Stability of Elastic Systems with Local Damping

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Overview

Control and Stabilization of Linear System

Consider the system

$$\frac{d}{dt}z(t) = Az(t) + Bu(t), \quad z(0) = z_0 \in Z. \tag{1}$$

- **Stabilization**: $\lim_{n\to\infty} \|z(t)\| = 0$.
- **Feedback**: $u(t) = Kz(t)$.
- **Closed-loop system**: $\frac{d}{dt}z(t) = (A + BK)z(t) \doteq \mathcal{A}z(t)$.

Stability of closed-loop system

Finite system: $\mathcal{A} \in M(n,n)$, $z(t) = e^{\mathcal{A}t} z_0$.

Infinite system: $\mathcal{A}$ generates a C_0 semigroup of contractions on H.

Definition

$e^{\mathcal{A}t}$ is

- logarithmically stable of order β if

$$\|e^{\mathcal{A}t}\mathcal{A}^{-1}\| \leq \frac{C}{[\log(t+1)]^{\beta}}, \qquad t \to \infty.$$

- polynomially stable of order β if

$$\|e^{\mathcal{A}t}\mathcal{A}^{-1}\| = \frac{C}{(t+1)^{\beta}}, \qquad t \to \infty.$$

- exponentially stable if there are constants $M, \omega > 0$ such that

$$\|e^{\mathcal{A}t}\| \leq M e^{-\omega t}, \qquad t \geq 0.$$

Damping mechanism

- Type : viscous damping, Boltzmann damping, Kelvin-Voigt damping, thermal damping...
- Position: global, local.

Wave equation with global viscous damping

$$\begin{cases} w_{tt} - \Delta w + a w_t = 0 & \text{in} \quad (0,\infty)\times\Omega, \\ w = 0 & \text{on} \quad (0,\infty)\times\Gamma, \end{cases}$$

The semigroup exponentially stable [Chen, Fulling. etc, 1991]

Wave equation with global Kelvin-Voigt damping

$$\begin{cases} w_{tt} - div(\nabla w + a\nabla w_t) = 0 & \text{in} \quad (0,\infty)\times\Omega, \\ w = 0 & \text{on} \quad (0,\infty)\times\Gamma, \end{cases}$$

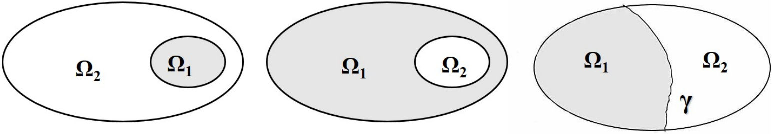
The semigroup is analytic and exponentially stable [Russell, Huang, 1988], [Chen, Liu and Liu, SICON, 1998]), [B.Z.Guo, J.M.Wang and G.D.Zhang,ZAMP, 2010].

Wave equation with local viscous damping

$$\begin{cases} w_{tt} - \Delta w + a(x) w_t = 0 & \text{in} \quad (0,\infty) \times \Omega, \\ w = 0 & \text{on} \quad (0,\infty) \times \Gamma, \end{cases}$$

where $a(x) = \begin{cases} b(x) \geq 0, & x \in \Omega_1, \\ 0, & x \in \Omega_2, \end{cases}$

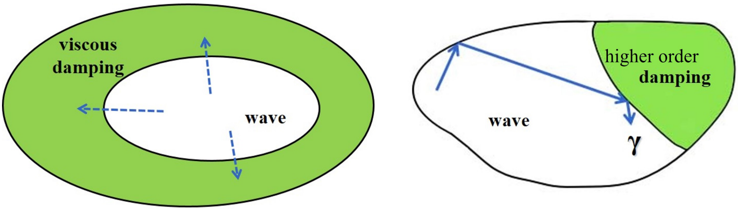
- Exponentially stable under Geometrical Control Condition [Bardos, Lebeau and Rauch, 1993]
- Logarithmically stable if damping subdomain is arbitrary [X.Fu, 2009,2011)



Wave equation with local Kelvin-Voigt damping

$$\begin{cases} w_{tt} - div(\nabla w + a(x)\nabla w_t) = 0 & \text{in} \quad (0,\infty)\times\Omega, \\ w = 0 & \text{on} \quad (0,\infty)\times\Gamma, \end{cases}$$

- 1-d, $a = 1$ on Ω_1 and $a = 0$ on Ω_2 $\Rightarrow$ Non-exponentially stable [Chen, Liu and Liu, 1998].
- $a(\cdot) \in C^2$, $a \geq 0$, GCC $\Rightarrow$ Exponentially stable [Liu and Rao, 2004].
- wave-heat transmission system $\Rightarrow$ Non-exponentially stable [Rauch, Zhang and Zuazua,2005], [Zhang and Zuazua, 2007].

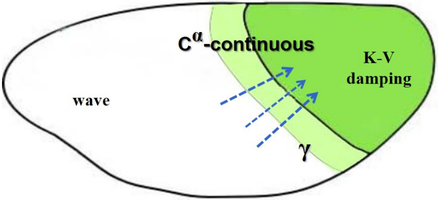


These results reveals that too strong control could destroy the property of the system, such as exp. stability. Stability and Regularity of wave equation with local K-V damping depends on

- location of the damping
- smoothness of the material coefficient near the interface

Problem:

- How about the case $a(\cdot) \in C(\overline{\Omega}) \backslash C^2(\overline{\Omega})$??
- The relationship between $a(\cdot)$ and stability, regularity of the semigroup?



Theorem

Let $\Omega \subset R^N$ be a bounded convex connected domain with partition $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ and $a = 1$ on Ω_1 and $a = 0$ on Ω_2.

(i) *The semigroup is not exponentially stable.*

(ii) *If $\partial\Omega_1$ and $\partial\Omega_2$ are convex curvilinear polygons or curved plane polyhedrons, Γ_1, $\gamma \neq \emptyset$ and $(m \cdot \nu_2)|_{\Gamma_2} \leq 0$, where $m(x) = x - x_0$, $x_0 \in \mathbb{R}^2$ or $\mathbb{R}^3$. Then, the semigroup is polynomially stable with order $\frac{1}{2}$.*

(ii)' *If $\partial\Omega = \Gamma_1$, $\partial\Omega_2 = \gamma$, Γ_1 and γ are of C^2 class, then the semigroup is polynomially stable with order 1.*

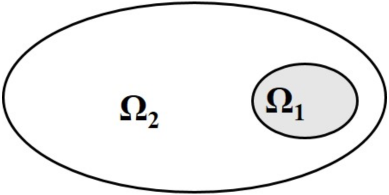
[Q.Z.,Nonal.Anal.RWA,2017], [Q.Z.,ZAMP,2018], [Z.Han,K.Yu,Q.Z., ZAMM, 2022]

Wave equation with local K-V damping with or without GCC

Theorem

Let $\Omega \subset R^N$ be a bounded convex connected domain with partition $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$. If $a(\cdot) \in C_0^\infty(\Omega)$ is nonnegative and supp $a \neq \emptyset$, then the semigroup is logarithmically stable of order $\frac{4}{5}$.

[L.Robbiano, Q. Z.]



String with local K-V damping

$$\begin{cases} w_{tt}(t,x) = [w_x(t,x) + a(x)w_{t,x}(t,x)]_x, & x \in (-1,1) \\ w(t,-1) = w(t,1) = 0, & t \in R^+ \\ w(0,x) = w_0(x), \quad w_t(0,x) = v_0(x), \end{cases} \tag{2}$$

where the damping coefficient function $a(x)$ satisfies $a(x) = 0$ for $x \in [-1,0]$, $a(x) > 0$ for $x \in (0,1]$, and for $\alpha > 0$,

$$\lim_{x \to 0^+} \frac{a(x)}{x^\alpha} = k > 0 \tag{H}$$

Previous results:

- If $a(\cdot)$ satisfies (H) with some $\alpha = 0$. then the semigroup of system (2) is polynomially stable with order 2 [Z. Liu and B. Rao, 2005]
- If $a(\cdot)$ satisfies (H) with $\alpha > 1$, then the real part of the eigenvalues of system (2) is unbounded below [M. Renardy, 2004].

Theorem

Suppose that $a(\cdot)$ satisfies (H) with some $\alpha \geq 0$. Then,

(i) *For $0 < \alpha < 1$, the semigroup of system (2) is polynomially stable of order $\frac{1}{1-\alpha}$*

(ii) *For $\alpha \geq 1$, the semigroup is exponentially stable.*

(iii) *For $\alpha > 1$, the semigroup is eventually differentiable.*

α value	Regularity	Stability
$(1, \infty)$	eventually differentiable	exponentially stable
1	**no smoothing**	exponentially stable
$(0, 1)$	**no smoothing**	polynomially stable of order $\frac{2-\alpha}{1-\alpha}$
0	no smoothing	optimal polynomially stable of order 2

[K.Liu,Z.Liu,Q.Z., ESIAM,COCV,2017], [Z.Liu,Q.Z.,SICON,2016], [Z.Han,Z.Liu,Q.Z.,ZAMM,2022], [M.Ghader,R.Nasser,A.Wehbe,MMAS,2021]

Lemma

Let $A : D(A) \subset H \to H$ generate a bounded C_0-semigroup e^{tA} on Hilbert space H. Assume that $i\omega \in \rho(A), \quad \forall\, \omega \in \mathbb{R}$. Then

(i) *(**Pazy**) e^{tA} is differentiable for $t > t_0 > 0 \Leftrightarrow$*

$$\rho(A) \supset \Theta_{\varsigma,b} = \{\lambda \in \mathbb{C} \,:\, \mathcal{R}e\lambda \geq \varsigma - b\ln|\mathcal{I}m\lambda|\}$$

$$\sup_{\lambda\in\Theta_{\varsigma,b},\,\mathcal{R}e\lambda<0} |\mathcal{I}m\lambda|^{-1}\|(\lambda I - A)^{-1}\|_{\mathcal{L}(H)} < \infty.$$

(ii) *(**Huang-Prüss**) e^{tA} is exponentially stable $\Leftrightarrow$*

$$\overline{\lim_{\omega\in\mathbb{R},|\omega|\to\infty}} \|(i\omega I - A)^{-1}\|_{\mathcal{L}(H)} < \infty.$$

(iii) *(**Borichev-Tomilov**) e^{tA} is polynomially stable of order $\frac{1}{\beta} \Leftrightarrow$*

$$\overline{\lim_{\omega\in\mathbb{R},|\omega|\to\infty}} |\omega|^{-\beta}\left\|(i\omega I - A)^{-1}\right\|_{\mathcal{L}(H)} < \infty.$$

(iv) *(**Batty-Duyckaerts**) e^{tA} is logarithmically stable of order $\frac{1}{\beta} \Leftrightarrow$*

$$\overline{\lim_{\omega\in\mathbb{R},|\omega|\to\infty}} e^{-c\,|\,\omega\,|^{\,\beta}}\left\|(i\omega I - A)^{-1}\right\|_{\mathcal{L}(H)} < \infty.$$

Lemma

Let $x^{\frac{\alpha}{2}} y' \in L^2(0,1)$ satisfy $y(1) = 0$. Assume that $\delta \geq \alpha - 2$ when $\alpha > 1$ and $\delta > -1$ when $0 \leq \alpha < 1$. Then there exists positive constant C, independent of y, such that

$$\int_0^1 x^\delta |y(x)|^2 dx \leq C \int_0^1 x^\alpha |y'(x)|^2 dx. \tag{3}$$

Lemma

Assume $\varepsilon > 0$ be arbitrary, function y satisfies $x^{\frac{1+\varepsilon}{2}} y' \in L^2(0,1)$ and $y(1) = 0$. Then for any $\delta \geq -1 + \varepsilon$, there exists positive constant C, independent of y, such that

$$\int_0^1 x^\delta |y(x)|^2 dx \leq C \int_0^1 x\, |y'(x)|^2 dx. \tag{4}$$

[V.D.Stepanov,Siberian Math.J.,1987], [M.Renardy,2004], [Z.Liu,Q.Z.,SICON,2016]

Abstract system with local damping

Consider an abstract system of second order equation:

$$\begin{cases} u_{tt} + Lu + Bu_t = 0, \quad t > 0, \\ u(0) = u_0, \;\; u_t(0) = u_1, \end{cases} \tag{5}$$

where $L : D(L) \subset H_0 \to H_0$ is a self-adjoint, positive, densely defined, linear (unbounded) operator with compact resolvent. B is nonnegative.

Example:

$$\begin{cases} u_{tt} - \Delta u + b_1(x)u_t - \operatorname{div}\left(b_2(x)\nabla u_t\right) = 0 & \text{in} \quad (0,\infty)\times\Omega, \\ u = 0 & \text{on} \quad (0,\infty)\times\partial\Omega, \\ u(0) = u_0, \quad u_t(0) = u_1 & \text{in} \quad \Omega, \end{cases}$$

where $b_1,\; b_2 \geq 0$.

The energy function is

$$E(t) = \frac{1}{2}(\|u_t\|^2 + \|L^{\frac{1}{2}}u\|^2).$$

and it is dissipated according to the following law:

$$\frac{d}{dt}E(t) = -\langle Bu_t,\, u_t\rangle.$$

When damping operator B is positive and $B \sim L^\alpha$ with $\alpha \in \mathbb{R}$, the associated C_0 semigroup e^{At} is

1. analytic for $\frac{1}{2} \leq \alpha \leq 1$,
2. of Gevrey class of order $\delta > \frac{1}{2\alpha}$ for $0 < \alpha < \frac{1}{2}$,
3. not differentiable for $\alpha = 0$,
4. exponentially stable for $0 \leq \alpha$,
5. polynomially stable with optimal decay rate $\frac{1}{2|\alpha|}$ for $\alpha < 0$

([G. Chen, D.L. Russell, Q. Appl. Math. 1981], [S.Chen, R.Triggiani, Pac. J.Math. 1989], [S. Chen, R. Triggiani, Proc. Am. Math. Soc. 1990], [F. Huang, Acta Math. Sci. 1985], [F. Huang, SICON, 1988], [F. Huang, K. Liu, Ann. Differ. Equ. 1988], [K. Liu, Z. Liu, JDE 1997], [Z.Liu, J. Yong, Adv. Differ. Equ. 1998], [Z.Liu, Q. Zhang, ZAMP, 2015], etc.)

Question: If the the influence of the dissipative operator B is not enough, one can expect a weaker decay rate.

□ Assume that the spectrum of L consists in a sequence of distinct eigenvalues $\{\lambda_k\}_{k\geq 1}$, with the least eigenvalue $\lambda_1 > 0$, numbered in an increasing order and $\lambda_k \to +\infty$ as $k \to \infty$, each eigenvalue λ_k having multiplicity $m_k \geq 1$.

□ We assume the operator B satisfies

$$\begin{cases} B : H_1 \doteq D(L^{\frac{1}{2}}) \to H_{-1} \doteq (H_1)' \text{ is bounded and self-adjoint,} \\ \langle B\phi, \phi\rangle \geq 0, \quad \forall\, \phi \in H_1. \end{cases} \tag{H1}$$

Previous results:

- Strong stability.
  [S.Chen, K. Liu, Z. Liu, SIAM J. Appl. Math.,1999]

- Exponential stability:

$$\frac{dU}{dt} = A_1 U(t) + B_1 U(t) \quad \text{for} \;\; t > 0.$$

where $A_1^* = -A_1$, compact resolvent,

$$\inf\{|\mu_j - \mu_k| \; : \; j, k = 1, 2, 3, \ldots, j \neq k\} = \gamma > 0. \qquad (*)$$

$$\|B_1\phi\| \geq \delta > 0, \quad \text{for any unit eigenfunction } \phi \text{ of } A_1. \qquad (**)$$

[G. Chen, S. A. Fulling, F. J. Narcowich, S. Sun, SIAM J. Appl. Math, 1991]

Main Results

We can reformulate abstract second order system in a semigroup setting on the Hilbert space

$$\mathbb{H} \doteq H_1 \times H_0,$$

$$(U_1,\ U_2)_{\mathbb{H}} = (L^{\frac{1}{2}} u_1,\ L^{\frac{1}{2}} u_2) + (v_1,\ v_2), \quad \forall\ U_i = (u_i,\ v_i) \in \mathbb{H},\ i = 1, 2.$$

Define an unbounded linear operator $A : D(A) \in \mathbb{H} \to \mathbb{H}$ by

$$\begin{cases} A(u, v) = (v,\ -Lu - Bv), \\ D(A) = \{(u, v) \in \mathbb{H} \mid v \in H_1,\ Lu + Bv \in H_0\}. \end{cases}$$

Then, the abstract system of second order can be written as a first-order linear evolution equation on the space $\mathbb{H}$:

$$\frac{dU}{dt} = AU(t) \quad \text{for} \ \ t > 0, \quad U(0) = U^0 \doteq (u_0, u_1) \in \mathbb{H}.$$

A generates a C_0 semigroup of contractions on $\mathbb{H}$.

Theorem.

Assume that (H1) holds. Let $k_0 \geq 1$ be an integer, $\gamma_1 \geq 0$, $\gamma_2 \in \mathbb{R}$. Suppose that

$$\lambda_* \doteq \inf_{k\geq 1} \frac{\lambda_k}{\lambda_{k+1}} > 0, \tag{H2}$$

$$\frac{\lambda_{k-1}}{\lambda_k - \lambda_{k-1}} + \frac{\lambda_{k+1}}{\lambda_{k+1} - \lambda_k} \leq C\,\lambda_k^{\gamma_1}, \quad \forall\, k \geq k_0. \tag{H3}$$

$$\beta_k \doteq \min\left\{\langle B\phi, \phi\rangle \mid \phi \in N(L - \lambda_k I),\ \|\phi\| = 1\right\} > 0, \quad \forall\, k \geq k_0. \tag{H4}$$

$$\lambda_k^{\gamma_2} \leq C\beta_k\,, \quad \forall\, k \geq k_0. \tag{H5}$$

Set $m \doteq \max\{3 - 2\gamma_2 + 4\gamma_1,\ 1 + 4\gamma_1\}$. Then for all $(u_0, u_1) \in D(A)$, the solution $e^{At}(u_0, u_1)$ to the abstract system (5) satisfies

$$\|e^{At}(u_0, u_1)\|_{\mathbb{H}} \leq \frac{C}{t^{\frac{1}{m}}}\|(u_0, u_1)\|_{D(A)}, \quad \forall\, t \geq 1. \tag{6}$$

[O.Kavian,Q.Z.,JMAA,2022]

Application I: Wave Equation with Local Damping

Consider

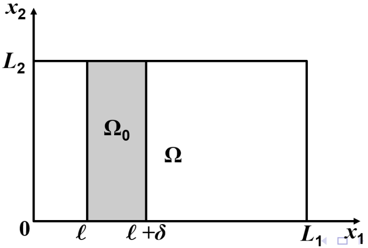
$$\begin{cases} u_{tt} - \Delta u + a(x)(-\Delta)^{\theta} u_t = 0 & \text{in} \quad (0,\infty) \times \Omega, \\ u = 0 & \text{on} \quad (0,\infty) \times \partial\Omega, \\ u(0) = u_0, \quad u_t(0) = u_1 & \text{in} \quad \Omega, \end{cases} \tag{7}$$

where $0 \leq \theta \leq 1$ is a real number,

$$\Omega = (0, L_1) \times \cdots \times (0, L_N) \subset \mathbb{R}^N, \quad L_j > 0,\ 1 \leq j \leq N. \tag{8}$$

Assume function $a(\cdot) = \chi_{\Omega_0}$ and

$$\Omega_0 = (\ell, \ell + \delta) \times (0, L_2) \times \cdots \times (0, L_N), \quad 0 \leq \ell < \ell + \delta \leq L_1. \tag{9}$$



Corollary

Let domains Ω, Ω_0 *be defined by* (8) *and* (9) *and satisfy*

$$\frac{L_i^2}{L_j^2} \in \mathbb{Q} \qquad \textit{for } \ 1 \leq i < j \leq N. \tag{10}$$

Then, the solution of (7) *satisfies*

$$\|(u(t),\, u_t(t))\|_{\mathbb{H}} \leq \frac{C}{t^{\frac{1}{7-2\theta}}} \|A(u_0, u_1)\|_{\mathbb{H}}, \quad \forall\, t \geq 1,$$

where $\mathbb{H} = H_0^1(\Omega) \times L^2(\Omega)$.

Corollary

Let domains Ω, Ω_0 satisfy (8), (9) and

$$\frac{L_i^2}{L_j^2} \text{ is an algebraic number of degree greater or equal to 2.} \tag{11}$$

The solution of (7) satisfies

$$\|(u(t),\, u_t(t))\|_{\mathbb{H}} \leq \frac{C}{t^{\frac{1}{11-2\theta+\varepsilon}}}\|A(u_0, u_1)\|_{\mathbb{H}}, \quad \forall\, t \geq 1.$$

Application II: Euler-Bernoulli Beam with Local Damping

Consider

$$\begin{cases} u_{tt} + \partial_{xxxx} u + a(x)(\partial_{xxxx})^{\theta} u_t = 0 & \text{in} \quad (0,\infty) \times (0,\pi), \\ u(0) = u_{xx}(0) = u(\pi) = u_{xx}(\pi) = 0 & \text{in} \quad (0,\infty), \\ u(0) = u_0, \quad u_t(0) = u_1 & \text{in} \quad (0,\pi), \end{cases} \tag{12}$$

where $0 \leq \theta \leq 1$ is a real number, function $a(\cdot) = \chi_{(\ell,\ell+\delta)}$ with $0 \leq \ell < \ell + \delta \leq \pi$. Set

$$H_0 = L^2(0,\pi), \quad H_1 = H^2(0,\pi) \cap H_0^1(0,\pi),$$

and operators

$$Lu = \partial_{xxxx} u, \quad D(L) = \left\{ u \in H_1 \mid u \in H^4(0,\pi), u_{xx}(0) = u_{xx}(\pi) = 0 \right\},$$

$$Bu = a(x)(\partial_{xxxx})^{\theta} u, \quad D(B) = H^2(0,\pi) \cap H_0^1(0,\pi).$$

Then L is a selfadjoint, positive operator with a compact resolvent. B satisfies condition (H1).

Corollary

Assume that $a(\cdot) = \chi_{(\ell,\ell+\delta)}$. *Then the solution of* (12) *satisfies*

$$\|(u(t),\, u_t(t))\|_{\mathbb{H}} \leq \frac{C}{t^{\frac{1}{4-2\theta}}} \|A(u_0, u_1)\|_{\mathbb{H}}, \quad \forall\, t \geq 1,$$

where $\mathbb{H} = H_1 \times H_0$.

Thanks For Your Attention!