Geometric guidance strategies for terminal angle and time control problems

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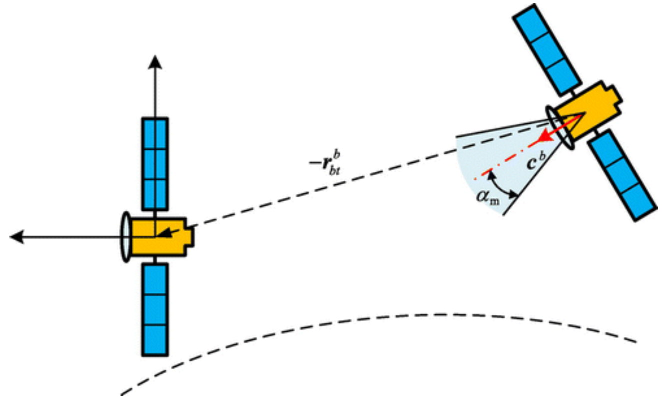
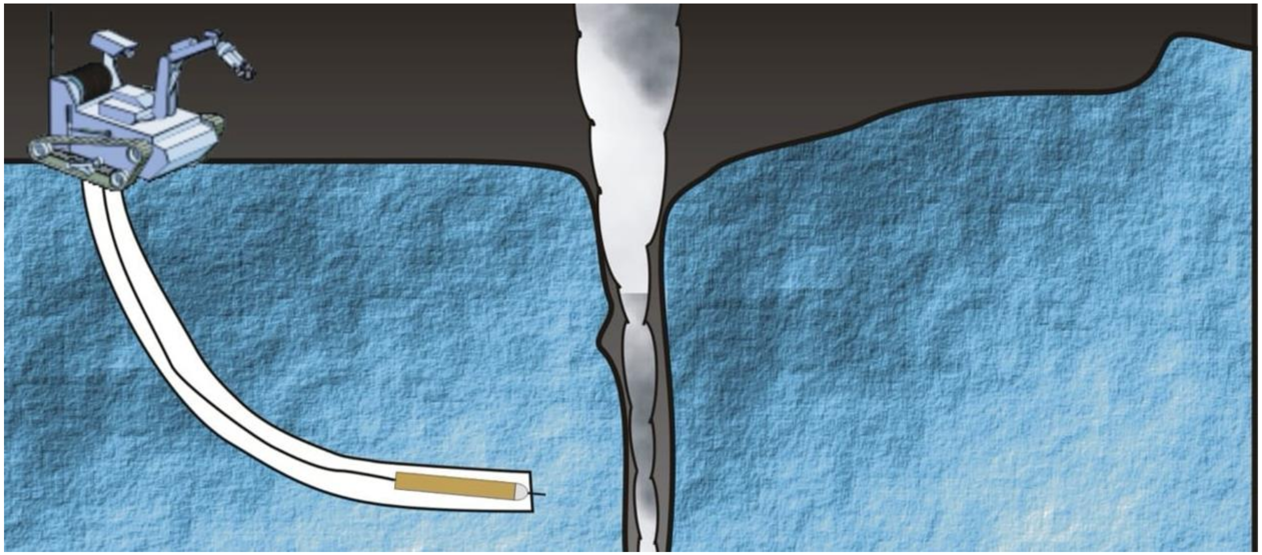
Outline

- ❑ Background & Problem description
- ❑ Geometric rule of the circle involute
- ❑ Trajectory design
- ❑ Guidance law design
- ❑ Simulation results

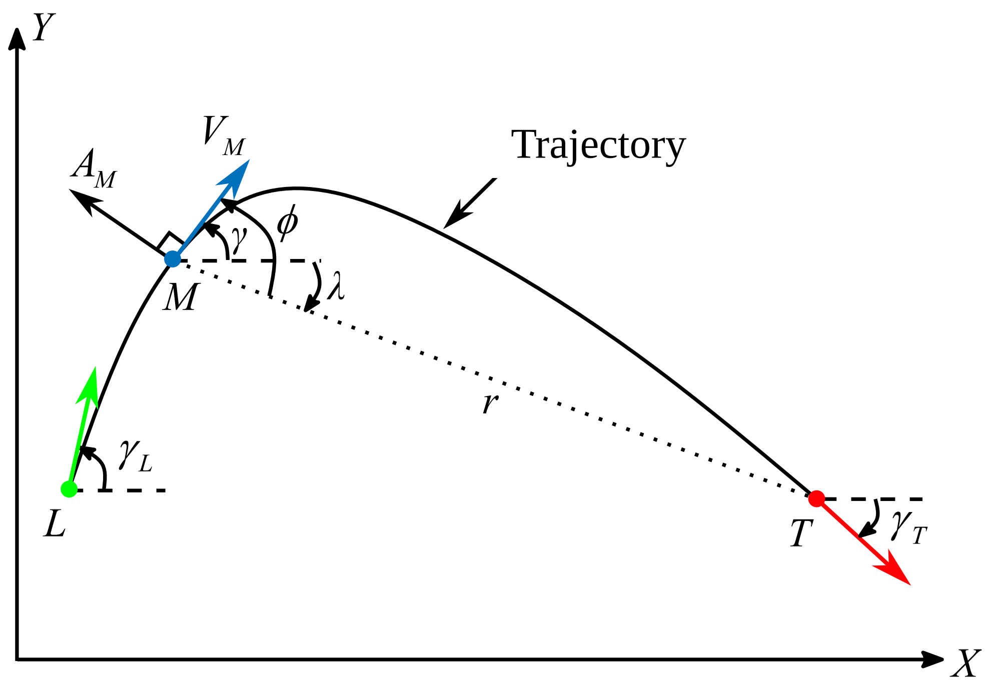
Trajectory with different constraints

Impact angle, impact time, and field of view





Problem description



Planar engagement geometry

Kinematic equations

$$
\begin{cases}
\dot{x}_M = V_M \cos\gamma \\
\dot{y}_M = V_M \sin\gamma \\
\dot{\gamma} = \dfrac{A_M}{V_M} \\
\dot{r} = -V_M \cos(\gamma - \lambda) \\
r\dot{\lambda} = -V_M \sin(\gamma - \lambda)
\end{cases}
$$

Guidance objectives

$$
\begin{cases}
r_f \to 0 \\
t_f \to t_{des} \\
\gamma_f \to \gamma_T \\
|\phi| \le \phi_{\max}
\end{cases}
$$

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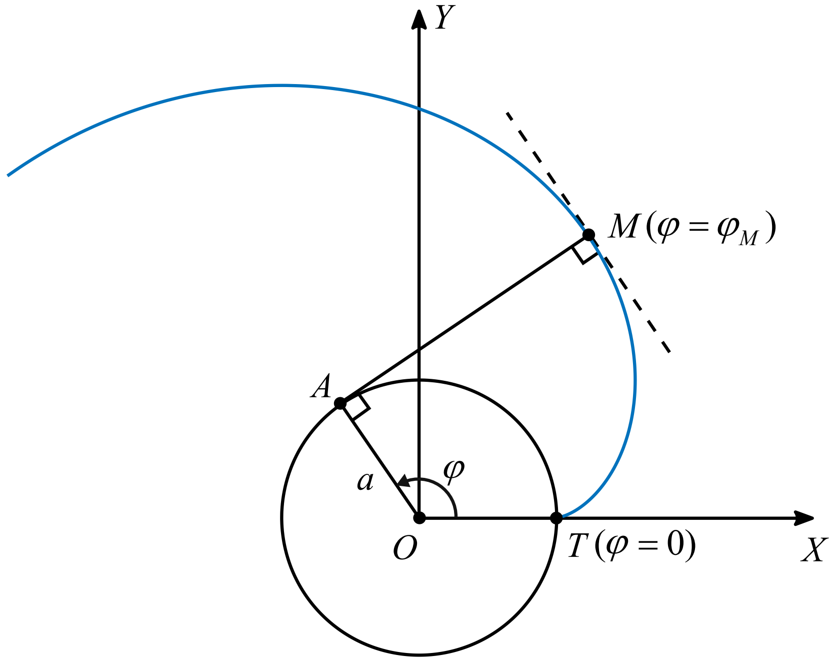
Involute of a circle

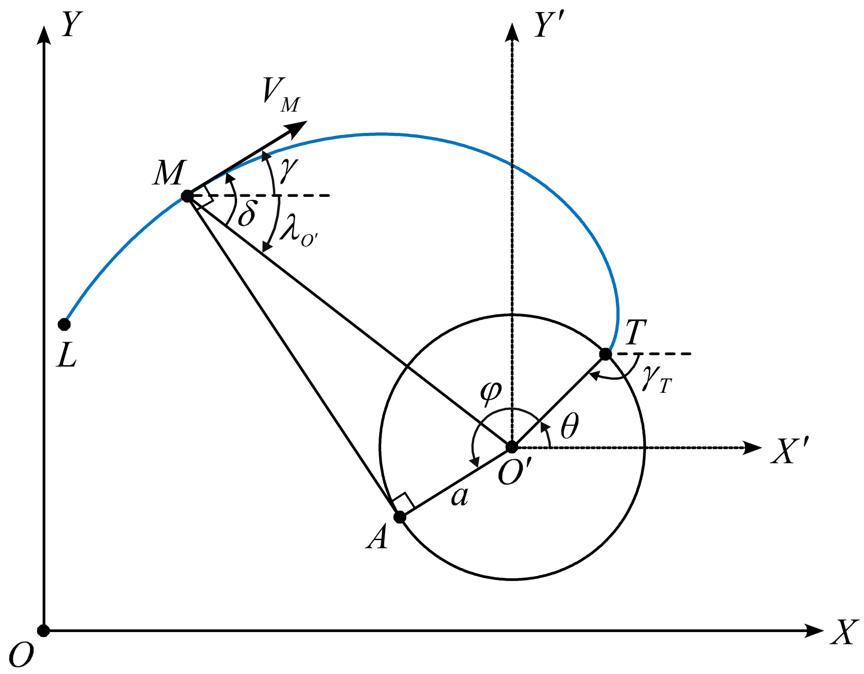


Parametric equations

$$\begin{cases} x = a\left(\cos\varphi + \varphi\sin\varphi\right) \\ y = a\left(\sin\varphi - \varphi\cos\varphi\right) \end{cases}$$

- ❑ The involute is tangent to the horizontal axis at one end
- ❑ The normal line to the involute is tangent to the circle
- ❑ Radius of curvature: $MA = \overset{\frown}{TA} = a\varphi_M$
- ❑ The arc length of an involute is given by $S(\varphi) = (1/2)a\varphi^2$

Parametric equations for the rotated involute



1

$$\begin{cases} x = a\left(\cos\varphi + \varphi\sin\varphi\right) \\ y = a\left(\sin\varphi - \varphi\cos\varphi\right) \end{cases}$$

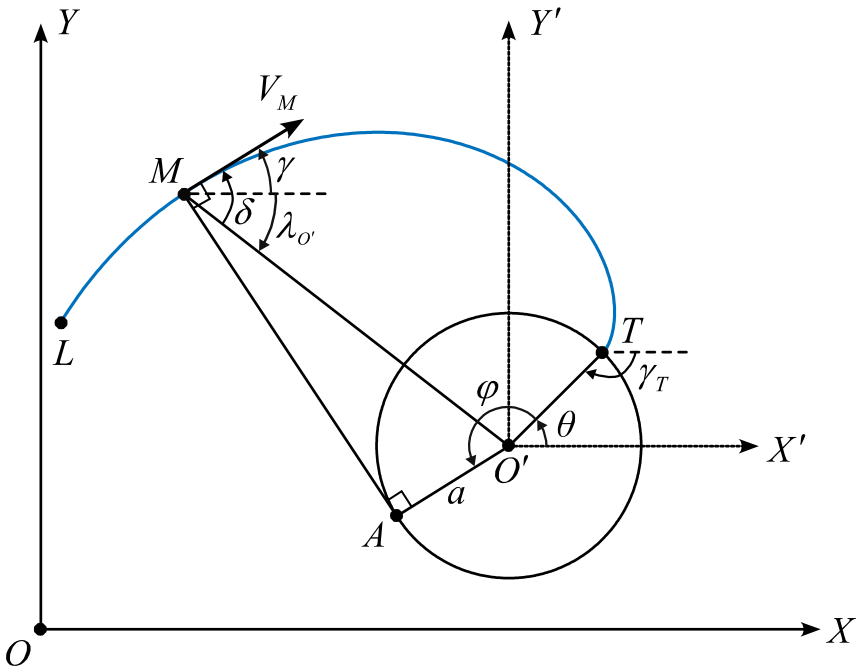
center of the circle $O'(x_{O'}, y_{O'})$

rotation angle $\theta = \gamma_T + \pi$

2

$$\begin{cases} x = x_{O'} + a(\cos(\varphi+\theta) + \varphi\sin(\varphi+\theta)) \\ y = y_{O'} + a(\sin(\varphi+\theta) - \varphi\cos(\varphi+\theta)) \end{cases}$$

Geometric Rule



$$\delta = \angle MO'A$$

$$\tan\delta = \tan\angle MO'A = \frac{MA}{O'A} = \frac{a\varphi}{a} = \varphi = \gamma - \gamma_T$$

❑ The geometric rule for an involute trajectory with an impact angle γ_T can be written as $\tan\delta = \gamma - \gamma_T$, where $\delta = \gamma - \lambda_{O'}$

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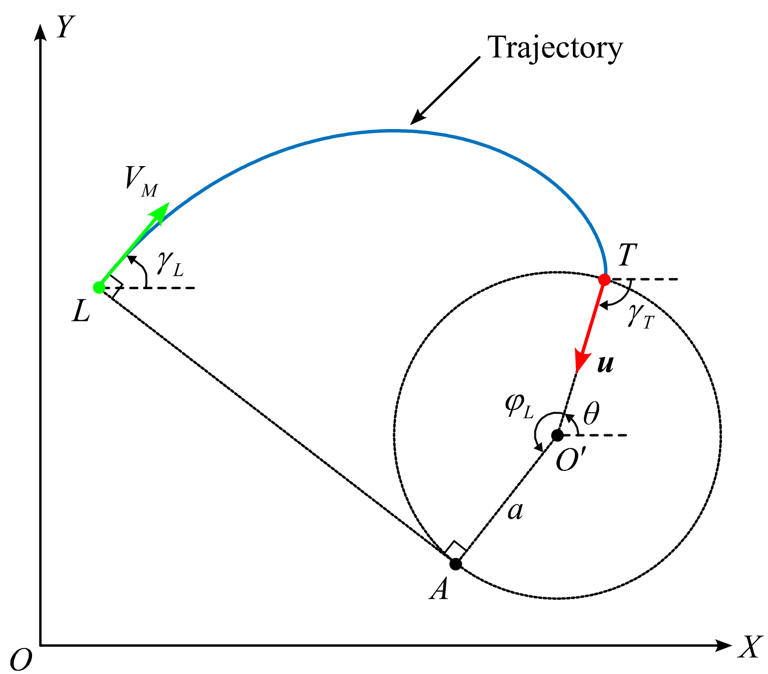
- ❑ **Guidance law design**
- ❑ **Analysis of achievable ranges**

One-stage trajectory for impact angle constraint

Goal: find the value of a



1

$$\begin{cases} x = x_{O'} + a(\cos(\varphi+\theta) + \varphi\sin(\varphi+\theta)) \\ y = y_{O'} + a(\sin(\varphi+\theta) - \varphi\cos(\varphi+\theta)) \end{cases}$$

$$\theta = \gamma_T + \pi$$

$$\varphi_L = \gamma_L - \gamma_T$$

$$\mathbf{O}' = \mathbf{T} + a\mathbf{u} = (x_T + a\cos\gamma_T, y_T + a\sin\gamma_T)$$

2

$$\begin{cases} x_L = x_T + a\cos\gamma_T - a\cos\gamma_L - a(\gamma_L - \gamma_T)\sin\gamma_L \\ y_L = y_T + a\sin\gamma_T - a\sin\gamma_L + a(\gamma_L - \gamma_T)\cos\gamma_L \end{cases}$$

Eliminating the variable a

3

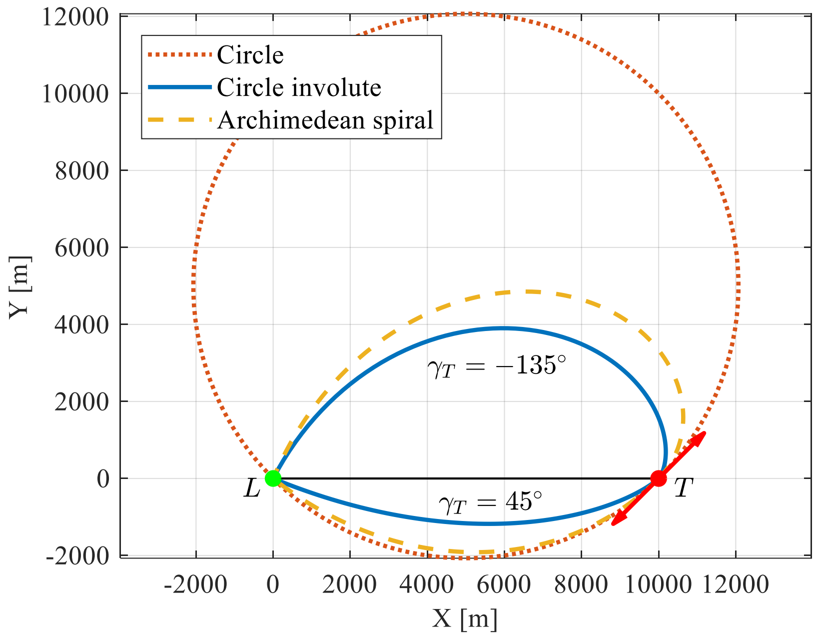
$$(x_L - x_T)(\sin\gamma_T - \sin\gamma_L + (\gamma_L - \gamma_T)\cos\gamma_L) \\ -(y_L - y_T)(\cos\gamma_T - \cos\gamma_L - (\gamma_L - \gamma_T)\sin\gamma_L) = 0$$

Solve above equation to get γ_L

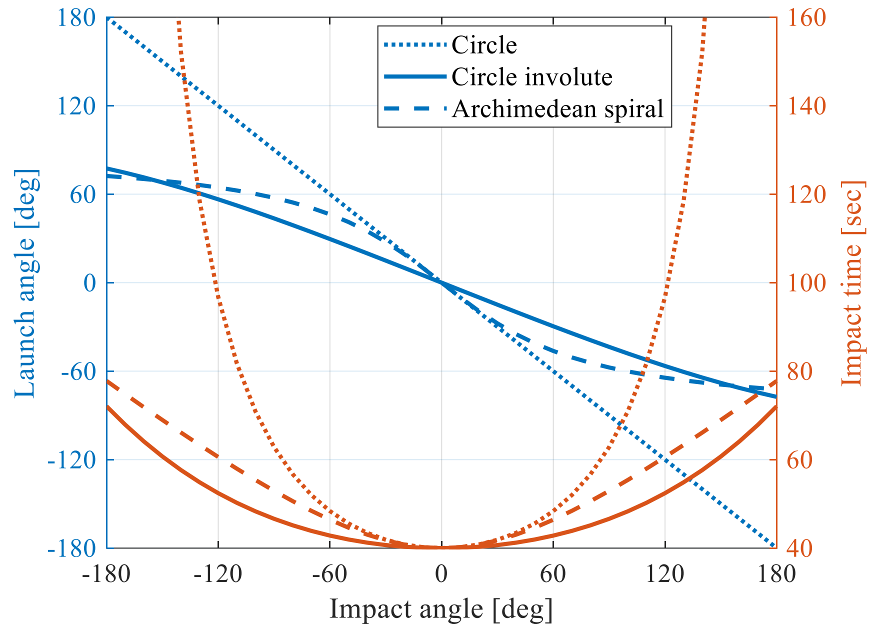
4

$$a = \frac{x_L - x_T}{\cos\gamma_T - \cos\gamma_L - (\gamma_L - \gamma_T)\sin\gamma_L}$$

Comparison with other curves*



Trajectories of different kinds of curves

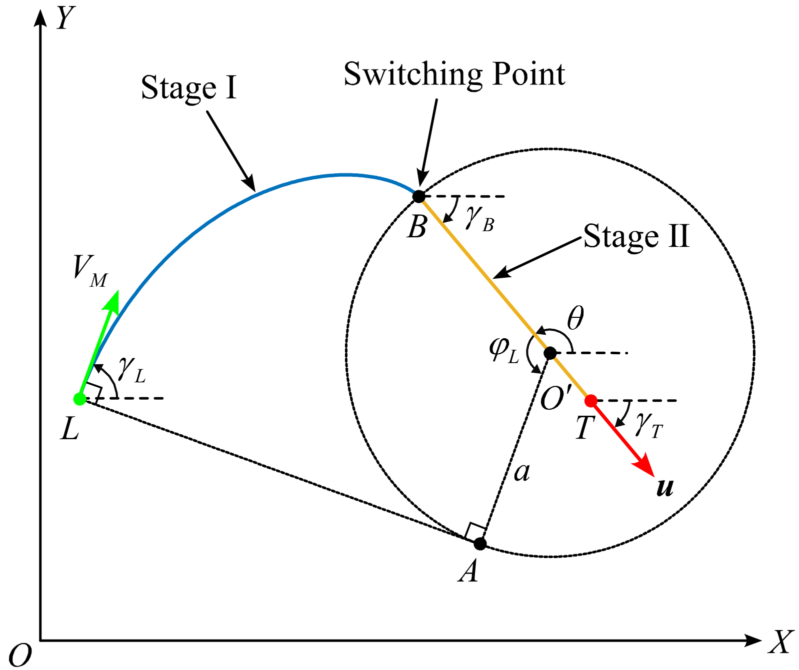


Variations of launch angle and impact time

- ❑ The involute has the shortest path and the shortest flight time.
- ❑ Under the same impact angle, the involute has the smallest launch angle.

* R. Tsalik and T. Shima, Inscribed Angle Guidance, *Journal of Guidance, Control, and Dynamics*, vol. 38, no. 1, pp. 30-40, 2015.
* T. Tripathy and T. Shima, Archimedean Spiral-Based Intercept Angle Guidance, *Journal of Guidance, Control, and Dynamics*, pp. 1-11, 2018.

Two-stage trajectory for angle and time constraints



Geometry of the two-stage trajectory

Stage I: $S_1 = \frac{1}{2}a\varphi_L{}^2 = \frac{1}{2}a(\gamma_L - \gamma_T)^2$

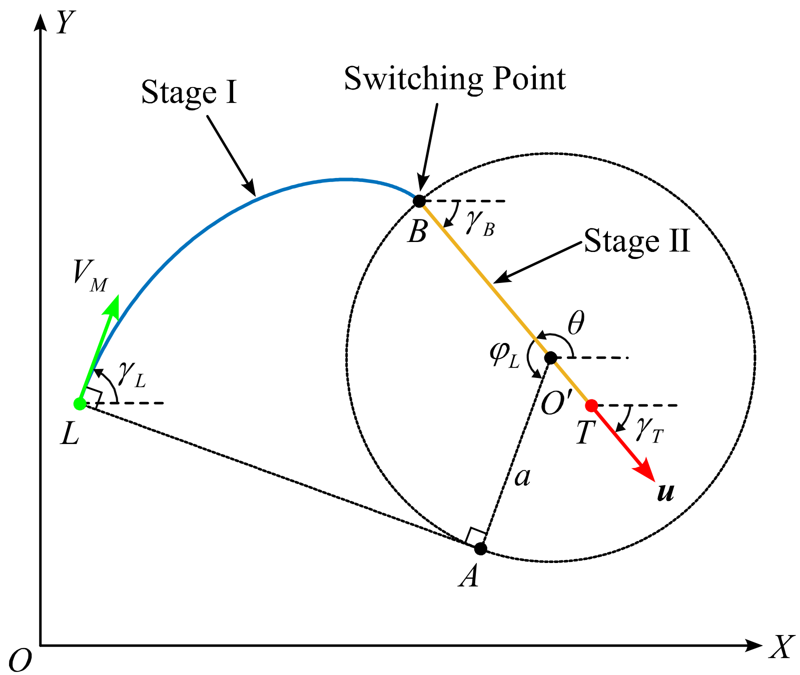
Stage II: $S_2 = b$

$$t_f = \frac{S_1 + S_2}{V_M} = \frac{a(\gamma_L - \gamma_T)^2 + 2b}{2V_M}$$

Goal: find the value of *a* and *b*

Two-stage trajectory for angle and time constraints

Goal: find the value of *a* and *b*



Geometry of the two-stage trajectory

1

$$\begin{cases} x = x_{O'} - a(\cos(\varphi+\gamma_T) + \varphi\sin(\varphi+\gamma_T)) \\ y = y_{O'} - a(\sin(\varphi+\gamma_T) - \varphi\cos(\varphi+\gamma_T)) \end{cases}$$

$$\boldsymbol{B} = \boldsymbol{T} - b\boldsymbol{u} = (x_T - b\cos\gamma_T, y_T - b\sin\gamma_T)$$

$$\boldsymbol{O}' = \boldsymbol{B} + a\boldsymbol{u} = (x_T + (a-b)\cos\gamma_T, y_T + (a-b)\sin\gamma_T)$$

2

$$\begin{cases} x = x_T + (a-b)\cos\gamma_T - a(\cos(\varphi+\gamma_T) + \varphi\sin(\varphi+\gamma_T)) \\ y = y_T + (a-b)\sin\gamma_T - a(\sin(\varphi+\gamma_T) - \varphi\cos(\varphi+\gamma_T)) \end{cases}$$

$\boldsymbol{L}(x_L, y_L)$

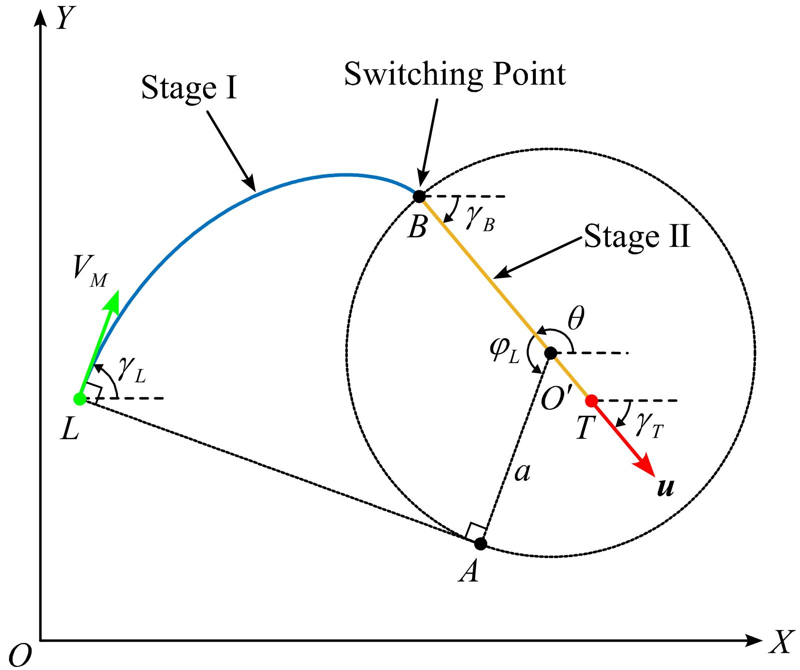
$\varphi_L = \gamma_L - \gamma_T$

3

$$\begin{cases} x_L = x_T + (a-b)\cos\gamma_T - a\cos\gamma_L - a(\gamma_L - \gamma_T)\sin\gamma_L \\ y_L = y_T + (a-b)\sin\gamma_T - a\sin\gamma_L + a(\gamma_L - \gamma_T)\cos\gamma_L \end{cases}$$

Two-stage trajectory for angle and time constraints

Goal: find the value of *a* and *b*



Geometry of the two-stage trajectory

3

$$\begin{cases} x_L = x_T + (a-b)\cos\gamma_T - a\cos\gamma_L - a(\gamma_L-\gamma_T)\sin\gamma_L \\ y_L = y_T + (a-b)\sin\gamma_T - a\sin\gamma_L + a(\gamma_L-\gamma_T)\cos\gamma_L \end{cases}$$

4

$$a = \frac{(x_T-x_L)\sin\gamma_T - (y_T-y_L)\cos\gamma_T}{(\gamma_L-\gamma_T)\cos(\gamma_L-\gamma_T) - \sin(\gamma_L-\gamma_T)}$$

$$b = \frac{(x_T-x_L)\sin\gamma_L - (y_T-y_L)\cos\gamma_L - a(\gamma_L-\gamma_T-\sin(\gamma_L-\gamma_T))}{\sin(\gamma_L-\gamma_T)}$$

$$t_f = \frac{a(\gamma_L-\gamma_T)^2 + 2b}{2V_M}$$

5

$$\frac{(x_T-x_L)\sin\gamma_T - (y_T-y_L)\cos\gamma_T}{(\gamma_L-\gamma_T)\cos(\gamma_L-\gamma_T) - \sin(\gamma_L-\gamma_T)} \times \left[\frac{(\gamma_L-\gamma_T)^2}{2} - \frac{\gamma_L-\gamma_T-\sin(\gamma_L-\gamma_T)}{\sin(\gamma_L-\gamma_T)}\right] + \frac{(x_T-x_L)\sin\gamma_L - (y_T-y_L)\cos\gamma_L}{\sin(\gamma_L-\gamma_T)} - V_M t_f = 0$$

Solve above equation to get γ_L

6

$$\gamma_L,\ a,\ b$$

Two-stage trajectory for angle and time constraints

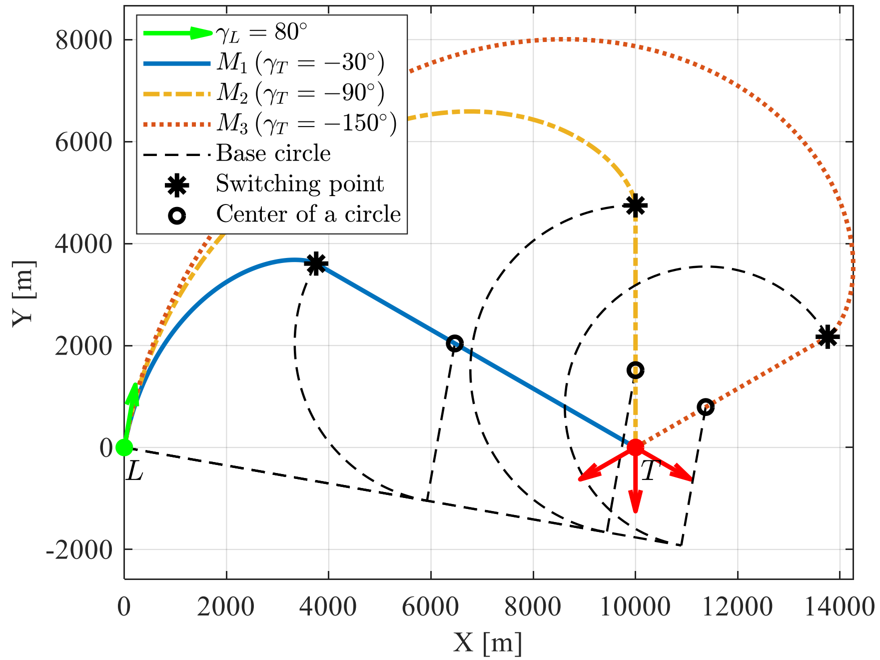


Illustration of two-stage trajectories with different impact angles and the same launch angle

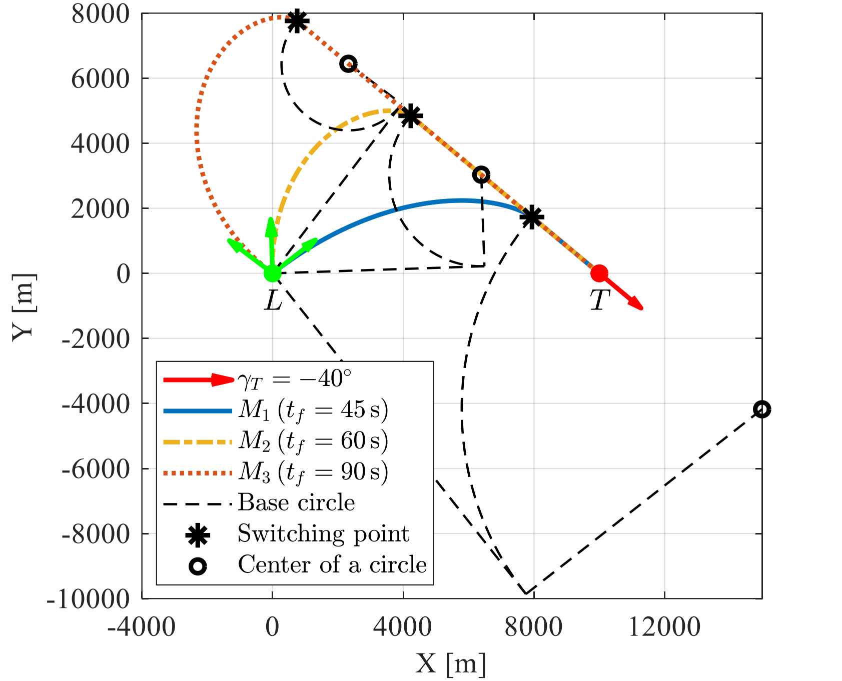
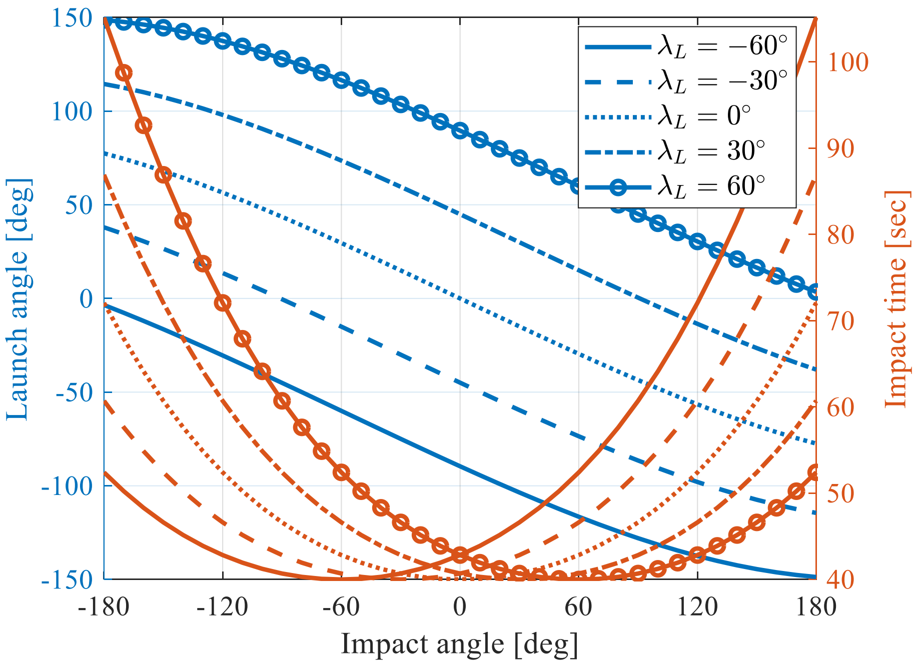
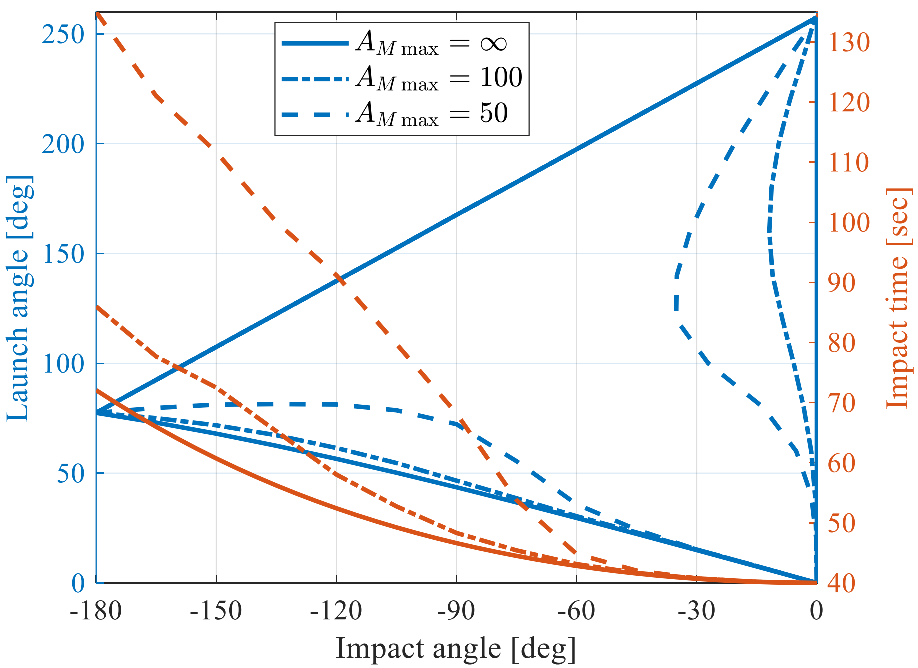


Illustration of two-stage trajectories with different impact times and the same impact angle

Analysis of achievable constraints



Variations of the launch angle and impact time with respect to impact angles for different initial LOS angles.



Capture regions under different acceleration limits

3D implementation

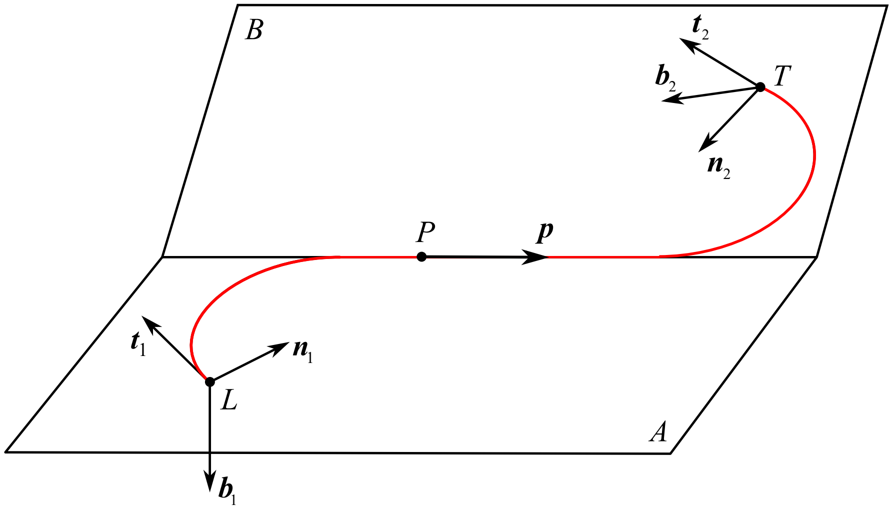


Illustration of 3D implementation

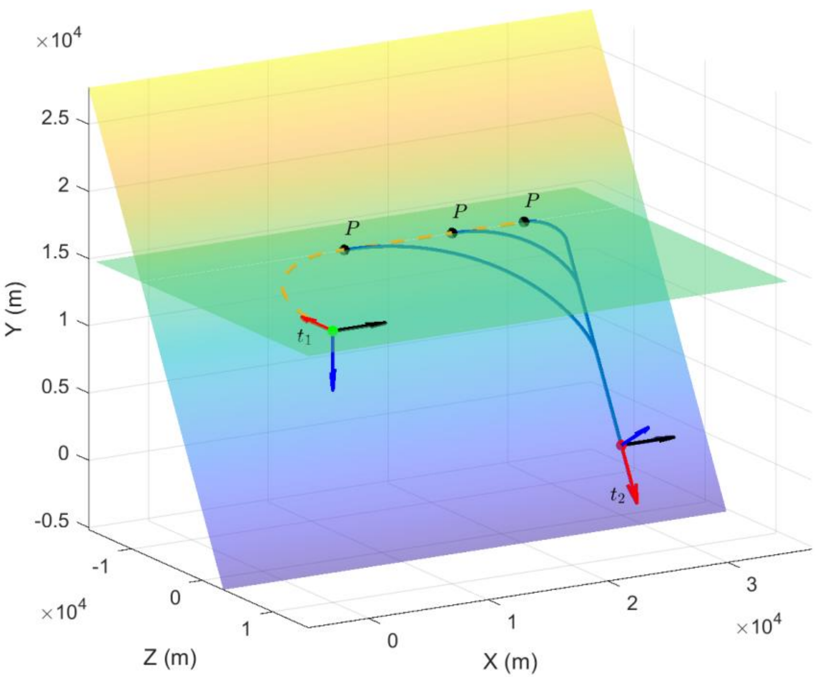


Illustration of 3D trajectories with different impact times and the same impact angle

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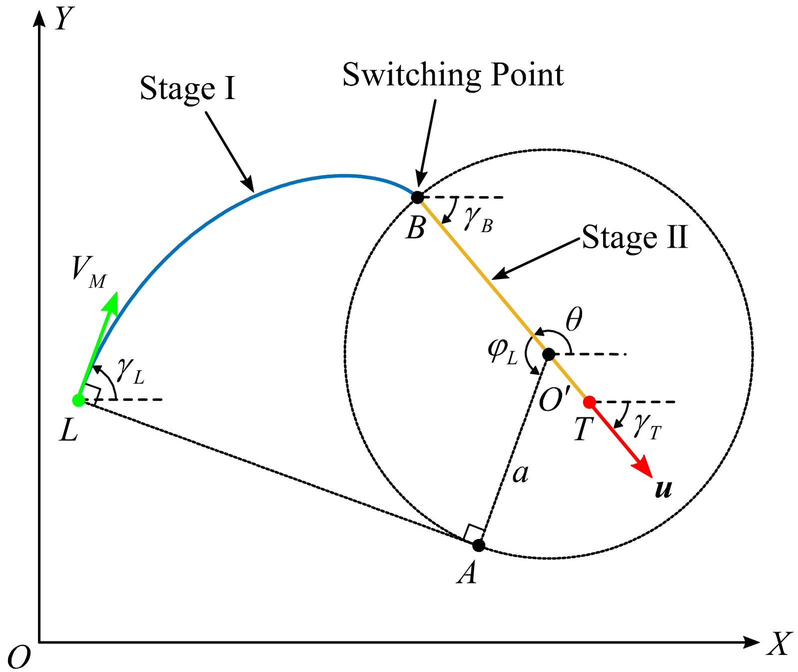
Two-stage Involute-based Guidance (TIG) Law

Based on the curvature of the involute



Geometry of the two-stage trajectory

Stage I: $A_M = -\frac{V_M^{\ 2}}{a\varphi} = -\frac{V_M^{\ 2}}{a(\gamma-\gamma_T)}$

Stage II: $A_M = N V_M \dot{\lambda}$

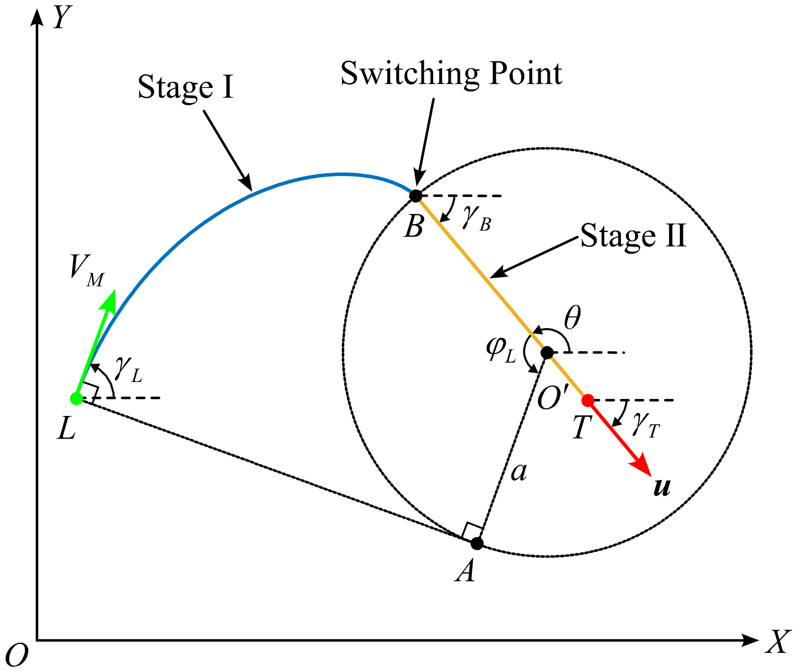
TIG law

$$A_M = \begin{cases} -\frac{V_M^{\ 2}}{a(\gamma-\gamma_T)}, & \text{if } |\varphi| > \zeta \\ N V_M \dot{\lambda}, & \text{if } |\varphi| \le \zeta \end{cases}$$

- ❑ The successful implementation of the TIG law depends on the assumption that the vehicle is on the involute at the initial moment.
- ❑ The TIG law cannot compensate the external disturbances.

Robust Two-stage Involute-based Guidance (RTIG) Law

Based on the geometric rule of the involute



Geometry of the two-stage trajectory

External disturbances*: $\dot{\gamma} = \frac{A_M + w}{V_M}$

Geometric rule: $\tan\delta = \gamma - \gamma_T$

RTIG Stage I

$$s_1 = \tan\delta - \gamma + \gamma_T$$

$$\dot{s}_1 = \frac{\sec^2\delta - 1}{V_M} A_M + \frac{\sec^2\delta - 1}{V_M} w - \dot{\lambda}_{O'} \sec^2\delta$$

$$A_M = \frac{V_M}{\sec^2\delta - 1} \dot{\lambda}_{O'} \sec^2\delta - k_1 \text{sign}(s_1) \quad (k_1 > 0)$$

Stability analysis

$$W = (1/2) s_1^{\,2}$$

$$\dot{W} = s_1 \dot{s}_1 = s_1(-k_1 \text{sign}(s_1) + w) \frac{\sec^2\delta - 1}{V_M}$$

$$\leq -|s_1|(k_1 - |w|) \frac{\sec^2\delta - 1}{V_M}$$

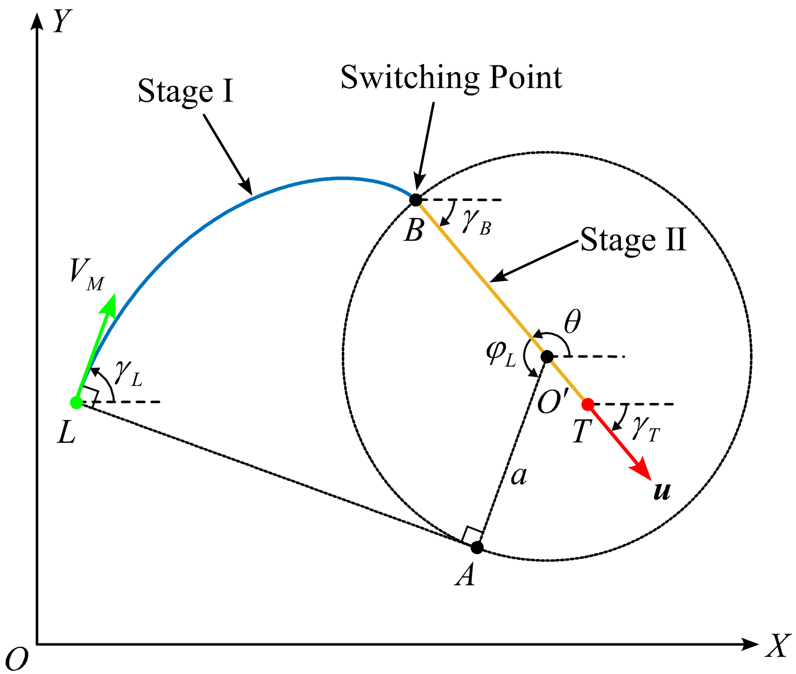
⬇

$$k_1 > \max(|w|)$$

* T. Tripathy and T. Shima, *JGCD*, 2019.

Robust Two-stage Involute-based Guidance (RTIG) Law

Based on the geometric rule of the involute



Geometry of the two-stage trajectory

External disturbances*: $\dot{\gamma} = \frac{A_M + w}{V_M}$

Geometric rule: $\tan\delta = \gamma - \gamma_T$

RTIG Stage II

$$s_2 = \dot{\lambda}$$

$$\dot{s}_2 = \frac{\dot{r}}{rV_M} A_M + \frac{\dot{r}}{rV_M} w - \frac{2\dot{r}\dot{\lambda}}{r}$$

$$A_M = 2V_M\dot{\lambda} + k_2 \mathrm{sign}(s_2) \quad (k_2 > 0)$$

Stability analysis

$$k_2 > \max(|w|)$$

RTIG law

$$A_M = \begin{cases} \dfrac{V_M}{\sec^2\delta - 1}\dot{\lambda}_{O'}\sec^2\delta - k_1 \mathrm{sign}(s_1), & \text{if } |\varphi| > \zeta \\ 2V_M\dot{\lambda} + k_2 \mathrm{sign}(s_2) & , \text{if } |\varphi| \le \zeta \end{cases}$$

$k_1 > \max(|w|)$, $k_2 > \max(|w|)$

* T. Tripathy and T. Shima, *JGCD*, 2019.

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- ❑ **Guidance law design**
- ❑ **Simulation results**

Simulation results - TIG law

1. Scenario

$L = (0,0)$ km, $T = (10,0)$ km

$V_M = 250$ m/s, $A_{M\max} = 50$ m/s^2

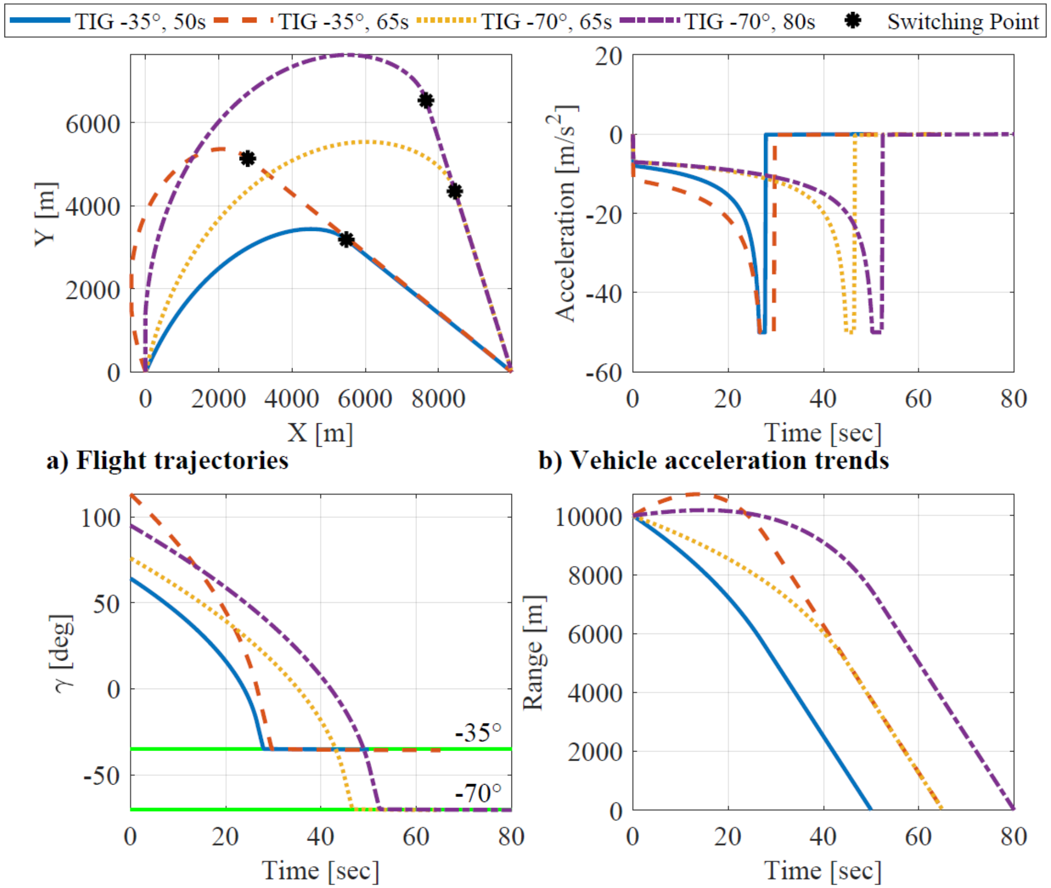
2. Trajectory parameters

Guidance law	γ_T, deg	t_{des}, sec	γ_L, deg	a, m	b, m
TIG	−35	50	64.19	4539.4	5698.1
TIG	−35	65	113.14	2105.8	9211.6
TIG	−70	65	75.94	3519.2	4834.2
TIG	−70	80	94.97	3091.1	7187.9

3. Guidance law

$$A_M = \begin{cases} -\dfrac{V_M^{\ 2}}{a(\gamma - \gamma_T)}, & \text{if } |\varphi| > \zeta \\ NV_M\dot{\lambda}, & \text{if } |\varphi| \le \zeta \end{cases}$$

$(N = 3)$



a) Flight trajectories
b) Vehicle acceleration trends
c) Flight-path angle variations
d) Relative range histories

- ❑ All the missions succeed to arrive the target at the desired impact time and angle.
- ❑ For the same γ_T, a longer t_{des} results in larger values of b and γ_L.

Simulation results - RTIG vs TIG

1. Scenario

$L=(0,0)$ km, $T=(10,0)$ km

$V_M=250$ m/s, $A_{M\max}=50$ m/s^2

$w=[-10,10]$ m/s^2, $\Delta\gamma_L=10$ deg, $\tau=0.1$s

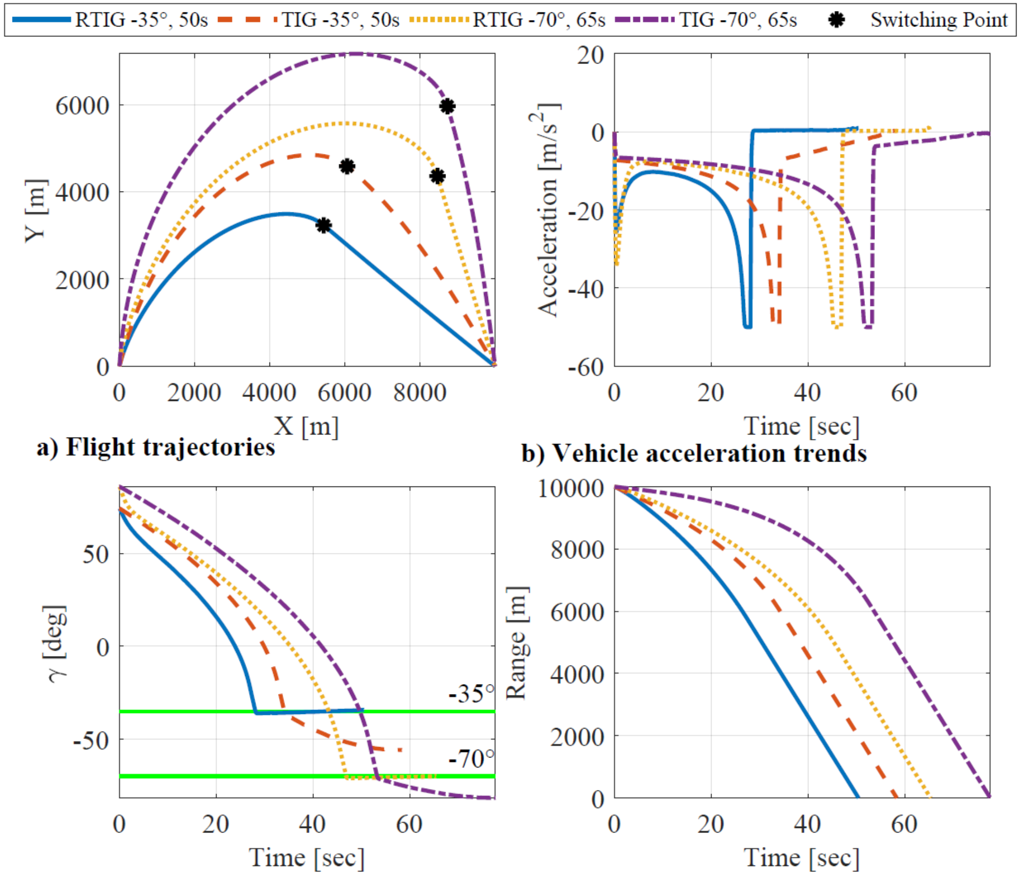
2. Trajectory parameters

Guidance law	γ_T, deg	t_{des}, sec	γ_L, deg	a, m	b, m
RTIG	−35	50	64.19	4539.4	5698.1
TIG					
RTIG	−70	65	75.94	3519.2	4834.2
TIG					

3. Guidance law

$$A_M=\begin{cases}\dfrac{V_M}{\sec^2\delta-1}\dot{\lambda}_{O'}\sec^2\delta-k_1\mathrm{sign}(s_1), & \text{if } |\varphi|>\zeta\\ 2V_M\dot{\lambda}+k_2\mathrm{sign}(s_2), & \text{if } |\varphi|\le\zeta\end{cases}$$

$(k_1=k_2=30)$



a) Flight trajectories
b) Vehicle acceleration trends
c) Flight-path angle variations
d) Relative range histories

❑ Only by using the RTIG law, vehicles can satisfy both angle and time constraints.
❑ The acceleration has a dip at the beginning due to the initial heading error.

Simulation results - Monte Carlo Simulations of the RTIG Law

1. Scenario

$L = (0,0)$ km, $T = (10,0)$ km

$V_M = 250$ m/s, $A_{M\max} = 50$ m/s^2

$w = [-10,10]$ m/s^2, $\Delta\gamma_L = 10$ deg, $\tau = 0.1$s

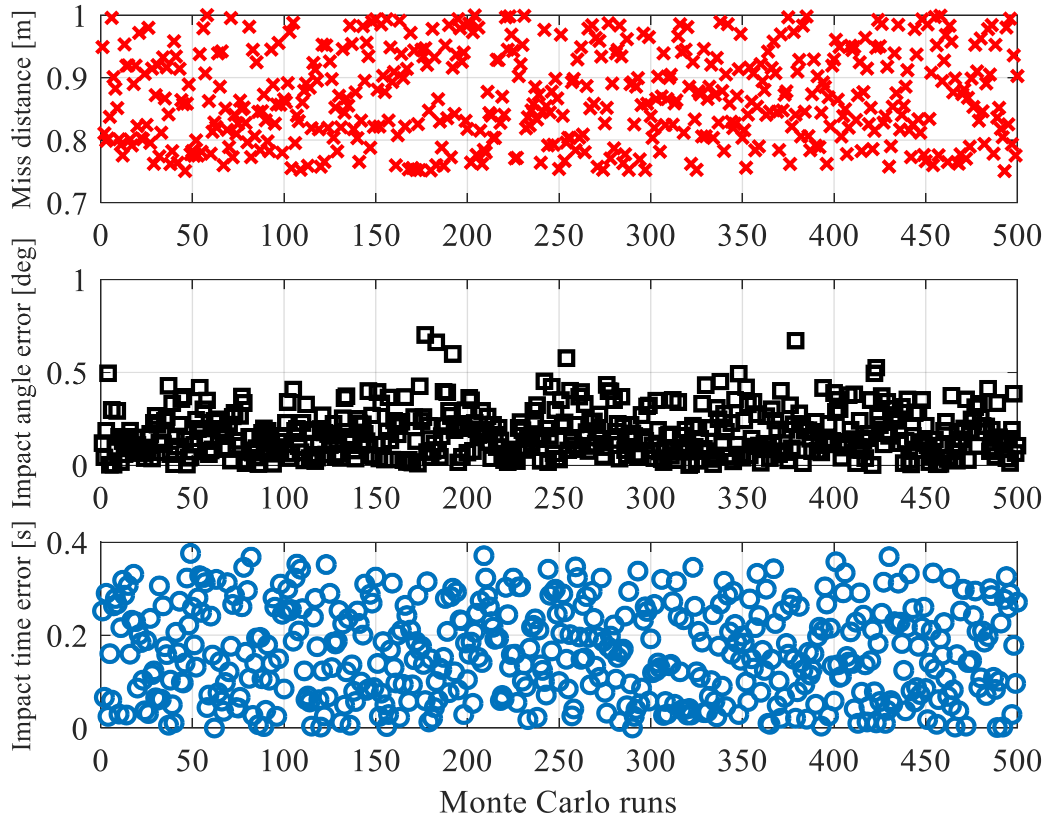
2. Trajectory parameters

Guidance law	γ_T, deg	t_{des}, sec	γ_L, deg	a, m	b, m
RTIG / TIG	−35	50	64.19	4539.4	5698.1
RTIG / TIG	−70	65	75.94	3519.2	4834.2

3. Guidance law

$$A_M = \begin{cases} \dfrac{V_M}{\sec^2\delta - 1}\dot{\lambda}_{O'}\sec^2\delta - k_1 \mathrm{sign}(s_1), & \text{if } |\varphi| > \zeta \\ 2V_M\dot{\lambda} + k_2 \mathrm{sign}(s_2), & \text{if } |\varphi| \le \zeta \end{cases}$$

$(k_1 = k_2 = 30)$



❑ The impact angle errors are around 0.15 deg, and the impact time errors are around 0.16s, which show the strong robustness of the RTIG law against external disturbances and initial heading errors.

Simulation results - RTIG vs Elliptic Guidance*

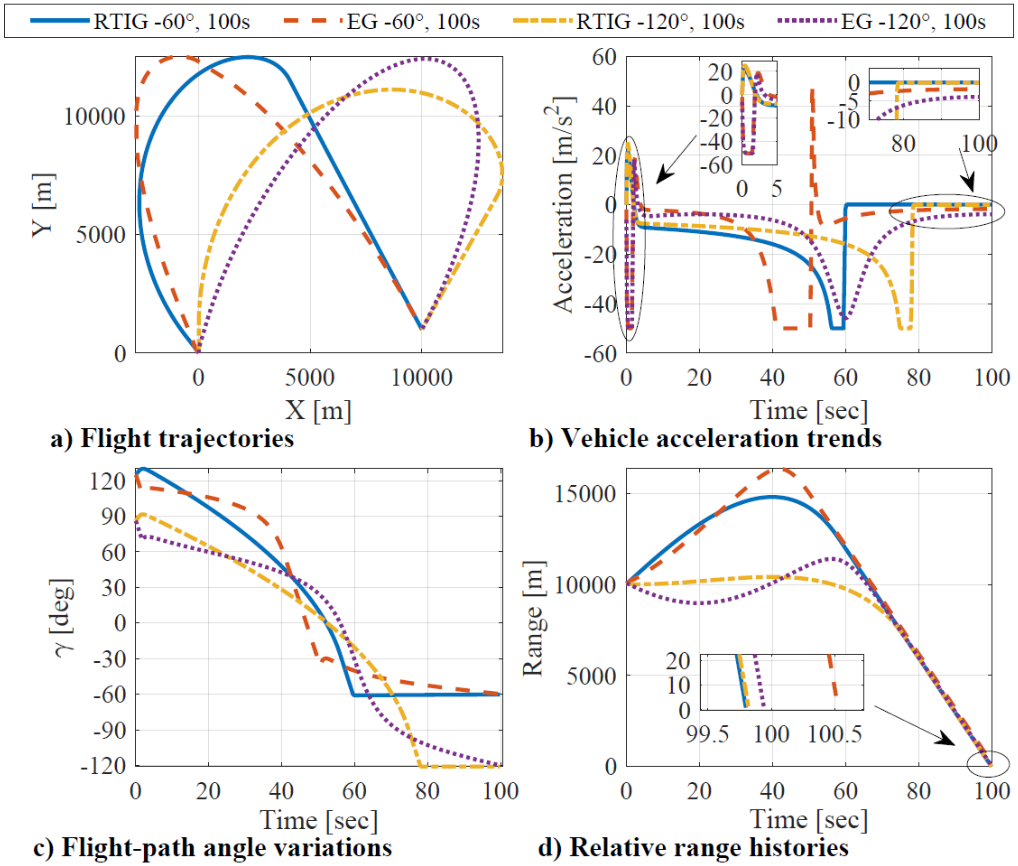
1. Scenario

$L = (0,0)$ km, $T = (10,1)$ km

$V_M = 300$ m/s, $A_{M\max} = 50$ m/s^2

2. Trajectory parameters

Guidance law	γ_T, deg	t_{des}, sec	γ_L^*, deg	γ_L, deg
RTIG	−60	100	134.5	124.9
EG			115.3	
RTIG	−120	100	95	86
EG			77	



a) Flight trajectories
b) Vehicle acceleration trends
c) Flight-path angle variations
d) Relative range histories

- ❑ The initial acceleration of the EG law is saturated while the RTIG law is not.
- ❑ For the EG law and RTIG law, the maximum curvature appears at the vertex of the ellipse and the involute's end, respectively.

* R. Livermore, R. Tsalik, T. Shima, "Elliptic Guidance," *Journal of Guidance Control and Dynamics*, 2018.

Conclusions

- A circle involute-based two-stage guidance law that achieves the desired impact time and angle is designed, which does not involve the model linearization, time-to-go estimation, and numerical optimization routine.
- To avoid complicated parameter calculations, a simple method for finding the explicit two-stage trajectory is provided. Moreover, near-zero acceleration is created and maintained in the second stage.
- The robustness of the guidance law against external disturbances and heading errors is ensured via providing a geometric rule and forcing it in the first stage, while eliminating the line-of-sight rate in the second stage.

Z. Wang, Q. Hu, T. Han and M. Xin, "**Two-Stage Guidance Law With Constrained Impact via Circle Involute**," in ***IEEE Transactions on Aerospace and Electronic Systems***, vol. 57, no. 2, pp. 1301-1316, April 2021.

Thanks for your attention!