Control and stabilization of geometrically exact beams

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Information

Few words about the contents:

- **geometrically exact beams** (GEB)
- **networks** of GEB
- **well-posedness** (for $C_t^0(H_x^1)$, $C_t^0(H_x^2)$ or $C_{x,t}^1$ solutions)
- exponential **stabilization**, by means of a **boundary feedback control** (for one beam and some networks)
- exact **controllability** of a **nodal profile** (for networks)

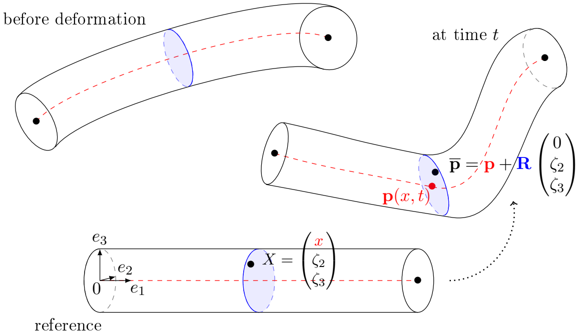
Notation for the cross product. For any $u, \zeta \in \mathbb{R}^3$ we also write $u \times \zeta$ as $\widehat{u}\,\zeta = u \times \zeta$, meaning that

$$\widehat{u} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix},$$

while u is recovered by means of the operator $\mathrm{vec}(\cdot)$ acting on skew-symmetric matrices as follows: $\mathrm{vec}(\widehat{u}) = u$.

What is a geometrically exact beam?

"Geometrically exact beam"
"Nonlinear Timoshenko beam"
"Geometrically nonlinear beam"



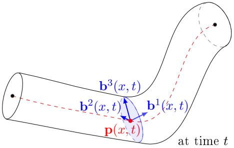
Small strains BUT large motions.

linear constitutive law $\leftarrow$ $\rightarrow$ nonlinear governing system

The mathematical model 1.

Framework 1. The state is $(\mathbf{p}, \mathbf{R})$, expressed in some fixed coordinate system $\{e_j\}_{j=1}^3$,

- centerline's position $\mathbf{p}(x,t) \in \mathbb{R}^3$
- cross sections' orientation given by the columns $\mathbf{b}^j$ of $\mathbf{R}(x,t) \in \mathrm{SO}(3)$



looks like a quasilinear wave equation

Set in $(0,\ell) \times (0,T)$, the governing system reads (freely vibrating beam)

$$\boxed{\begin{bmatrix} \partial_t & \mathbf{0} \\ (\partial_t \widehat{\mathbf{p}}) & \partial_t \end{bmatrix} \left[\begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \mathbf{M} v \right] = \begin{bmatrix} \partial_x & \mathbf{0} \\ (\partial_x \widehat{\mathbf{p}}) & \partial_x \end{bmatrix} \left[\begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} z \right]}$$

given $\mathbf{M}(x), \mathbf{C}(x) \in \mathbb{S}^6_{++}$ the mass and flexibility matrices and $\Upsilon_c(x) \in \mathbb{R}^3$ the curvature before deformation, and where v, s depend on $(\mathbf{p}, \mathbf{R})$:

$$v = \begin{bmatrix} \mathbf{R}^\intercal \partial_t \mathbf{p} \\ \mathrm{vec}\,(\mathbf{R}^\intercal \partial_t \mathbf{R}) \end{bmatrix} \quad \text{and} \quad z = \mathbf{C}^{-1} \begin{bmatrix} \mathbf{R}^\intercal \partial_x \mathbf{p} - e_1 \\ \mathrm{vec}\,(\mathbf{R}^\intercal \partial_x \mathbf{R}) - \Upsilon_c \end{bmatrix}.$$

Notation. Cross-product: $\widehat{u}\,\zeta = u \times \zeta$ and $\mathrm{vec}\,(\widehat{u}) = u$

$\mathrm{SO}(3)$: rotation matrices. $\mathbb{S}^n_{++}$: positive definite symmetric matrices of size n.

The mathematical model 2.

Framework 2. The state is $y = \begin{bmatrix} v \\ z \end{bmatrix}$, expressed in the moving basis $\{\mathbf{b}^j\}_{j=1}^3$,

- linear and angular velocities $v(x,t) \in \mathbb{R}^6$
- internal forces and moments $z(x,t) \in \mathbb{R}^6$

Set in $(0,\ell) \times (0,T)$, the governing system reads (freely vibrating beam)

$$\boxed{\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} \partial_t y - \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \partial_x y - \begin{bmatrix} \mathbf{0} & \begin{matrix} \widehat{\Upsilon}_c & \mathbf{0} \\ \widehat{e}_1 & \widehat{\Upsilon}_c \end{matrix} \\ \begin{matrix} \widehat{\Upsilon}_c & \widehat{e}_1 \\ \mathbf{0} & \widehat{\Upsilon}_c \end{matrix} & \mathbf{0} \end{bmatrix} y = - \begin{bmatrix} \begin{matrix} \widehat{v}_2 & \mathbf{0} \\ \widehat{v}_1 & \widehat{v}_2 \end{matrix} & \begin{matrix} \mathbf{0} & \widehat{z}_1 \\ \widehat{z}_1 & \widehat{z}_2 \end{matrix} \\ \mathbf{0} & \begin{matrix} \widehat{v}_2 & \widehat{v}_1 \\ \mathbf{0} & \widehat{v}_2 \end{matrix} \end{bmatrix} \begin{bmatrix} \mathbf{M}v \\ \mathbf{C}z \end{bmatrix}}$$

denoting by v_1, z_1 and v_2, z_2 the first and last 3 components of v, z.

$$\Updownarrow$$

$$\boxed{\partial_t y + A(x)\partial_x y + \overline{B}(x) y = \overline{g}(x,y).}$$

Notation. Cross-product: $\widehat{u}\,\zeta = u \times \zeta$ and $\mathrm{vec}\,(\widehat{u}) = u$

The mathematical model 3.

Two frameworks:

1. GEB. Quasilinear
second-order
(Reissner '81, Simo '85)
'Wave-like'

linked by a nonlinear transformation:

$$\mathcal{T} \colon (\mathbf{p}, \mathbf{R}) \longmapsto \begin{bmatrix} \mathbf{I}_6 & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{R}^\intercal \partial_t \mathbf{p} \\ \mathrm{vec}\left(\mathbf{R}^\intercal \partial_t \mathbf{R}\right) \\ \mathbf{R}^\intercal \partial_x \mathbf{p} - e_1 \\ \mathrm{vec}\left(\mathbf{R}^\intercal \partial_x \mathbf{R}\right) - \Upsilon_c \end{bmatrix} = y$$

2. IGEB. Semilinear (quadratic)
first-order hyperbolic
(Hodges '03)
'Hamiltonian framework' (Simo '88)

The mathematical model for **networks**

The states are now $(\mathbf{p}_i, \mathbf{R}_i)_{i\in\mathcal{I}}$ and $(y_i)_{i\in\mathcal{I}}$, with $y_i = \begin{bmatrix} v_i \\ z_i \end{bmatrix}$.

Transmission conditions at a multiple node n (where several beams meet):

- Rigid joint. Any two incident beams i, j remain attached to each other $\mathbf{p}_i = \mathbf{p}_j$ and without changing the respective angles between them $\mathbf{R}_i R_i^\intercal = \mathbf{R}_j R_j^\intercal$.
- Kirchhoff condition. In the fixed basis, the internal forces and moments exerted by the incident beams at the node are balanced with the external load.

$\rightarrow$ derive the corresponding transmission conditions for the IGEB model:

- Continuity of velocities. For any two incident beams i, j,

$$\boxed{\begin{bmatrix} R_i & \mathbf{0} \\ \mathbf{0} & R_i \end{bmatrix} v_i = \begin{bmatrix} R_j & \mathbf{0} \\ \mathbf{0} & R_j \end{bmatrix} v_j}$$

- Corresponding Kirchhoff condition. For q_n the external load applied at the node n, expressed in the body-attached basis,

$$\boxed{\sum_{\text{incident beam } i} \tau_i^n \begin{bmatrix} R_i & \mathbf{0} \\ \mathbf{0} & R_i \end{bmatrix} z_i = q_n}$$

Boundary condition at a simple node n: for the incident beam i,

$$\boxed{\tau_i^n z_i = q_n, \quad \text{or} \quad v_i = q_n}$$

Well-posedness

Based on **Li-Jin '01** and **Bastin-Coron '16 and '17**:

- $(y_i)_{i\in\mathcal{I}} \in \prod_{i=1}^{N} C^1([0,\ell_i]\times[0,T];\mathbb{R}^{12})$ semi-global in time
- $(y_i)_{i\in\mathcal{I}} \in C^0([0,T), \prod_{i\in\mathcal{I}} H^k(0,\ell_i;\mathbb{R}^{12}))$ local in time, with $q_n = -K_n v_i$ where $K_n \in \mathbb{R}^{6\times 6}$

Requires some properties of the transmission for the system in diagonal form conditions.

Assumption 1

Let $m \in \{1,2,\ldots\}$ be given. For all $i \in \mathcal{I}$, we suppose that

- $\mathbf{C}_i, \mathbf{M}_i \in C^m([0,\ell_i];\mathbb{S}^6_{++})$;
- for $\Theta_i := (\mathbf{C}_i^{1/2}\mathbf{M}_i\mathbf{C}_i^{1/2})^{-1}$, there exists $U_i, D_i \in C^m([0,\ell_i];\mathbb{R}^{6\times 6})$ such that $\Theta_i = U_i^\intercal D_i^2 U_i$ in $[0,\ell_i]$, where $D_i(x) \in \mathbb{S}^6_{++}$ diagonal & consists of the square roots of the eigenvalues of $\Theta_i(x)$, and $U_i(x)$ is unitary.

i.e. enough regularity of $\mathbf{C}_i, \mathbf{M}_i$ and the eigenvalues and eigenvectors of $(\mathbf{C}_i^{1/2}\mathbf{M}_i\mathbf{C}_i^{1/2})^{-1}$.

Well-posedness: inverting the **transformation**

Two frameworks:

1. GEB. Quasilinear
second-order
(Reissner '81, Simo '85)
'Wave-like'

linked by a nonlinear transformation:

$$\mathcal{T}\colon \begin{cases} \mathbb{R}^3 \times \mathrm{SO}(3) \longrightarrow \mathbb{R}^{12} \\ (\mathbf{p}, \mathbf{R}) \longmapsto \begin{bmatrix} \mathbf{I}_6 & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{R}^\intercal \partial_t \mathbf{p} \\ \mathrm{vec}\,(\mathbf{R}^\intercal \partial_t \mathbf{R}) \\ \mathbf{R}^\intercal \partial_x \mathbf{p} - e_1 \\ \mathrm{vec}\,(\mathbf{R}^\intercal \partial_x \mathbf{R}) - \Upsilon_c \end{bmatrix} =: y \end{cases}$$

2. IGEB. Semilinear (quadratic)
first-order hyperbolic
(Hodges '03)
'Hamiltonian framework' (Simo '88)

Well-posedness: inverting the transformation

We do the presentation for a single beam.

The GEB model

$$(1)\quad \begin{cases} \begin{bmatrix} \partial_t & \mathbf{0} \\ (\partial_t \widehat{\mathbf{p}}) & \partial_t \end{bmatrix} \left[\begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \mathbf{M} v \right] = \begin{bmatrix} \partial_x & \mathbf{0} \\ (\partial_x \widehat{\mathbf{p}}) & \partial_x \end{bmatrix} \left[\begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} z \right] & \text{in } (0,\ell)\times(0,T) \quad \text{(1a)} \\ (\mathbf{p}, \mathbf{R})(0,t) = (f^{\mathbf{p}}, f^{\mathbf{R}}) & t \in (0,T) \quad \text{(1b)} \\ z(\ell,t) = -Kv(\ell,t) & t \in (0,T) \quad \text{(1c)} \\ (\mathbf{p}, \mathbf{R})(x,0) = (\mathbf{p}^0, \mathbf{R}^0)(x) & x \in (0,\ell) \quad \text{(1d)} \\ (\partial_t \mathbf{p}, \mathbf{R}W)(x,0) = (\mathbf{p}^1, w^0)(x) & x \in (0,\ell), \quad \text{(1e)} \end{cases}$$

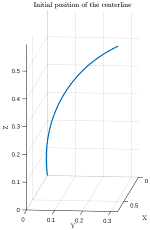
and its IGEB counterpart

$$(2)\quad \begin{cases} \partial_t y + A(x)\partial_x y + \overline{B}(x) y = \overline{g}(x,y) & \text{in } (0,\ell)\times(0,T) \quad \text{(2a)} \\ v(0,t) = \mathbf{0} & \text{for } t \in (0,T) \quad \text{(2b)} \\ z(\ell,t) = -Kv(\ell,t) & \text{for } t \in (0,T) \quad \text{(2c)} \\ y(x,0) = y^0(x) & \text{for } x \in (0,\ell). \quad \text{(2d)} \end{cases}$$

- clamped at $x = 0$
- force opposing velocities applied at $x = \ell$

Notation: $y = \begin{bmatrix} v \\ z \end{bmatrix}$.

Well-posedness: inverting the transformation



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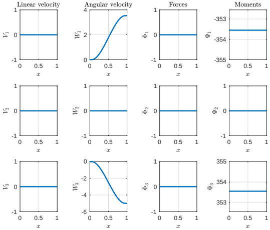


Theorem

Assume that $(f^{\mathbf{p}}, f^{\mathbf{R}}) = (\mathbf{p}^0, \mathbf{R}^0)(0)$ *holds with*

$\mathbf{M}, \mathbf{C} \in C^1([0,\ell]; \mathbb{R}^{6\times 6})$

$R \in C^2([0,\ell]; \mathrm{SO}(3))$

$(\mathbf{p}^0, \mathbf{R}^0) \in C^2([0,\ell]; \mathbb{R}^3 \times \mathrm{SO}(3))$

$\mathbf{p}^1, w^0 \in C^1([0,\ell]; \mathbb{R}^3)$

$$y^0 := \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^{-1} \end{bmatrix} \begin{bmatrix} (\mathbf{R}^0)^\intercal \mathbf{p}^1 \\ (\mathbf{R}^0)^\intercal w^0 \\ (\mathbf{R}^0)^\intercal \frac{\mathrm{d}}{\mathrm{d}x}\mathbf{p}^0 - e_1 \\ \mathrm{vec}\left((\mathbf{R}^0)^\intercal \frac{\mathrm{d}}{\mathrm{d}x}\mathbf{R}^0\right) - \Upsilon_c \end{bmatrix}.$$

Then,
if there exists a unique solution $y \in C^1([0,\ell]\times[0,T]; \mathbb{R}^{12})$ *to* (2) *with initial data* y^0 *(for some* $T > 0$*),*

$\implies$ *there exists a unique solution* $(\mathbf{p}, \mathbf{R}) \in C^2([0,\ell]\times[0,T]; \mathbb{R}^3 \times \mathrm{SO}(3))$ *to* (1) *with initial data* $(\mathbf{p}^0, \mathbf{R}^0, \mathbf{p}^1, w^0)$ *and boundary data* $(f^{\mathbf{p}}, f^{\mathbf{R}})$*, and* $y = \mathcal{T}(\mathbf{p}, \mathbf{R})$*.*

Well-posedness: inverting the transformation

The GEB model

$$
(1)\quad \begin{cases}
\begin{bmatrix} \partial_t & \mathbf{0} \\ (\partial_t \widehat{\mathbf{p}}) & \partial_t \end{bmatrix} \left[\begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \mathbf{M} v \right] = \begin{bmatrix} \partial_x & \mathbf{0} \\ (\partial_x \widehat{\mathbf{p}}) & \partial_x \end{bmatrix} \left[\begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} z \right] & \text{in } (0,\ell)\times(0,T) \quad \text{(1a)} \\
(\mathbf{p}, \mathbf{R})(0,t) = (f^{\mathbf{P}}, f^{\mathbf{R}}) & t \in (0,T) \quad \text{(1b)} \\
z(\ell, t) = -K v(\ell, t) & t \in (0,T) \quad \text{(1c)} \\
(\mathbf{p}, \mathbf{R})(x,0) = (\mathbf{p}^0, \mathbf{R}^0)(x) & x \in (0,\ell) \quad \text{(1d)} \\
(\partial_t \mathbf{p}, \mathbf{R} W)(x,0) = (\mathbf{p}^1, w^0)(x) & x \in (0,\ell), \quad \text{(1e)}
\end{cases}
$$

and its IGEB counterpart

$$
(2)\quad \begin{cases}
\partial_t y + A(x)\partial_x y + \overline{B}(x) y = \overline{g}(x,y) & \text{in } (0,\ell)\times(0,T) \quad \text{(2a)} \\
v(0,t) = \mathbf{0} & \text{for } t \in (0,T) \quad \text{(2b)} \\
z(\ell,t) = -K v(\ell,t) & \text{for } t \in (0,T) \quad \text{(2c)} \\
y(x,0) = y^0(x) & \text{for } x \in (0,\ell). \quad \text{(2d)}
\end{cases}
$$

last 6 equations

The transformation $\mathcal{T} \colon E_1 \to E_2$ is well defined for

$$E_1 = \left\{ (\mathbf{p}, \mathbf{R}) \in C^2 \left([0,\ell] \times [0,T]; \mathbb{R}^3 \times \mathrm{SO}(3) \right) : \boxed{\text{some constraints}} \right\}$$

last 6 equations

$$E_2 = \left\{ y \in C^1 \left([0,\ell] \times [0,T]; \mathbb{R}^{12} \right) : \boxed{\text{some constraints}} \right\}.$$

Well-posedness: inverting the transformation

Notation. $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $s = \mathbf{C}z = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$.

Idea of the proof.

- A solution y to (2) always belongs to E_2 since we maintained the link between the initial and boundary data of (1) and (2);
- $\mathcal{T} \colon E_1 \to E_2$ **is bijective**: let $y = \begin{bmatrix} v \\ z \end{bmatrix} \in E_2$
  - $\mathcal{T}(\mathbf{p}, \mathbf{R}) = y$ is equivalent to the linear PDE systems

$$\begin{cases} \partial_t \mathbf{p} = \mathbf{R} v_1 & \text{in } (0,\ell)\times(0,T) \\ \partial_x \mathbf{p} = \mathbf{R}(s_1 + e_1) & \text{in } (0,\ell)\times(0,T) \\ \mathbf{p}(0,0) = \mathbf{p}^0(0). \end{cases} \qquad \begin{cases} \partial_t \mathbf{R} = \mathbf{R}\widehat{v}_2 & \text{in } (0,\ell)\times(0,T) \\ \partial_x \mathbf{R} = \mathbf{R}(\widehat{s}_2 + \widehat{\Upsilon}_c) & \text{in } (0,\ell)\times(0,T) \\ \mathbf{R}(0,0) = \mathbf{R}^0(0) \end{cases}$$

  - Quaternions.

$$\mathbf{R} = (q_0^2 - \langle q\,, q\rangle)\mathbf{I}_3 + 2qq^\intercal + 2q_0\widehat{q} \quad \leftrightarrow \quad \mathbf{q} = \begin{bmatrix} q_0 \\ q \end{bmatrix}, \ \|\mathbf{q}\| \equiv 1$$

$\Updownarrow$ **Equivalence: Lemma.**

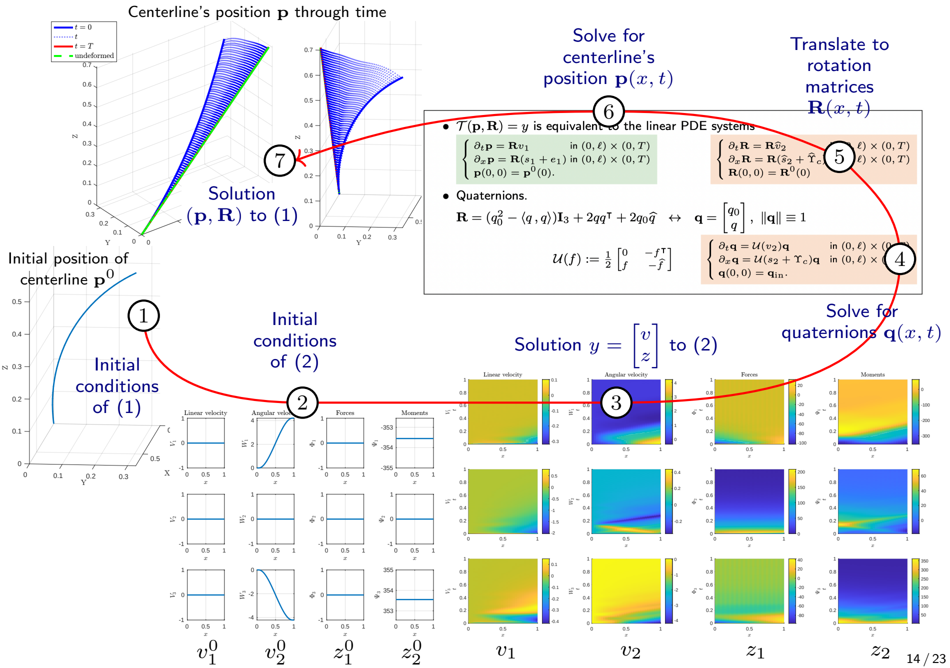
$$\mathcal{U}(f) := \frac{1}{2}\begin{bmatrix} 0 & -f^\intercal \\ f & -\widehat{f} \end{bmatrix} \qquad \begin{cases} \partial_t \mathbf{q} = \mathcal{U}(v_2)\mathbf{q} & \text{in } (0,\ell)\times(0,T) \\ \partial_x \mathbf{q} = \mathcal{U}(s_2 + \Upsilon_c)\mathbf{q} & \text{in } (0,\ell)\times(0,T) \\ \mathbf{q}(0,0) = \mathbf{q}_{\mathrm{in}}. \end{cases}$$

Well-posedness: Lemma.

$\Longrightarrow$ provides $(\mathbf{p}, \mathbf{R})$, candidate to be solution (1);

- The rest of governing, boundary and initial conditions of (2) lead to those of (1);
- Uniqueness comes from that of (2) and bijectivity of $\mathcal{T}$.

An illustration



Exponential **stabilization** via **boundary** feedback

The GEB model

$$\begin{cases} \begin{bmatrix} \partial_t & \mathbf{0} \\ (\partial_t \widehat{\mathbf{p}}) & \partial_t \end{bmatrix} \left[\begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \mathbf{M} v \right] = \begin{bmatrix} \partial_x & \mathbf{0} \\ (\partial_x \widehat{\mathbf{p}}) & \partial_x \end{bmatrix} \left[\begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} z \right] & \text{in } (0,\ell)\times(0,T) \quad \text{(1a)} \\ (\mathbf{p}, \mathbf{R})(0,t) = (f^{\mathbf{p}}, f^{\mathbf{R}}) & t \in (0,T) \quad \text{(1b)} \\ z(\ell,t) = -Kv(\ell,t) & t \in (0,T) \quad \text{(1c)} \\ (\mathbf{p}, \mathbf{R})(x,0) = (\mathbf{p}^0, \mathbf{R}^0)(x) & x \in (0,\ell) \quad \text{(1d)} \\ (\partial_t \mathbf{p}, \mathbf{R}W)(x,0) = (\mathbf{p}^1, w^0)(x) & x \in (0,\ell), \quad \text{(1e)} \end{cases}$$

and its IGEB counterpart

$$\begin{cases} \partial_t y + A(x)\partial_x y + \overline{B}(x) y = \overline{g}(x,y) & \text{in } (0,\ell)\times(0,T) \quad \text{(2a)} \\ v(0,t) = \mathbf{0} & \text{for } t \in (0,T) \quad \text{(2b)} \\ z(\ell,t) = -Kv(\ell,t) & \text{for } t \in (0,T) \quad \text{(2c)} \\ y(x,0) = y^0(x) & \text{for } x \in (0,\ell). \quad \text{(2d)} \end{cases}$$

Theorem

Suppose $R \in C^2([0,\ell]; \mathrm{SO}(3))$, *Assumption 1* $(m=2)$, *and* $K \in \mathbb{S}^6_{++}$.
Then, there exist $\varepsilon > 0$, $\beta > 0$ *and* $\eta \geq 1$ *such that for any*

$$\|y^0\|_{H^1} \leq \varepsilon$$

satisfying the zero-order compatibility conditions, there exists a unique solution $y \in C^0([0,+\infty); H^1(0,\ell;\mathbb{R}^{12}))$ *to* (2), *and*

$$\|y(\cdot,t)\|_{H^1} \leq \eta e^{-\beta t} \|y^0\|_{H^1}, \qquad \textit{for all } t \geq 0.$$

We use a quadratic Lyapunov functional $\overline{\mathcal{L}} = \sum_{\alpha=0}^{1} \int_0^\ell \langle \partial_t^\alpha y, \overline{Q} \partial_t^\alpha y \rangle dx$ characterized by

$$\boxed{\overline{Q} = \rho Q^{\mathcal{P}} + w \begin{bmatrix} \mathbf{0} & \mathbf{C}^{-1/2}(\mathbf{C}^{1/2}\mathbf{M}\mathbf{C}^{1/2})^{1/2}\mathbf{C}^{1/2} \\ \mathbf{C}^{-1/2}(\mathbf{C}^{1/2}\mathbf{M}\mathbf{C}^{1/2})^{1/2}\mathbf{C}^{1/2} & \mathbf{0} \end{bmatrix}.}$$

Remark. In the diagonal system, the Lyapunov functional is characterized by

$$Q = \begin{bmatrix} (\rho + w)\mathbf{I}_6 & \mathbf{0} \\ \mathbf{0} & (\rho - w)\mathbf{I}_6 \end{bmatrix} Q^{\mathcal{D}}.$$

$Q^{\mathcal{P}}(x)$, $Q^{\mathcal{D}}(x)$ are the matrices defining the beam energy.

Exponential stabilization via boundary feedback

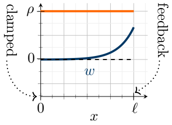
What kind of w and ρ?

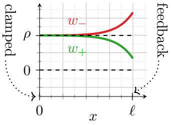
Overall, one should find an increasing and nonnegative $w \in C^1([0,\ell])$ and $\rho > 0$ s.t.

$$w(\ell) < \chi\rho, \qquad \frac{\mathrm{d}w}{\mathrm{d}x} > \eta|w| \quad \text{in } (0,\ell).$$

with $\chi, \eta > 0$ depending on the feedback, Lyapunov functional, beam parameters.

$w = q - q(0)$

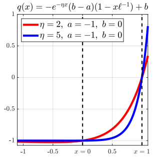




$$w_- = \rho + w$$
$$w_+ = \rho - w$$

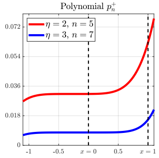
E.g., for any $\rho > 0$, with a, b such that $0 \leq a < b < (a + \chi\rho)$,

$$q(x) = a + e^{-\eta(\ell - x)} x\ell^{-1}(b - a)$$



E.g., with polynomial degree n large enough for $2\eta\ell < n$ and $\chi\rho > \frac{1}{2^n}$ to be satisfied

$$q(x) = \frac{1}{2^n} + \left(\frac{1}{2} + \frac{\eta}{n}(x - \ell)\right)^n =: p_n^+(x)$$



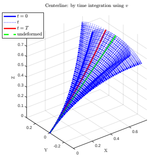
Exponential stabilization via boundary feedback

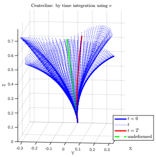
Using the transformation:

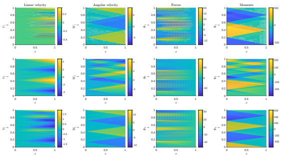
Choose the initial and boundary data of (1) in such a way that the corresponding data for (2) fulfills the stability requirements.

Corollary

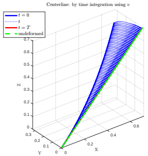
Exists a unique global in time solution $(\mathbf{p}, \mathbf{R}) \in C^2([0,\ell] \times [0,+\infty); \mathbb{R}^3 \times \mathrm{SO}(3))$ to (1) with decay in terms of velocities and strains.

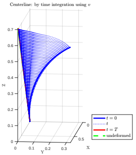


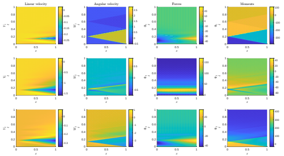




effect of feedback

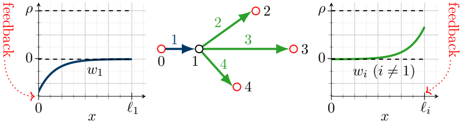






Exponential stabilization via boundary feedback

Networks:



Theorem

Suppose Assumption 1 $(m = 2)$, $R_i \in C^2([0, \ell_i]; \mathrm{SO}(3))$.

If $K_1 = \mathbf{0}_6$ *at the multiple node* $n = 1$, *and* $K_n \in \mathbb{S}^6_{++}$ *for all simple nodes* n, *then the steady state* $y \equiv 0$ *of the IGEB network is locally* H^1 *exponentially stable.*

For single beam and network,

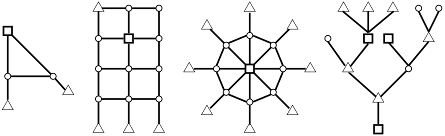
- quadratic nonlinearity $\Rightarrow$ only local in time solution, and need at least H^1_x spatial regularity
- the linearized system is not homogeneous: hence the stabilization result is proved not only by looking at the boundary/nodal conditions, but also by looking the governing system.

For star-shaped networks, when trying to remove one control, we see the limitation of just using the extended Lyapunov functional from the one beam case.

Exact controllability of nodal profiles

Control of nodal profiles:

Square $\square$ are the "charged nodes", where the state should meet some profiles
Triangles $\triangle$ are the "controlled nodes".



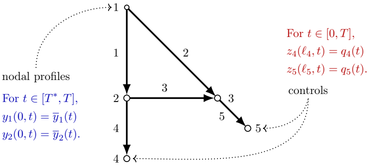
Travelling time:

Let us denote the eigenvalues of A_i by $\{\lambda_i^k\}_{k=1}^{12}$ (there are negative and positive eigenvalues).

We may define, for any $i \in \mathcal{I}$, the travelling time $T_i > 0$ by

$$T_i = \int_0^{\ell_i} \left| \min_{k \in \{1,\ldots,12\}} \frac{1}{\lambda_i^k(x)} \right| dx;$$

Exact controllability of nodal profiles



Theorem (A-shaped network)

Suppose $R_i \in C^2([0,\ell_i]; \mathrm{SO}(3))$ *and Assumption 1* $(m=2)$. *Then, for any*

$$T > T^* > \max\{T_1, T_2\} + \max\{T_4, T_5\}.$$

there exists $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$, *for some* $\delta, \gamma > 0$, *and*

- *(i) for all initial - boundary data satisfying the first-order compatibility conditions and* $\|y_i^0\|_{C_x^1} + \|q_n\|_{C_t^1} \leq \delta$, *and*
- *(ii) for all nodal profiles* $\overline{y}_1, \overline{y}_2 \in C^1([T^*, T]; \mathbb{R}^{12})$, *satisfying* $\|\overline{y}_i\|_{C_t^1} \leq \gamma$ *and the transmission conditions at the node* $n = 1$,

there exist controls $q_4, q_5 \in C^1([0,T]; \mathbb{R}^6)$ *with* $\|q_i\|_{C_t^1} \leq \varepsilon$, *such that the IGEB network admits a unique solution* $(y_i)_{i \in \mathcal{I}} \in \prod_{i \in \mathcal{I}} C^1([0,\ell_i] \times [0,T]; \mathbb{R}^{12})$, *which fulfills* $\|y_i\|_{C_x^1} \leq \varepsilon$ *and*

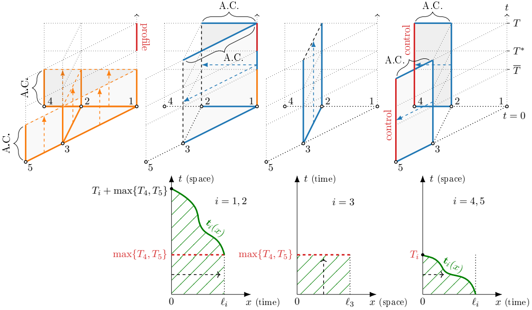
$$y_i(0,t) = \overline{y}_i(t) \quad \text{for all } i \in \{1,2\},\ t \in [T^*, T].$$

Constructive method of by Li and collaborators; notably here Zhuang '18 and '21.

Exact controllability of nodal profiles

$$\partial_t y_i + A_i \partial_x y_i + \overline{B}_i y_i = \overline{g}(x, y_i) \qquad \partial_x y_i + A_i^{-1} \partial_t y_i + A_i^{-1} \overline{B}_i y_i = A_i^{-1} \overline{g}_i(\cdot, y_i)$$

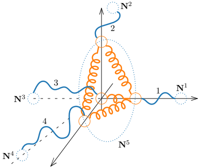
Step 1 $\rightarrow$ **Step 2** $\rightarrow$ **Step 3** $\rightarrow$ **Step 4**



$$\mathbf{t}_i(x) = T_i + \max\{T_4, T_5\} + \int_0^x \min_{1 \le k \le 12} \frac{1}{\lambda_i^k(\xi)}\, d\xi, \qquad \mathbf{t}_i(x) = T_i + \int_0^x \min_{1 \le k \le 12} \frac{1}{\lambda_i^k(\xi)}\, d\xi,$$

Outlook

- **Single beam**:
  - Leave the free beam setting (gravity, aerodynamic forces) and also apply different boundary conditions (rotating beam). Need some work on the transformation.
  - Well-posedness and stabilization with Kelvin-Voigt damping. Relax the smallness assumption on the initial data.
- **Networks**:
  - More general junction conditions for networks of geometrically exact beams: mass-spring junction.



  - Stabilization of star-shaped network: removing one control.
  - Nodal profile control: theorem with general conditions sufficient for obtaining nodal profile controllability for any network.

Thank you for your attention!

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