Geodesics and Laplacian on 3D contact sub-Riemannian manifolds

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Contact structure and REEB vector field in 3D.

Let M be a 3D manifold, a contact structure is a rank 2 sub-bundle $D \subset TM$ which is defined by $D = \ker \alpha$ with α a 1-form and $\alpha \wedge d\alpha$ a volume form. We orient M by $\alpha \wedge d\alpha$ independently of the choice of α.

If α is given, the *Reeb vector field* $\vec{R}$ is defined by

$$\alpha(\vec{R}) = 1,\ d\alpha(\vec{R}, .) = 0$$

$\vec{R}$ is transversal to D.

Hamiltonian interpretation of $\vec{R}$:

$\Sigma = D^\perp \setminus 0$ is a symplectic 4D sub-cone of $T^\star M$. If $\rho : \Sigma \to \mathbb{R}^+$ is defined by $\rho(s\alpha) = |s|$, the Hamiltonian dynamics of ρ in Σ is homogeneous of degree 0 and the orbits project onto the Reeb orbits. We denote by $R_t,\ t \in \mathbb{R}$ the Reeb flow on Σ or the sphere bundle $S(\Sigma)$.

Sub-Riemannian metric

We give a smooth metric g on D. This defines a distance on M by minimizing the lengths of "horizontal" curves (tangent to D at every point). The geodesics with speed 1 are the projections on M of the Hamiltonian orbits of the dual metric

$$\frac{1}{2}g^\star(x,\xi) = \frac{1}{2}\|\xi_{|D_x}\|_g^2$$

sitting inside $U^\star X := \{g^\star = 1\}$. Assuming that D is oriented, we get a choice of the contact form α_g satisfying

$$(d\alpha_g)_{|D} = v_g$$

This defines a Reeb vector field $\vec{R}_g$ which will be the main actor in what follows.

The fibers of the unit cotangent bundle $U^\star M$ are 2D-cylinders. The Reeb flow plays the role of a **compactification of the geodesics flow**: the co-sphere bundle

$$S(T^\star M) = U^\star M \cup S(\Sigma)$$

identifies as a disjoint union of $U^\star M$ and the sphere bundle of $\Sigma = D^\perp$. The geodesic flow lives on $U^\star M$ while the Reeb flow lives on $S(\Sigma)$.

Example 1: Heisenberg H^3/Γ

We consider the presentation of H^3 as $\mathbb{R}^3$ equipped with the group law

$$(x, y, z) \star (x', y', z') = (x + x', y + y', z + z' - xy')$$

We choose the lattice

$$\Gamma := \{(x, y, z) | (x, y) \in (\sqrt{2\pi}\mathbb{Z})^2, \ z \in 2\pi\mathbb{Z}\}$$

Our sR manifold is then $M := \mathbb{R}^3/\Gamma$ with the orthonormal basis for D given by

$$X = \partial_x, Y = \partial_y - x\partial_z$$

We have $\alpha_g = dz + xdy$ and $\vec{R}_g = \partial_z$.

The spectrum of $\Delta = -(X^2 + Y^2)$ is explicitely computable: one gets the union of the eigenvalues of the flat torus $\mathbb{R}^2/\sqrt{2\pi}\mathbb{Z}^2$ and the set of integers

$$m(2l+1),\ m = 1, \cdots,\ l = 0, \cdots$$

with multiplicities $2m$.

The lengths spectrum (the set of lengths of closed geodesics) is the set of

$$2\pi\sqrt{2n},\ n \in \mathbb{N}$$

Example 2: magnetic fields over a Riemannian surface

Let $\pi : M \to X$ be a principal S^1-bundle on an oriented Riemannian surface (X, h) i.e. M is equipped with a free action of S^1 and $X = M/S^1$.

We assume that this bundle is equipped with an Hermitian connection ∇ whose horizontal distribution is our D. If the curvature of the connection does not vanish, the distribution D is contact.

We take for g the pull-back on D of the metric h by π.

The curvature of ∇ is a 2-form B (the magnetic field) and one introduces the magnetic scalar $b = B/dx_h$. The sR metric is invariant by the S^1 action, this gives an invariant momentum e.

The geodesics of (M, D, g) of momentum e project onto the trajectories on X with the magnetic field b and the electric charge e.

The Reeb vector field is

$$\vec{R} = b\partial_\theta - \vec{b}$$

where $\vec{b}$ is the horizontal lift of the Hamiltonian vector field of b w.r. to the symplectic form B on X.

Example 3: Liouville

Again (X, h) is a Riemannian surface. M is the unit cotangent bundle. $D = \ker \lambda$ where λ is the restriction to M of the Liouville 1-form $\xi dx + \eta dy$. We take onto D the restriction of the “Sasaki metric” to D so that $\alpha_g = \lambda$. Then the Reeb vector field is the geodesic flow of h.

Spiraling

In the Heisenberg case, the geodesics are helices spiraling around vertical axis which are the Reeb orbits. This can be seen as follows

$$g^\star = \xi^2 + (\eta - x\zeta)^2 = |\zeta|(u^2 + v^2)$$

with $\{u, v\} = 1$ and $\{\zeta, u^2 + v^2\} = 0$. $I = u^2 + v^2$ is an harmonic oscillator. The dynamics is decoupled and if we remember that $\rho := |\zeta|$ is the Reeb Hamiltonian, we see that

$$g^\star = \rho I$$

which allows to compute the geodesic flow quite easily because $\{\rho, I\} = 0$.

Taking $\sigma \in \Sigma$ and $(u, v) \in \mathbb{R}^2$ as coordinates in $T^\star M$, the geodesic flow is given by

$$\vec{G} = \rho \frac{\partial}{\partial \theta} + \frac{1}{2} I \vec{\rho}$$

with $\rho I = 1$.

$$G_t(\sigma, u + iv) = \left(R_{It/2}(\sigma), e^{it/I}(u + iv) \right)$$

As $I \to 0$, $\rho \to \infty$, the geodesics are spiraling very fast around the Reeb orbits while moving slowly along the Reeb orbits.

Birkhoff normal forms.

We assume for simplicity that the bundle $D \to M$ is topologically trivial. This holds for the magnetic sR if X is a torus and for the Liouville sR if the surface X is orientable. Then

Theorem 1 (CdV-Hillairet-Trélat) *There exists an homogeneous canonical transformation $\chi : C \to C'$ with C a conic neighborhood of Σ in $T^\star M \setminus 0$ and C' a conic neighborhood of $\Sigma \times 0$ in $\Sigma \times \mathbb{R}^2$, with $\mathbb{R}^2_{u,v}$ equipped with the symplectic form $dv \wedge du$, so that $\chi_{|\Sigma} = \mathrm{Id} \times 0$ and*

$$g^\star \circ \chi^{-1} = \sum_{j=1}^{\infty} \rho_j(\sigma) I^j + O\left(I^2 \left(I/\rho\right)^\infty\right)$$

with ρ_j homogeneous of degree $2 - j$, $\rho_1 = \rho$ with ρ the Reeb Hamiltonian and $I = u^2 + v^2$.

From the Birkhoff normal form, one gets the following asymptotics for the geodesic flow:

$$\vec{G} \sim \frac{1}{I}\frac{\partial}{\partial\theta} + \frac{1}{2}I\vec{\rho}$$

which gives the following approximation of the geodesic flow as $I \to 0$:

$$G_t(\sigma, u+iv) \sim \left(R_{It/2}(\sigma), e^{it/I}(u+iv)\right)$$

which is valid for $t = O(1/I) >> 1$.

Note that it is well known in the 2D magnetic example: when e is large the magnetic orbits spiral around the level lines of the magnetic field.

Periodic geodesics.

Let us assume that the BNF is convergent in some cone around Σ, ie there exists a smooth homogeneous function F so that $F(\sigma, I) \circ \chi = g^\star$ where $F = \rho I + \cdots$.

Assume also that the Reeb flow has a ND closed orbit of period T_0, then, for each $l \in \mathbb{N}$ and $k >> 1$, there exists a closed geodesic $\gamma_{k,l}$ of approximate length

$$L_{k,l} \sim 2\sqrt{klT_0\pi}$$

The geodesics $\gamma_{k,l}$ cover l-times the Reeb closed orbit and spiral very fast around it.

To prove this we start by defining, for I small, a smooth family of periodic orbits of $\frac{1}{I}F(\sigma, I)$. Then we look at the return map for the angle θ.

Conjecture 1 *The existence of closed geodesics spiraling around a ND periodic Reeb orbit stil holds even if the BNF is not convergent.*

The Laplacian

We choose a smooth density $|dx|$ on M. The Laplacian Δ is the Friedrichs extension on $L^2 = L^2(M, |dx|)$ of $Q(f) = \int_M g^\star(df)|dx|$. If (X, Y) is an ONB of D (locally), $\Delta = X^\star X + Y^\star Y = -X^2 - Y^2 +$ l.o.t. The symbol of Δ is $g^\star$ which vanishes exactly on Σ. The Laplacian is sub-elliptic (Hörmander) and hence has a discrete spectrum (λ_k) with an o.n.b. of eigenfunctions of (ϕ_k).

Weyl law

If $N(\lambda) = \#\{j | \lambda_k \leq \lambda\}$, we have, as $\lambda \to +\infty$,

$$N(\lambda) \sim \frac{\int_M \alpha_g \wedge d\alpha_g}{32} \lambda^2$$

Note that this asymptotics is independent of the measure $|dx|$. If $M = S^3$, $\int_M \alpha_g \wedge d\alpha_g$ is the (inverse of) the asymptotic Hopf invariant (Arnold 86) of the Reeb vector field, which is hence a spectral invariant.

Quantum limits.

What is a QL?

We consider, for a ΨDO A of degree 0 and ppal symbol $a : S(T^\star M) \to \mathbb{R}$, the quantities $\langle A\phi_k | \phi_k \rangle$. Taking subsequences, they converge to $\int_{S(T^\star M)} a d\mu$ for some probability measures μ. Such μ's are called Quantum limits.

Theorem 2 (CdV-Hillairet-Trélat) *Let us decompose the sphere bundle $S(T^\star M)$ as the union of the unit bundle $U^\star M := \{g^\star = 1\}$ and the sphere bundle of the characteristic manifold $S(\Sigma)$.*

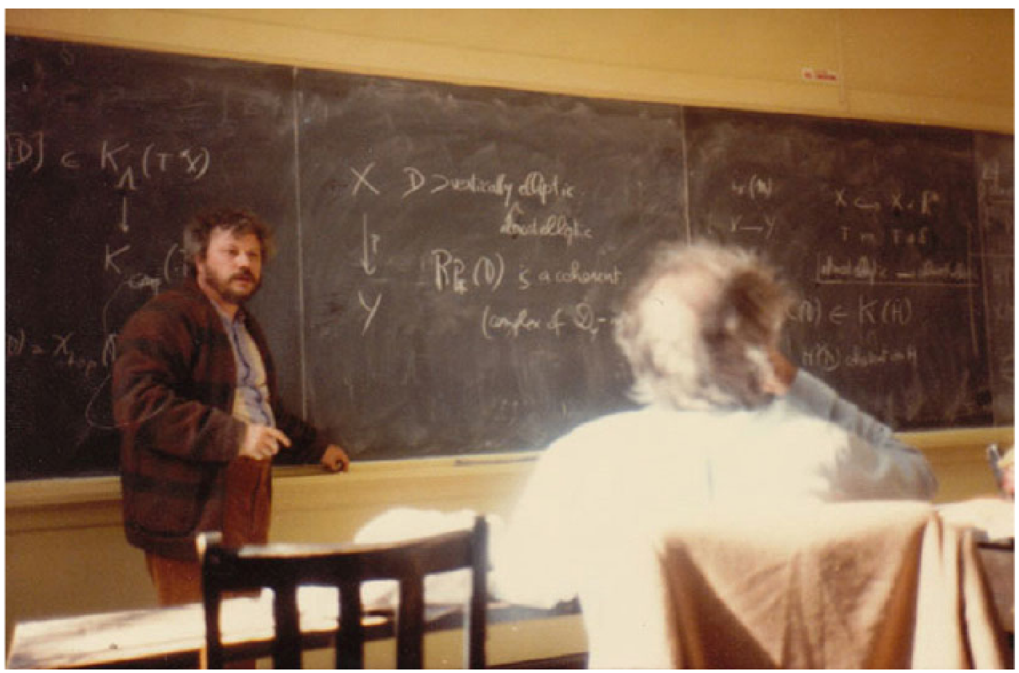
1. *Any QL μ (a probability measure on $S(T^\star M)$), can be uniquely written as the sum $\mu = \mu_0 + \mu_\infty$ where μ_∞ is supported by $S(\Sigma)$ and is invariant by the Reeb flow, while $\mu_0(S(\Sigma)) = 0$ and μ_0 is invariant by the geodesic flow.*

2. *If (ϕ_j) is an ONB of eigenfunctions, there exists a subsequence (ϕ_{j_k}) of density 1, so that all corresponding QL's are supported by $S(\Sigma)$ and are invariant by Reeb.*

3. *If the Reeb flow is ergodic, then we have QE for any real eigenbasis with the limit measure the normalized Liouville measure on $\rho^{-1}(1) \sim S(\Sigma)$.*

Toeplitz operators.

In order to quantize the BNF, we need to quantize the Reeb Hamiltonian as a function on the symplectic cone Σ: this can be done by using the Toeplitz quantization as introduced by Louis Boutet de Monvel and Victor Guillemin.



To any symplectic sub-cone of $T^\star M$, they associate an Hilbert subspace $\mathcal{H}_\Sigma$ of $L^2(M)$ which consists of distributions whose WF is included in Σ. If $\Pi : L^2(M) \to \mathcal{H}_\Sigma$ is the orthogonal projection and A a ΨDO on M, they define a "Toeplitz operator" $T_A := \Pi A$ on $\mathcal{H}_\Sigma$. The set of operators T_A is a graded algebra. The principal symbol of T_A is the restriction to Σ of the symbol of A. The rules of calculus are the same as those of ΨDO 's.

QBNF.

Theorem 3 *We use a FIO associated to the canonical transform* χ *to transform* Δ *into*

$$\Delta_0 = \sum_{j=1}^{\infty} R_j \otimes \Omega^j + R_\infty$$

where the R_j *are Toeplitz operators on* Σ *of degree* $1-j$, R_0 *is elliptic with symbol* $|\rho|$, Ω *is an harmonic oscillator on* $\mathbb{R}^2$ *and* R_∞ *is smoothing along* Σ *(the symbol is* $O((I/\rho)^\infty)$*).*

The Landau operators Δ_l's .

It follows that we have, for each value of $l \in \mathbb{N}$ a sequence of eigenvalues (μ_j^l), $j = 1, \cdots$, which are the eigenvalues of the Toeplitz operator of degree 1

$$\Delta_l := \sum_{j=1}^{\infty} (2l+1)^j R_j$$

modulo smoothing operators.

The ppal symbol of Δ_l is $(2l+1)\rho$.

We call such a sequence of eigenvalues a Landau cluster. Such eigenvalues are approximations of some exact eigenvalues of Δ: for each l, there exists a sequence λ_{k_j} of eigenvalues of Δ with $\lambda_{k_j} = \mu_j^l + O(j^{-\infty})$.

Trace formulae

What is a trace formula?

We express the trace of a suitable function $f(\Delta)$ into two ways:

$$\text{Trace}(f(\Delta)) = \sum_k f(\lambda_k)$$

and

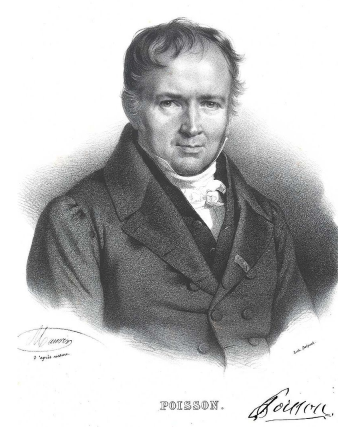
$$\text{Trace}(f(\Delta)) = \int_M K(x,x)|dx|$$

where K is the Schwartz kernel of $f(\Delta))$ computed in a direct way using group theory (Poisson, Selberg), PDE theory, . . .

One get the heat trace using $f : x \to \exp(-tx)$, the Shrödinger trace $f : x \to \exp(-zx)$, $\Re(z) > 0$, the wave trace $f : x \to \exp(-it\sqrt{x}))$, . . .

Flat tori and Riemannian case Recall the following formula : $\Gamma \subset \mathbb{R}^d$ is a lattice and $\Gamma^\star$ the dual lattice.

$$\sum_{\gamma^\star \in \Gamma^\star} e^{-4\pi^2 \|\gamma^\star\|^2 t} = \frac{1}{(4\pi t)^{d/2}} \left(\sum_{\gamma \in \Gamma} e^{-\|\gamma\|^2/4t} \right)$$



This follows from the Poisson summation formula.

This formula has a spectral interpretation: we consider the flat torus $\mathbb{R}^d/\Gamma$: the lhs is the trace of $\exp(-t\Delta)$ while the rhs is a sum on the periodic geodesics.

This Schrödinger trace formula for the flat tori was extended as an asymptotic formula for compact Riemannian manifolds by myself in 73'; the wave trace was used by Chazarain and Duistermaat-Guillemin shortly after. Both approaches show that the length spectrum (the set of lengths of closed geodesics) is generically a spectral invariant.

The Schrödinger trace in the Riemannian case is, as $z \to 0$, $\Re(z) > 0$,

$$\sum_k e^{-\lambda_k z} \sim \sum_\gamma a_\gamma(z) e^{-L_\gamma^2/4z}$$

with $a_\gamma(z)$ of polynomial growth w.r.t. to $1/z$ [CdV73].

The wave trace says that the singular support of the Schwartz distribution

$$\sum_k e^{-it\sqrt{\lambda_k}}$$

is the set of length of periodic geodesics [Chazarain, Duistermaat-Guillemin 74].

Both hold generically.

We will try to get such formulae for sR manifolds.

For $\Re(z) > 0$, we have

$$\sum_{m=1}^{\infty} 2m \sum_{l=0}^{\infty} e^{-(2l+1)mz} = \frac{\pi^2}{4z^2} - \frac{1}{2z} + \frac{\pi^2}{z^2} \sum_{n=1}^{\infty} \frac{1}{1+\cosh(2\pi^2 n/z)}$$

This formula is obtained using the Poisson summation formula. It can be interpreted as a trace formula for the Laplacian of a quotient of the 3D Heisenberg group (example 1).

The lhs is the spectral part. The rhs is the dynamical part: it contains the Weyl formula, the lengths of periodic geodesics and the periods of the Reeb vector field.

Melrose Trace formulae

Melrose trace formula (1984) says that the Duistermaat-Guillemin trace formula for the wave trace distribution

$$W(t) := \sum_k \exp(-it\sqrt{\lambda_k})$$

holds outside $t = 0$:

the singular support of $W(t)$ is included in the length spectrum and, assuming some ND assumption, the ppal term is given explicitely in terms of the Poincaré map and the Morse index of the periodic geodesic.

Boutet-Guillemin Trace formulae for Δ_l

Boutet de Monvel and Guillemin showed that the D-G wave trace formula extends to the case of elliptic self-adjoint Toeplitz operators of degree 1, hence it works for the Δ_l's. The corresponding orbits are the closed Reeb orbits. The periods are the $T_\gamma/(2l+1)$ if the T_γ are the Reeb periods.

Geometric interpretation of the Schrödinger Trace formula for H^3/Γ

For $\Re(z) > 0$, recall the formula

$$Z_0(z) := \sum_{m=1}^{\infty} 2m \sum_{l=0}^{\infty} e^{-(2l+1)mz} = \frac{\pi^2}{4z^2} - \frac{1}{2z} + \frac{\pi^2}{z^2} \sum_{n=1}^{\infty} \frac{1}{1+\cosh(2\pi^2 n/z)}$$

The lhs is obtained by removing from the trace of $\exp(-z\Delta)$ for the sR manifold H^3/Γ the trace of $\exp(-z\Delta_T)$, where T is the flat torus $\mathbb{R}^2/\sqrt{2\pi}\mathbb{Z}^2$.

As $z = t$ is real and $t \to 0^+$, we get

$$Z_0(t) = \frac{\pi^2}{4t^2} - \frac{1}{2t} + \frac{2\pi^2}{t^2} \sum_{n=1}^{\infty} a_n e^{-2\pi^2 n/t} + \cdots$$

This gives the heat expansion and the Weyl law. Moreover the exponentially small terms are of the form $\exp(-L_n^2/4t)$ with L_n's the lengths of periodic geodesics $2\pi\sqrt{2n}$.

As $\Re(z) \to 0$, we get a dense family of poles

$$z_{n,l} = 2\pi n/(2l+1)$$

which correspond to the periods of the Hamiltonian fields of $(2l+1)\rho$, the ppal symbols of the Δ_l's.

This formula contains an heat trace expansion and a wave trace singularity along the imaginary axis.

This leads to

Conjecture 2 *The Reeb periods are spectral invariants of the sR laplacian.*

First hint: Melrose formula $\rightarrow$ closed geodesics $\rightarrow$ Reeb orbits

Second hint: $\Delta \sim \oplus \Delta_l$. BdM-G formula for each $\Delta_l \rightarrow$ Reeb orbits.

Thanks for your attention