Asymptotic behavior of solutions for some diffusion problems on metric graphs

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Outline

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1. Classical diffusion and the asymptotic behaviour
2. Heat equation on graphs
3. Nonlocal diffusion on graphs
4. A nonlinear problem

Few words about diffusion problems

Heat equation:

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u(0) = u_0. \end{cases}$$

For any $u_0 \in L^1(\mathbb{R})$ the solution $u \in C([0,\infty), L^1(\mathbb{R}))$ is given by:

$$u(t,x) = (G(t,\cdot) * u_0)(x)$$

where

$$G(t,x) = (4\pi t)^{-1/2} \exp(-\frac{|x|^2}{4t})$$

Decay of solutions, $1 \le p \le q \le \infty$:

$$\|u(t)\|_{L^q(\mathbb{R})} \lesssim t^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L^p(\mathbb{R})}$$

$L^1 - L^2$ decay

$$u_t = Au$$

When the Kernel is not explicit it is sufficient to analyze the bilinear form associate with the operator

$$a(u, v) = (-Au, v)$$

• $\|e^{-tA}\|_{L^1-L^2} \leq Ct^{-\alpha/4}\ \forall t > 0$
equivalent with a Nash inequality

$$\|u\|_2^{2+4/\alpha} \leq ca(u, u)\|u\|_1^{4/\alpha}$$

• $\|e^{-tA}\|_{L^1-L^2} \leq Ct^{-\alpha/4}\ \forall t \in (0, 1)$
equivalent with a Nash inequality

$$\|u\|_2^{2+4/\alpha} \leq c[\|u\|_2^2 + a(u, u)]\|u\|_1^{4/\alpha}$$

• More variants $t > t_0$, $u_0 \in L^1 \cap L^2$, etc... : Davies, Carlen-Kusuoka-Stroock 87, Coulhon 96, Varopoulos 92 book, Ouhabaz' book, etc...

Asymptotics

Theorem

For any $u_0 \in L^1(\mathbb{R})$ and $p \geq 1$ we have

$$t^{\frac{1}{2}(1-\frac{1}{p})} \|u(t) - MG_t\|_{L^p} \to 0,$$

where $M = \int u_0$.

Proof:

$$(G_t * u_0)(x) - MG_t(x) = \frac{1}{(4\pi t)^{1/2}} \int_{\mathbb{R}^d} (\exp(-\frac{|x-y|^2}{4t}) - \exp(-\frac{|x|^2}{4t}))u_0(y)dy$$

+ Taylor expansion with integral reminder, etc...
Remark 1: assuming $u_0 \in L^1(1+|x|)$ gives a better convergence result: extra $t^{1/2}$
Remark 2: More terms can be obtained using decomposion on Dirac deltas
$u_0 = M\delta_0 + M_1\delta_0' + ...$ Duoandikoetxea + Zuazua, CRAS 92

Scaling Arguments $u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x)$

Why?

- Leaves invariant the equation : $(u_\lambda)_t = (u_\lambda)_{xx}$, keeps the mass
- $u(t) - MG_t \to 0$ in $L^1(\mathbb{R}^N) \Leftrightarrow u_\lambda(1) \to f_M$ in $L^1(\mathbb{R})$ as $\lambda \to \infty$

How it works (Four step method: Vazquez-Kamin)

- scaling - write the equation for u_λ
- estimates and compactness of $\{u_\lambda\}$
- passage to the limit $u_\lambda \to U$, $u_\lambda(1) \to U(1)$
- identification of the limit: $U_t = U_{xx}$,
  $U(0) = M\delta_0$: $\lim_{t\to 0} \int_{\mathbb{R}} U(t,x)\varphi(x)dx = M\varphi(0)$

Remarks

- A good method for nonlinear problems
- It works for the first term in the asymptotic expansion
- For more terms possible connection with the "correctors" in homogenization

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- $\Gamma = \Gamma_f \cup \Gamma_\infty$ a metric graph with at least one infinite edge: $|E_{\Gamma_\infty}| = N \geq 1$

$$\begin{cases} \mathbf{u}_t(t,x) - \Delta_\Gamma \mathbf{u}(t,x) = 0, & x \in \Gamma, t > 0, \\ \mathbf{u}(0,x) = \mathbf{u}_0(x), & x \in \Gamma. \end{cases} \tag{1}$$

- Kirchhoff-type coupling at nodes:

1. continuity
2. sum of the derivatives from the left $=$ sum of the derivatives from the right
3. when Γ has only one infinite edge: classical Laplacian on $(0,\infty)$ with Neumann boundary condition

Theorem

Let $\mathbf{u}_0 \in L^1(\Gamma)$ *and* $\mathbf{u}$ *the solution of the problem* (1). *Then for any* $1 \le p \le \infty$

$$t^{\frac{1}{2}(1-\frac{1}{p})}\|\mathbf{u}(t) - \mathbf{U}_M(t)\|_{L^p(\Gamma_\infty)} \to 0, \qquad \text{as } t \to \infty, \tag{2}$$

and

$$t^{\frac{1}{2}}\|\mathbf{u}(t) - \mathbf{U}_M(t)\|_{L^\infty(\Gamma_f)} \to 0, \qquad \text{as } t \to \infty, \tag{3}$$

where M *is the total mass of the initial datum* $\mathbf{u}_0$ *and*

$$\mathbf{U}_M(t,x) = \frac{2M}{N} t^{-\frac{1}{2}} \begin{cases} G(x/\sqrt{t}), & x \in \Gamma_\infty, \\ G(0), & x \in \Gamma_f. \end{cases} \tag{4}$$

Here $G(s)$ *is given by the classical Gaussian profile,*

$$G(s) = \frac{1}{\sqrt{4\pi}} e^{-\frac{s^2}{4}}.$$

Scale everything $\mathbf{u}_\lambda(t,x) = \lambda \mathbf{u}(t,\lambda x)$

$\mathbf{u}:\Gamma \to \mathbb{R}$ then $\mathbf{u}_\lambda$ should be defined correspondingly

$$\mathbf{u}_\lambda : \Gamma^\lambda \to \mathbb{R}$$

$\Gamma = (V,E)$, $I_e = [0,l_e]$ or $I_e = [0,\infty)$
$\Gamma^\lambda = (V,E^\lambda)$,

$$I_e^\lambda = \begin{cases} [0, l_e/\lambda], & \text{if } l_e < \infty, \\ [0,\infty), & \text{if } l_e = \infty. \end{cases}$$

$\Gamma = \Gamma_f \cup \Gamma_\infty \qquad \Gamma^\lambda = \Gamma_f^\lambda \cup \Gamma_\infty$
As $\lambda \to \infty$ Γ_f^λ collapses to one point while Γ_∞ remains invariant

Passing to the limit

$u_\lambda \to \mathbf{U}$ in $C((\tau,T),L^1(\Gamma_\infty))$

Limit equation for the profile $\mathbf{U}$

• $\varphi : C([0,\infty),H^1(\Gamma_\infty))$ such that $\varphi^e(t,0)=\varphi^{e'}(t,0)$ for all $e,e'\in\Gamma_\infty$ + trivial extension on Γ_f^λ

$$\int_0^\infty\int_{\Gamma_\infty}\Big(\mathbf{U}(t,x)\varphi_t(t,x)-\mathbf{U}_x(t,x)\varphi_x(t,x)\Big)dxdt+M\varphi(0,0)=0.$$

Identification of the limit

$\mathbf{U}_M=(U^1,\dots,U^N)$ given by

$$U_M^k(t,x)=\frac{2M}{N}G_t(x),\quad k=1,\dots,N,$$

On Γ_f: a little more work

Outline

Nonlocal on graphs

LI, J.D. Rossi, A. San Antolin, Ann. Mat. Pura Appl. 2021

$$\begin{cases} \mathbf{u}_t(t,x) = \int_\Gamma J(d(x,y))(\mathbf{u}(t,y) - \mathbf{u}(t,x))dy, & x \in \Gamma, t > 0, \\ \mathbf{u}(0,x) = \mathbf{u}_0(x), & x \in \Gamma. \end{cases} \quad (5)$$

Technical assumptions

- $J \in L^1(\mathbb{R}, 1+|x|^2)$ with $J(z) = J(-z)$,
- $J(z)$ is non-increasing for $z > 0$,
- J is nonnegative in $\mathbb{R}$ and positive in neighborhood of the origin.

Two objectives

- approximate the heat model with nonlocal models
- long time behaviour for the nonlocal model

Relaxation limit

$$\begin{cases} \mathbf{u}^\varepsilon_t(t,x) = \varepsilon^{-3} \displaystyle\int_\Gamma J\Big(\frac{d(x,y)}{\varepsilon}\Big)(\mathbf{u}^\varepsilon(t,y) - \mathbf{u}^\varepsilon(t,x))dy, & x \in \Gamma, t > 0, \\ \mathbf{u}^\varepsilon(0,x) = \mathbf{u}_0(x), & x \in \Gamma. \end{cases} \qquad (6)$$

Theorem

For any $\mathbf{u}_0 \in L^2(\Gamma)$ *it holds that*

$$\mathbf{u}^\varepsilon \rightharpoonup \mathbf{u} \ \text{ weakly in } L^2([0,\infty), L^2(\Gamma)),$$

where $\mathbf{u} \in C([0,\infty), L^2(\Gamma))$ *is the unique solution of the heat equation with the same initial condition.*

• In the limit as $\varepsilon \to 0$ we recover the Kirchoff conditions on the nodes without assuming any condition on $\mathbf{u}^\varepsilon$ (not even continuity).

• the symmetry of J and its second momentum play a role

Outline

A convection diffusion problem on a star shaped tree

C. Cazacu, L. I., A. Pazoto, J. Rossi, NoDEA 2022

$$\begin{cases} \partial_t u_i(t,x) + \partial_x(f(u_i(t,x)) = \partial_{xx} u_i(t,x), \quad t>0,\ x<0,\ i\in\{1,\dots,n\}, \\ \partial_t u_j(t,x) + \partial_x(f(u_j(t,x)) = \partial_{xx} u_j(t,x), \quad t>0,\ x>0,\ j\in\{n+1,\dots,n+m \\ u_i(t,0) = u_j(t,0), \qquad t>0,\ i,j\in\{1,\dots,n+m\}, \\ \displaystyle\sum_{i=1}^{n}\big(f(u_i) - \partial_x u_i\big)(t,0) = \sum_{j=n+1}^{n+m}\big(f(u_j) - \partial_x u_j\big)(t,0), \quad t>0, \\ u_i(0,x) = u_{0i}(x), \qquad x<0,\ i\in\{1,\dots,n\}, \\ u_j(0,x) = u_{0j}(x), \qquad x>0,\ j\in\{n+1,\dots,n+m\}. \end{cases} \tag{7}$$

$$\frac{d}{dt}\Big(\sum_{i=1}^{n}\int_{-\infty}^{0} u_i(t,x)dx + \sum_{j=n+1}^{n+m}\int_{-\infty}^{0} u_j(t,x)dx\Big) = (n-m)f(0).$$

Theorem (Global well-posedness)

Let $f \in C^1(\mathbb{R})$ *satisfying*

$$(n-m)\limsup_{x\to-\infty} f(x) \geq 0 \geq (n-m)\liminf_{x\to\infty} f(x). \tag{8}$$

For any $\mathbf{u}_0 \in L^2(\Gamma)\cap L^\infty(\Gamma)$ *there exists a unique solution satsisfying*

$$\mathbf{u} \in C([0,\infty); L^2(\Gamma)) \cap L^\infty((0,\infty)\times\Gamma) \cap L^2((0,\infty); H^1_c(\Gamma)).$$

Moreover, if $\underline{M} \leq u_{0k}(x) \leq \overline{M}$ *for all* $k \in \{1,\dots,n+m\}$, *satisfying*

$$(n-m)f(\underline{M}) \geq 0 \geq (n-m)f(\overline{M}). \tag{9}$$

then

$$\underline{M} \leq u_k(t,x) \leq \overline{M}, \quad \forall t>0,\ x\in I_k,\ 1\leq k\leq n+m.$$

- $f(s) = \pm|s|^{q-1}s$ where $\pm(n-m)\leq 0$
- $f(s) = |s|^p$ with $p>1$ is not covered for all initial data in $L^2(\Gamma)\cap L^\infty(\Gamma)$ but works for nonnegative (nonpositive) solutions if $n-m\leq 0$ (respectively $n-m\geq 0$)

why such restrictions

For any function ρ satisfying $\rho(0) = \rho'(0) = 0$ the following holds

$$\frac{d}{dt}\int_\Gamma \rho(\mathbf{u}(t))dx + \int_\Gamma (\partial_x \mathbf{u}(t))^2 \rho''(\mathbf{u}(t))dx = \int_\Gamma f(\mathbf{u}(t))\rho''(\mathbf{u}(t))\partial_x \mathbf{u}(t)dx$$
$$= (n-m)\int_0^{\mathbf{u}(t,0)} f(s)\rho''(s)ds, \text{ for a.e. } t \in (0,T),$$

- $\rho(s) = s^2$, $f(s) = |s|^p s$ then

$$\frac{d}{dt}\int_\Gamma \mathbf{u}^2(t)dx + 2\int_\Gamma (\partial_x \mathbf{u}(t))^2 dx = 2(n-m)\int_0^{\mathbf{u}(t,0)} |s|^p s ds$$

Asymptotic behaviour

Let $f(s) = -|s|^{q-1}s$, with $q \geq 2$, and $(n-m)x \geq 0$. For any initial datum $\mathbf{u}_0 \in L^1(\Gamma)$ denoting by M the total mass of the initial datum, i.e.,

$$M = \int_\Gamma \mathbf{u}_0(x)dx := \sum_{i=1}^{n} \int_{-\infty}^{0} u_{0i}(x)dx + \sum_{j=n+1}^{n+m} \int_0^\infty u_{0j}(x)dx,$$

the solution satisfies:

• if $q > 2$

$$t^{\frac{1}{2}(1-\frac{1}{p})}\|\mathbf{u}(t) - \mathbf{u}_M(t)\|_{L^p(\Gamma)} \to 0, \text{ as } t \to \infty, \ 1 \leq p < \infty, \qquad (10)$$

where $\mathbf{u}_M(t) = (u_{M,k}(t))_{k=1}^{m+n}$ is given by

$$u_{M,k}(t,x) = \frac{2M}{m+n}\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}},$$

- if $q = 2$ and $\mathbf{u}_0$ is nonnegative (or nonpositive) then

$$t^{\frac{1}{2}(1-\frac{1}{p})}\|\mathbf{u}(t) - \mathbf{u}_M(t)\|_{L^p(\Gamma)} \to 0, \text{ as } t \to \infty, \ 1 \leq p < \infty, \tag{11}$$

where $\mathbf{u}_M(t) = (u_{M,k}(t))_{k=1}^{m+n}$ is given by

$$u_{M,k}(t,x) = \frac{1}{\sqrt{t}} f_M(\frac{x}{\sqrt{t}})$$

where for a given parameter M

$$f_M(y) = \frac{\alpha_{n,m,M} e^{-\frac{y^2}{4}}}{1 + \alpha_{n,m,M} \int_{-\infty}^{y} e^{-s^2/4} ds} \tag{12}$$

and α_M is the constant obtained as the unique solution in the interval $(-\frac{1}{2\sqrt{\pi}}, \infty)$ of the equation

$$|1 + \alpha\sqrt{\pi}|^{n-m}|1 + 2\alpha\sqrt{\pi}|^m = e^M.$$

Asymptotic behaviour: Self similar profiles

- Long time behaviour by scaling as in the linear case when $q \geq 2$
- no idea how it works for $q < 2$: Hyperbolic regime, Oleinik estimates: $u_x \leq /t$
- difficulty for $q = 2$: the uniqueness of the solutions of the limit

Self-similar profiles: $q > 2$, linear problem in the limit

- $$u_{M,k}(t,x) = \frac{2M}{m+n}\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}, \qquad x \in I_k.$$

$$\begin{cases} \partial_t u_i(t,x) - \partial_{xx} u_i(t,x) = 0, & t>0, x<0, i \in \{1,\dots,n\}, \\ \partial_t u_j(t,x) - \partial_{xx} u_j(t,x) = 0, & t>0, x>0, j \in \{n+1,\dots,n+m\}, \\ u_i(t,0) = u_j(t,0), & i,j \in \{1,\dots,n+m\}, \\ \sum_{i=1}^{n} \partial_x u_i(t,0) = \sum_{j=n+1}^{n+m} \partial_x u_j(t,0). \end{cases} \tag{13}$$

For any $\boldsymbol{\varphi} = (\varphi_l)_{l=1}^{n+m} \in BC(\Gamma)$, with $\boldsymbol{\varphi}(0) := \varphi_i(0) = \varphi_j(0)$

$$\lim_{t\to 0} \sum_{i=1}^{n} \int_{-\infty}^{0} u_i(t,x)\varphi_i(x)dx + \sum_{j=n+1}^{n+m} \int_0^\infty u_j(x)\varphi_j(t,x)dx = M\boldsymbol{\varphi}(0)$$

- For any $\boldsymbol{\varphi} = (\varphi_k)_{k=1}^{n+m} \in C_c([0,\infty), D(\Delta_\Gamma)) \cap C_c^1([0,\infty), L^2(\Gamma))$.

$$\int_0^\infty \int_\Gamma \mathbf{u}_M(t,x)(\partial_t \boldsymbol{\varphi} + \partial_{xx}\boldsymbol{\varphi})dxdt + M\boldsymbol{\varphi}(0,0) = 0,$$

The nonlinear case

$$\begin{cases} \partial_t u_i(t,x) - \partial_{xx} u_i(t,x) - \partial_x(|u|^{q-1}u) = 0, & t>0,\ x<0,\ i\in\{1,\dots,n\}, \\ \partial_t u_j(t,x) - \partial_{xx} u_j(t,x) - \partial_x(|u|^{q-1}u) = 0, & t>0,\ x>0,\ i\in\{n+1,\dots,n \\ u_i(t,0) = u_j(t,0), \quad t>0,\ i,j\in\{1,\dots,n+m\}, \\ \sum_{i=1}^{n} \left(\partial_x u_i(t,0) + |u_i|^{q-1}u_i(t,0)\right) = \sum_{j=n+1}^{n+m} \left(\partial_x u_j(t,0) + |u_j|^{q-1}u_j(t,0)\right),\ t>0, \end{cases} \tag{14}$$

with the initial datum a delta at the origin taken in the sense of measures

$$\lim_{t\to 0} \sum_{i=1}^{n} \int_{-\infty}^{0} u_i(t,x)\varphi_i(x)dx + \sum_{j=n+1}^{n+m} \int_{0}^{\infty} u_j(x)\varphi_j(t,x)dx = M\boldsymbol{\varphi}(0) \tag{15}$$

for any $\boldsymbol{\varphi} = (\varphi_l)_{l=1}^{n+m} \in BC(\Gamma)$, with $\boldsymbol{\varphi}(0) := \varphi_i(0) = \varphi_j(0)$

Symmetry reduces the problem to a 1-d problem

$$\begin{cases} \partial_t u - \partial_{xx} u - \partial_x(|u|^{q-1}u) = 0, & t>0, x\neq 0,\\ u(t,0-) = u(t,0+), & t>0,\\ n(\partial_x u + |u|^{q-1}u)(t,0-) = m(\partial_x u + |u|^{q-1}u)(t,0+), & t>0, \end{cases} \tag{16}$$

with the initial datum u_0 satisfying $(n1_{\mathbb{R}_-} + m1_{\mathbb{R}_+})u_0 = M\delta_0$ in the sense of bounded measures:

$$\lim_{t\to 0} n\int_{-\infty}^{0} u(t,x)\varphi(x)dx + m\int_0^{\infty} u(t,x)\varphi(x)dx = M\varphi(0) \tag{17}$$

for any $\varphi \in BC(\mathbb{R})$.

• when $n=m$ previous results for nonnegative/nonpositive solutions of Liu - Pierre JDE'84 (no diffusion), Escobedo Vazquez Zuazua ARMA'93 (diffusion, $q>1$), Ana Carpio '96 (changing sign solutions for uniform tail solutions)

• the difficulty when $m\neq n$: lack of a comparison principle as in EVZ93 since if U is a solution then $U(x+\alpha)$ does not solve the same problem

• when $q=2$ we can prove uniqueness using an adapted Hoph-Cole transform

The case $q = 2$

To fix the ideas we cheat a little :) (skip the approximation procedure + regularity)

$$v(t,x) = \int_{-\infty}^{x} u(t,y)(n1_{\mathbb{R}_-} + m1_{\mathbb{R}_+})dy$$

satisfies

$$\lim_{t\downarrow 0} v(t,x) = MH(x)$$

and

$$w(t,x) = \begin{cases} e^{v(t,x)/n}, & x<0, \\ e^{v(t,x)/m}, & x>0. \end{cases} \underset{t\downarrow 0}{\to} w_0(x) = \begin{cases} 1, & x<0, \\ e^{M/m}, & x>0, \end{cases}$$

solves

$$\begin{cases} \partial_t w - \partial_{xx} w = 0, & t>0, x\neq 0, \\ w^n(t,0-) = w^m(t,0+), & t>0, \\ \dfrac{w_x}{w}(t,0-) = \dfrac{w_x}{w}(t,0-)(t,0+) = u(t,0), & t>0, \\ w(0,x) = w_0(x). \end{cases}$$

Solve separately two heat equations with Neumann boundary condition on half lines

$$w_R(t,x)=\int_0^\infty (K_t(x-y)+K_t(x+y))w_{0R}(y)dy-2\int_0^t K_{t-\tau}(x)w_{R,x}(\tau,0)d\tau,$$

and

$$w_L(t,x)=\int_{-\infty}^0 (K_t(x-y)+K_t(x+y))w_{0L}(y)dy+2\int_0^t K_{t-\tau}(x)w_{L,x}(\tau,0)d\tau.$$

At $x=0$ we get

$$w_R(t,0)=1-\frac{1}{\sqrt{\pi}}\int_0^t \frac{w_R(\tau,0)u(\tau,0)}{\sqrt{t-\tau}}d\tau$$

and

$$w_L(t,0)=e^{M/m}+\frac{1}{\sqrt{\pi}}\int_0^t \frac{w_L(\tau,0)u(\tau,0)}{\sqrt{t-\tau}}d\tau.$$

New functions $\eta_R(t) = \sqrt{t}u(t,0)w_R(t,0)$ and $\eta_L(t) = \sqrt{t}u(t,0)w_L(t,0)$

$$0 \leq \eta_L(t) \leq CMe^{M/m}, \qquad 0 \leq \eta_R(t) \leq CMe^{M/m}.$$

Both of them are bounded functions in $[0,\infty)$ and they satisfy the system

$$\eta_R(t) = \sqrt{t}u(t,0)(e^{M/m} - (\mathcal{L}\eta_R)(t)),$$

$$\eta_L(t) = \sqrt{t}u(t,0)(1 + (\mathcal{L}\eta_L)(t)),$$

$$(\mathcal{L}\eta)(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{\eta(s)}{\sqrt{t-s}\sqrt{s}} ds.$$

Finaly: $\eta_L \in L^\infty(0,T)$ satisfies

$$\Big(1 + (\mathcal{L}\eta)(t)\Big)^{\frac{n}{m}} + \mathcal{L}\Big(\eta(t)\big(1 + (\mathcal{L}\eta)(t)\big)^{\frac{n}{m}-1}\Big) = e^{M/m}, \ \forall \ t \in (0,T). \qquad (18)$$

Uniqueness $\eta_L = c_{n,m,M}$ ($\mathcal{L}(1) = \sqrt{\pi}$, $\mathcal{L}$ is monotone)

To be continued...

- try different couplings
- incorporate transport terms even in the local problem:

$$u_t^e + \alpha_e u_x^e - u_{xx}^e = 0$$

Strongly related with the asymptotic behaviour as $t \to \infty$ of

$$u_t + (\sigma u)_x - u_{xx} = 0, \quad \sigma = \begin{cases} \sigma_-, & x < 0, \\ \sigma_+, & x > 0. \end{cases}$$

- different non symmetric J : a transport term should appear
- countable infinite edges
- nonlinear nonlocal convection diffusion (some progress by Dragos Manea)
- BBM-B on graphs: asymptotics (some progress with Nicola de Nitti)
- fractional laplacian type equations on similar structures

To be continued...

Uniqueness when no symmetry occurs

$$\begin{cases} \partial_t u_i - \partial_{xx} u_i - \alpha_i \partial_x (u_i^2) = 0, & t > 0,\ x < 0,\ i \in \{1, \dots, n\}, \\ \partial_t u_j - \partial_{xx} u_j - \alpha_j \partial_x (u_j^2) = 0, & t > 0,\ x > 0,\ i \in \{n+1, \dots, n+m\}, \\ u_i(t,0) = u_j(t,0), & t > 0,\ i,j \in \{1, \dots, n+m\}, \\ \sum_{i=1}^{n} \left(\partial_x u_i(t,0) + \alpha_i u_i^2(t,0) \right) = \sum_{j=n+1}^{n+m} \left(\partial_x u_j(t,0) + \alpha_j u_j^2(t,0) \right),\ t > 0, \end{cases} \tag{19}$$

with the initial datum a delta at the origin with the initial datum a delta at the origin taken in the sense of measures

$$\lim_{t \to 0} \sum_{i=1}^{n} \int_{-\infty}^{0} u_i(t,x) \varphi_i(x) dx + \sum_{j=n+1}^{n+m} \int_{0}^{\infty} u_j(x) \varphi_j(t,x) dx = M \boldsymbol{\varphi}(0) \tag{20}$$

Thanks for your attention!