. Motivation
2. Well-posedness and further properties
3. Controllability results
4. Work in progress

Motivation

- In the 1940's and 50's, the interest in studying the propagation of pressure waves of small amplitude in bubbly liquids appeared.
- The reason was to determine whether it was possible to take advantage of these acoustical properties to control the sound produced by propellers, both of surface ships and submerged ship.
- A vast literature on the subject deals with theoretical and experimental studies of the various aspects of propagation of pressure waves of small amplitude in bubbly liquids[1]

[1] L. V. Wijngaar. One-dimensional flow of liquids containing small gas bubbles. *Annual review of fluid Mechanics*, 4:369–396, 1972.

- The acoustic planar propagation perpendicular to and along the X-axis (i.e., 1D flow) in bubbly liquids is given by the following equation[2]

$$u_{tt}-(1-2\varepsilon(\beta-1))u_{xx}+(\varepsilon(u_{xx})^2)_{tt}-\frac{1}{Re}u_{xxt}-a_0^2u_{xxtt}=0,\ t\geq 0,\ x\in\mathbb{R},$$

- $Re=\dfrac{c_eL}{\delta}$: Reynolds number
- $c_e>0$: adiabatic sound speed
- δ: diffusivity of the sound
- L: macroscopic length
- a_0: Knudsen number (corresponds to the dimensionless bubble radius)
- $\beta=\dfrac{\gamma+1}{2}>1$: coefficient of nonlinearity
- γ: adiabatic index of the liquid

[2] P. M. Jordan, R. S. Keiffer, and G. Saccomandi. Anomalous propagation of acoustic traveling waves in thermoviscous fluids under the Rubin-Rosenau-Gottlieb theory of dispersive media. *Wave Motion*, 51(2):382–388, 2014.

- In this talk we focus on the linearized version, that is, $\varepsilon = 0$.

- The linearized equation is known as the viscous (or dissipative) van Wijngaarden–Eringen equation

$$(1) \qquad u_{tt} - u_{xx} - \frac{1}{Re} u_{xxt} - a_0^2 u_{xxtt} = 0, \quad t \geq 0, \quad x \in \mathbb{R},$$

- Chaotic dynamics:

$$\text{whenever } a_0 < 1 \quad \text{and} \quad \frac{\sqrt{5}}{6} < a_0 Re < \frac{1}{2}$$

the equation (1) admits a uniformly continuous semigroup which is Devaney chaotic on an isomorphic copy of the sequence space $c_0^2(\mathbb{N}_0)$[3].

[3]Conejero, J Alberto and Lizama, Carlos and Murillo-Arcila, Marina. On the existence of chaos for the viscous Van Wijngaarden–Eringen equation, *Chaos, Solitons & Fractals*, 89:100–104, 2016

- Our objective: Study the controllability properties from the boundary of the van Wijngaarden–Eringen equation

$$(2) \qquad \begin{cases} y_{tt} - y_{xx} - (Re)^{-1} y_{xxt} - a_0^2 y_{xxtt} = 0, & (0,1)\times(0,T), \\ y(0,t) = 0, \quad y(1,t) = h(t), & t \in (0,T), \\ y(x,0) = y^0(x), \quad y_t(x,0) = y^1(x), & x \in (0,1). \end{cases}$$

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Homogeneous equation

- First order Cauchy problem on $H_0^1(0,1) \times H_0^1(0,1)$:

$$\begin{cases} Y_t = AY(t), & t \in (0,T), \\ Y(0) = Y_0, \end{cases} \tag{3}$$

where $Y = (y, y_t)^\mathsf{T}$ and $Y_0 = (y^0, y^1)^\mathsf{T}$.

- The operator A is defined by

$$A := \begin{pmatrix} 0 & I \\ (I - a_0^2 \partial_{xx})^{-1}\partial_{xx} & (Re)^{-1}(I - a_0^2\partial_{xx})^{-1}\partial_{xx} \end{pmatrix},$$

with $D(A) = H^2(0,1) \cap H_0^1(0,1) \times H^2(0,1) \cap H_0^1(0,1)$.

Proposition

The operator A generates a strongly continuous semigroup of contractions in the energy space $H_0^1(0,1) \times H_0^1(0,1)$.

Theorem

For every $(y^0, y^1) \in H_0^1(0,1) \times H_0^1(0,1)$, there exists a unique $(y, y_t) \in C([0,T]; H_0^1(0,1) \times H_0^1(0,1))$ solution of (3).

Spectral analysis

For $s \geq 0$, we consider the space

$$\mathcal{H}^s(\Omega) = \left\{ y : \Omega \to \mathbb{R} \middle| \sum_{k \geq 1} k^{2s} |a_k|^2 < \infty \right\}.$$

Endowed with the scalar product

$$\langle y, z \rangle_s = \sum_{k \geq 1} k^{2s} \widehat{y}_k \widehat{z}_k,$$

$\mathcal{H}^s(\Omega)$ is a Hilbert space. In addition,

- For $s \leq \frac{1}{2}$, $\mathcal{H}^s(\Omega) = H^s(\Omega)$.
- For $\frac{1}{2} < s \leq \frac{3}{2}$, $\mathcal{H}^s(\Omega) = H_0^s(\Omega)$.
- For $\frac{3}{2} < s \leq 2$, $\mathcal{H}^s(\Omega) = H^s(\Omega) \cap H_0^1(\Omega)$.

In particular, $\mathcal{H}^1(\Omega) = H_0^1(\Omega)$, $\mathcal{H}^2(\Omega) = H^2(\Omega) \cap H_0^1(\Omega)$.
We denote by $\mathcal{H}^{-s} = (\mathcal{H}^s(\Omega))'$ the dual space of $\mathcal{H}^s(\Omega)$ with respect to the pivote space $\mathcal{H}^0(\Omega) = L^2(\Omega)$.

Proposition

Let $a_0 < 1$ and $a_0 R_e \notin \left(\frac{\sqrt{5}}{6}, \frac{1}{2}\right)$. For every $(y^0, y^1) \in \mathcal{H}^s(0,1) \times \mathcal{H}^s(0,1)$, $s \geq 0$, the solution y of (3) belongs to $C^1([0,T]; \mathcal{H}^s(0,1))$ and

$$y(x,t) = \sum_{n\geq 0} \sqrt{2}\sin(n\pi x)\, e^{\alpha_n t}\left(c_n \cos(\beta_n t) + \frac{d_n - \alpha_n c_n}{\beta_n}\sin(\beta_n t)\right),$$

where $y^0 = \sum_{n\geq 0} c_n \sqrt{2}\sin(n\pi x)$, $y^1 = \sum_{n\geq 0} d_n \sqrt{2}\sin(n\pi x)$, and the coefficients α_n and β_n are given by

$$\alpha_n = -\frac{(\mathrm{Re})^{-1} n^2\pi^2}{2(1 + a_0^2 n^2 \pi^2)}, \quad \beta_n = \frac{\sqrt{4n^2\pi^2 - n^4\pi^4\left(\mathrm{Re}^{-2} - 4a_0^2\right)}}{2(1 + a_0^2 n^2\pi^2)}.$$

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Remark

Let us observe that

$$\lim_{n\to+\infty}(\alpha_n-\beta_n)=\frac{-(Re)^{-1}-\sqrt{4a_0^4-(Re)^{-2}}}{2a_0^2}$$

Therefore, the control properties of (3) are expected to deeply differ from the ones for the wave equation. The purpose of this talk is to discuss the impact on controllability of such unusual spectrum.

Similar results:

- E. Cerpa and E. Crépeau. Improved Boussinesq equation, 2018.
- L. Rosier and P. Rouchon. Wave equation with structural damping, 2007.
- S. Micu. Linearized Benjamin–Bona–Mahony equation, 2001.

Proposition

Let $a_0 \mathrm{Re} \notin \left(\frac{\sqrt{5}}{6}, \frac{1}{2} \right)$, and $\left(y^0, y^1\right) \in \mathcal{H}^s(\Omega) \times \mathcal{H}^s(\Omega)$. If $s > \frac{3}{2}$, then $y_x(1, \cdot) \in C\left(\mathbb{R}^+\right)$.

Finally, for the controlled equation, we have the following

Proposition

Let $a_0 \mathrm{Re} \notin \left(\frac{\sqrt{5}}{6}, \frac{1}{2} \right)$, $\left(y^0, y^1\right) \in L^2(\Omega) \times L^2(\Omega)$ and $h \in H^2(0, T)$. Then, there exists a unique $y \in C^1\left([0, T]; L^2(\Omega)\right)$ solution of (2).

Adjoint problem

To study the controllability properties of the van Wijngaarden–Eringen equation, we introduce the adjoint system associated to the equation, that is

$$
\begin{cases}
z_{tt} - z_{xx} + (\mathrm{Re})^{-1} z_{xxt} - a_0^2 z_{xxtt} = 0, & (0,1) \times (0,T), \\
z(0,t) = z(1,t) = 0, & t \in (0,T), \\
z(x,T) = z_T^0(x), z_t(x,T) = z_T^1(x), & x \in (0,1),
\end{cases} \tag{4}
$$

Proposition

Assume that $a_0 \mathrm{Re} \notin \left(\frac{\sqrt{5}}{6}, \frac{1}{2}\right)$. For every $\left(z_T^0, z_T^1\right) \in \mathcal{H}^2(\Omega) \times \mathcal{H}^2(\Omega)$, there exists a unique $z = z(x,t)$ solution of (4) which belongs to $C^1\left([0,T]; \mathcal{H}^2(\Omega)\right)$ and

(5)

$$z(x,t) = \sum_{n\geq 0} \sqrt{2}\sin(n\pi x) \cdot e^{\alpha_n (T-t)} \left(e^{i\beta_n (T-t)} \left(\frac{\widehat{a}_n}{2} - \frac{i\widehat{b}_n}{2} \right) + e^{-i\beta_n (T-t)} \left(\frac{\widehat{a}_n}{2} + \frac{i\widehat{b}_n}{2} \right) \right),$$

where $z_T^0 = \sum_{n\geq 0} \widehat{c}_n \sqrt{2}\sin(n\pi x)$, $z_T^1 = -\sum_{n\geq 0} \widehat{d}_n \sqrt{2}\sin(n\pi x)$, and the coefficients α_n y β_n are given by

$$\alpha_n = -\frac{(\mathrm{Re})^{-1} n^2\pi^2}{2\left(1 + a_0^2 n^2 \pi^2\right)}, \quad \beta_n = \frac{\sqrt{4n^2\pi^2 - n^4\pi^4\left(\mathrm{Re}^{-2} - 4a_0^2\right)}}{2\left(1 + a_0^2 n^2 \pi^2\right)},$$

where $\widehat{c}_n = \left(\frac{\widehat{a}_n}{2} - \frac{i\widehat{b}_n}{2}\right)$, $\widehat{d}_n = \left(\frac{\widehat{a}_n}{2} + \frac{i\widehat{b}_n}{2}\right)$.

Controllability results

- Let us remember our equation

$$(6)\qquad \begin{cases} y_{tt} - y_{xx} - (Re)^{-1} y_{xxt} - a_0^2 y_{xxtt} = 0, & (0,1)\times(0,T), \\ y(0,t) = 0, \quad y(1,t) = h(t), & t \in (0,T), \\ y(x,0) = y^0(x), \quad y_t(x,0) = y^1(x), & x \in (0,1). \end{cases}$$

- The set of reachable states

$$R\left(T; \left(y^0, y^1\right)\right) := \{(y(T), y_t(T)) : y \text{ solution of } (6)\}$$

Lemma

The equation (6) is null controllable if and only if for every $\left(y^0, y^1\right) \in L^2(\Omega) \times L^2(\Omega)$, there exists $h = h(t) \in H^2(0, T)$ such that the solution $z = z(x, t)$ of (5) satisfies

$$\begin{aligned}
&\int_\Omega y^1(x)\left(z(x,0) - a_0^2 \partial_{xx} z(x,0)\right) dx \\
&- \int_\Omega y^0(x)\left(z_t(x,0) + (\mathrm{Re})^{-1} \partial_{xx} z(x,0) - a_0^2 \partial_{xx} z_t(x,0)\right) dx \\
&= \int_0^T \left(h(t) + (\mathrm{Re})^{-1} \dot{h}(t) + a_0^2 \ddot{h}(t)\right) z_x(1,t) dt,
\end{aligned}$$

for every $\left(z_T^0, z_T^1\right) \in \mathcal{H}^2(\Omega) \times \mathcal{H}^2(\Omega)$.

Lack of null and exact controllability

Theorema

Assume that $a_0\mathrm{Re} \notin \left(\frac{\sqrt{5}}{6}, \frac{1}{2}\right)$. Then, the equation (6) is neither exact nor null controllable in $L^2(\Omega) \times L^2(\Omega)$.

Main ingredients of proof:

- Spectrally controllable: if any finite linear combination of eigenvectors can be steered to zero by a control input $h = h(t)$.
- Paley–Wiener Theorem: Let $F \in L^2(\mathbb{R}^+)$ and ξ belonging to the upper half plane. Then, the Fourier transform of F, defined by

$$f(\xi) = \int_0^\infty F(x)\, e^{ix\xi} dx,$$

is a holomorphic function.

- Prove that the system is not spectrally controllable
- Conclusion: That is, there exist initial conditions (y^0, y^1) (nontrivial) such that for any boundary control function h, the associated solution (y, y_t) of the system (6) is not identically zero at time T.
- Namely, the system (6) is not null controllable. On the other hand, since the exact controllability implies the null controllability and the latter one fails, we finally obtain that the system (6) is not exact controllable. The proof is finished.

Approximate controllability

Theorem

Assume that $a_0\mathrm{Re} \notin \left(\frac{\sqrt{5}}{6}, \frac{1}{2}\right)$. The equation (6) is approximately controllable in $L^2(\Omega) \times L^2(\Omega)$ for any time $T > 0$.

Work in Progress

- Numerical simulation for the approximate controllability result: We are working in the numerical simulation using the approach given by R. Glowinski and J.L. Lions[4]

- We define the operator L as follows

$$\begin{aligned} L : L^2(0, T) &\to L^2(\Omega) \times L^2(\Omega) \\ v &\mapsto \{-y_t(T), y(T)\}, \end{aligned}$$

where $v(t) = h(t) + (\mathrm{Re})^{-1}\dot{h}(t) + a_0^2\ddot{h}(t)$.

- Let φ be the solution of (4), and let $f = \{f^0, f^1\} \in H^2(\Omega) \cap H_0^1(\Omega) \times H^2(\Omega) \cap H_0^1(\Omega)$, with f^0 and f^1 the initial datum of (4).

[4]R. Glowinski and J.L. Lions. Exact and approximate controllability for distributed parameter systems, *Acta numerica*, Volume 4, 159–328, 1995

- From the controllability Lemma, we have

$$\begin{aligned}\langle Lv, f\rangle &= \int_\Omega y_t(T)\left(f^0 - a_0^2\partial_{xx}f^0\right)dx \\ &\quad - \int_\Omega y(T)\left(f^1 + (\mathrm{Re})^{-1}\partial_{xx}f^0 - a_0^2\partial_{xx}f^1\right)dx \\ &= 0,\end{aligned}$$

for every $v \in L^2(0,T)$.

- Let y be the solution of

$$(7) \qquad \begin{cases} y_{tt} - y_{xx} - (\mathrm{Re})^{-1} y_{xxt} - a_0^2 y_{xxtt} = 0, & (0,1)\times(0,T), \\ y(0,t) = 0, y(1,t) = v, & t \in (0,T), \\ y(x,0) = y_t(x,0) = 0, & x \in (0,1). \end{cases}$$

- Then, we have

$$\langle Lv, f\rangle = \int_0^T \varphi_x(1,t)\cdot v(t)\,dt.$$

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- Let L^* be the operator such that $\langle Lv, f \rangle = \langle v, L^* f \rangle$. Then L^* is given by

$$\begin{aligned} L^* : L^2(\Omega) \times L^2(\Omega) &\to L^2(0,T) \\ \{\varphi_t(T), \varphi(T)\} &\mapsto \varphi_x(1,t). \end{aligned}$$

- If $\Lambda = LL^*$, we obtain

$$\Lambda f = \{-y_t(T), y(T)\}.$$

- Besides, the operator Λ is definite positive (thanks to the approximate controllability result) and self-adjoint.

- Finally, consider the functional J_ε,

$$J_\varepsilon(h) = \frac{1}{2}\int_0^T |h(t) + (\mathrm{Re})^{-1}\dot{h}(t) + a_0^2\ddot{h}(t)|^2 dt + \frac{\varepsilon}{2}\|(y(T) - z_0, y_t(T) - z_1)\|_{L^2(\Omega)\times L^2(\Omega)},$$

where $(z_0, z_1) \in L^2(\Omega) \times L^2(\Omega)$ is the "target".

- Since the equation (6) is approximately controllable, the functional J_ε is coercive, continuous and strictly convex.

- Then, there exists a unique minimizer $\widehat{h}$ of J_ε, and we have the following:

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Theorem

Let $\widehat{h}$ be the minimizer of $J_\varepsilon(h)$. Then, there exists $\varphi \in C^1\left([0,T];\left(H^2(\Omega)\cap H_0^1(\Omega)\right)^2\right)$, which solves

$$\begin{cases} \varphi_{tt} - \varphi_{xx} + (\mathrm{Re})^{-1}\varphi_{xxt} - a_0^2\varphi_{xxtt} = 0, & (0,1)\times(0,T), \\ \varphi(0,t) = \varphi(1,t) = 0, & t\in(0,T), \\ \varphi(x,T) = f^0(x), \varphi_t(x,T) = f^1(x), & x\in(0,1), \end{cases}$$

where f^0 and f^1 fulfilling

$$\varepsilon\left(y(T) - z_0\right) = -\left(f^1 + (\mathrm{Re})^{-1}\partial_{xx}f^0 - a_0^2\partial_{xx}f^1\right),$$

$$\varepsilon\left(y_t(T) - z_1\right) = f^0 - a_0^2\partial_{xx}f^0,$$

and $\varphi_x(1,t) = h(t) + (\mathrm{Re})^{-1}\dot{h}(t) + a_0^2\ddot{h}(t)$.

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- We define the operator $U : H^2(0,1) \cap H_0^1(0,1) \times H^2(0,1) \cap H_0^1(0,1) \to L^2(0,1) \times L^2(0,1)$ given by

$$U\begin{pmatrix} f^0 \\ f^1 \end{pmatrix} = \frac{1}{\varepsilon}\begin{pmatrix} I - a_0^1\partial_{xx} & 0 \\ 0 & I - a_0^2\partial_{xx} \end{pmatrix}\begin{pmatrix} f^0 \\ f^1 \end{pmatrix} + \frac{1}{\varepsilon}\begin{pmatrix} 0 \\ (Re)^{-1}\partial_{xx} f^0 \end{pmatrix} + \Lambda\begin{pmatrix} f^0 \\ f^1 \end{pmatrix}.$$

- It can be proved that U is a coercive operator.

- Consequence: The conjugate gradient method converges!!!

- The lack of null and exact controllability is also fulfilled in the case of interior control. The previous results of non exact controllability lead us to look another type of control, for instance, moving control.

- Is it possible to obtain the exact controllability of (6) with a moving control?

- Similar questions arise in the multidimensional case.

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