Rigidity results for the Robin $p-$Laplacian

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The Poisson problem with Robin boundary conditions

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and Lipschitz set. We consider the following problem for the $p-$Laplace operator:

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + \beta |u|^{p-2}u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (P)$$

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There exists a unique, positive, weak solution to (P).

The symmetrized problem

Let $\Omega^\sharp$ be the ball satisfying $|\Omega| = |\Omega^\sharp|$. We consider

$$\begin{cases} -\Delta_p v = f^\sharp & \text{in } \Omega^\sharp \\ |\nabla v|^{p-2}\dfrac{\partial v}{\partial \nu} + \beta |v|^{p-2} v = 0 & \text{on } \partial\Omega^\sharp, \end{cases} \qquad (P^*)$$

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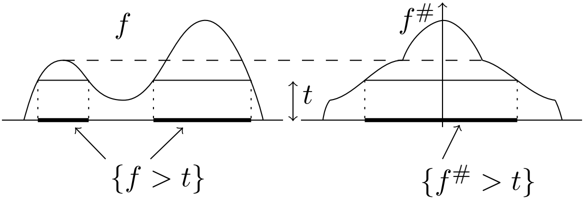

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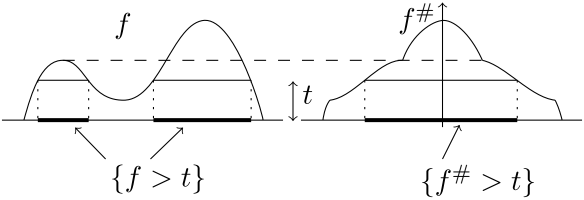

Can we compare the solutions u and v? Which is the right way to compare them?

The Dirichlet case

G. Talenti- 1976, Ann. Scuola Sup. Pisa (linear case)
G. Talenti- 1979, Ann. Mat. Pura Appl. (nonlinear case)

$$\begin{cases} -\Delta_p u_D = f & \text{in } \Omega \\ u_D = 0 & \text{on } \partial\Omega. \end{cases} \qquad \begin{cases} -\Delta_p v_D = f^\sharp & \text{in } \Omega^\sharp \\ v_D = 0 & \text{on } \partial\Omega^\sharp. \end{cases}$$

$$u_D^\sharp(x) \le v_D(x), \ \ \forall x \in \Omega^\sharp$$

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Question

- Does $u^\sharp \le v$ hold also in the Robin case?
- Does a weaker result hold?

Applications

- The ball maximizes every L^k norm of the solutions:

$$u_D^\sharp(x) \leq v_D(x) \Longrightarrow \|u_D\|_{L^k(\Omega)} = \|u_D^\sharp\|_{L^k(\Omega^\sharp)} \leq \|v_D\|_{L^k(\Omega^\sharp)}$$

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- This gives sharp a priori estimates on the L^k-norm of the solution to (P)
- when $f \equiv 1$ we recover the Saint-Venaint inequality

$$T(\Omega) = \int_\Omega u_D\,dx \leq \int_{\Omega^\sharp} v_D\,dx = T(\Omega^\sharp)$$

Applications

Another proof of the Faber-Krahn inequality, for all p, n:

$$\Lambda_p(\Omega) = \inf_{v \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla v|^p}{\int_\Omega |v|^p}$$

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$$w^\sharp \leq z \Longrightarrow \int_{\Omega^\sharp} (w^\sharp)^{p-1} z \leq \left(\int_{\Omega^\sharp} (w^\sharp)^p \right)^{\frac{p-1}{p}} \left(\int_{\Omega^\sharp} z^p \right)^{\frac{1}{p}} \leq \int_{\Omega^\sharp} z^p$$

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Remark

It is sufficient $\|w\|_{L^p(\Omega)} \leq \|z\|_{L^p(\Omega^\sharp)}$ to prove the Faber-Khran

The linear case

A. Alvino-C. Nitsch-C. Trombetti, Comm. Pure Appl. Math. 2022

Let u and v the solution respectively to

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta v = f^\sharp & \text{in } \Omega^\sharp, \\ \dfrac{\partial v}{\partial \nu} + \beta v = 0 & \text{on } \partial\Omega^\sharp, \end{cases}$$

If $f \in L^2(\Omega)$, $f > 0$ then

$$\|u\|_{L^{k,1}(\Omega)} \leq \|v\|_{L^{k,1}(\Omega^\sharp)} \qquad \forall 0 < k \leq \frac{n}{2n-2}$$

$$\|u\|_{L^{2k,2}(\Omega)} \leq \|v\|_{L^{2k,2}(\Omega^\sharp)} \qquad \forall 0 < k \leq \frac{n}{3n-4}$$

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Moreover, if $f \equiv 1$

$$u^\sharp(x) \leq v(x), \qquad n = 2,$$

$$\|u\|_{L^k(\Omega)} \leq \|v\|_{L^k(\Omega)}, \qquad n \geq 2, \quad k = 1, 2.$$

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V. Amato-A. Gentile- A. L. M., Ann. Mat. Pura Appl. 2022

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Definition

$$\|u\|_{L^{p,q}} = \begin{cases} p^{\frac{1}{q}} \left(\int_0^\infty t^q \mu(t)^{\frac{q}{p}} \, \frac{dt}{t} \right)^{\frac{1}{q}} & 0 < q < \infty \\ \sup\limits_{t>0} \left(t^p \mu(t) \right) & q = \infty \end{cases}$$

where $\mu(t) = |\{u > t\}|$, is the *distibution function* of u.

Remark

If $p = q$, we recover the classical L^p norm, as a consequence of the Cavalieri principle:

$$\int_\Omega |u|^p = p \int_0^{+\infty} t^{p-1} \mu(t)$$

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- The ball maximizes the $p-$Torsion in any dimension;
- The alternative proof of the Faber-Krahn inequality holds if $p \geq n$.

Characterize the equality case

Can we obtain some information if the equality holds in one of the previous estimates?

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In the Dirichlet case

A. Alvino-P. L. Lions-G. Trombetti (Proc. Roy. Soc. Edinburgh Sect. A, 1986)

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$$\begin{cases} -\Delta u_D = f & \text{in } \Omega, \\ u_D = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta v_D = f^\sharp & \text{in } \Omega^\sharp, \\ v_D = 0 & \text{on } \partial\Omega^\sharp, \end{cases}$$

If $u_D^\sharp(x) = v_D(x)$ for almost every $x \in \Omega^\sharp$, then

$$\Omega = \Omega^\sharp + x_0, \quad u_D(\cdot) = u_D^\sharp(\cdot + x_0), \quad f(\cdot) = f^\sharp(\cdot + x_0)$$

In the Robin Case

- Linear case: A. L. M., G. Paoli, to appear on J. Geom. Anal.
  we study the case $n = 2$, $f = 1$, for which a pointwise comparison holds;
- Nonlinear case: A. L. M., G. Paoli, preprint
  we treat the general $p-$Laplace case.

V. Amato-A. Gentile-A. L. M., Ann. Mat. Pura Appl. 2022

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A. L. M.-G. Paoli, preprint

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$$\|u\|_{L^{pk,p}(\Omega)} = \|v\|_{L^{pk,p}(\Omega^\sharp)}, \quad \text{for some } k \in \left]0, \frac{n(p-1)}{(n-2)p+n}\right]$$

then, there exists $x_0 \in \mathbb{R}^n$ such that

$$\Omega = \Omega^\sharp + x_0, \qquad u(\cdot + x_0) = v(\cdot), \qquad f(\cdot + x_0) = f^\sharp(\cdot).$$

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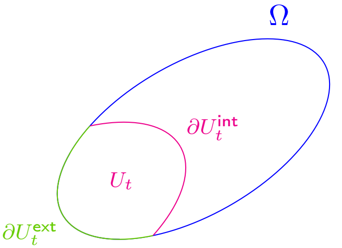
sketch of the proof for $n = 2, p = 2, f \equiv 1$

Some Notation



$$U_t = \{x \in \Omega : u(x) > t\},$$
$$\partial U_t^{int} = \partial U_t \cap \Omega,$$
$$\partial U_t^{ext} = \partial U_t \cap \partial\Omega.$$

- $\mu(t) = |\{x \in \Omega \ : \ u(x) > t\}|,$
- $\phi(t) = |\{x \in \Omega^\sharp \ : \ v(x) > t|\}, \quad V_t = \left\{x \in \Omega^\sharp : v(x) > t\right\}.$

Some properties

- Let us denote by $u_m = \min_\Omega u$ and $v_m = \min_{\Omega^\sharp} v$ that are achieved on the boundary. Since $\beta > 0$, we have that $u_m, v_m > 0$.

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- v is radial and decreasing
  V_t is a ball concentric to $\Omega^\sharp$ and strictly contained in it
- It holds that $v_m \geq u_m$. Indeed:

$$\begin{aligned} v_m \mathsf{P}(\Omega^\sharp) &= \int_{\partial\Omega^\sharp} v(x)\, d\mathcal{H}^1 = \frac{1}{\beta}\int_{\Omega^\sharp} dx = \frac{1}{\beta}\int_\Omega dx \\ &= \int_{\partial\Omega} u(x)\, d\mathcal{H}^1 \geq u_m \mathsf{P}(\Omega) \geq u_m \mathsf{P}(\Omega^\sharp). \end{aligned}$$

M.-Paoli, preprint

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Talenti comparison for Robin: case $n = 2$ and $f \equiv 1$

$$\|u(x)\|_{L^{2k,2}(\Omega)} \leq \|v(x)\|_{L^{2k,2}(\Omega^\sharp)}. \tag{1}$$

Lemma: Talenti comparison [ANT]

Recalling $\mu(t) = |\{u > t\}|, \qquad \phi(t) = |\{v > t\}|$, it holds

$$4\pi \leq \left(-\mu'(t) + \frac{1}{\beta} \int_{\partial U_t^{ext}} \frac{1}{u} \, d\mathcal{H}^1 \right) \tag{2}$$

and

$$4\pi = \left(-\phi'(t) + \frac{1}{\beta} \int_{\partial V_t \cap \partial \Omega^\sharp} \frac{1}{v} \, d\mathcal{H}^1 \right). \tag{3}$$

- From (2) and (3) one can prove (1).
- These (in)equalities are the key to the rigidity result.

Lemma: idea of the proof

The key points in proving this inequality (2) are:

- the isoperimetric inequality applied on the super level set of u and v resp. U_t and V_t;
- the Hölder inequality applied on the function g

$$g(x) = \begin{cases} |\nabla u| & \text{if } x \in \partial U_t^{int}, \\ \beta u & \text{if } x \in \partial U_t^{ext}. \end{cases}$$

that satisfies

$$\int_{\partial\{u>t\}} g(x)\, d\mathcal{H}^1 = \int_{\{u>t\}} dx = \mu(t).$$

Sketch of the proof of the Rigidity result

- From the hypothesis

$$\|u(x)\|_{L^{2k,2}(\Omega)} = \|v(x)\|_{L^{2k,2}(\Omega^\sharp)}.$$

one can prove the equality in the Talenti comparison, i.e.

$$4\pi = \left(-\mu'(t) + \frac{1}{\beta}\int_{\partial U_t^{ext}} \frac{1}{u}\, d\mathcal{H}^1\right),$$

for almost every $t \in [0, u_M]$, where $u_M = \max_{\Omega} u$.

Sketch of the proof of the Rigidity result

Step 1: We prove that every super level set is a ball.

- Equality in the Talenti comparison implies that

$$2\sqrt{\pi}\mu(t)^{\frac{1}{2}} = P(U_t), \quad \text{for a. e. } t$$

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  that means that a.e level set is a ball.

- For all $t \in [u_m, u_M)$, there exists $\{t_k\}$ s.t.
  1. $t_k \to t$;
  2. $t_k > t_{k+1}$;
  3. $\{u > t_k\}$ is a ball for all k.

  Then, since $\{u > t\} = \cup_k \{u > t_k\}$, we have that $\{u > t\}$ is a ball for all t.
  In particular, Ω is a ball!

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$$w^\sharp(x) = z(x), \quad \text{in } \Omega^\sharp.$$

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Step 2**:** We prove that the level sets are concentric balls.

In order to do that, we show that $u - u_m$ solves

$$\begin{cases} -\Delta_p(u-u_m) = f & \text{in } \Omega, \\ u - u_m = 0 & \text{on } \partial\Omega, \end{cases}$$

and if one compares $w = u - u_m$ with the solution to

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If one can prove a rigidity result for the Dirichlet $p-$Laplacian, we achieve **Step** 2.

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Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and Lipschitz set. Let $f \in L^{p'}(\Omega)$ be a positive function and let w and z be weak solutions respectively to

$$\begin{cases} -\Delta_p w = f & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta_p z = f^\sharp & \text{in } \Omega^\sharp \\ z = 0 & \text{on } \partial\Omega^\sharp. \end{cases}$$

If $w^\sharp(x) = z(x)$, for all $x \in \Omega^\sharp$, then there exists $x_0 \in \mathbb{R}^n$ such that

$$\Omega = \Omega^\sharp + x_0, \qquad w(\cdot + x_0) = z(\cdot), \qquad f(\cdot + x_0) = f^\sharp(\cdot).$$

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This result seems new in the literature!

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Main differences with ALT

- Solutions to $p-$Laplace equation are not, in general, continuous or $C^{1,\alpha}(\bar{\Omega})$;
- The proof of the rigidity results by A-L-T strongly relies on the fact that they are dealing with a linear operator and the high regularity of the solutions that can be lost for a generic p.
- To overcome this regularity issue, we show that w satisfies the Brothers-Ziemer result:

Brothers, Ziemer, J. Reine Angew. Math.

Let $w \in W_0^{1,p}(\Omega)$, let

$$w_M := \begin{cases} \|w\|_\infty & \text{if } w \in L^\infty(\Omega) \\ +\infty & \text{otherwise.} \end{cases} .$$

If

$$\int_\Omega |\nabla w|^p = \int_\Omega |\nabla w^\sharp|^p,$$

and

$$\left| \left\{ \, |\nabla w^\sharp| = 0 \, \right\} \cap \left\{ \, 0 < w^\sharp < w_M \, \right\} \right| = 0$$

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then, $w = w^\sharp$ up to a translation.

So $w = u - u_m$ is radial and decreasing, and $u = w + u_m$ too!

Open Problems

- Generalize the rigidity results in the anisotropic setting or to the mixed boundary condition setting, for which Talenti-type results are proved in

  1. R. Sannipoli, *Nonlinear Anal.* (2022),
  2. A. Alvino, C. Chiacchio, C. Nitsch, C. Trombetti, *J. Math Pures Appl.*, (2021).

- Is it true that the ball maximizes every L^k norm of the Torsion function (with Robin boundary conditions) in any dimension?
  A first evidence is contained in "R. Sannipoli *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*" where it is proved that the ball is a critical shape for every L^k norm.

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Thank you for your attention!