Asymptotics for partially dissipative hyperbolic systems without Fourier analysis

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Partially dissipative hyperbolic systems

Partially dissipative hyperbolic systems

$$\partial_t U + \sum_{j=1}^{d} A_j(U)\partial_{x_j} U = -BU, \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}_+.$$

- $U = (U_1, U_2)(t,x) \in \mathbb{R}^n$ $(n \geq 2)$: the unknown
  $U_1 \in \mathbb{R}^{n_1}, U_2 \in \mathbb{R}^{n_2}$, $1 < n_1, n_2 < n$, $n_1 + n_2 = n$.
- A_i: $n \times n$ smooth symmetric matrics
- B: $n \times n$ symmetric matrix

$$B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix},$$

where the $n_2 \times n_2$ matrix D satisfies the strong dissipativity condition

$$(DU_2, U_2) \geq \kappa |U_2|^2.$$

Example: Compressible Euler system with damping

The compressible Euler system with damping

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) + \nabla P(\rho) = -\lambda \rho u. \end{cases} \tag{1}$$

- ρ: Density.
- u: Velocity.
- $P(\rho)$: Pressure.
- $\lambda > 0$: Friction coefficient.
- For γ-law $P(\rho) = A\rho^\gamma$ with $A > 0$ and $\gamma > 1$, let $c = \sqrt{\frac{\partial P(\rho)}{\partial \rho}}$ be the sound speed. Then System (1) can be symmetrized in terms of (c, u) :

$$\begin{cases} \partial_t c + u \cdot \nabla c + \dfrac{\gamma - 1}{2} c \mathrm{div}\, u = 0, \\ \partial_t u + u \cdot \nabla u + \dfrac{\gamma - 1}{2} c \mathrm{div}\, c = -\lambda u. \end{cases}$$

Progress: Local theorem without relaxation

General hyperbolic system of balance laws

$$\partial_t U + \sum_{j=1}^{m} \partial_{x_j} F_j(U) = 0.$$

where $\partial_U F_j(U) = A_j(U)$.

• Local well-posedness:

Kato (1975), Majda (1984), Hormander (1996), et. al. : H^s $(s > \frac{d}{2} + 1)$

Iftimie (1999) , Bahouri-Chemin-Danchin (2011), et. al. : $B_{2,1}^{\frac{d}{2}+1}$

• Finite-time singularities even for small and smooth initial data:

Dafermos (2010), et. al.

Progress: Global theorem with relaxation

Dissipative hyperbolic systems

$$\partial_t U + \sum_{j=1}^{d} A_j(U)\partial_x U = -BU.$$

B can stabilize hyperbolic systems:

- Full dissipative case $(BU, U) \geq \kappa |U|^2$.

  B provides exponential decay of U.

- Partially dissipative case $(BU, U) \geq \kappa |U_2|^2$.

  Missing the dissipation of U_1, i.e., lack of coercivity. However, the coupling of (A, B) can complete the dissipation and provides algebraic decay of U.

Progress: Global theorem with relaxation

- Shizuta-Kawashima condition (SK) (Shizuta-Kawashima (1985)):

$$\{\text{eigenvectors of } A(\xi)\} \cap \mathrm{Ker}(B) = \{0\} \quad \forall \xi \in \mathbb{R}^d,$$

where $A(\xi) = \sum_{j=1}^{d} \xi_j A_j$, and ξ is the Fourier variable.

- Kalman rank condition (K) (Beauchard-Zuazua (2011)):

The (SK) is equivalent to (K):

$$\mathrm{Rank}[B, A(\xi)B, ..., A(\xi)^{n-1}B] = n.$$

Under (SK) or (K), there exists a Lyapunov functional $\mathcal{L}_\xi(t) \sim |\widehat{U}|^2$ such that

$$\frac{d}{dt}\mathcal{L}_\xi(t) + \min\{1, |\xi|^2\}\mathcal{L}_\xi(t) \lesssim 0.$$

When $d = 1$, $A(\xi) = \xi A$ where ξ is scalar. Then (SK) and (K) do not involve the Fourier variable ξ.

Progress: Global theorem with relaxation and (SK)/(K)

- Global well-posedness for small initial perturbations:
  - Wang-Yang (2001), Sideris-Thomases-Wang (2003): Damped Euler in H^s $(s > \frac{d}{2} + 1)$
  - Hanouzet-Natalini (2003), Yong (2004), Beauchard-Zuazua (2011), et. al.: H^s $(s > \frac{d}{2} + 1)$
  - Xu-Kawashima (2014) : $B^{\frac{d}{2}+1}_{2,1}$
  - Crin Barat-Danchin (2022): $\dot{B}^{\frac{d}{2}}_{2,1} \cap \dot{B}^{\frac{d}{2}+1}_{2,1}$
- Long time asymptotics for small initial perturbations:

Bianchini-Hanouzet-Natalini (2007), Beauchard-Zuazua (2011), Kwwashima-Yong (2009), Xu-Kawashima (2015) , Crin Barat-Danchin (2022), et. al.$\cdots$

All the above results depend heavily on Fourier analysis tools.

Hypocoercivity for partially dissipative hyperbolic systems with Fourier analysis

We recall the Fourier hypocoercivity method by Beauchard-Zuazua (2011). Applying the Fourier transform for linear systems yields

$$\partial_t \widehat{U} + i|\xi| A(\xi)\widehat{U} + B\widehat{U} = 0.$$

Taking the inner product with $\widehat{U}$ and taking the real part, we get

$$\frac{1}{2}\frac{d}{dt}|\widehat{U}|^2 + \kappa|\widehat{U}_2|^2 = 0.$$

This only provides the dissipation of $|U_2| \sim |B\widehat{U}|$. In addition, we need the cross energy:

$$\frac{d}{dt}\mathrm{Re}\sum_{k=1}^{n-1} \varepsilon_k < BA_\omega^{k-1}\widehat{U} \cdot BA_\omega^k \widehat{U} > + |\xi| \sum_{k=1}^{n-1} \varepsilon_k |BA_\omega^k \widehat{U}|^2 = \mathcal{R}.$$

Here $A_\omega = \sum_{j=1}^d \omega_j A_j$ with $\omega|\xi| = \xi$. The cross energy can complete the dissipation of U_1, and the reminder $\mathcal{R}$ is bounded by the direct dissipation of U_2.

Hypocoercivity with Fourier analysis

Choosing suitable small constants ε_k we can estimate the reminder $\mathcal{R}$:

$$\mathcal{R} \leq C\varepsilon_0 \max\{|\xi|, |\xi|^{-1}\}|B\widehat{U}|^2.$$

Define the Lyapunov functional

$$\mathcal{L}_\xi(t) \triangleq |\widehat{U}|^2 + \min\left\{\frac{1}{|\xi|}, |\xi|\right\}\mathrm{Re}\sum_{k=1}^{n-1} \varepsilon_k < BA_\omega^{k-1}\widehat{U} \cdot BA_\omega^k \widehat{U} >.$$

Then since ε_k are suitable small, we have

$$\frac{d}{dt}\mathcal{L}_\xi(t) + \min\{1, |\xi|^2\} \sum_{k=0}^{n-1} \varepsilon_k |BA_\omega^k \widehat{U}|^2 \lesssim 0, \quad \text{and} \quad \mathcal{L}_\xi(t) \sim |\widehat{U}|^2.$$

According to (K) condition, it holds that

$$\sum_{k=0}^{n-1} \varepsilon_k |BA_\omega^k \widehat{U}|^2 \sim |\widehat{U}|^2 \sim \mathcal{L}_\xi(t).$$

Hypocoercivity with Fourier analysis

Then Grönwall's inequality leads to the different behaviors in low and high frequencies:

$$|\widehat{U}| \lesssim e^{-|\xi|^2 t}|\widehat{U_0}|, \qquad \|U^\ell(t)\|_{L^2} \leq Ct^{-d/4}\|U_0\|_{L^1},$$
$$|\widehat{U}| \lesssim e^{-\gamma_* t}|\widehat{U_0}|, \qquad \|U^h(t)\|_{L^2} \leq Ce^{-\gamma_* t}\|U_0\|_{L^2}.$$

Theorem 1

(Bianchini-Hanouzet-Natalini (2007), Beauchard-Zuazua (2011)).

It holds that

$$\begin{cases} \|U(t)\|_{L^2} \lesssim (1+t)^{-\frac{d}{4}}\|U_0\|_{H^1\cap L^1}, \\ \|U_2(t)\|_{L^2} + \|\nabla U(t)\|_{L^2} \lesssim (1+t)^{-\frac{d}{4}-\frac{1}{2}}\|U_0\|_{H^1\cap L^1}. \end{cases}$$

The L^1-assumption can be replaced by the weaker $\dot{B}^{-\frac{d}{2}}_{2,1}$-assumption.

Motivations

Fourier analysis methods have known a growing importance in the study of linear and nonlinear PDEs. However,

- Fourier transform is defined in $\mathbb{R}^d$ or $\mathbb{T}^d$. It is not easy to apply it to initial boundary value problem.
- Fourier analysis usually depends on the linearization. Some effects on nonlinear terms may not be got, for example, the compressible Euler system with the nonlinear damping $\rho|u|^{r-1}u$.
- It is not well-suited to analyze numerical schemes.

Our goal is to study the large-time behaviors of partially dissipative hyperbolic systems without using any Fourier analysis tools.

Main results: Asymptotics for partially dissipative hyperbolic systems

Linear partially dissipative hyperbolic systems in 1D

We first consider linear partially dissipative hyperbolic systems on the real line:

$$\begin{cases} \partial_t U + A\partial_x U + BU = 0, \\ U_0(x,t) = U_0(x), \qquad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \end{cases} \tag{2}$$

where A, B are constant symmetric matrices and the matrix B satisfies

$$B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}, \qquad (DU_2, U_2) \geq \kappa |U_2|^2.$$

To highlight the partially dissipative structure, we decompose $U = (U_1, U_2)$ where $U_1 \in \mathbb{R}^{n_1}$ and $U_2 \in \mathbb{R}^{n_2}$. The couple (U_1, U_2) satisfies

$$\begin{cases} \partial_t U_1 + A_{1,1}\partial_x U_1 + A_{1,2}\partial_x U_2 = 0, \\ \partial_t U_2 + A_{2,1}\partial_x U_1 + A_{2,2}\partial_x U_2 = -DU_2, \end{cases} \quad \text{where } A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix},$$

with the initial data $(U_1, U_2)(x,t) = (U_{1,0}, U_{2,0})(x)$.

Recall that 1D (SK) and (K) do not involve the Fourier variable ξ.

Lemma

(Beauchard-Zuazua (2011))

The following assertions are equivalent:

- (A, B) *satisfies the Kalman rank condition* (K)*: the Kalman matrix*

$$\mathcal{K} := \begin{pmatrix} B \\ BA \\ \dots \\ BA^{n-1} \end{pmatrix} \text{ has the rank } n.$$

- (A, B) *satisfies the Shizuta-Kawashima* (SK) *condition:*

$$\mathrm{Ker}(B) \cap \{\text{eigenvectors of } A\} = \{0\}.$$

- *For any* $y \in \mathbb{C}$*,*

$$\Big(\sum_{k=0}^{n-1} |BA^k y|^2\Big)^{\frac{1}{2}} \text{ defines a norm, and } \quad \sum_{k=0}^{n-1} |BA^k y|^2 \sim |y|^2.$$

Hyperbolic hypocoercivity without Fourier analysis

Inspired by Hérau-Nier (2004, 2007), Villani (2010), Porretta-Zuazua (2016) on the study of Fokker-Planck type equations, we adapt the Fourier hyperbolic hypocoercivity method to a Fourier-free framework.

We explain our method for the toy model (linearized damped Euler system):

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x u + v = 0, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ (u,v)(x,0) = (u_0, v_0)(x). \end{cases}$$

Hyperbolic hypocoercivity without Fourier analysis

- H^1-energy:

$$\frac{1}{2}\frac{d}{dt}\|(u,v)\|_{L^2}^2 + \|v\|_{L^2}^2 = 0,$$
$$\frac{1}{2}\frac{d}{dt}\|\partial_x(u,v)\|_{L^2}^2 + \|\partial_x v\|_{L^2}^2 = 0.$$

- Cross energy

$$\frac{d}{dt}\int v\partial_x u + \|\partial_x u\|_{L^2}^2 - \|\partial_x v\|_{L^2}^2 + \int v\partial_x u = 0.$$

- Time-weighted energy

$$\frac{1}{2}\frac{d}{dt}(t\|\partial_x(u,v)\|_{L^2}^2) + t\|\partial_x v\|_{L^2}^2 = \frac{1}{2}\|\partial_x(u,v)\|_{L^2}^2.$$

Hyperbolic hypocoercivity without Fourier analysis

Define the time-weighted Lyapunov functional

$$\mathcal{L}(t) := \frac{1}{2}\|(u,v)\|_{H^1}^2 + \varepsilon_1 \int v\partial_x u + \varepsilon_2 \frac{1}{2} t \|\partial_x(u,v)\|_{L^2}^2.$$

Adjusting two small constants $\varepsilon_1, \varepsilon_2$, we have $\mathcal{L}(t) \sim \|(u,v)\|_{H^1}^2 + \varepsilon_2 t\|\partial_x(u,v)\|_{L^2}^2$ and

$$\frac{d}{dt}\mathcal{L}(t) + \|v\|_{H^1}^2 + \|\partial_x u\|_{L^2}^2 + t\|\partial_x v\|_{L^2}^2 \lesssim 0.$$

Integration in time gives rise to

$$\|\partial_x(u,v)\|_{L^2} \lesssim (1+t)^{-\frac{1}{2}}.$$

In addition, since the equation of v can be viewed as a damped equation with the source $-\partial_x u$, we also obtain the $(1+t)^{-\frac{1}{2}}$ decay of v in L^2.

Hyperbolic hypocoercivity

For general systems, we introduce the time-weighted Lyapunov functional

$$\mathcal{L}(t) := \|U(t)\|_{H^1}^2 + c_0 t\|\partial_x U(t)\|_{L^2}^2 + \sum_{k=1}^{n-1} \varepsilon_k \big(BA^{k-1}U, BA^k\partial_x U\big)_{L^2},$$

and apply Kalman rank condition to the dissipation

$$\sum_{k=0}^{n-1} \varepsilon_k \|BA^k \partial_x U(t)\|_{L^2}^2 \sim \|\partial_x U\|_{L^2}^2.$$

Theorem 2

(Crin Barat-Shou-Zuazua, 2023) *Assume that (A, B) satisfies the Kalman rank condition and let $U_0 \in H^1(\mathbb{R})$. Then the solution U to System* (2) *satisfies*

$$\|U_2(t)\|_{L^2} + \|\partial_x U(t)\|_{L^2} \le C(1+t)^{-\frac{1}{2}} \|U_0\|_{H^1}.$$

Space-weighted estimates

Theorem 2 does not provide the decay of the solution U in L^2. Under space-weighted conditions on the initial data, we get the decay rates of U in L^2 for any order.

Under the space-weighted condition $\||x|^\mu U_0\|_{L^2} + \||x|^\mu \partial_x U_0\|_{L^2} < \infty$, we can carry out hypocoericivity H^1-estimates on $|x|^\mu U$ developed in Theorem 2. Then the Caffarelli-Kohn-Nirenberg inequality implies

$$\|u(t)\|_{L^2} \lesssim \|\partial_x u(t)\|_{L^2}^{\frac{\mu}{1+\mu}} \||x|^\mu u(t)\|_{L^2}^{\frac{1}{1+\mu}} \lesssim \|\partial_x u(t)\|_{L^2}^{\frac{\mu}{1+\mu}}.$$

Then the Lyapunov inequality gives

$$\frac{d}{dt}\|(u,v)\|_{H^1}^2 + \|(u,v)\|_{H^1}^{2+\frac{2}{\mu}} \lesssim 0.$$

Solving the above differential inequality, we have

$$\|(u,v)\|_{H^1}^2 \lesssim t^{-\mu}.$$

This give batter decay of $\|(u,v)\|_{L^2}$.

Space-weighted estimates

Theorem 3

(Crin Barat-Shou-Zuazua, 2023) *Let* $\mu > \frac{1}{2}$*, and* U *be the global solution to System* (2)*. Assume* $A_{1,1} = 0$*,* $(DU_2, U_2) \geq \kappa |U_2|^2$ *with* $\kappa \geq \kappa_0$ *for some positive constant* κ_0 *and that the initial data* U_0 *satisfies*

$$X_0 := \|U_0\|_{H^1} + \||x|^{\mu} U_0\|_{L^2} + \||x|^{\mu} \partial_x U_0\|_{L^2} < \infty.$$

Then the solution U *of System* (2) *satisfies*

$$\begin{cases} \|U_1(t)\|_{L^2} \leq C(1+t)^{-\frac{\mu}{2}} X_0, \\ \|U_2(t)\|_{L^2} + \|\partial_x U(t)\|_{L^2} \leq C(1+t)^{-\frac{\mu}{2}-\frac{1}{2}} X_0. \end{cases}$$

Application I: Asymptotics for compressible Euler system with damping

Compressible Euler system with damping

Our methods can be applied to some concrete nonlinear systems, for example, the compressible Euler system with (linear) damping on the real line:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x P(\rho) = -\lambda \rho u, \\ (\rho, u)(x,0) = (\rho_0, u_0)(x), \qquad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \end{cases} \tag{3}$$

where the general pressure P satisfies

$$P(\rho) \in C^\infty(\mathbb{R}_+), \qquad P'(\rho) > 0.$$

Damped compressible Euler system

Theorem 4

(Crin Barat-Shou-Zuazua, 2023) *Let $\bar{\rho} > 0$ be a given constant. Assume that the initial data (ρ_0, v_0) fulfills*

$$\|(\rho_0 - \bar{\rho}, v_0)\|_{H^2} << 1. \tag{4}$$

Then the damped compressible Euler system (3) *admits a unique global solution (ρ, v) which satisfies*

$$\|v(t)\|_{L^2} + \|\partial_x(\rho - \bar{\rho}, v)(t)\|_{L^2} \lesssim (1+t)^{-\frac{1}{2}}. \tag{5}$$

For $\mu > \frac{1}{2}$, if $|x|^\mu(\rho_0 - \bar{\rho}, u_0) \in L^2(\mathbb{R})$, $|x|^\mu \partial_x(\rho_0 - \bar{\rho}, u_0) \in L^2(\mathbb{R})$ and $\lambda \geq \lambda_0$ with some $\lambda_0 > 0$, then the solution (ρ, v) satisfies

$$\begin{cases} \|(\rho - \bar{\rho})(t)\|_{L^2} \lesssim (1+t)^{-\frac{\mu}{2}}, \\ \|v(t)\|_{L^2} + \|\partial_x(\rho - \bar{\rho}, v)(t)\|_{L^2} \lesssim (1+t)^{-\frac{\mu}{2}-\frac{1}{2}}. \end{cases}$$

Application II: Asymptotics for nonlinearly damped p-system

Nonlinearly damped p-system

Our methods can be applied to some concrete nonlinear systems, for example, the nonlinearly damped p-system on the real line:

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x u + |v|^{r-1} v = 0, \\ (u, v)(x, 0) = (u_0, v_0)(x), \qquad (x, t) \in \mathbb{R} \times \mathbb{R}_+, \end{cases} \tag{6}$$

with $1 < r < 3$. System (6) is a a simplified version of the nonlinearly damped compressible Euler system, and can be used to simulate gas-networks.

Classical Fourier analysis may not be applied to analyze the weak dissipation effects caused by $|v|^{r-1}v$ in (6).

Nonlinearly damped p-system

The energy of (6) reads

$$\frac{1}{2}\frac{d}{dt}\|(u,v)\|_{L^2}^2 + \int |v|^{r-1}|v|^2 = 0,$$
$$\frac{1}{2}\frac{d}{dt}\|(\partial_x u,\partial_x v)\|_{L^2}^2 + r\int |v|^{r-1}|\partial_x v|^2 = 0.$$

The above dissipation degenerates near $v=0$ when we use it to control $\|(u,v)\|_{H^1}^2$.

We develop time-space weighted hypocoercivity estimates on the Lyapunov functional

$$\mathcal{L}(t) := \int \frac{1}{2}w_1(t+|x|)(|u^2+|v|^2) - \varepsilon\int w_1'(t+|x|)v\int_{-\infty}^{x} u$$
$$-\varepsilon\int \frac{1}{2}w_1''(t+|x|)\Big|\int_{-\infty}^{x} u\Big|^2 + \varepsilon\int w_2(t+|x|)\Big|\int_{-\infty}^{x} u\Big|^{r+1},$$

where $w_1(s) = \log^{2q}(a+s)$, $w_2(s) = \frac{\log^{2q-r+1}(a+s)}{|a+s|^r}$ for all $q>0$ some suitable large constant $a>0$. This leads to obtain the logarithmic time-decay rates of solutions for weighted initial data.

Asymptotics for nonlinearly damped p-system

Theorem 5

(Crin Barat-Shou-Zuazua, 2023) *For any* $q > 0$, *suppose that the initial data* (u_0, v_0) *satisfies*

$$u_0 \in L^1(\mathbb{R}), \quad \log^q(1+|x|)(u_0, v_0) \in L^2(\mathbb{R}),$$

Then the solution (u, v) *to*

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x u + |v|^{r-1} v = 0, \\ (u, v)(x, 0) = (u_0, v_0)(x), \end{cases}$$

satisfies

$$\|(u, v)(t)\|_{L^2} \le \frac{C}{\log^q(1+t)}.$$

Extensions

We explain some possible extensions.

- Non-symmetric relaxation matrix. For example, Euler-Maxwell system.
- Our methods may be applied to the initial boundary value problem.
- Our methods can be applied to the compressible Euler system with damping in any dimension.
- It is difficult to study high-dimensional general systems due to the Fourier variable ξ in (SK) condition. However, we may consider some new stability condition to promise Euler-like structure.
- Euler system with nonlinear damping. The main difficulty comes from the nonlinear terms $u \cdot \nabla \rho, u \cdot \nabla u$ which can not be controlled by nonlinear dissipation. One may establish faster decay rates to get time integrability.
- Numerical analysis with finite difference approximation.

Thank you for your attention!