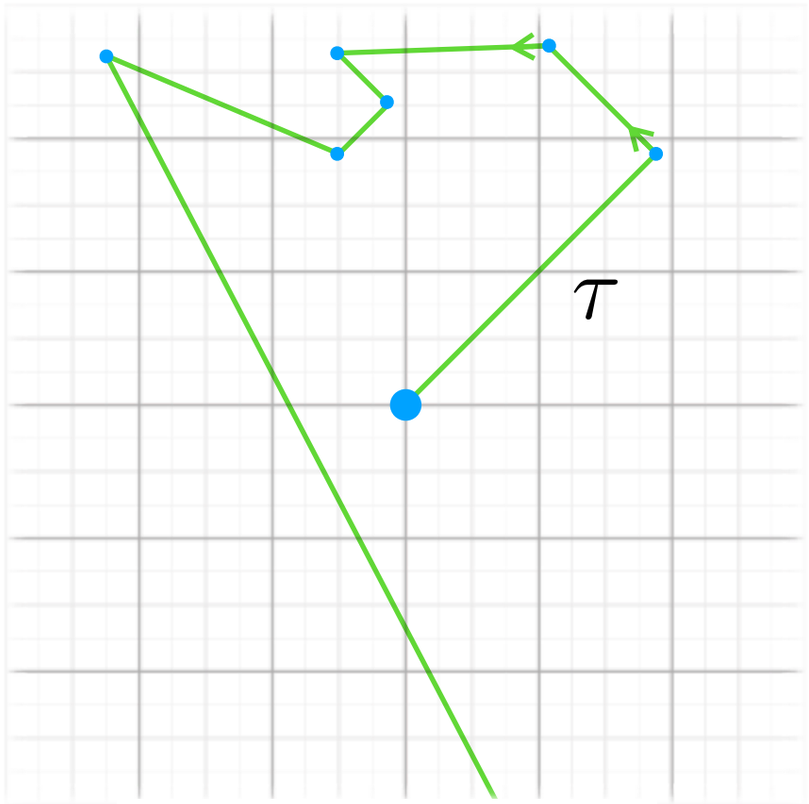
Building bridges Lévy-walk way:
an approach to home-range search and foraging

Vasily Zaburdaev & Sergey Denisov

Lévy walks



Shlesinger and Klafter (1982)

VZ, SD and Klafter, RMP (2015)

J. Klafter (TAU)

M. Shlesinger (ONR)

$$\psi(\tau) = \frac{1}{\tau_0} \frac{\gamma}{(1+\tau/\tau_0)^{1+\gamma}}, \quad \gamma > 0$$

$$v_0 = \text{const}$$

$$l_i = v_0 \tau_i$$

Central Limit Theorem

$$\langle l^2 \rangle < \infty, \quad \gamma > 2$$

$$\sum_{i=1}^{N} \vec{l}_i \rightarrow \text{ Gaussian distribution}$$

Generalized Central Limit Theorem

Lévy (1937), Gnedenko and Kolmogorov (1954)

$$\langle l^2 \rangle = \infty, \quad 0 < \gamma < 2$$

$$\sum_{i=1}^{N} \vec{l}_i \rightarrow \text{ Lévy distribution}$$

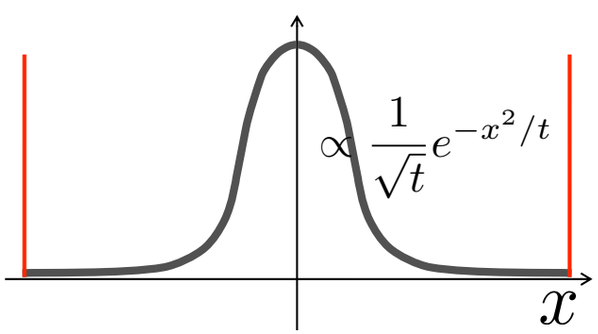
in 1D: $\tilde{L}(k) \propto e^{-K_\gamma |k|^\gamma}, \quad L_\gamma(x) \sim \frac{A}{|x|^{1+\gamma}}$

Lévy walks

VZ, SD and Klafter, RMP (2015)

$P(x)$

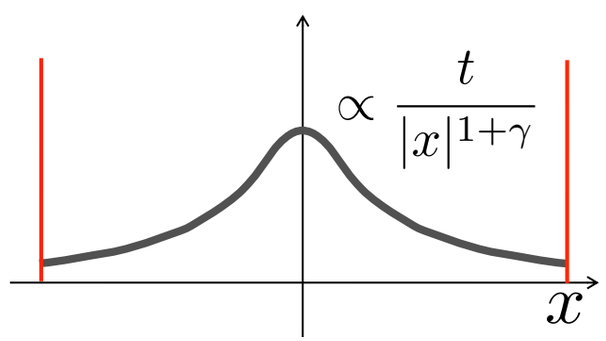
$(\gamma > 2)$



$\langle \mathbf{r}^2(t) \rangle \propto t$

"Gaussian"
(telegrapher's eq.)

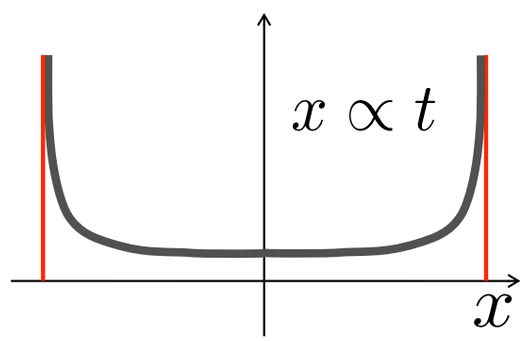
$(1 < \gamma < 2)$



$\langle \mathbf{r}^2(t) \rangle \propto t^{3-\gamma}$

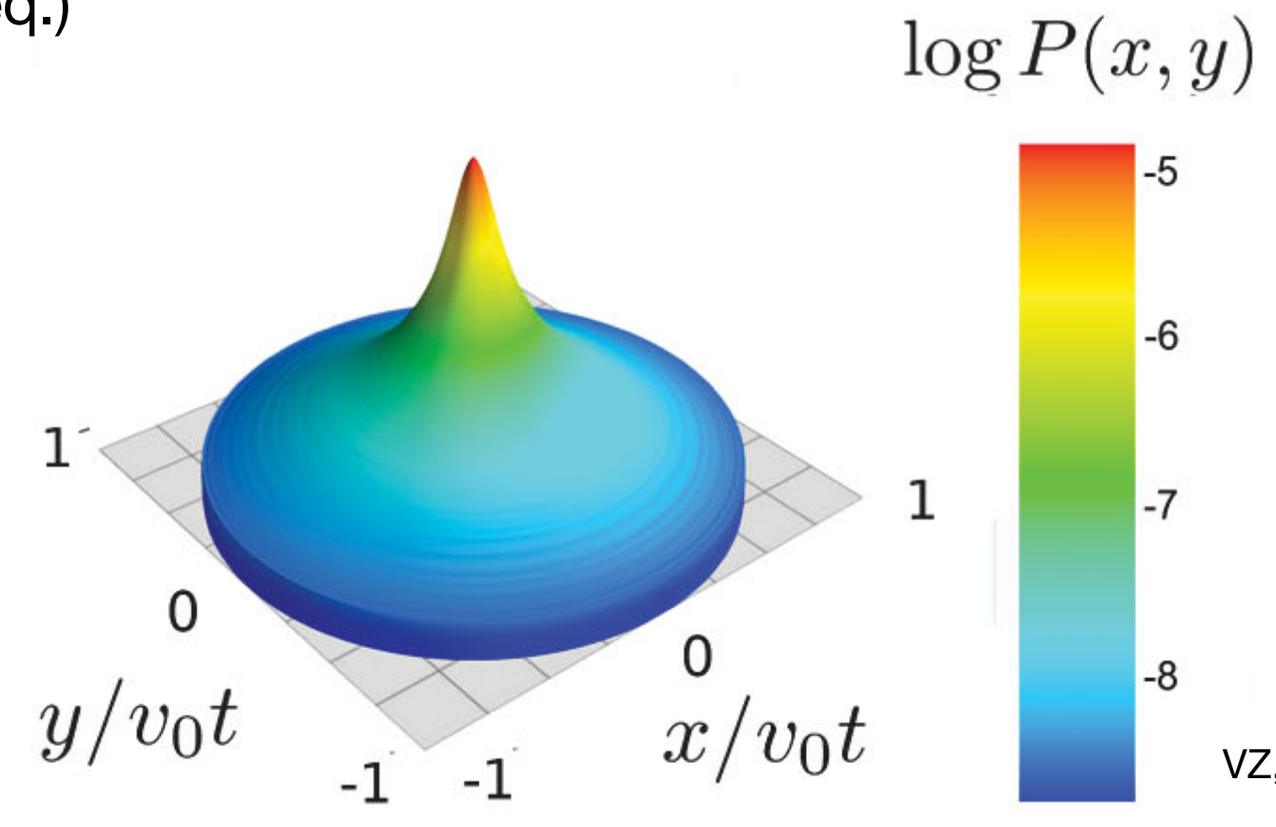
"Lévy"

$(0 < \gamma < 1)$



$\langle \mathbf{r}^2(t) \rangle \propto t^2$

Lamperti

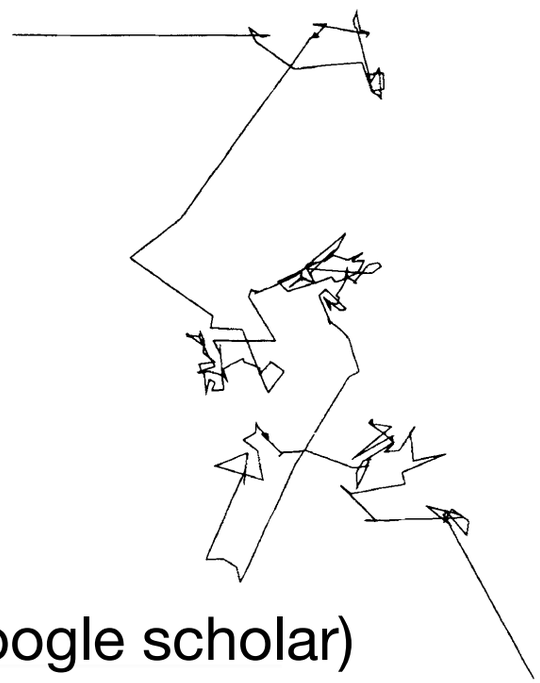


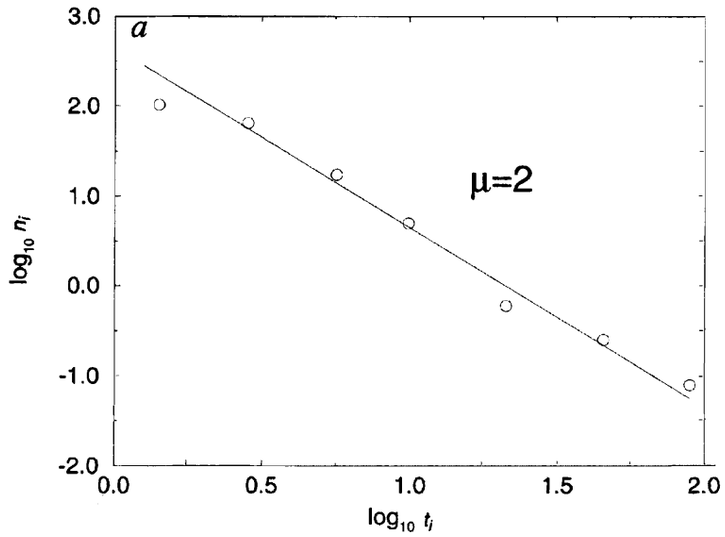
VZ, Fouxon, SD, Barkai (2016)

Lévy walks in physics

- Transport of light in atomic vapours and Lévy glass
- One-dimensional non-linear maps
- Transport in single particle and many particle Hamiltonian systems
- Cold atoms in optical lattices
- Blinking quantum dots
- ...

Lévy walks in biology





nature 1439 citations (Google scholar)

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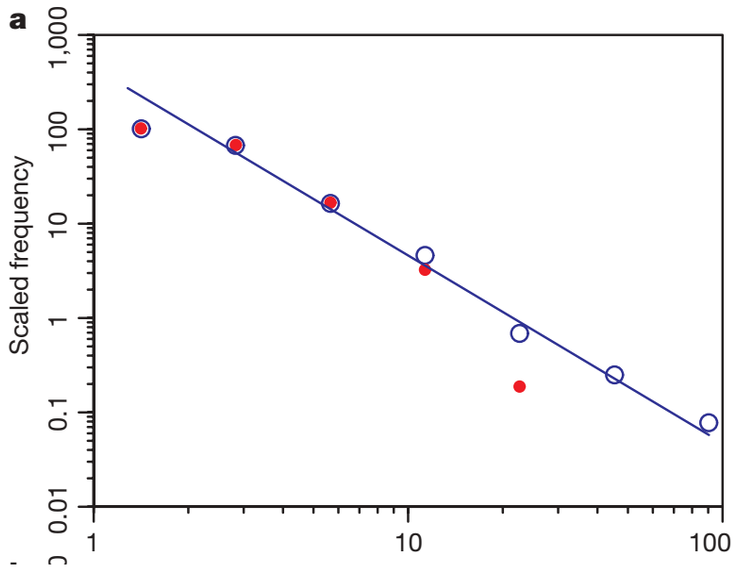
Lévy flight search patterns of wandering albatrosses

G. M. Viswanathan, V. Afanasyev, S. V. Buldyrev, E. J. Murphy, P. A. Prince & H. E. Stanley

Nature 381, 413–415 (1996) | Cite this article

2657 Accesses | 865 Citations | 12 Altmetric | Metrics



nature 922 citations (Google scholar)

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Published: 25 October 2007

Revisiting Lévy flight search patterns of wandering albatrosses, bumblebees and deer

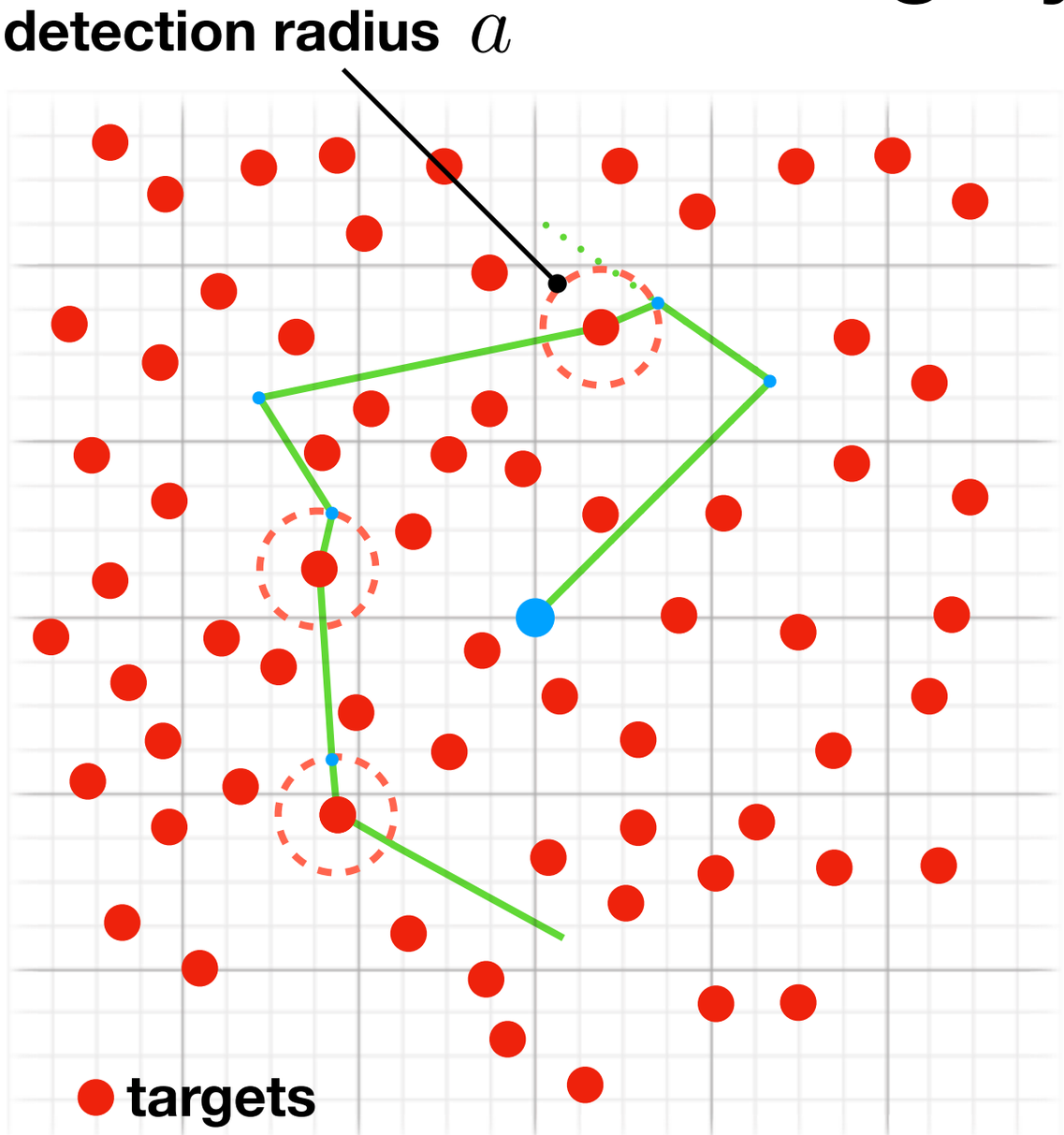
Andrew M. Edwards ✉, Richard A. Phillips, Nicholas W. Watkins, Mervyn P. Freeman, Eugene J. Murphy, Vsevolod Afanasyev, Sergey V. Buldyrev, M. G. E. da Luz, E. P. Raposo, H. Eugene Stanley & Gandhimohan M. Viswanathan

Nature 449, 1044–1048 (2007) | Cite this article

5408 Accesses | 597 Citations | 20 Altmetric | Metrics

Lévy walks in biology

Searching by Lévy walks



	destructive search	non-destructive search
high-density	?	?
low-density	?	?

$$\textbf{Search efficiency} = \frac{\textbf{Number of found targets}}{\textbf{Total distance (time) traveled}} = \eta \cdot \lambda = (2a\rho)^{-1}$$

Shlesinger and Klafter (1986), in *On Growth and Form*
Viswanathan et al. (1999)

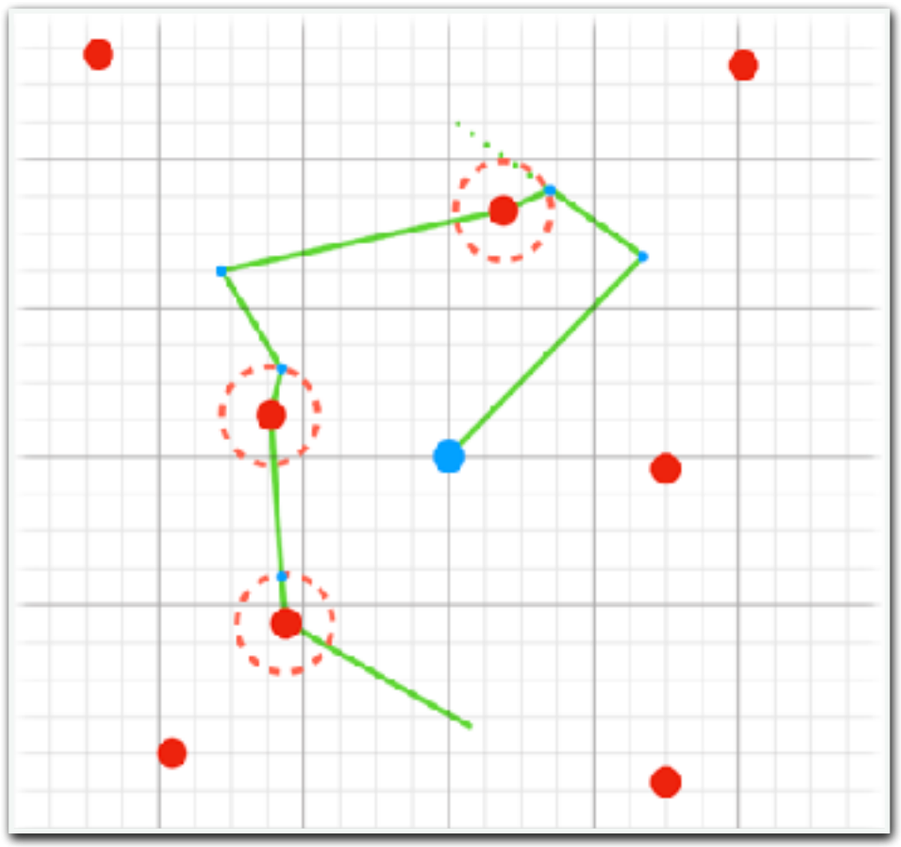
Optimality of Lévy walks for search

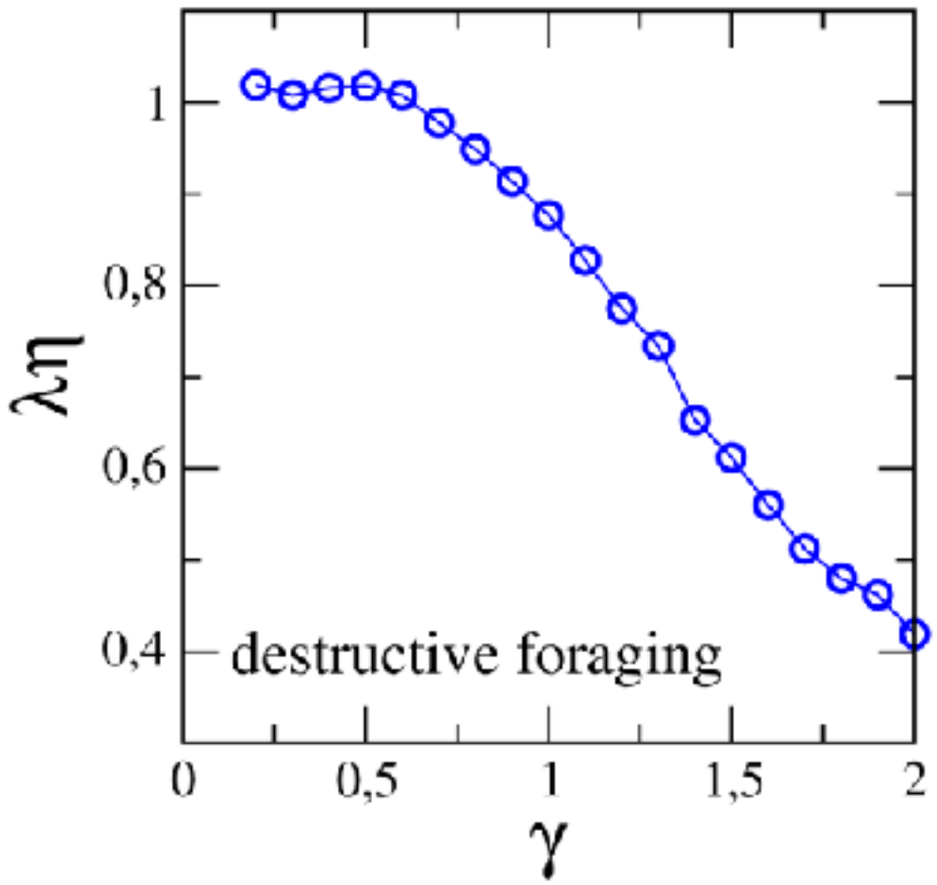
Viswanathan et al. (1999)

	destructive search	non-destructive search
high-density	*independent of* γ	
low-density	$\gamma \to 0$	

$$\psi(\tau) = \frac{1}{\tau_0} \frac{\gamma}{(1+\tau/\tau_0)^{1+\gamma}}, \quad \gamma > 0$$

change γ and follow efficiency η





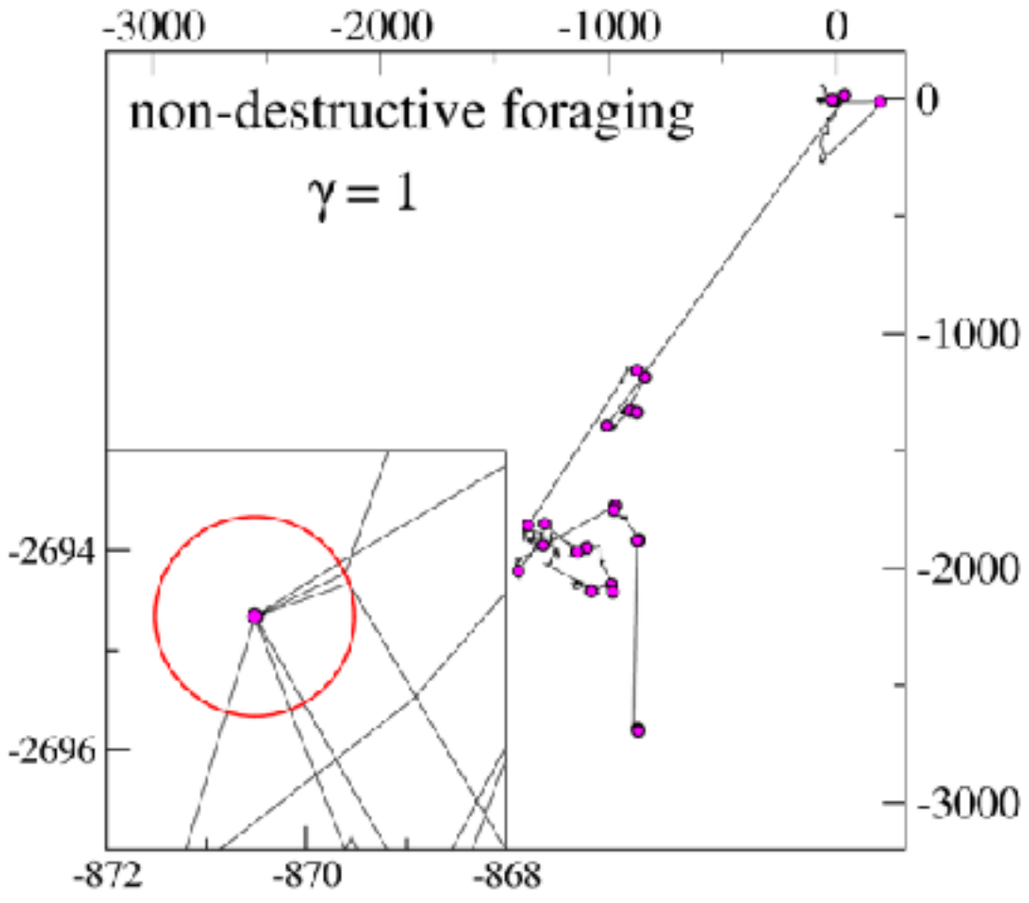
Optimality of Lévy walks for search

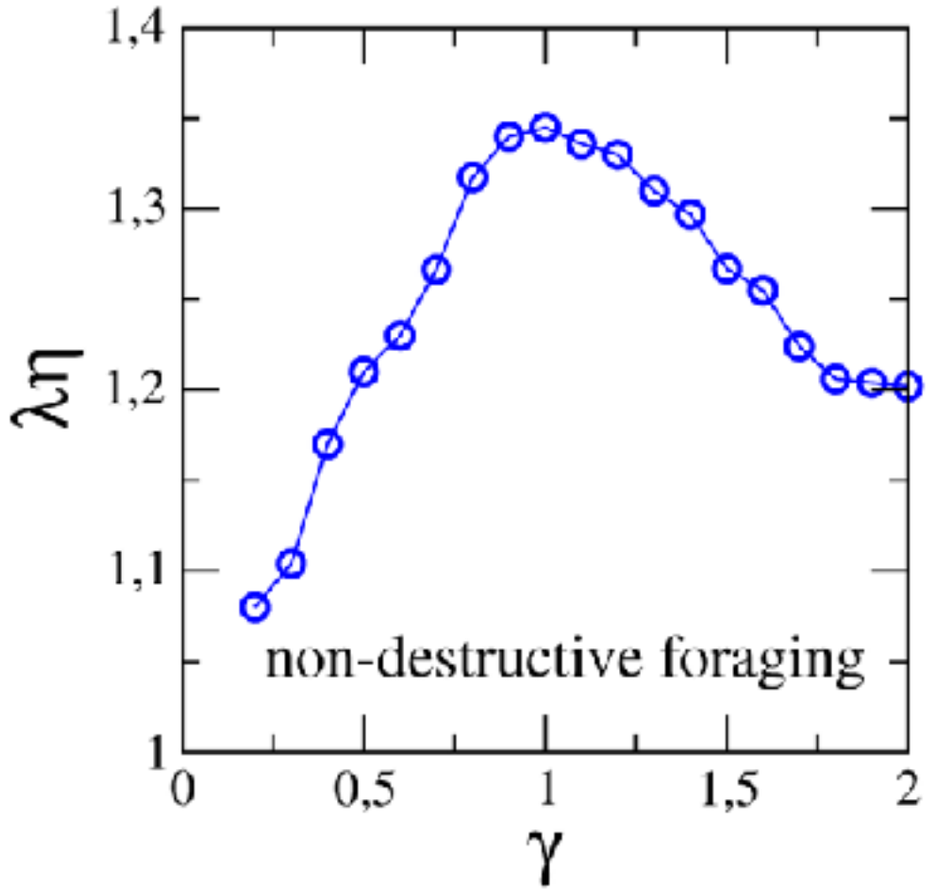
Viswanathan et al. (1999)

	destructive search	non-destructive search
high-density	*independent of* γ	
low-density	$\gamma \to 0$	$\gamma \simeq 1$

$$\psi(\tau) = \frac{1}{\tau_0} \frac{\gamma}{(1+\tau/\tau_0)^{1+\gamma}}, \quad \gamma > 0$$

change γ and follow efficiency η





Optimality of Lévy walks?

nature 1543 citations (Google scholar)

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nature › letters › article

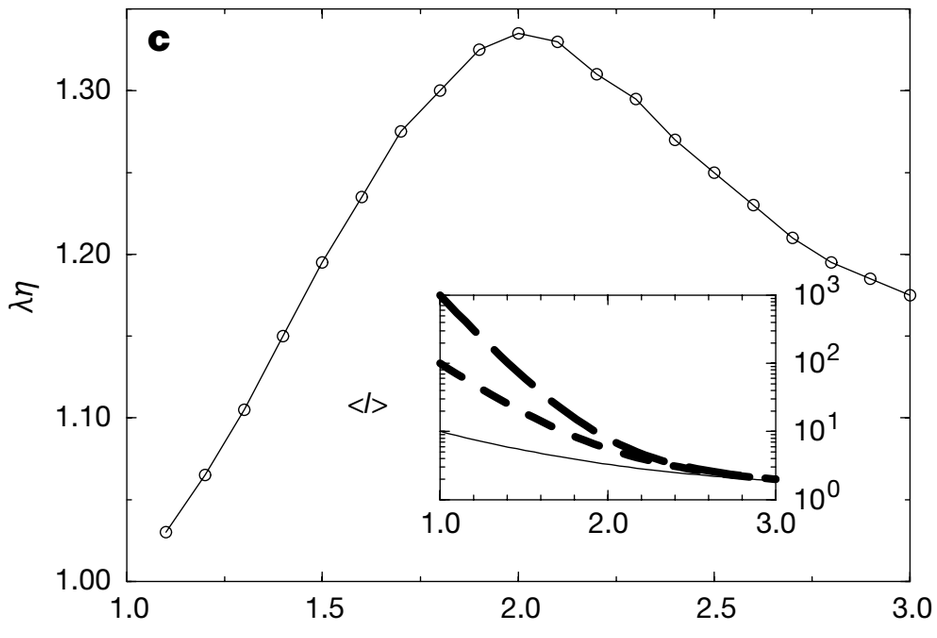
Published: 28 October 1999

Optimizing the success of random searches

G. M. Viswanathan ✉, Sergey V. Buldyrev, Shlomo Havlin, M. G. E. da Luz, E. P. Raposo & H. Eugene Stanley

Nature **401**, 911–914 (1999) | Cite this article

7009 Accesses | **960** Citations | **18** Altmetric | Metrics



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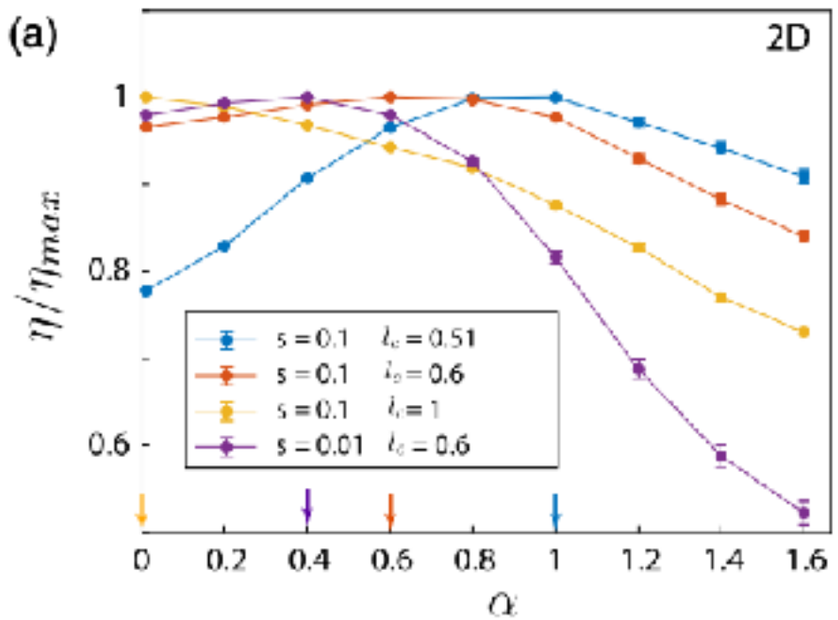
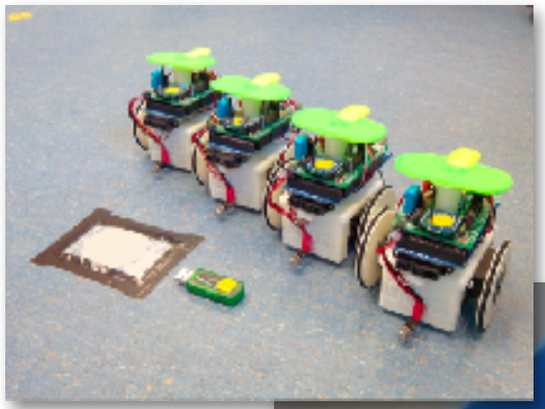
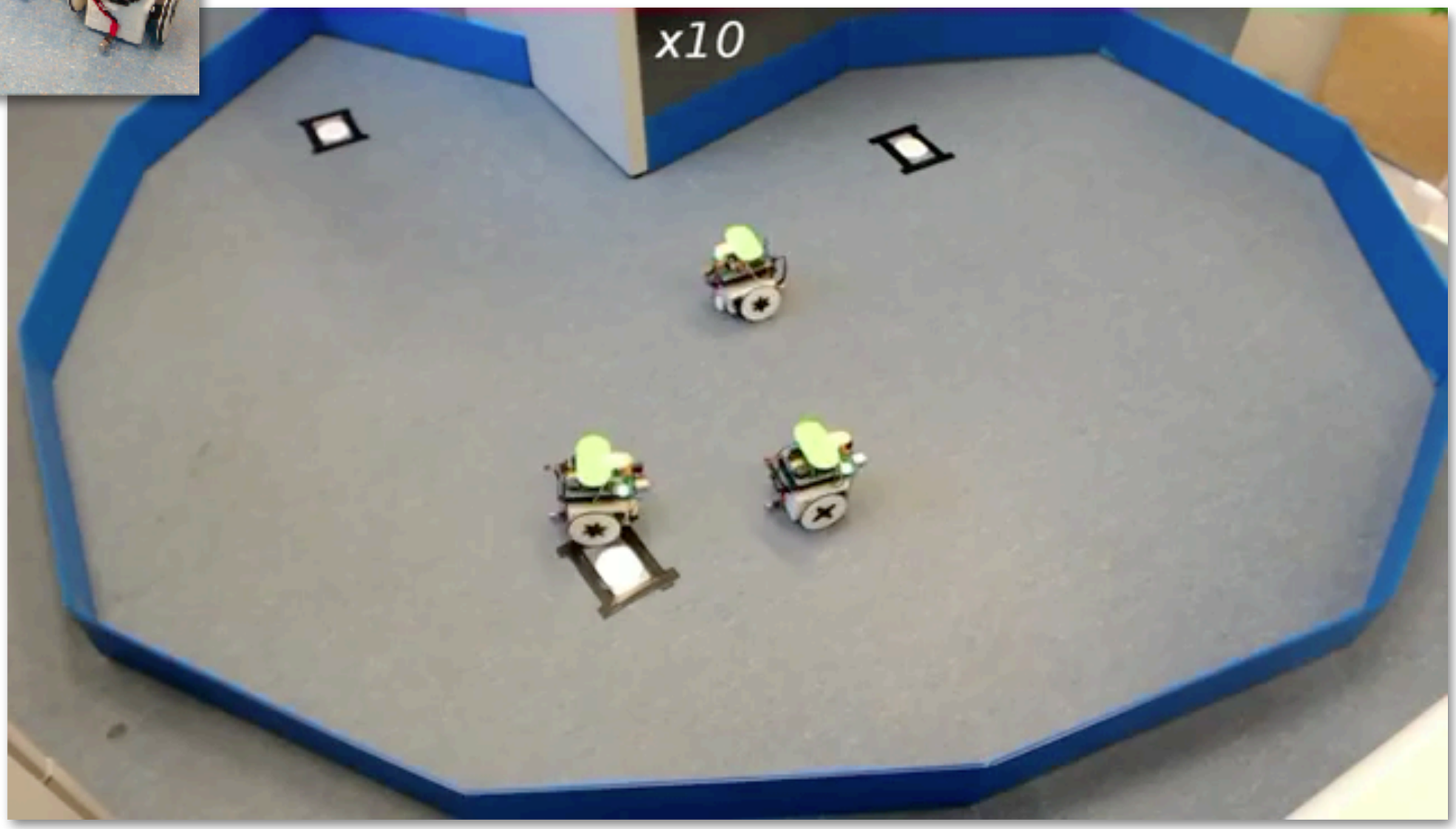
Inverse Square Lévy Walks are not Optimal Search Strategies for $d \geq 2$

Nicolas Levernier, Johannes Textor, Olivier Bénichou, and Raphaël Voituriez
Phys. Rev. Lett. **124**, 080601 – Published 26 February 2020



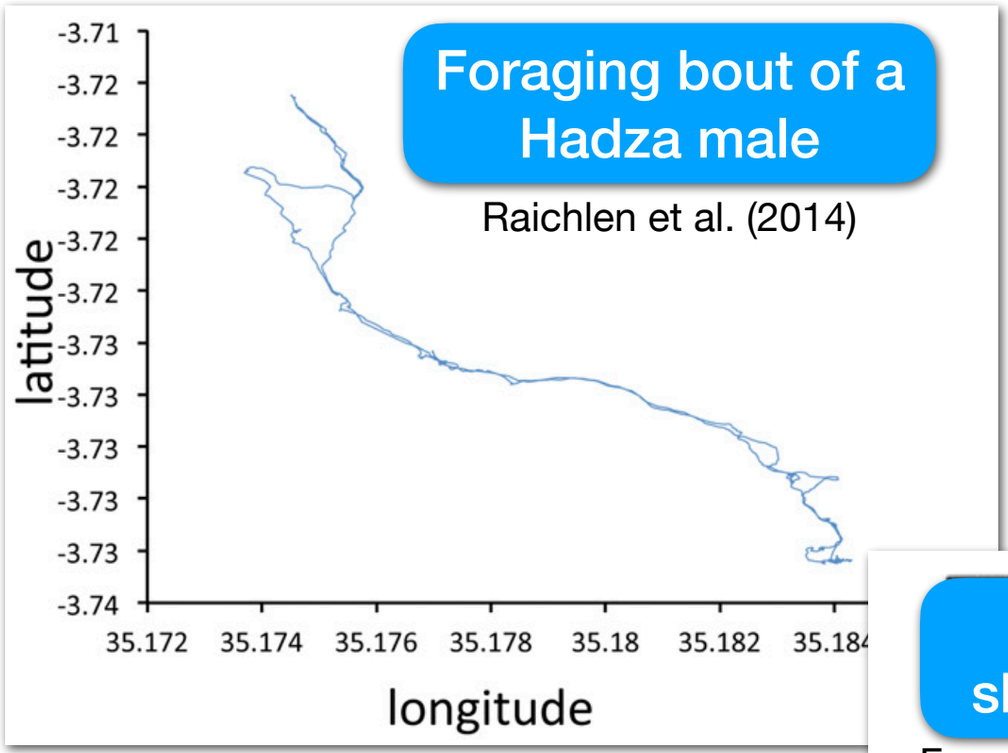
Do robots dream of Lévy walks?

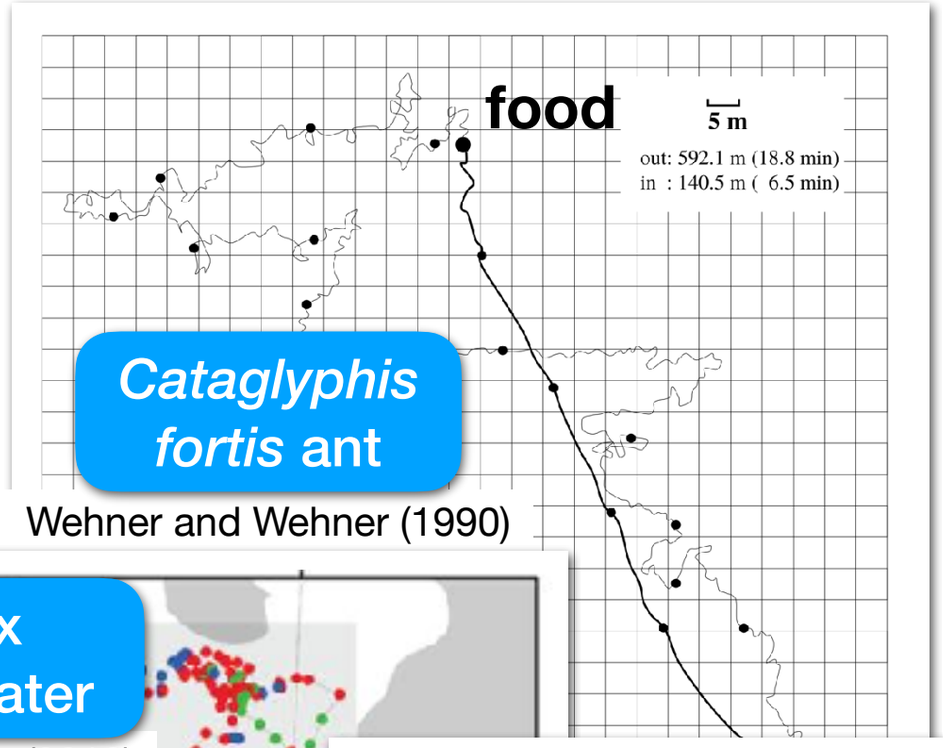


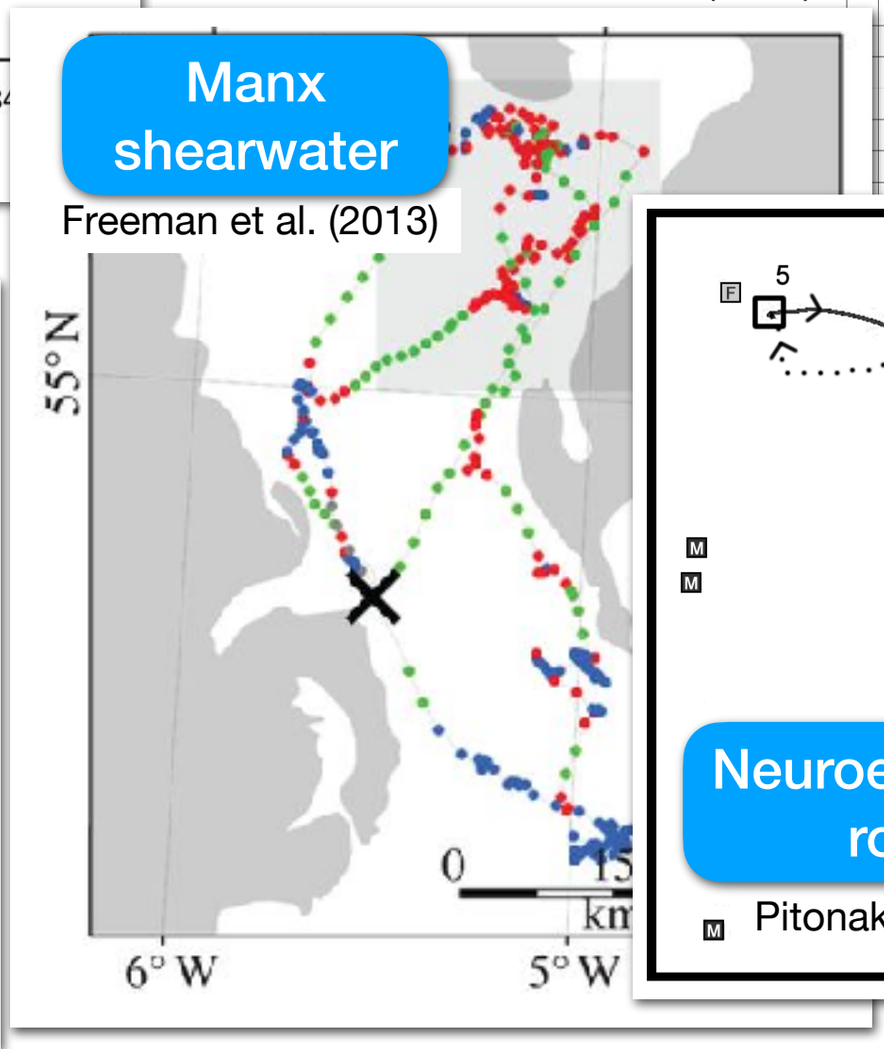


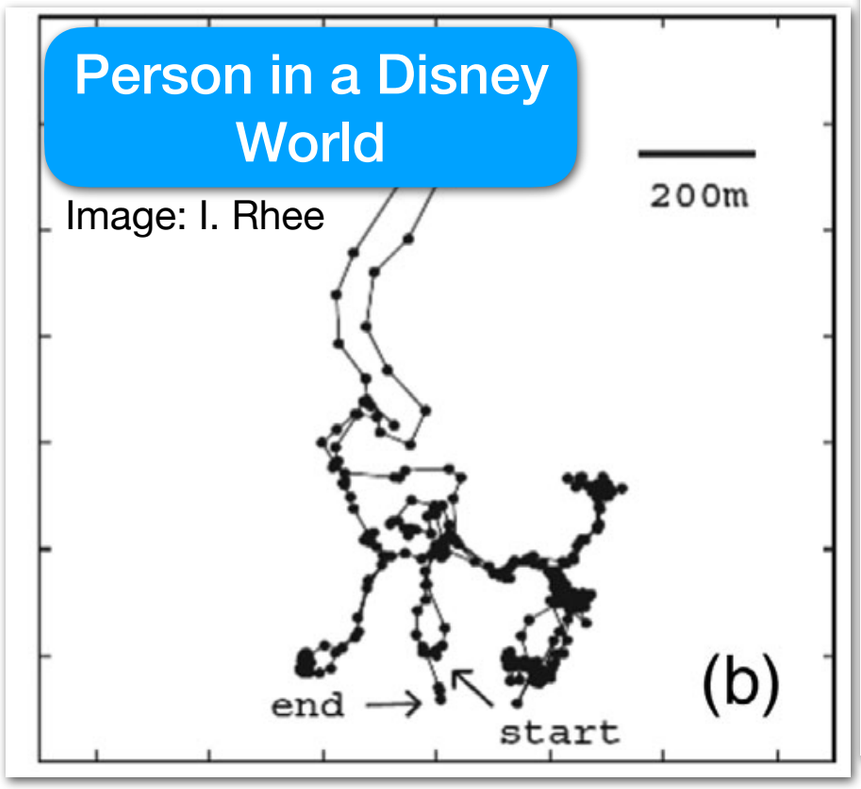
Grupo de Neurocomputaci´on Biol´ogica,
Escuela Polit´ecnica Superior Universidad Aut´onoma de Madrid

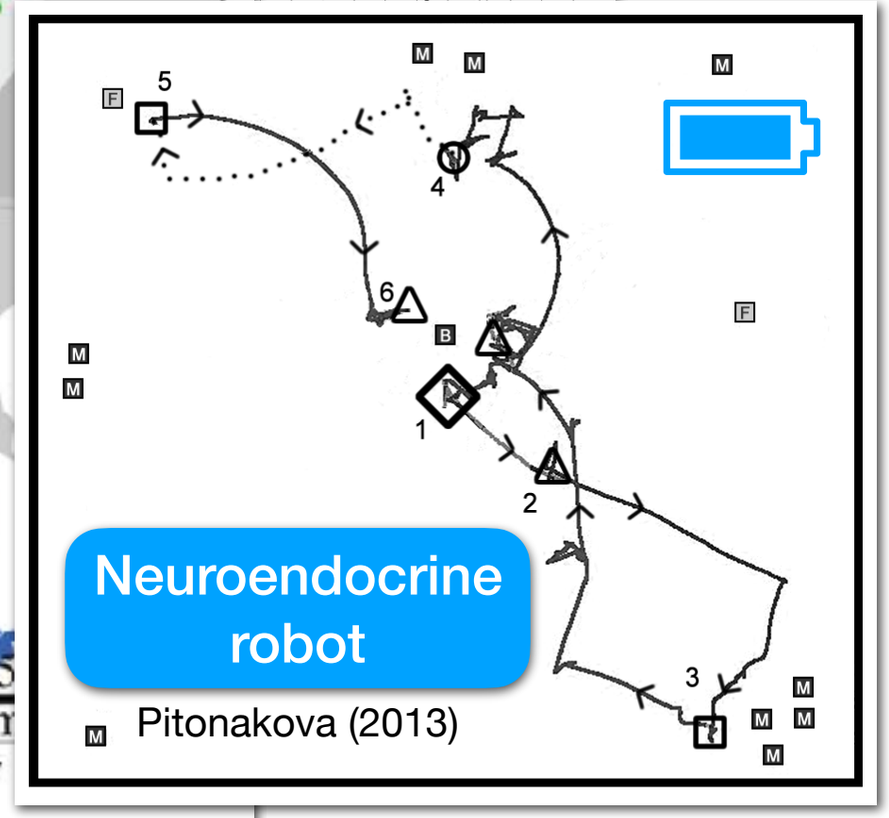
… but is it really a simple Lévy walk?











… but is it really a simple Lévy walk?

Foraging bout of a Hadza male

Raichlen et al (2014)

- **Home range (Ecology)**

Horne et al. (2007) Fieberg and Börger (2012) Powell and Mitchell (2012) Fagan et al. (2013)

- **Energy constraint: "*Staying alive*" (Robotics)**

Silverman et al. (2002) Mei et al. (2005) Wawerla & Vaughan (2007) Wang et al. (2008) Rappaport (2016)

Person in a Disney World

Image: I. Rhee

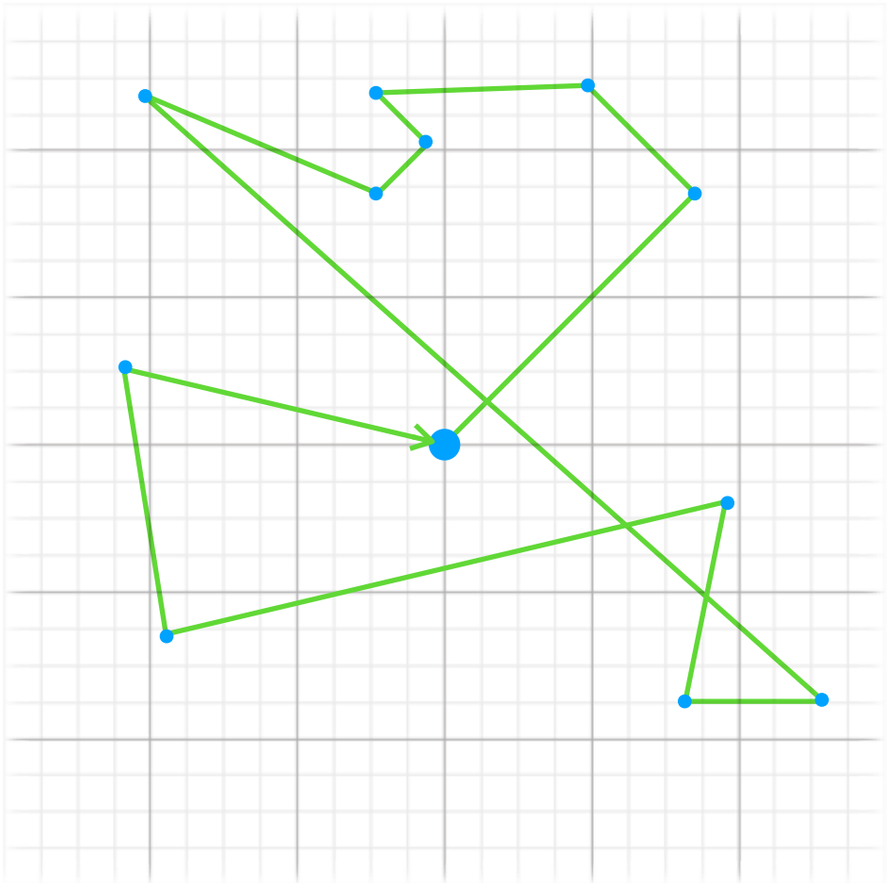
Freeman et al (2013)

Neuroendocrine robot

Pitonakova (2013)

It is a Lévy-walk bridge

(*Lévy walk trajectory of a <u>fixed length</u> T which <u>starts and ends at the same point</u>*)



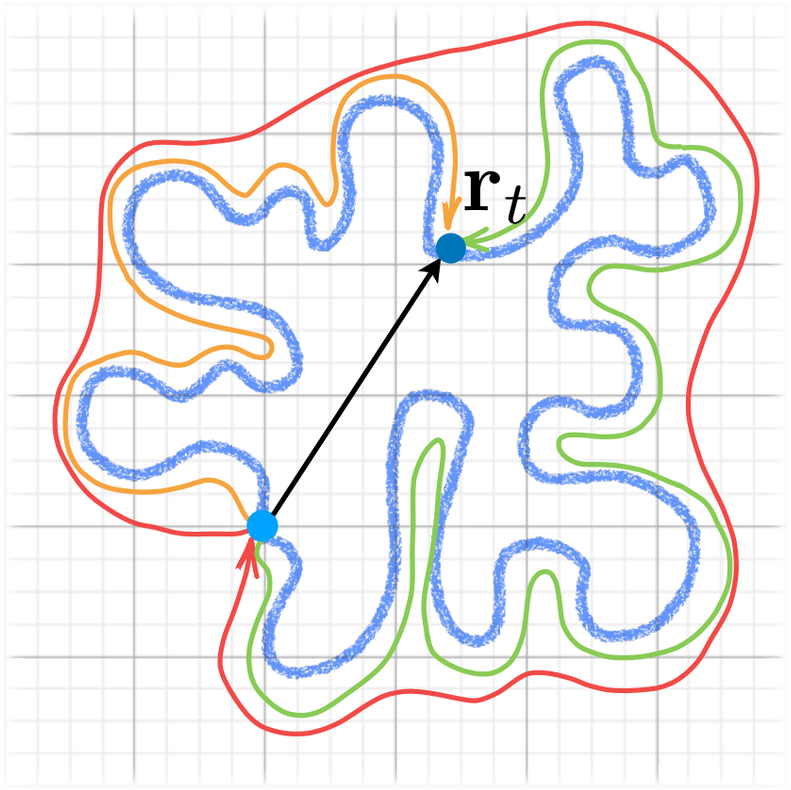
$$\psi(\tau) = \frac{1}{\tau_0}\frac{\gamma}{(1+\tau/\tau_0)^{1+\gamma}}$$

$$v_0 = \text{const}$$

$$T = \text{const}$$

How to describe Lévy-walk bridges?

Random walk (Brownian) bridges

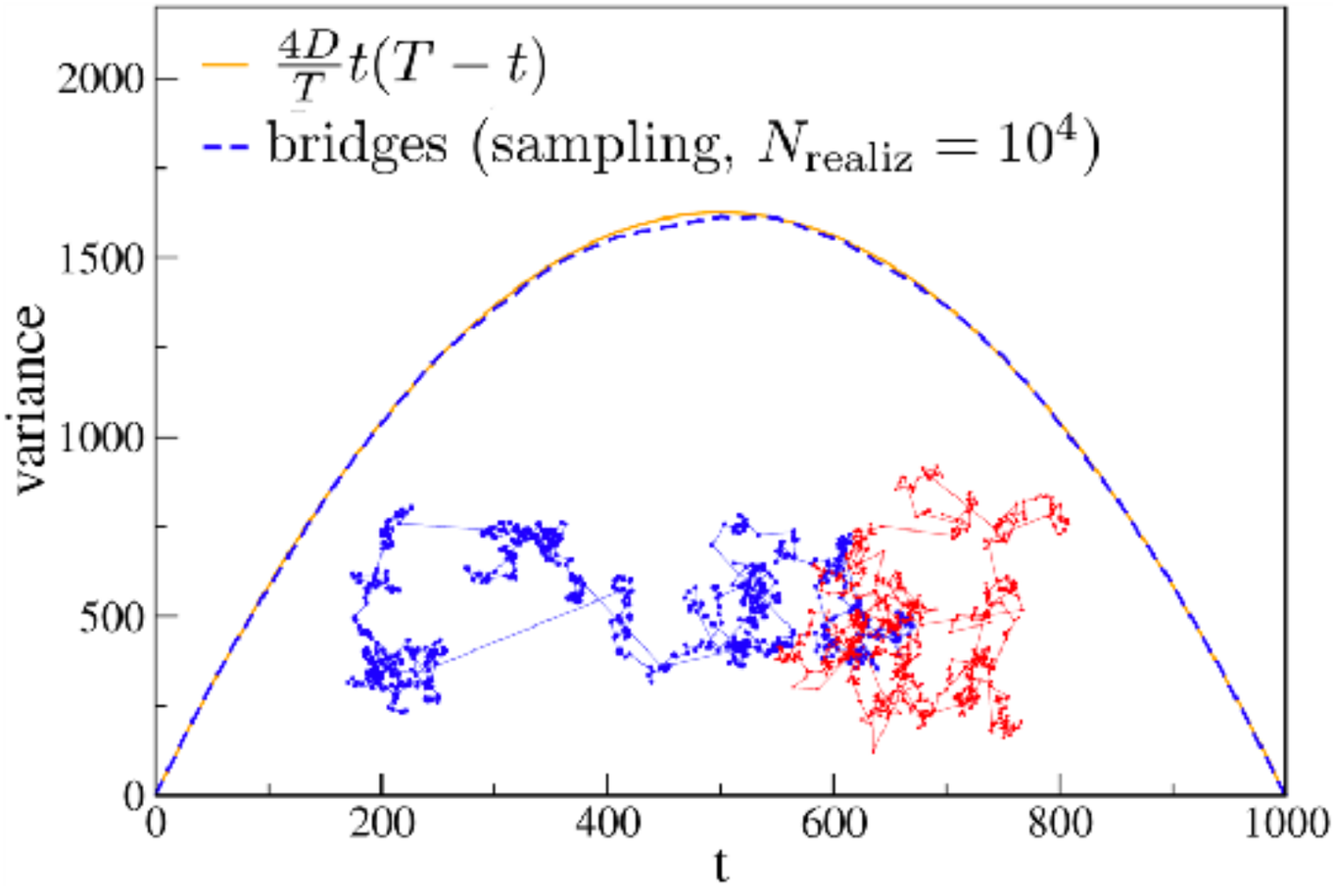


$$\rho^B(\mathbf{r}_t = \mathbf{r}) =$$

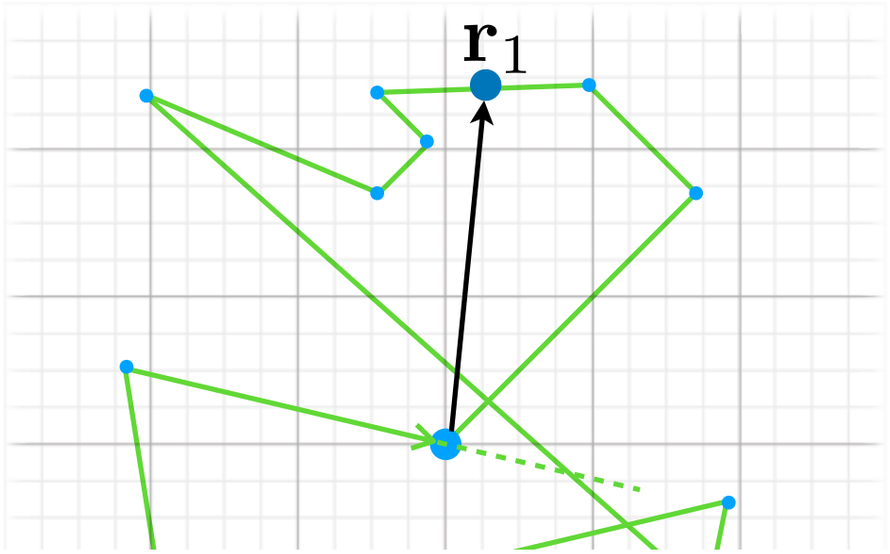
For a “normal” random walk ρ is a Gaussian

ρ^B is Gaussian too

$$\sigma_t^2 = \frac{\sigma_{0\to t}^2 \sigma_{0\to T-t}^2}{\sigma_{0\to T}^2} \quad \Rightarrow \quad \langle \mathbf{r}^2(t) \rangle \propto \frac{t(T-t)}{T}$$



Theory of Lévy-walk bridges



Memory effects - at every point there are particles flying for a different time and thus their next turning event depends on pre-history.

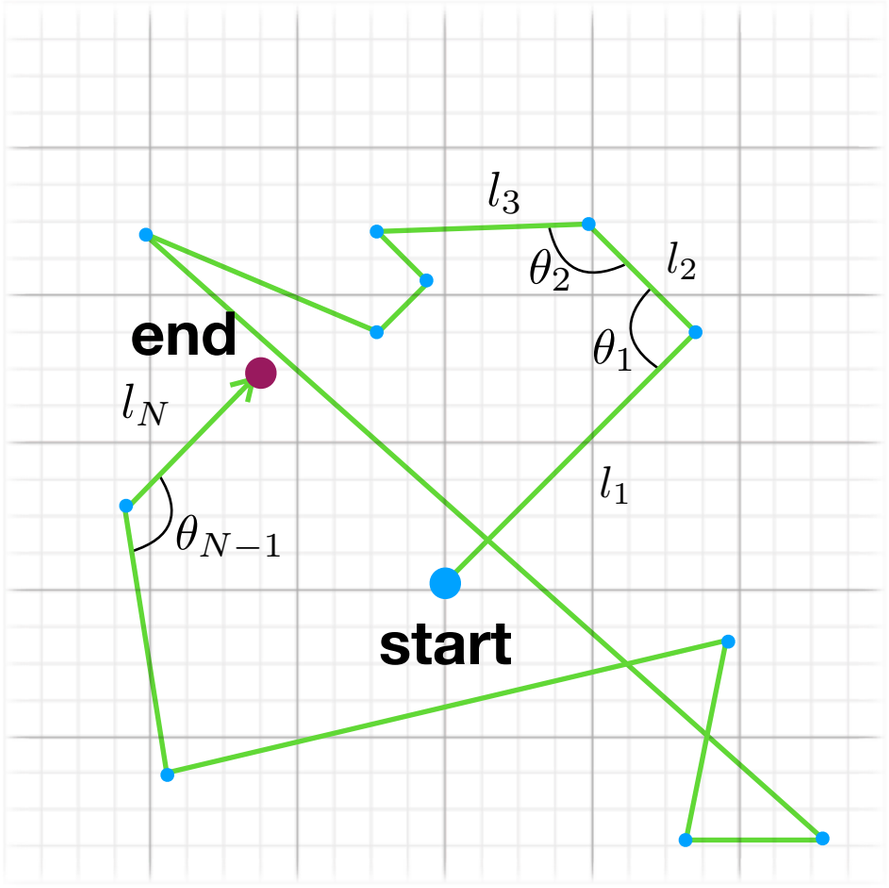
$$P^{LW}(\mathbf{r}_1, t_1) = \left. \frac{P(\mathbf{r}_1, t_1, \mathbf{r}_2, t_2 | \mathbf{r}_0 = 0)}{P(\mathbf{r} = 0, t = T | \mathbf{r}_0 = 0)} \right|_{\mathbf{r}_2 = 0, t_2 = T}$$

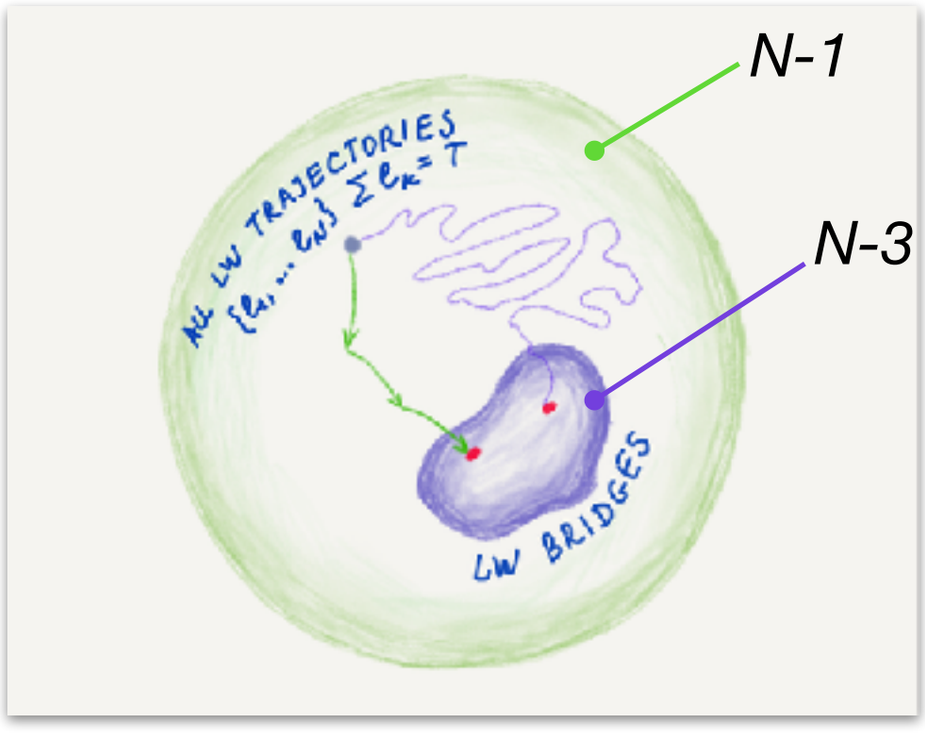
$$P(\mathbf{k}, \mathbf{k}_1, s, s_1) = \frac{1}{1 - \int \psi(s_1 + i(\mathbf{k}+\mathbf{k}_1)\mathbf{v}) h(\mathbf{v}) d\mathbf{v}} \left(\frac{\int \Psi(s + i\mathbf{k}\mathbf{v}) h(\mathbf{v}) d\mathbf{v} \int \psi(s_1 + i(\mathbf{k}+\mathbf{k}_1)\mathbf{v}, s + i\mathbf{k}\mathbf{v}) h(\mathbf{v}) d\mathbf{v}}{1 - \int \psi(s + i\mathbf{k}\mathbf{v}) h(\mathbf{v}) d\mathbf{v}} + \int \Psi(s_1 + i(\mathbf{k}+\mathbf{k}_1)\mathbf{v}, s + i\mathbf{k}\mathbf{v}) h(\mathbf{v}) d\mathbf{v} \right)$$

$$\langle \mathbf{r}^2(s, s_1) \rangle = - \Delta_{\mathbf{k}} \int d\mathbf{k}_1 P(\mathbf{k}, \mathbf{k}_1, s, s_1) \Big|_{\mathbf{k}=0}$$

How to build Lévy-walk bridges?

Construction of Lévy-walk bridges





Recipe 📖

1) Generate a set of Lévy distributed segments $\{l_k\}$

$$\sum_k l_k = v_0 T \qquad \forall l_k < v_0 T/2$$

(the last segment can be truncated)

2) Generate a set of uniformly distributed angles $\{\theta_k\}$

Goal: connect start and end points

"Planar or Random Linkage Problem"

3) Minimize start to end distance:

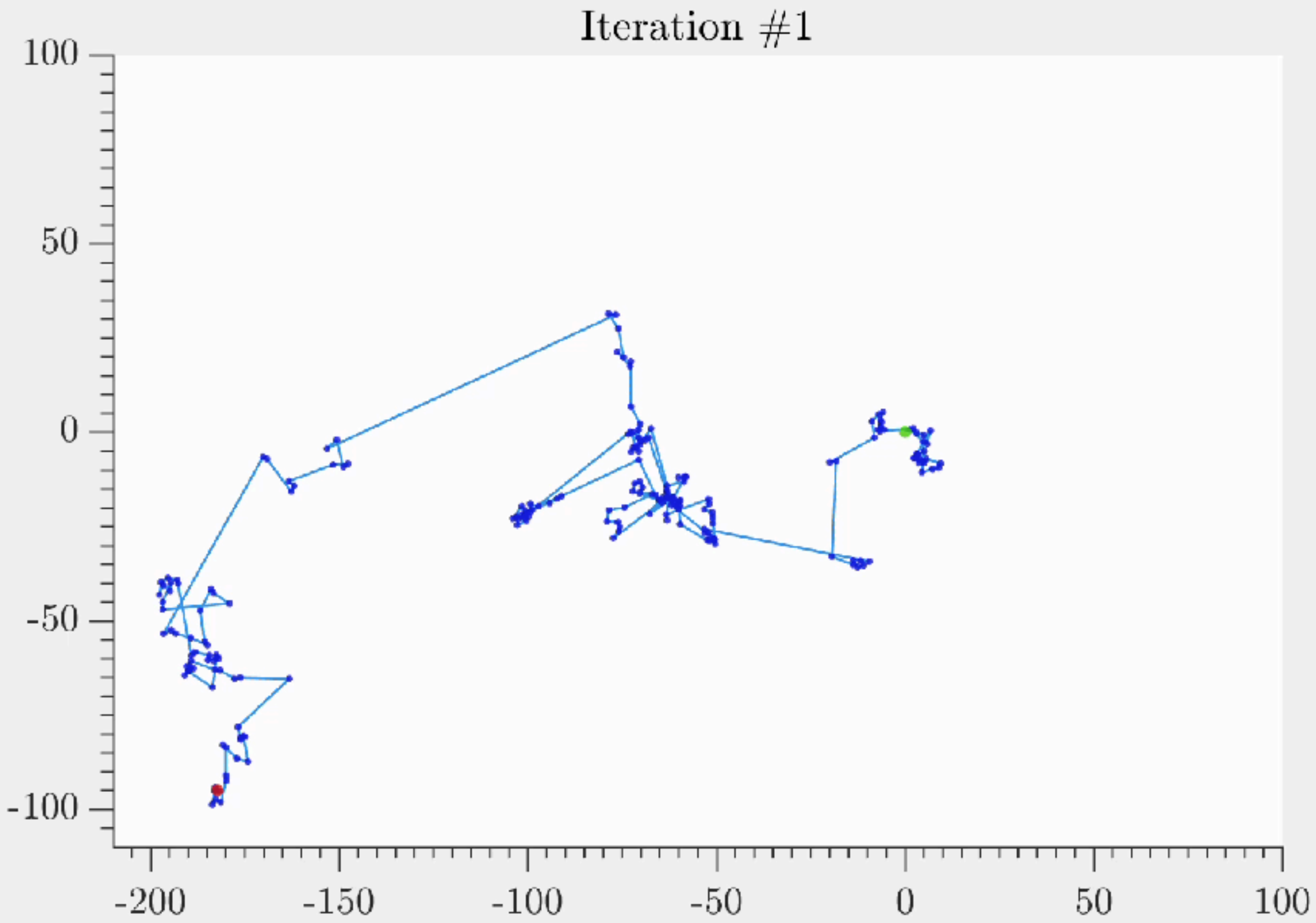
$$\Delta = \left(\sum_k l_k \cos\theta_k\right)^2 + \left(\sum_k l_k \sin\theta_k\right)^2$$

Levenberg–Marquardt algorithm

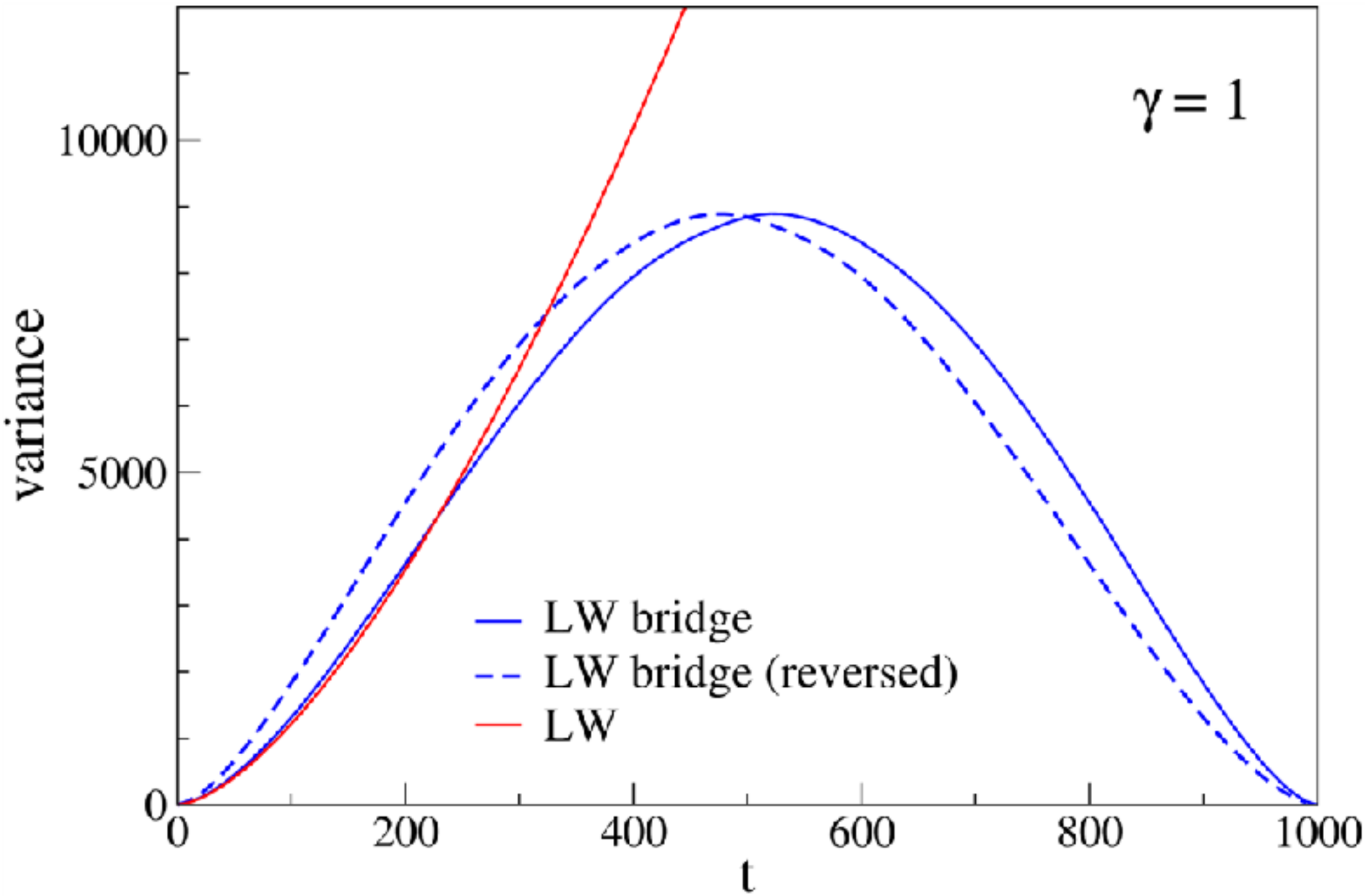
Monte Carlo

or actually both!

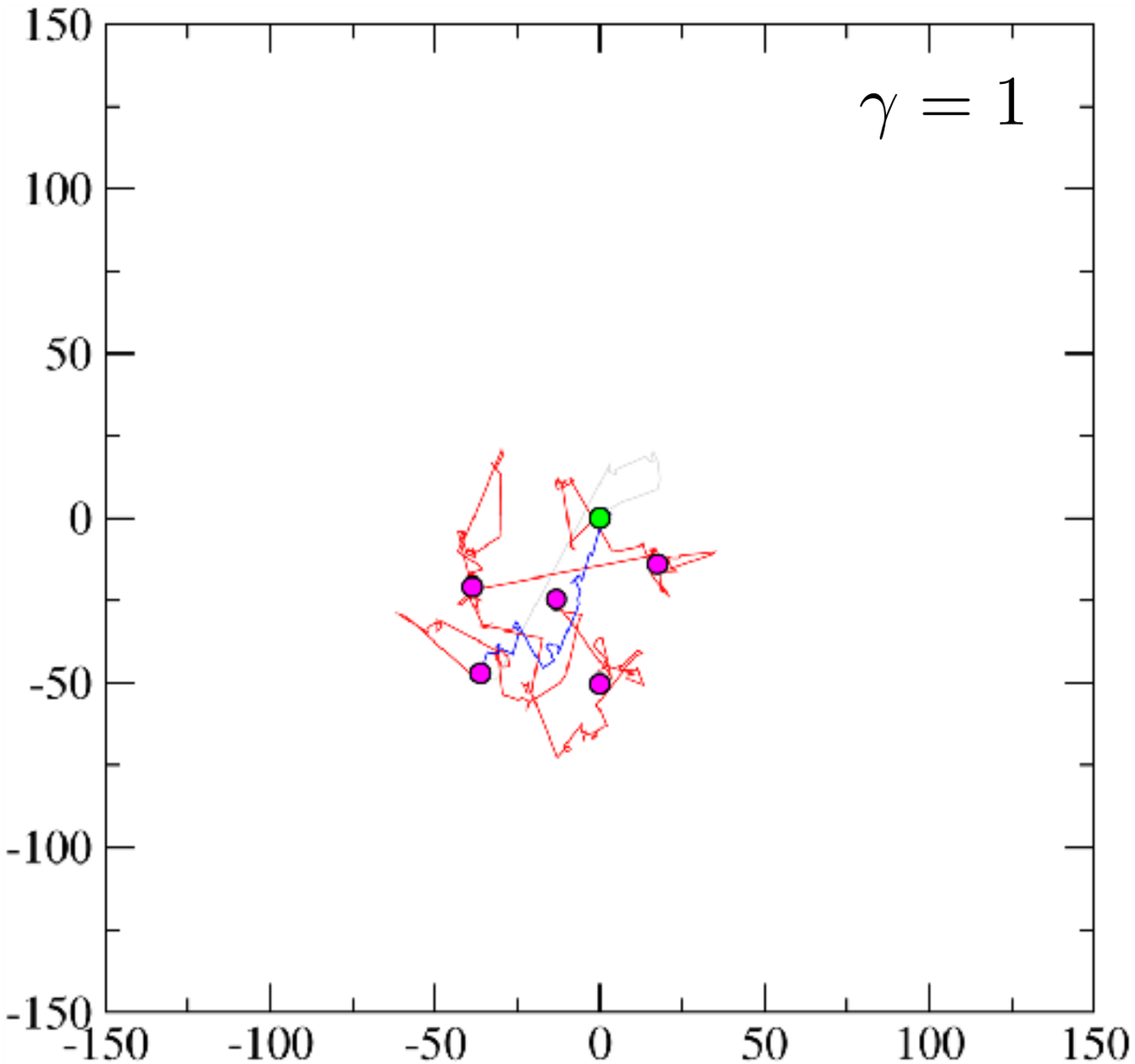
Construction of Lévy-walk bridges



LW bridges: variance, time reversal asymmetry



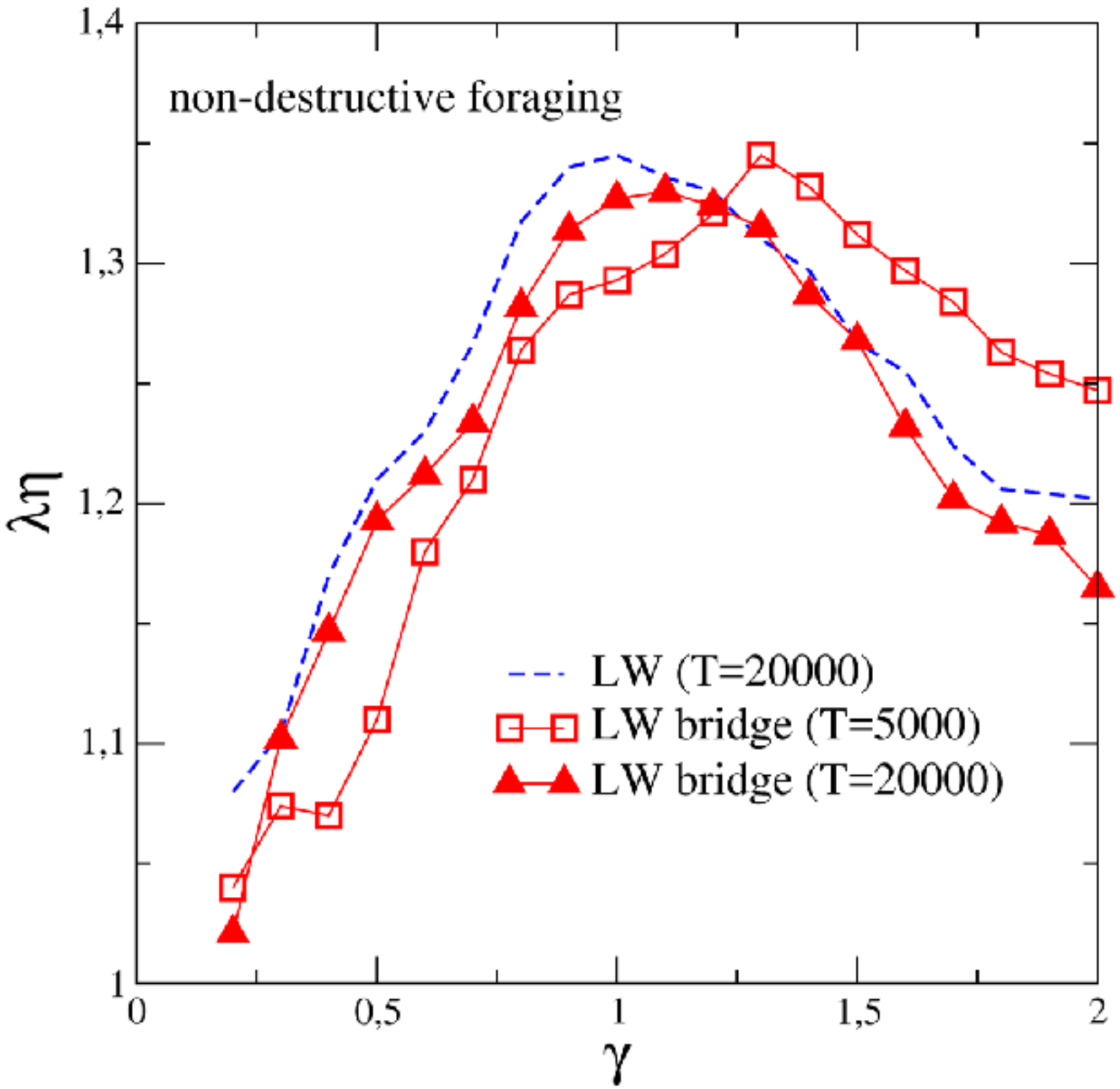
Lévy-walk bridges and search



"Foraging" strategy

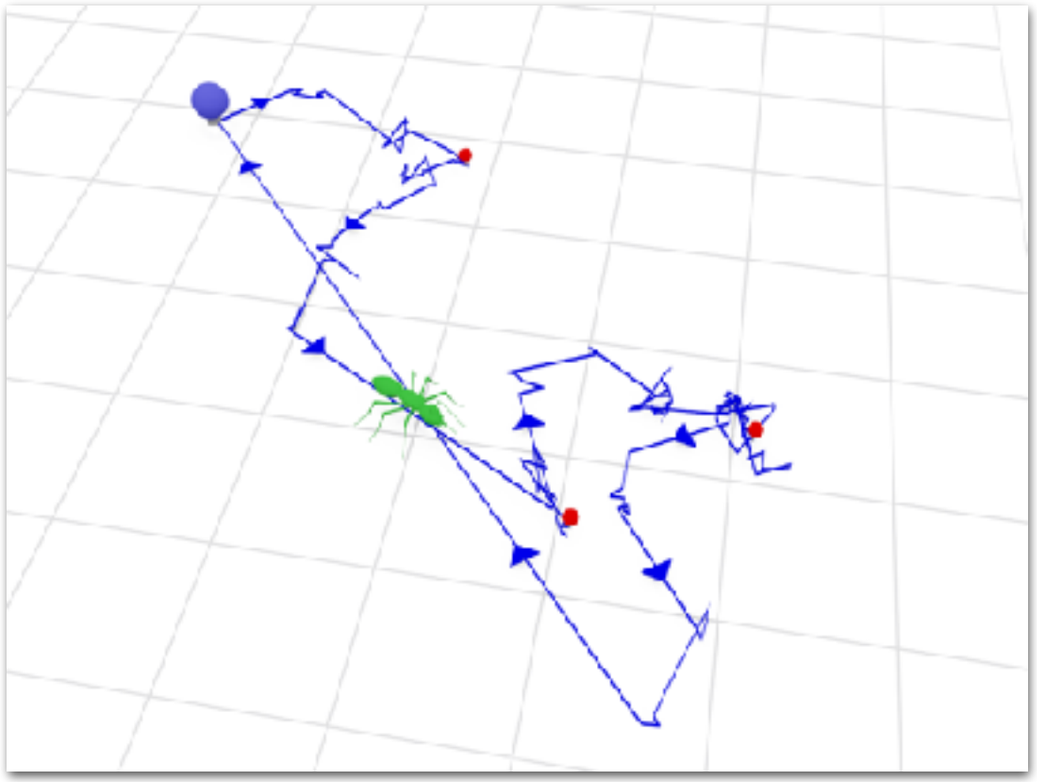
1) Generate LW bridge
2) If target detected:
   check if there is enough time
3) Go and collect target
4) Generate new bridge

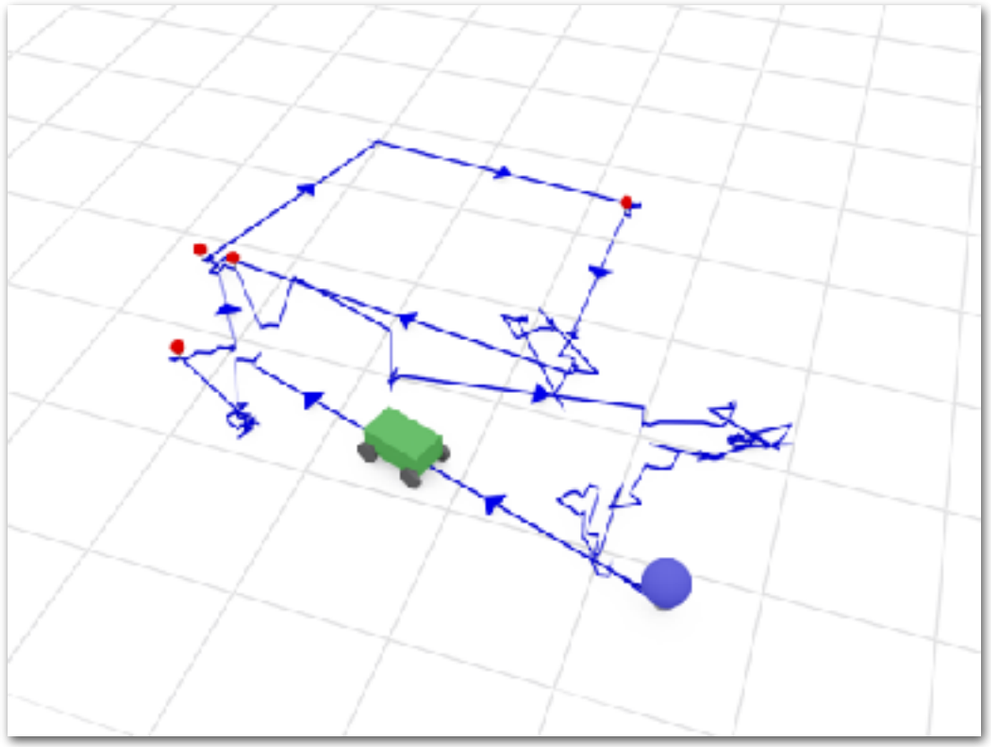
First results for Lévy-walk bridge search



Summary

- Lévy-walk bridges: trajectories complying with home range concept and fulfilling the energetic constraint;
- Analytical description of Lévy-walk bridges is possible via two-point density;
- An efficient sampling by minimisation algorithm;

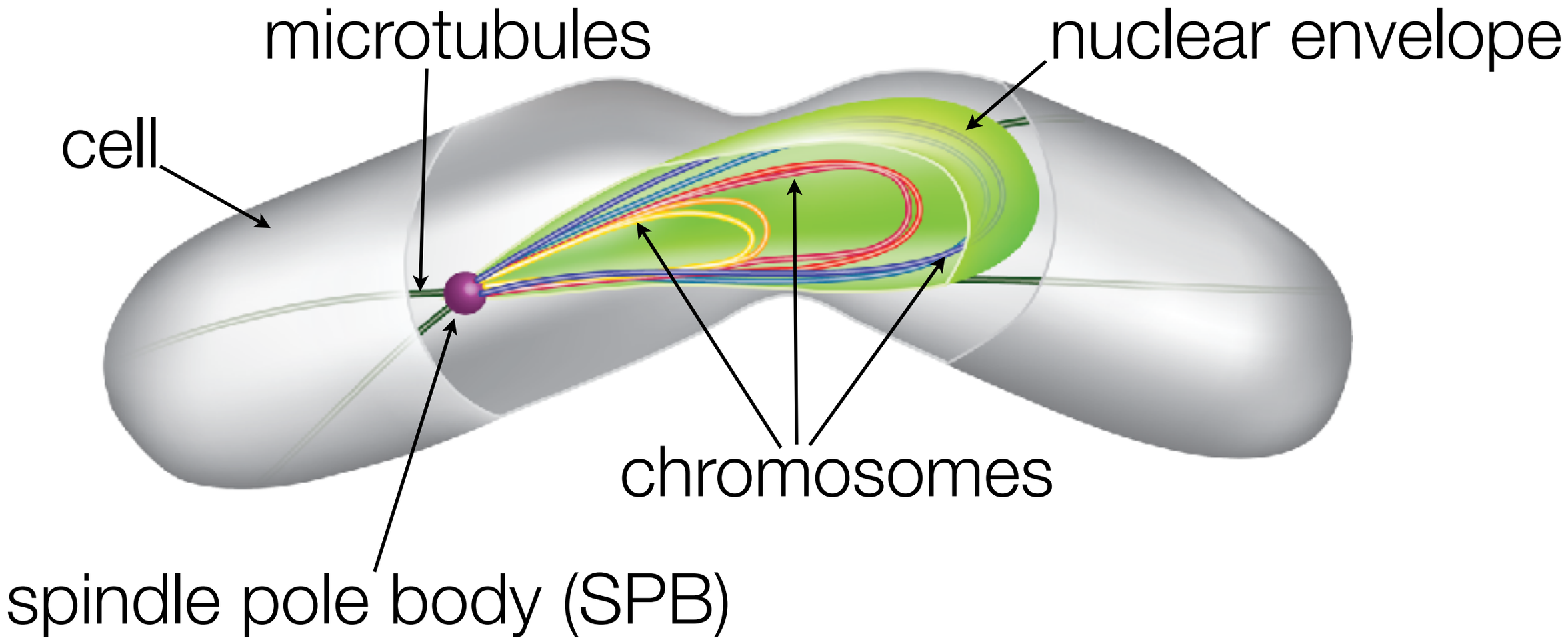




& Outlook

- Theory of Lévy-walk bridges;
- Theory of doing search by Lévy-walk bridges;
- Explore further possible Lévy-walk bridge setups and search strategies…

Other “bridges” in biology



Brownian bridges in external field, Fermi statistics, ASEP with reflecting boundaries, Bethe ansatz…