Fractional Pohozaev identities and applications

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Some classical results

Let Ω be an open set of $\mathbb{R}^n$, $n \geq 2$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a given real valued function. Consider the PDE

$$-\Delta u = f(x, u) \quad \text{in} \quad \Omega \quad \& \quad u = 0 \quad \text{on} \quad \partial\Omega \tag{1}$$

- Effect of Ω on existence/nonexistence of solutions?
- It is well-known that $f(x, t) = g(x) \in L^2(\Omega)$ existence follows in any bounded Ω from Riez representation theorem.

- $f(x,t) = |t|^{p-2}t$ and Ω bounded, compactness argument gives existence of solutions as long as $1 < p < 2n/(n-2)$

$$\min_{\mathcal{H}_0^1(\Omega)} \left\{ \frac{1}{2} \int_\Omega |\nabla u|^2 \,\mathrm{dx} + \frac{1}{p} \int_\Omega |u|^p \, dx \right\}$$

- For $p \geq 2n/(n-2)$ it is well known that minimization argument fails because $\mathcal{H}_0^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$ is <u>not</u> compact (for $p > \frac{2n}{n-2}$, the functional above is not even well-defined).

- What happens if $p \geq \frac{2n}{n-2}$? For instance, if $f(x,t) = t^{\frac{n+2}{n-2}}$? This types of nonlinearity are related to the Yamabe problem in differential geometry: Let $(\mathcal{M}, g)$ be a closed smooth Riemannian manifold. Is there any positive and smooth function f on $\mathcal{M}$ such that the Riemannian metric fg has constant scalar curvature ? This leads to solve equations of the types

$$-\Delta u = u^{\frac{n+2}{n-2}} \quad \text{in} \quad \mathcal{M}.$$

- Because of lack of compactness, one cannot expect a general existence result for equations with nonlinearity $f(x,t) = |t|^{p-2}t$.
- Pohozaev $\rightarrow\ \exists\,\Omega/\ -\Delta u = |u|^{p-2}u$ in Ω, $u = 0$ on $\partial\Omega$ has no solution.
- Assume $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ solves

$$-\Delta u = f(u) \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega$$

- Multiply the equation with $x \cdot \nabla u$ and integrate over Ω, yields

$$\boxed{(2-n)\int_\Omega uf(u)\,\mathrm{dx} + 2n\int_\Omega F(u)\,\mathrm{dx} = \int_{\partial\Omega}(\frac{\partial u}{\partial \nu})^2(x\cdot\nu)\mathrm{dx}} \qquad (2)$$

with $F(t) = \int_0^t f(s)ds$. The identity above is called Pohozaev identity.

- Apply Pohozaev with $f(t) = |t|^{p-2}t$, gives

$$\boxed{u \equiv 0 \quad \text{if} \quad p \geq \frac{2n}{n-2} \quad \text{and} \quad x \cdot \nu \geq 0 \text{ on } \partial\Omega}.$$

- Nonexistence holds in starshaped domains for any nonlinearity f satisfying

$$\frac{2-n}{2n} u f(u) + \int_0^u f(s) ds \geq 0$$

- Nonexistence in non necessarity starshaped domains can be obtained via

$$\boxed{\left(\frac{1}{p} - \frac{1}{2}\right) \int_\Omega |u|^p divY \, \mathrm{dx} + \int_\Omega dY(\nabla u) \cdot \nabla u \, \mathrm{dx} = \int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu}\right)^2 (Y \cdot \nu) \mathrm{dx}} \quad (3)$$

  for all $Y \in C^1(\overline{\Omega}, \mathbb{R}^n)$.

- Further applications: Unique continuation, simplicity results ...

Fractional Laplacian

- Fractional laplacian via Fourier transform

$$\mathcal{F}((-\Delta)^s u)(x) = |x|^{2s} \mathcal{F} u(x)$$

- Via singular integral

$$(-\Delta)^s u(x) := c_{n,s} p.v \int_{\mathbb{R}^n} \frac{u(x) - u(x+y)}{|y|^{n+2s}} \mathrm{dy}.$$

Fractional Laplacian

- In weak form,

$$\left\langle (-\Delta)^s u, w \right\rangle = \frac{c_{n,s}}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x)-u(y))(w(x)-w(y))}{|x-y|^{n+2s}} \mathrm{dxdy}$$

  it appears as the *first variation* of the nonlocal Dirichlet energy

$$[u]^2_{H^s(\mathbb{R}^n)} := \frac{c_{n,s}}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x)-u(y))(w(x)-w(y))}{|x-y|^{n+2s}} \mathrm{dxdy}$$

- Motivation: Fractional powers of Laplacian appear as *infinitesimal generator of stable Lévy processes*.

Pohozaev identity for the fractional Laplacian

- Ros-Oton and Serra '14: let Ω bounded and $\partial\Omega \in C^{1,1}$. Any bounded weak solution of

$$(-\Delta)^s u = f(u) \text{ in } \Omega, \; u = 0 \text{ in } \mathbb{R}^n \setminus \Omega.$$

satisfies

$$\boxed{(2s-n)\int_\Omega u f(u)\,\mathrm{dx} + 2n\int_\Omega F(u)\,\mathrm{dx} = \Gamma^2(1+s)\int_{\partial\Omega} \Big(\frac{u}{d^s}\Big)^2 x\cdot\nu\,\mathrm{dx}} \quad (4)$$

- Here $d = dist(\cdot, \partial\Omega)$ is the distance function to the boundary and Γ is the usual gamma function.
- The function u/d^s represents the nonlocal counterpart of the classical normal derivative $\frac{\partial u}{\partial \nu}$.
- Apply Pohozaev with $f(u) = |u|^{p-2}u$ gives

$$\boxed{\Big[\frac{2n}{p} - (n-2s)\Big]\int_\Omega |u|^p\,\mathrm{dx} = \int_{\partial\Omega} (u/d^s)^2 (x\cdot\nu)\mathrm{dx}} \quad (5)$$

- Conclusion (Ros-Oton & Serra '14): Ω starshaped and $\partial\Omega \in C^{1,1}$. Then the equation

$$(-\Delta)^s u = |u|^{p-2}u \text{ in } \Omega, \;\; u = 0 \text{ on } \mathbb{R}^n \setminus \Omega$$

  does not admits a nontrivial solution in the following two cases.
  - $p > \frac{2n}{n-2s}$ or,
  - $p = \frac{2n}{n-2s}$ and $u \geq 0$ $(or \leq 0)$ in Ω.

Further applications

(1) **Unique continuation from the boundary**. If $p \neq \frac{2n}{n-2s}$, then

$$\left[\frac{u}{d^s}\Big|_{\partial\Omega} = 0\right] \Longrightarrow \left[u \equiv 0 \quad \text{in} \quad \mathbb{R}^n\right].$$

(2) **Symmetry breaking results**, conjecture by *Bañuelos & Kulczycki*: second fractional eigenfunctions of balls cannot be radial

- 1d case: Kwasnicki '12
- $s = 1/2$ & $n \leq 9$ or $n \leq 2$ & $s \in (0,1)$: Dyda et al '17
- 3d cases: Fereirra '19
- General case: Fall et all '20
- Annulus case: D. and Sven '22

(3) **simplicity results**: If u solves

$$u \text{ radial}, \ (-\Delta)^s u = \lambda u \ \text{ in } \ B, \ \ u = 0 \ \text{ in } \ B^c, \ \int_B u^2 \, \mathrm{dx} = 1$$

then u is unique. If not, one must has $w/d^s = 0$ on ∂B for some w solving the equation. A contradiction occurs by Pohozaev.

- Proof of Pohozaev identity: follows from the integration by parts formula (Ros-Oton & Serra)

$$\int_\Omega x \cdot \nabla u (-\Delta)^s u \, \mathrm{dx} = \frac{2s-n}{2} \int_\Omega u (-\Delta)^s u \, \mathrm{dx} - \frac{\Gamma^2(1+s)}{2} \int_{\partial\Omega} (\frac{u}{d^s})^2 x \cdot \nu \, \mathrm{dx}$$

under reasonable assumption on u.The proof of this is very difficult and lengthy. We shall see an alternative proof later .

Generalized fractional Pohozaev identity

- Recall that if u is a classical solution to the equation

$$-\Delta u = |u|^{p-2}u \; \Omega, \;\; u = 0 \text{ on } \partial\Omega,$$

then

$$\boxed{(\frac{1}{p} - \frac{1}{2}) \int_{\Omega} |u|^p divY \, \mathrm{dx} + \int_{\Omega} dY(\nabla u) \cdot \nabla u \, \mathrm{dx} = \int_{\partial\Omega} (\frac{\partial u}{\partial \nu})^2 (Y \cdot \nu) \mathrm{dx}} \quad (6)$$

for all $Y \in C^1(\overline{\Omega}, \mathbb{R}^n)$.

- What is the fractional version of (6) ? We give the answer to this question below.

First we prove the following more general identity

Theorem (D-Fall-Weth, '21)

For all $u : \mathbb{R}^n \to \mathbb{R}$ such that $(-\Delta)^s u \in L^\infty(\Omega) \cap C^\alpha_{loc}(\Omega)$ with $u = 0$ in $\mathbb{R}^n \setminus \Omega$, we have

$$\boxed{2\int_\Omega \nabla u \cdot Y(-\Delta)^s u \, dx = -\Gamma^2(1+s)\int_{\partial\Omega}\Big(\frac{u}{d^s}\Big)^2 Y \cdot \nu \, dx - \mathcal{E}_Y(u,u),} \tag{7}$$

for any vector field $Y \in C^{0,1}(\mathbb{R}^n, \mathbb{R}^n)$.

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for any vector field $Y \in C^{0,1}(\mathbb{R}^n, \mathbb{R}^n)$. The remainder term $\mathcal{E}_Y$ is given by

$$\mathcal{E}_Y(u,u) = \iint_{\mathbb{R}^n\times\mathbb{R}^n} (u(x)-u(y))^2 K_Y(x,y)\, dxdy. \tag{8}$$

with an explicit kernel K_Y.

- Applying the theorem to bounded weak soluions (classical solutions) of

$$(-\Delta)^s u = f(u) \quad \text{in } \Omega, \text{ and } u = 0 \text{ on } \mathbb{R}^n \setminus \Omega$$

gives the *generalized Pohozaev identity*

$$\boxed{\Gamma^2(1+s)\int_{\partial\Omega}\Big(\frac{u}{d^s}\Big)^2 Y\cdot\nu\,\mathrm{dx} = 2\int_\Omega F(u) divY\,\mathrm{dx} - \mathcal{E}_Y(u,u)} \qquad (9)$$

- $Y \equiv \mathrm{id}_{\mathbb{R}^n}$ gives

$$K_Y(x,y) = (n-2s)\frac{c_{n,s}}{2}|x-y|^{-n-2s}$$

in this case, identity (9) reduces to the Pohozaev identity obtained by Ros-Oton and Serra.

Applications

(1) **Nonexistence results**:

- Assume $Y : \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$a\|x\|^2 \geq [dY(h)\cdot x]\cdot x \geq b\|x\|^2, \;\; \forall\, h \in \mathbb{R}^n, \;\; \forall\, x \in \mathbb{R}^n, \tag{10}$$

- One checks that

$$K_Y(x,y) \geq \Big[2nb - (n+2s)a\Big]\frac{1}{|x-y|^{n+2s}} \quad \& \quad divY(x) \leq na$$

- Identity (9) applied with $f(u) = |u|^{p-2}u$ becomes

$$\boxed{\Gamma^2(1+s)\int_{\partial\Omega}\Big(\frac{u}{d^s}\Big)^2 Y\cdot\nu\,\mathrm{dx} \leq \Big[\frac{2na}{p} - (2nb-(n+2s)a\Big]\int_{\Omega}|u|^p\,\mathrm{dx}}$$

Thus we have the following result

Corollary

Let $Y : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field that satisfies (10). Assume there exist a bounded open set Ω of class $C^{1,1}$ such that

$$Y \cdot \nu \geq 0 \quad \text{on} \quad \partial\Omega. \tag{11}$$

Then, the equation

$$(-\Delta)^s u = |u|^{p-2} u \ \text{ in } \ \Omega, \ \ u = 0 \ \text{ in } \ \mathbb{R}^n \setminus \Omega \tag{12}$$

does not have a nontrivial solution if $\boxed{p > \dfrac{2na}{n(2b-a) - 2as}}$.

Recap: The Dirichlet problem (12) is not solvable in open sets of class $C^{1,1}$ with the property that there exists a vector field satisfying (10) and $Y \cdot \nu \geq 0$ on $\partial\Omega$.

(2) **Hadamard formula for Dirichlet simple eigenvalues**:

- Let $\lambda(\Omega)$ be a simple eigenvalue of Ω with respect to $(-\Delta)^s$. Then, the functional

$$\Omega \mapsto \lambda(\Omega) \quad \text{is shape differentiable}$$

- Given a family of diffeomorphisms $(\Phi_\varepsilon)_{\varepsilon\in(-1,+1)}$, the map

$$(-1,+1) \ni \varepsilon \mapsto \lambda(\varepsilon) := \lambda\big(\Phi_\varepsilon(\Omega)\big) \quad \text{is derivable at } 0.$$

and

$$\boxed{\frac{d\lambda(\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=0} = -\frac{\Gamma(1+s)^2\int_{\partial\Omega}(u/d^s)^2\, Y\cdot\nu\,\mathrm{dx}}{\int_\Omega u^2(x)\,\mathrm{dx}},} \tag{13}$$

- Formula (13) can be used to recover the so called *Faber Krahn inequality*: balls has the smallest eigenvalue among sets of fixed volume

- **Proof**: A standard argument based on the implicit function theorem gives the differentiability of $\lambda(\varepsilon)$.To obtain the Hadamard formula, we differentiate weakly the equation

$$(-\Delta)^s u_\varepsilon = \lambda(\varepsilon) u_\varepsilon \text{ in } \Phi_\varepsilon(\Omega) \ \& \ u_\varepsilon = 0 \text{ in } \mathbb{R}^n \setminus \Phi_\varepsilon(\Omega).$$

  and get

$$c_{n,s} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x)-u(y))(u_0'(x)-u_0'(y))}{|x-y|^{n+2s}} \mathrm{dxdy} + \mathcal{E}_Y(u,u)$$
$$= -\lambda' \int_\Omega u^2 \,\mathrm{dx} + 2\lambda \int_\Omega u_0' u \,\mathrm{dx} + \lambda \int_\Omega u^2 \mathrm{div}\, Y \,\mathrm{dx}$$

  where u_0' is the derivative at 0 of $\varepsilon \mapsto u_\varepsilon \circ \Phi_\varepsilon$

- Apply the *generalised Pohozaev identity* to derive the formula.

Generalised Pohozaev identity

(3) **Hardy inequalities via Pohozaev**

- Some classical Hardy inequalities

$$c_\Omega \int_\Omega \frac{u^2(x)}{d_\Omega^2(x)} dx \leq \int_\Omega |\nabla u|^2 \, dx \quad \forall \, u \in C_c^\infty(\Omega)$$

- If Ω is convex the sharp constant c_Ω is given by $c_\Omega = 1/4$.
- When $\Omega = \mathbb{R}^n$, we have

$$\Big(\frac{n-2}{2}\Big)^2 \int_{\mathbb{R}^n} \frac{u^2(x)}{|x|^2} dx \leq \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \quad \forall \, u \in C_c^\infty(\mathbb{R}^n) \; \& \; n > 2.$$

Applications

- Fractional versions of the above estimates read

$$c_\Omega \int_\Omega \frac{u^2(x)}{d_\Omega^{2s}(x)}dx \leq \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}}\,dxdy \quad \forall\, u \in C_c^\infty(\Omega)$$

and

$$C_{n,s,2} \int_{\mathbb{R}^n} \frac{u^2(x)}{|x|^{2s}}dx \leq \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}}\,dxdy \quad \forall\, u \in C_c^\infty(\mathbb{R}^n)$$

- The last estimate holds with an explicit sharp constant $C_{n,s,2}$ (R. Franck & R. Seiringer, '08).

Applications

Using the *generalised Pohozaev identity*, we obtain a bounded region version of the result by R. Franck and R. Seiringer

Theorem (D. & N. De Nitti)

Let Ω bounded open of class $C^{1,1}$. There is an explicit constant $M_{n,s} > 0$ such that

$$M_{n,s} \int_\Omega \frac{u^2(x)}{|x|^{2s}} dx \leq \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x-y|^{n+2s}} dxdy - \int_{\mathbb{R}^n \setminus \Omega} \Big(\int_\Omega \frac{u(y)}{|x-y|^{n+2s}} dy \Big)^2 dx$$

for all $u \in C_c^{1,1}(\Omega)$.

Applications

- **Idea of the proof**: Uses the identity

$$
\begin{aligned}
b_{n,s,\theta}\int_{\Omega}\frac{u^2}{|x|^{\theta+2s}}\mathrm{dx} &= \int_{\Omega}\frac{(-\Delta)^s u^2}{|x|^{\theta}}\mathrm{dx} \\
&\leq 2\int_{\Omega}\frac{u^2(-\Delta)^s u}{|x|^{\theta}}\mathrm{dx} = \int_{\Omega}\frac{u^2}{|x|^{\alpha}}\,\frac{(-\Delta)^s u}{|x|^{\theta-\alpha}}\mathrm{dx} \\
&\leq 2\left(\int_{\Omega}\frac{u^2}{|x|^{2\alpha}}\mathrm{dx}\right)^{\frac{1}{2}}\left(\int_{\Omega}\frac{|(-\Delta)^s u|^2}{|x|^{2(\theta-\alpha)}}\mathrm{dx}\right)^{\frac{1}{2}}.
\end{aligned}
\tag{14}
$$

  $\forall\, u \in C_c^{1,1}(\Omega),\ \forall\, \theta \geq 0.$

- Choose $s = s'/2$, $\theta = \alpha = 2s$ yields

$$
\left(\frac{b_{n,s'/2,s'}}{2}\right)^2\int_{\Omega}\frac{u^2(x)}{|x|^{2s'}}\mathrm{dx} \leq \int_{\Omega}|(-\Delta)^{s'/2}u|^2\,\mathrm{dx}.
$$

Applications

- Identity (14) follows by choosing $u = u^2$ and $w = \frac{\zeta_k}{(\varepsilon^2+|x|^2)\theta/2}$ (for a suitable cut-off function ζ_k supported in Ω) in

$$\int_\Omega x\cdot\nabla u(-\Delta)^s w\mathrm{d}x + \int_\Omega x\cdot\nabla w(-\Delta)^s u\mathrm{d}x$$
$$= -(n-2s)\frac{c_{n,s}}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{(u(x)-u(y))(w(x)-w(y))}{|x-y|^{n+2s}}\mathrm{d}x dy$$

which holds for all $u, w \in C_c^{1,1}(\Omega)$ and then pass to the limit. In order to do this, the *generalised integration by parts formula* is needed.

Generalised Pohozaev identity: Behind the scene

- Particular case $Y \equiv \mathrm{id}_{\mathbb{R}^n}$.
- Start with the identity

$$(-\Delta)^s(x \cdot \nabla u) = 2s(-\Delta)^s u + x \cdot \nabla(-\Delta)^s u \quad \text{for all } u \in C_c^\infty(\Omega) \tag{15}$$

- Multiply (15) with u and integrate by parts yields

$$2\int_\Omega x \cdot \nabla u(-\Delta)^s u \, \mathrm{dx} = (2s - n)\int_\Omega u(-\Delta)^s u \, \mathrm{dx} \ \forall \ u \in C_c^\infty(\Omega) \tag{16}$$

- If u is a bounded weak solution of $(-\Delta)^s u = f(u)$ in Ω with f locally Lipschitz, then by regularity $u \in C^\infty_{loc}(\Omega)$ and hence $u\zeta_k \in C_c^\infty(\Omega)$ for a suitable cut-off function ζ_k that vanishes near the boundary.

Generalised Pohozaev identity: Behind the scene

- Apply (16) to $u\zeta_k$ and expand

$$2\int_\Omega \zeta_k\, x\cdot\nabla(\zeta_k u)(-\Delta)^s u\,\mathrm{dx} + 2\int_\Omega x\cdot\nabla u\Big\{u(-\Delta)^s\zeta_k - \mathcal{I}_s(\zeta_k,u)\Big\}\mathrm{dx}$$
$$= -(2s-n)\int_\Omega (u\zeta_k)(-\Delta)^s(u\zeta_k)\,\mathrm{dx}$$

- Take the limit by the dominated convergence theorem.

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Thanks for listening !