On the estimation of the fundamental frequency

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(joint work with Jimmy Lamboley)

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October 1, 2021

Plan of the talk

1. Introduction
   - What is shape optimization ?
   - What is a Blaschke–Santaló diagram ?
2. Study of the $(P, \lambda_1, |\cdot|)$ Diagram
   - The case of open sets
   - The case of planar convex sets
   - An application of the main theorem
   - Sketches of proofs
   - Numerical results
3. Some Perspectives and current research projects

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What is shape optimization ?

Let $J : \Omega \subset \mathbb{R}^n \longmapsto J(\Omega) \in \mathbb{R}$ be a given shape functional. We are interested in studying problems of the type:

$$\inf_{\Omega \in \mathcal{F}_{ad}} J(\Omega),$$

where $\mathcal{F}_{ad}$ is a class of subsets of $\mathbb{R}^n$.
Many question are to be asked:

- Does the problem admit a solution ?
- Can we prove some qualitative results on the optimal set (regularity, symmetries...) ?
- Can we prove that a given shape is a "local" solution for the problem ?
- How can we use numerical simulations to find approximations of the optimal shapes ?

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What is a Blaschke–Santaló diagram ?

Definition

We consider three homogeneous functionals (J_1, J_2, J_3) and a class $\mathcal{F}_{ad}$ of subsets of $\mathbb{R}^n$, with $n \geq 1$.
A Blaschke–Santaló diagram of the triplet (J_1, J_2, J_3) for the class $\mathcal{F}_{ad}$ can be defined as the set of points:

$$\mathcal{D}_{\mathcal{F}_{ad}} := \left\{ \Big(J_1(\Omega), J_2(\Omega)\Big) \;\Big|\; J_3(\Omega) = 1 \text{ and } \Omega \in \mathcal{F}_{ad} \right\} \subset \mathbb{R}^2,$$

- Can we give an explicit characterization of the diagram ?
- If not, what can we say about it ?
- Can we conjecture new inequalities ?
- How can we use numerical simulations to obtain a good description of these diagrams ?

A non-exhaustive list of classical and recent works

Purely geometrical diagrams

- W. Blaschke (1915) and L. Santaló (1961)
- M.A. Hernandez Cifre and authors (...)
- A. Delyon, A. Henrot and Y. Privat (2018) and (2020) $(|\cdot|, d, r)$

Diagrams involving spectral and geometrical functionals

- Triplet $(\lambda_1, \lambda_2, |\cdot|)$:
  - D. Bucur, G. Buttazzo and I. Figueiredo (1999)
  - P. Antunes and A. Henrot (2010)
  - L. Brasco, C. Nitsch and A. Pratelli (2013)
- Triplet $(\lambda_1, T, |\cdot|)$, where T is the torsion functional.
  - I. Lucardesi and D. Zucco (2019)
  - G. Buttazzo, A. Pratelli and M. van den Berg (2019)
  - G. Buttazzo and A. Pratelli (2020)
- Triplet $(P, \lambda_1, |\cdot|)$:
  - P. Antunes and P. Freitas (2006)

The involved functionals

In this talk we are interested by studying the $(P, \lambda_1, |\cdot|)$-diagram.

For every open $\Omega \subset \mathbb{R}^n$, we take:

- $|\Omega|$ the **volume** and $P(\Omega)$ the **"perimeter"** of Ω.
- $\lambda_1(\Omega)$ **the first Dirichlet eigenvalue of the Laplace operator**. That is the smallest value λ such that there exists $u \in H_0^1(\Omega)\backslash\{0\}$ satisfying
$$-\Delta u = \lambda u$$
in the weak sense.
The first eigenvalue is also given by the following Rayleigh quotient:
$$\lambda_1(\Omega) = \inf_{u \in H_0^1(\Omega)\backslash\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx}.$$

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$(P, \lambda_1, |\cdot|)$ Diagram

- We are interested in the following set of points:

$$\mathcal{D}_{\mathcal{F}_{ad}} := \left\{ \Big(P(\Omega), \lambda_1(\Omega)\Big) \ \Big| \ |\Omega| = 1 \ \text{ and } \ \Omega \in \mathcal{F}_{ad} \right\},$$

  where $\mathcal{F}_{ad}$ is a class of subsets of $\mathbb{R}^d$ (for example: convex sets).
- all the sets considered in this talk are of unit measure.
- $\mathcal{K}_1$ is the set of planar convex domains of measure 1.

Remark

The characterization of the diagram implies the determination of all possible scale-invariant inequalities between these 3 quantities.

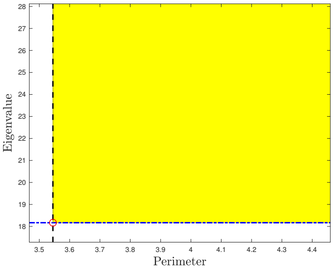
Plan of the talk

$(P, \lambda_1, |\cdot|)$-Diagram of open sets

Two fundamental inequalities

- $P(\Omega) \geq P(B)$,
- $\lambda_1(\Omega) \geq \lambda_1(B)$,

where Ω is an open set of unit volume and B is a ball of unit volume.



Theorem (F. - J. Lamboley (SIAM J. on Math. Analysis 2021))

$$\mathcal{D}_{\mathcal{O}^d} = \Big\{ \Big(P(B), \lambda_1(B)\Big) \Big\} \cup \Big]P(B), +\infty\Big[\times \Big]\lambda_1(B), +\infty\Big[$$

$\Rightarrow$ **No other inequalities !**

Question: What about the case of convex sets ?

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The case of planar convex sets

For every $\Omega \in \mathcal{K}$, we have the following inequalities:

- **Polya's** inequality (1959):

$$\lambda_1(\Omega) < \frac{\pi^2}{4}\left(\frac{P(\Omega)}{|\Omega|}\right)^2$$

- **Makai's** inequality (1960):

$$\lambda_1(\Omega) > \frac{\pi^2}{16}\left(\frac{P(\Omega)}{|\Omega|}\right)^2$$

- **Payne-Weinberger's** inequality (1970):

$$|\Omega|\lambda_1(\Omega) - |B|\lambda_1(B) \leq \left[\frac{1}{J_1^2(j_{0,1})} - 1\right]\left(\frac{P(\Omega)^2}{4\pi|\Omega|} - 1\right)$$

We recall that each inequality is represented in the diagram by a curve.

The case of planar convex sets

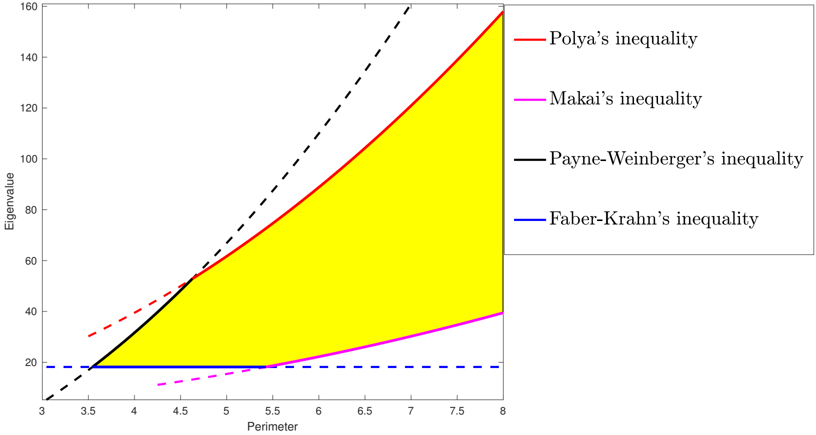

Figure: The smallest known region that contains the diagram

The case of planar Convex sets

P. Antunes - P. Freitas (2006)/ F. - J. Lamboley (2020)

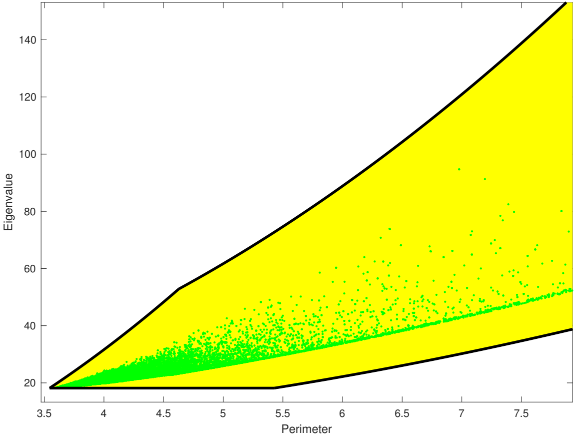

Figure: Approximation of the Blaschke–Santaló diagram $\mathcal{D}_{\mathcal{K}}$ obtained by generating 10^5 random convex polygons.

The case of planar convex sets

For every $x \geq P(B) = 2\sqrt{\pi}$, we define :

- $f/g(x) := \min/\max\{\lambda_1(\Omega) \mid \Omega \in \mathcal{K}_1 \text{ and } P(\Omega) = x\}$,

where $\mathcal{K}_1 := \{\Omega \in \mathcal{K} \mid |\Omega| = 1\}$.

Theorem (F - Lamboley (SIAM J. on Math. Analysis 2021))

1. *f and g are continuous and strictly increasing functions.*
2. $\mathcal{D}_\mathcal{K} = \left\{(x,y) \;\middle|\; x \geq 2\sqrt{\pi} \quad \text{and} \quad f(x) \leq y \leq g(x)\right\}$.
3. *For every $x > x_0$, let $\Omega \in \mathcal{K}$ such that $|\Omega| = 1$ and $\lambda_1(\Omega) = x$, then*
   - *if $P(\Omega) = g(x)$, then Ω is $C^{1,1}$,*
   - *if $P(\Omega) = f(x)$, then Ω is a polygon.*
4. $f(x) \underset{x\to+\infty}{\sim} \frac{\pi^2}{16}x^2 \quad \text{and} \quad g(x) \underset{x\to+\infty}{\sim} \frac{\pi^2}{4}x^2.$
5. $f'(x_0) = 0 \quad \text{and} \quad \limsup_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0} \geq \frac{\lambda_1(B)}{3\sqrt{\pi}}\left(\frac{\lambda_1(B)}{\pi} - 2\right).$

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Application to a regularity result

We recall the following regularity result:

Theorem (Lamboley-Novruzi-Pierre (ARMA 2012))

Let $\lambda > \lambda_1(B)$.

- *Any solution of*

$$\sup\{P(\Omega)\,|\,\Omega \in \mathcal{K}_1 \text{ et } \lambda_1(\Omega) = \lambda\}$$

  is Polygon.

- *Any solution of*

$$\inf\{P(\Omega)\,|\,\Omega \in \mathcal{K}_1 \text{ et } \lambda_1(\Omega) = \lambda\}$$

  is $C^{1,1}$.

Question: What happens if we exchange the role of λ_1 and P ?

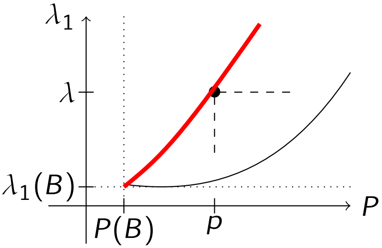
Equivalence between some shape optimization problems

Corollary

Let $p \geq 2\sqrt{\pi}$. The following problems are equivalent:

- $\sup\{\lambda_1(\Omega) \,|\, \Omega \in \mathcal{K}_1 \text{ and } P(\Omega) = p\}$
- $\sup\{\lambda_1(\Omega) \,|\, \Omega \in \mathcal{K}_1 \text{ and } P(\Omega) \geq p\}$
- $\inf\{P(\Omega) \,|\, \Omega \in \mathcal{K}_1 \text{ and } \lambda_1(\Omega) = \lambda\}$
- $\inf\{P(\Omega) \,|\, \Omega \in \mathcal{K}_1 \text{ and } \lambda_1(\Omega) \geq \lambda\}$

and the optimal domains are $C^{1,1}$.



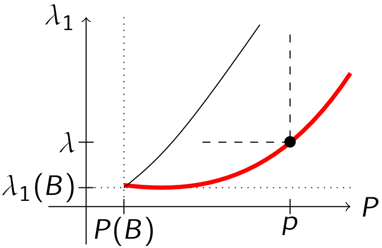
Equivalence between some shape optimization problems

Corollary

Let $p \geq P(B)$. The following problems are equivalent:

- $\inf\{\lambda_1(\Omega) \,|\, \Omega \in \mathcal{K}_1 \text{ and } P(\Omega) = p\}$
- $\inf\{\lambda_1(\Omega) \,|\, \Omega \in \mathcal{K}_1 \text{ and } P(\Omega) \geq p\}$
- $\sup\{P(\Omega) \,|\, \Omega \in \mathcal{K}_1 \text{ and } \lambda_1(\Omega) = \lambda\}$
- $\sup\{P(\Omega) \,|\, \Omega \in \mathcal{K}_1 \text{ and } \lambda_1(\Omega) \geq \lambda\}$

and the optimal domains are **polygonal**.



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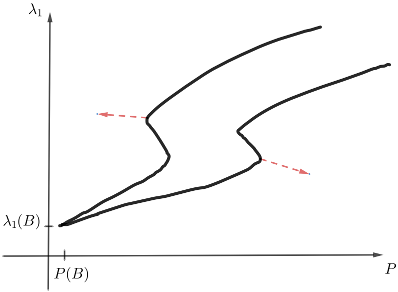
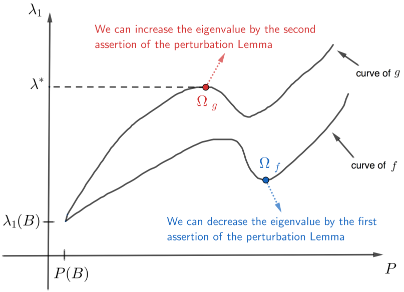
The continuity and monotonicity of the boundary

Lemma (Perturbation Lemma)

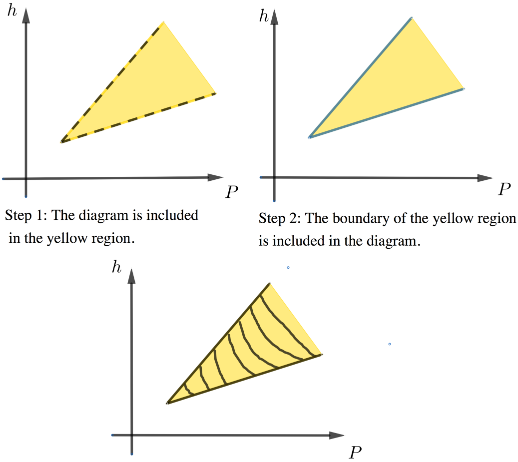
1. *The ball is the only local minimizer of the perimeter in* $\mathcal{K}_1$.
2. *A* $C^{1,1}$ *convex domain cannot be a local maximizer of* λ_1 *in* $\mathcal{K}_1$.
3. *The ball is the only local minimizer of* λ_1 *in* $\mathcal{K}_1$.
4. *There is no local maximizer of the perimeter in* $\mathcal{K}_1$.





The simple connectedness of the diagram

A first (unsuccessful) approach.



Step 1: The diagram is included in the yellow region.

Step 2: The boundary of the yellow region is included in the diagram.

Step 3: The yellow region is filled by continuous paths that connect the upper boundary to the lower one.

The simple connectedness of the diagram

1. Minkowski sum and continuous paths Γ_{K_0,K_1}.

$$\Gamma_{K_0,K_1} := \left\{ K_t := \frac{(1-t)K_0 \oplus tK_1}{\sqrt{|(1-t)K_0 \oplus tK_1|}} \ \middle| \ t \in [0,1] \right\},$$

where $A \oplus B := \{x+y \mid x \in A \text{ and } y \in B\}$ is the Minkowski sum of A and B.

2. Continuity of the paths for small perturbations.

$$\left(K_2^n \underset{n\to+\infty}{\longrightarrow} K_2 \ \text{ and } \ K_2^n \underset{n\to+\infty}{\longrightarrow} K_2 \right) \Rightarrow \sup_{t\in[0,1]} |\Gamma_{K_0,K_1}(t) - \Gamma_{K_0^n,K_1^n}(t)| \underset{n\to+\infty}{\longrightarrow} 0$$

3. The arcs go infinitely to the right when the perimeter increases

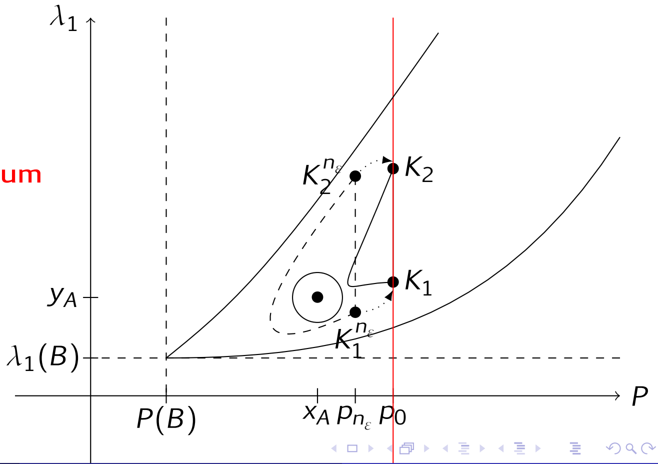
For every $K_0, K_1 \in \mathcal{K}_{1,p}$ (i.e. such that $P(K_0) = P(K_1) = p$):

$$\forall t \in [0,1], \quad P(K_t) \geq \frac{p}{2} \underset{p\to+\infty}{\longrightarrow} +\infty$$

4. Relevant paths and conclusion Assume that there exists $A \notin \mathcal{D}_{\mathcal{K}}$.

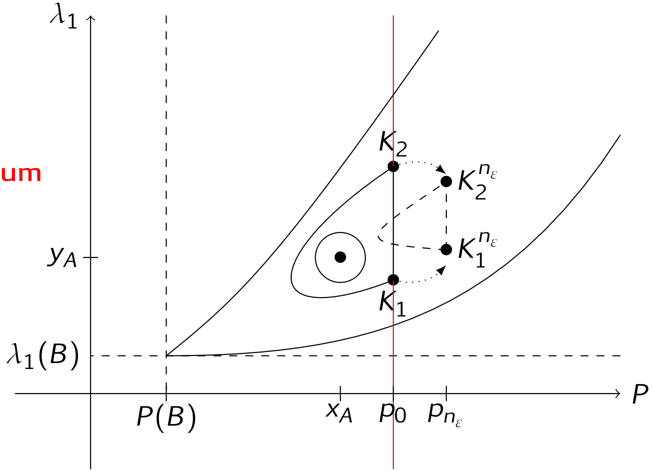
$$p_0 := \sup\left\{ p \in \mathbb{R} \;\middle|\; \exists K_1^p, K_2^p \in \mathcal{K}^2_{1,p} \text{ such that } A \text{ is in the interior of } \Gamma_{K_1,K_2} \right\}$$

If p_0 is not a maximum



The simple connectedness of the diagram

If p_0 is a maximum



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Extremal domains

⇒ different optimization methods are needed !

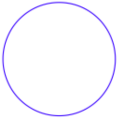
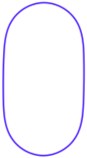
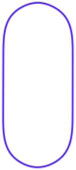
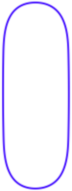
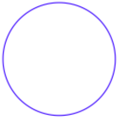
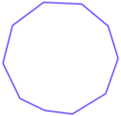
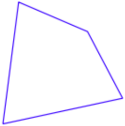
Problem	$p_0 = P(B) = 2\sqrt{\pi}$	$p_0 = 3.8$	$p_0 = 4$	$p_0 = 4.2$
Upper boundary				
Lower boundary				

Figure: Numerically obtained optimal shapes for different values of p_0.

Improved numerical description of the diagram

We combine the theoretical and numerical results:

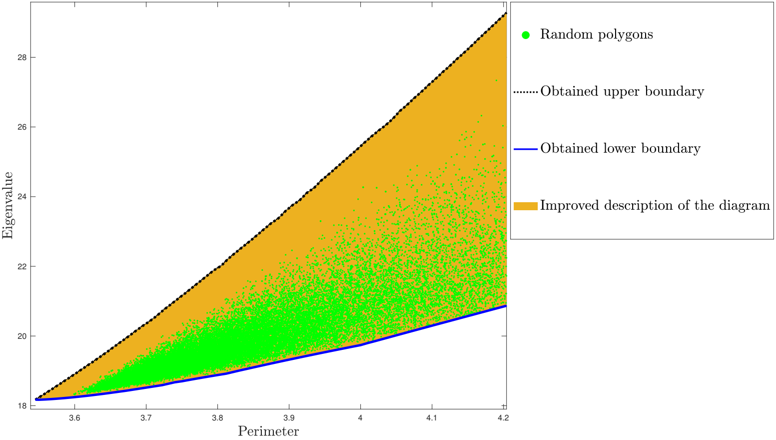


Figure: Improved $(P, \lambda_1, |\cdot|)$-diagram.

Plan of the talk

Some Perspectives and current research projects

Some perspectives:

- Apply these ideas to other diagrams.
- generalize to higher dimensions (the perturbation Lemma).
- There may be (or not) a link between diagrams of convex sets.
- Perform more numerical simulations for other diagrams.

Current research projects:

- Optimal placement and shape design of sensors.
- Testing and applying Machine Learning techniques in shape optimization.

Thank you for your attention !