Friedrich-Alexander-Universität
Naturwissenschaftliche Fakultät

Nodal Control and Probabilistic Constrained Optimization using the Example of Gas Networks

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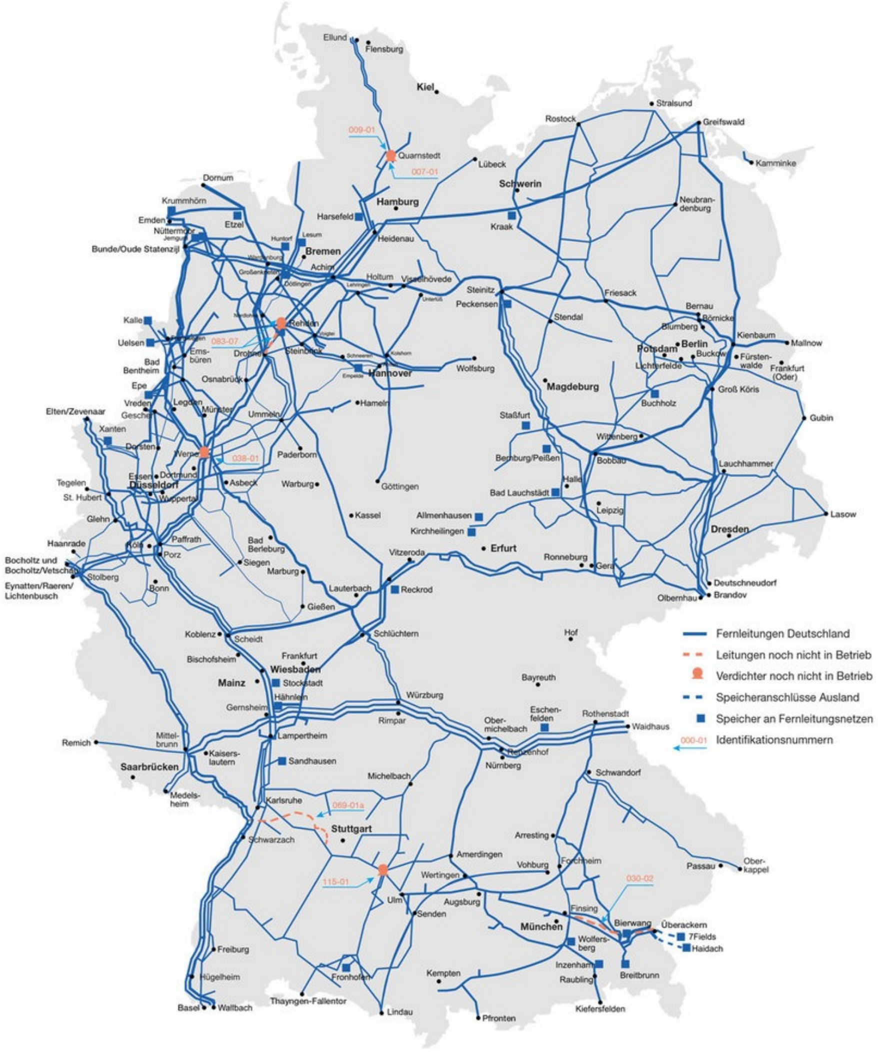
Motivation

Gas Transport Networks



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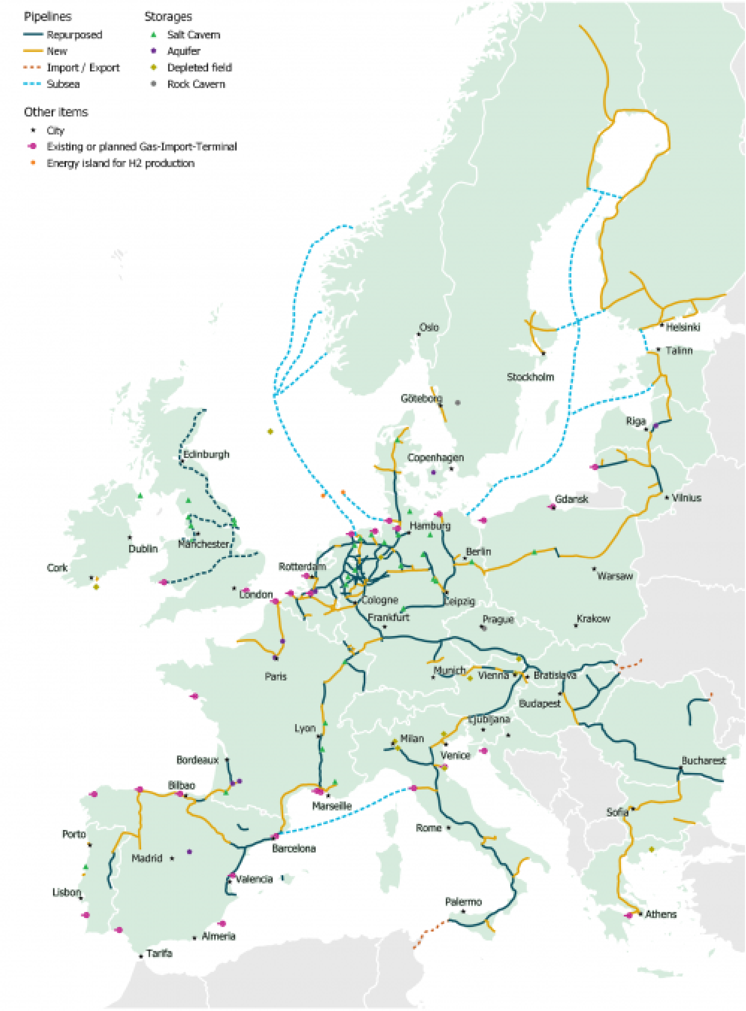

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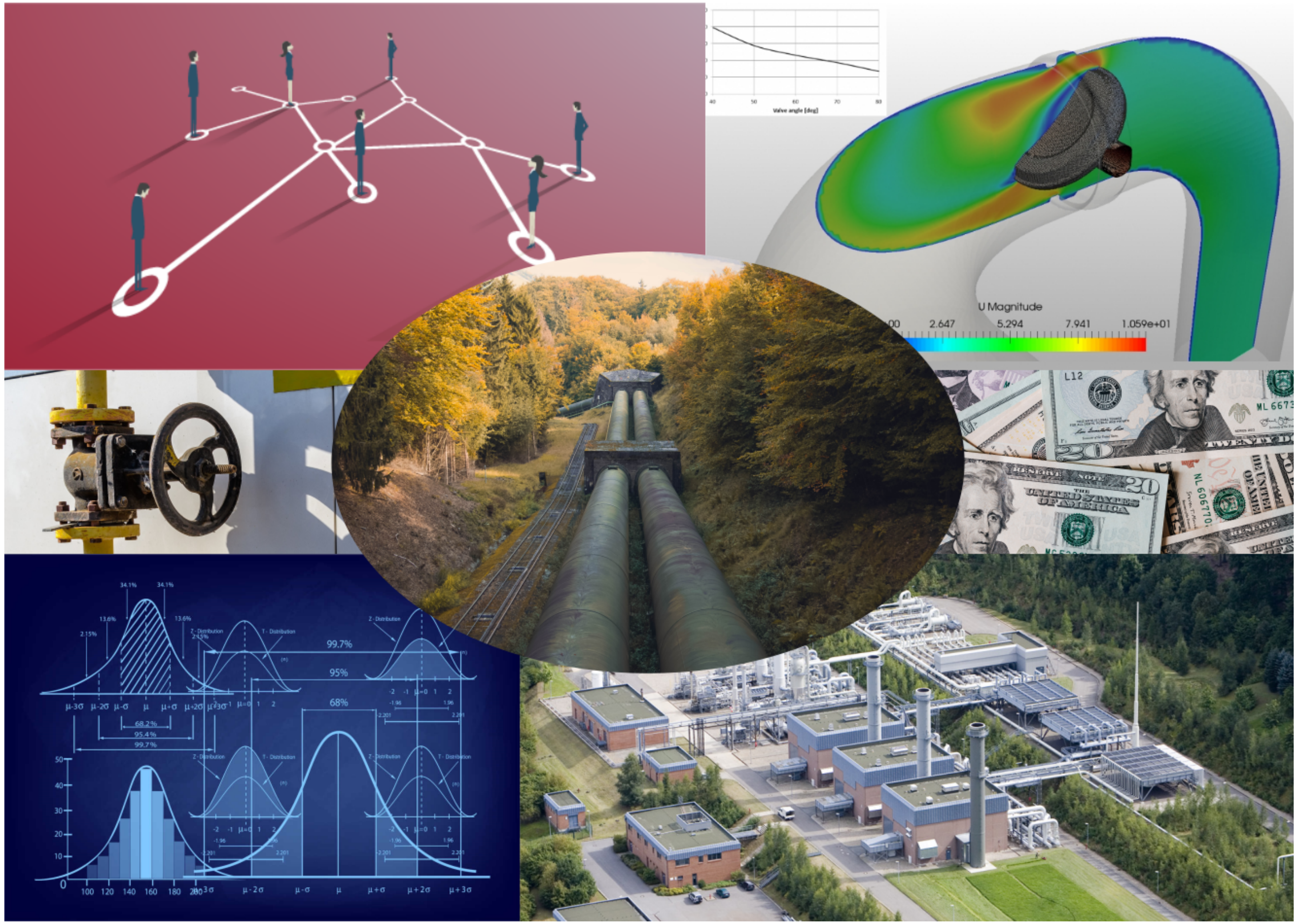
Motivation

Mathematical Gas Transport



Mathematical Gas Transport

Gas Transport in Pipeline Networks

For constant temperature the gas flow in pipelines is modeled by the isothermal Euler equations:

(ISO1)

$$\rho_t + q_x = 0,$$

$$q_t + \left(p + \frac{q^2}{\rho}\right)_x = -\frac{\lambda^F}{2D}\frac{q|q|}{\rho}.$$

Variable	Letter	Unit	Range
density	ρ	kg m^{-3}	$\mathbb{R}_{\geq 0}$
flow	q	kg s^{-1} m^{-2}	$\mathbb{R}$
pressure	p	Pa	$\mathbb{R}_{\geq 0}$
sound speed in the gas	c	m s^{-1}	$\mathbb{R}_{\geq 0}$
pipe friction coefficient	λ^F		$\mathbb{R}_{\geq 0}$
pipe diameter	D	m	$\mathbb{R}_+$

Inlet density & Gas outflow

$$\rho(t,0) = \rho_0(t),$$
$$q(t,L) = b(t).$$

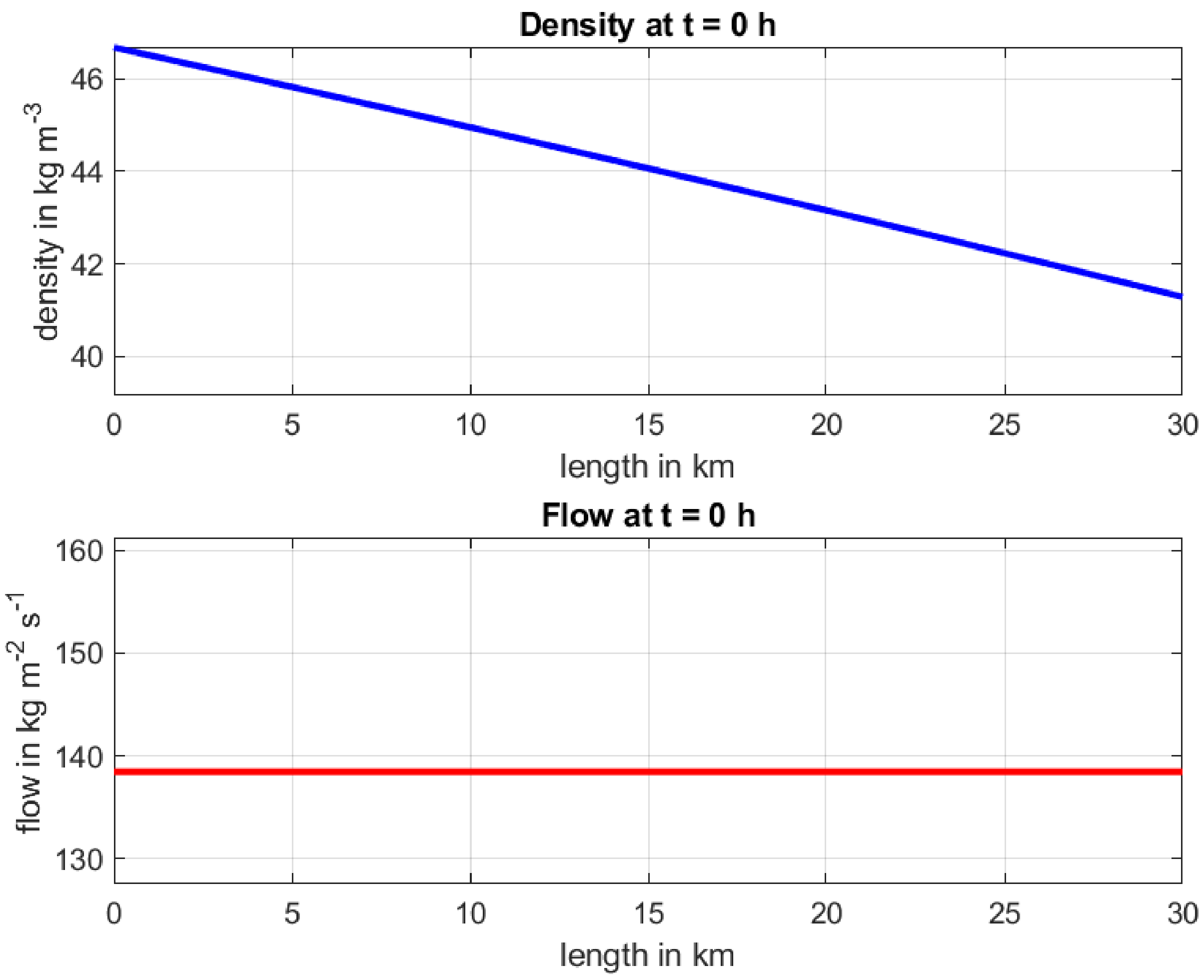
Initial condition

$$\rho(0,x) = \rho_{\text{ini}}(x),$$
$$q(0,x) = q_{ini}(x).$$

[Gugat and Ulbrich, 2018]: *Lipschitz solutions of initial boundary value problems for balance laws.* Math. Models Methods Appl. Sci., 28(5): 921–951

Mathematical Gas Transport

Gas Transport in Pipeline Networks



Mathematical Gas Transport

Gas Transport in Pipeline Networks

(ISO1) - quasilinear model

$$\rho_t + q_x = 0$$

$$q_t + \left(p + \frac{q^2}{\rho}\right)_x = -\frac{\lambda^F}{2D}\frac{q|q|}{\rho}$$

subsonic flow: $|v| = \left|\frac{q}{\rho}\right| \ll c$

(ISO2) - semilinear model

$$\rho_t + q_x = 0$$

$$q_t + p_x = -\frac{\lambda^F}{2D}\frac{q|q|}{\rho}$$

stationary flow: $\frac{\partial}{\partial t} \equiv 0$

(ISO4) - stationary model

$$q_x = 0$$

$$p_x = -\frac{\lambda^F}{2D}\frac{q|q|}{\rho}$$

Mathematical Gas Transport

Stationary Gas Transport in Pipeline Networks

- Consider a connected, directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V}$ and set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$
- We consider the stationary model (ISO4) for ideal gases on every edge $e \in \mathcal{E}$

(ISO4) for ideal gases

$$q_x = 0, \quad p_x = -\frac{c^2 \lambda^F}{2D} \frac{q|q|}{p}$$

solution of (ISO4)

$$q(x) = const., \quad p^2(x) = p(0)^2 - \frac{c^2 \lambda^F}{2D} q|q|\, x$$

Coupling conditions at the nodes:

Conservation of mass

$$\sum_{e \in \mathcal{E}_-(v)} q^e \left(\frac{D^e}{2}\right)^2 \pi = b^v + \sum_{e \in \mathcal{E}_+(v)} q^e \left(\frac{D^e}{2}\right)^2 \pi \quad \forall v \in \mathcal{V} \backslash \mathcal{V}_0.$$

Continuity in pressure

$$p^{e_1}(L^{e_1}) = p^{e_2}(0) \quad \forall e_1 \in \mathcal{E}_-(v),\ e_2 \in \mathcal{E}_+(v).$$

Mathematical Gas Transport

Stationary Gas Transport in Pipeline Networks

Boundary Conditions:

Inlet pressure

$$p^e(0) = p_0 \in \mathbb{R}_{\geq 0} \quad \forall e \in \mathcal{E}_+(v_0)$$

Gas outflow

$$q^e(L^e) = b^v \in \mathbb{R}_{\geq 0} \quad \forall e \in \mathcal{E}_-(v)$$

b^v represents the consumers gas demand

- Let $p \in \mathbb{R}^n$ be the vector of pressures at the nodes $v_1, \cdots, v_n$
- We assume box constraints for the pressures at the nodes: $p_i \in [p_i^{\min}, p_i^{\max}]$

Set of feasible loads

$$M := \left\{ b \in \mathbb{R}^n_{\geq 0} \;\middle|\; \begin{array}{l} (p,q) \in \mathbb{R}^n \times \mathbb{R}^n \text{ satisfies:} \\ \bullet \text{ stationary semilinear isothermal Euler equations,} \\ \bullet \text{ inlet pressure and gas outflow,} \\ \bullet \text{ conservation of mass and continuity in pressure,} \\ \bullet \text{ pressure bounds.} \end{array} \right\}$$

[Gugat, Hante, Hirsch-Dick, Leugering, 2015]: *Stationary states in gas networks.* Netw. Heterog. Media, 10(2): 295–320.

Mathematical Gas Transport

Stationary Gas Transport in Pipeline Networks

Assume that the consumers gas demand is random in the sense, that there is a random variable

$$\xi_b \sim \mathcal{N}(\mu, \Sigma),$$

on an appropriate probability space. We identify b with the image $\xi_b(\omega)$ for $\omega \in \Omega$.

For a given inlet pressure, can we guarantee, that every consumer receives their demanded gas, s.t. the gas pressure in the network is neither too high nor too low, in at least $\alpha\%$ of all scenarios?

$$\mathbb{P}\Big(\omega \in \Omega \mid \xi_b(\omega) \in M \Big) \geq \alpha$$

Mathematical Gas Transport

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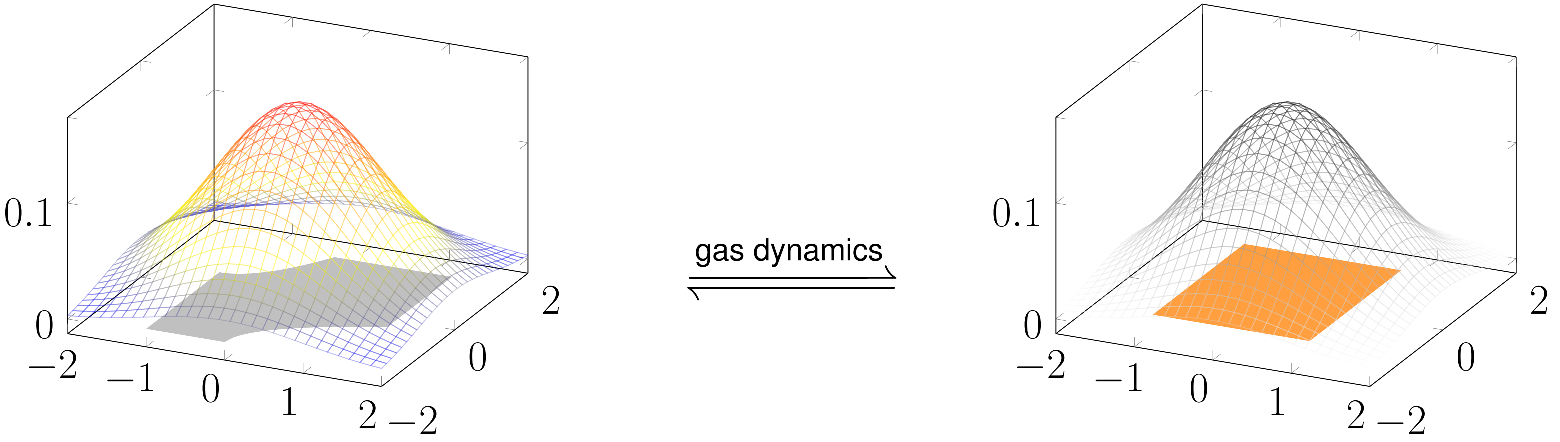


(a) Well-known distribution (colored), unknown set of feasible loads (grey)

(b) Unknown distribution (grey), well-known set of feasible pressures (orange)

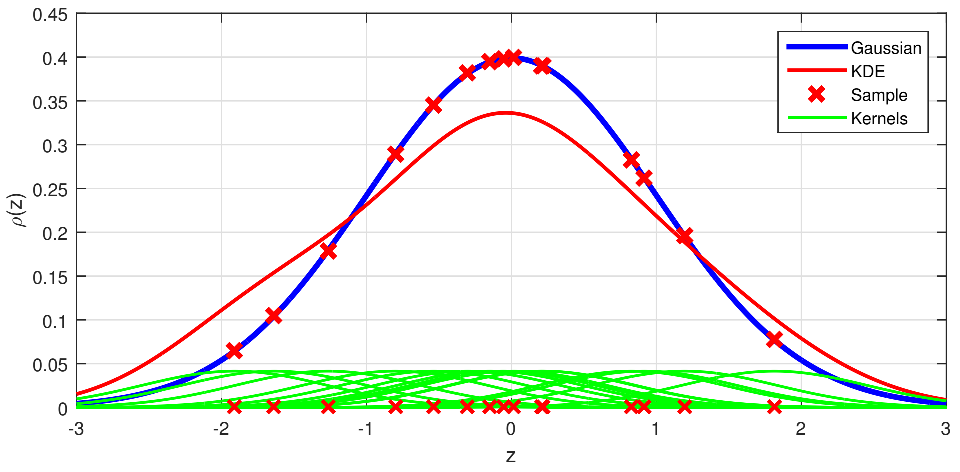
Mathematical Gas Transport

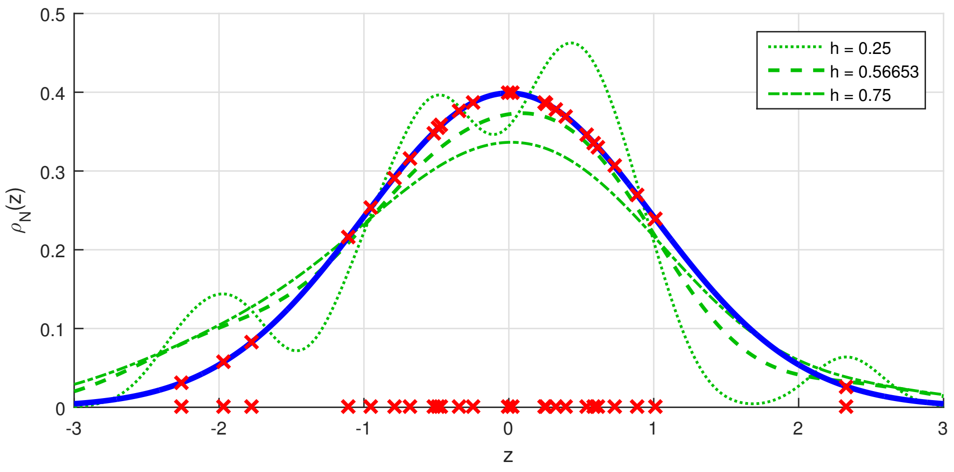
Stationary Gas Transport in Pipeline Networks

Definition: Kernel Density Estimator

Let $\mathcal{Y} = \{y_1, \cdots, y_N\} \subseteq \mathbb{R}^n$ be i.i.d. samples of the random variable Y, which has an absolutely continuous distribution function with probability density function ϱ. Let K be a kernel function.
Then the kernel density estimator ϱ_N corresponding to the bandwidth $h \in (0, \infty)$ is defined as

$$\varrho_N(z) = \frac{1}{Nh} \sum_{i=1}^{N} K\left(\frac{z - y_i}{h}\right).$$





[Gramacki 2018]: Nonparametric Kernel Density Estimation and its Computational Aspects. *Springer International Publishing*

Mathematical Gas Transport

Stationary Gas Transport in Pipeline Networks

- Let $\mathcal{B} = \{ b^{S,1}, \cdots, b^{S,N_{\mathsf{KDE}}} \} \subseteq \mathbb{R}^n_{\geq 0}$ be independent and identically distributed samples of the random variable ξ_b
- Let $\mathcal{P}_{\mathcal{B}} = \{ p(b^{S,1}), \cdots, p(b^{S,N_{\mathsf{KDE}}}) \} \subseteq \mathbb{R}^n$ be the corresponding pressures at the nodes (also independent and identically distributed)

Gaussian kernel

$$K(x) = \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x_j^2\right)$$

bandwidth matrix

$$H_{i,i} = h^2 \left(\Sigma_{N_{\mathsf{KDE}}}\right)_{i,i}$$

$$h = \left(\frac{4}{(n+2)N_{\mathsf{KDE}}}\right)^{\frac{1}{n+4}}$$

kernel density estimator

$$\varrho_{p,N_{\mathsf{KDE}}}(z) = \frac{1}{N_{\mathsf{KDE}} \det \sqrt{H_{j,j}}} \sum_{i=1}^{N_{\mathsf{KDE}}} \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{z_j - p_j(b^{S,i})}{\sqrt{H_{j,j}}}\right)^2\right)$$

[Schuster, Strauch, Gugat, Lang, 2022]: *Probabilistic Constrained Optimization on Flow Networks.* Optim. Eng. 23: 1–50

Mathematical Gas Transport

Stationary Gas Transport in Pipeline Networks

$$\begin{aligned}
\mathbb{P}_{N_{\mathsf{KDE}}}(p \in P_{\min}^{\max}) &= \int_{P_{\min}^{\max}} \varrho_{p,N_{\mathsf{KDE}}}(z)\, dz \\
&= \int_{P_{\min}^{\max}} \frac{1}{N_{\mathsf{KDE}} \prod_{j=1}^{n} h_j} \sum_{i=1}^{N_{\mathsf{KDE}}} \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{z_j - p_j(b^{S,i})}{h_j}\right)^2\right) dz \\
&= \frac{1}{N_{\mathsf{KDE}} \prod_{j=1}^{n} h_j} \sum_{i=1}^{N_{\mathsf{KDE}}} \prod_{j=1}^{n} \int_{p_j^{\min}}^{p_j^{\max}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{z_j - p_j(b^{S,i})}{h_j}\right)^2\right) dz_j
\end{aligned}$$

Mathematical Gas Transport

Stationary Gas Transport in Pipeline Networks

$$\mathbb{P}_{N_{\mathsf{KDE}}}(p \in P_{\min}^{\max}) = \int_{P_{\min}^{\max}} \varrho_{p,N_{\mathsf{KDE}}}(z)\, dz$$
$$= \int_{P_{\min}^{\max}} \frac{1}{N_{\mathsf{KDE}} \prod_{j=1}^{n} h_j} \sum_{i=1}^{N_{\mathsf{KDE}}} \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} \left(\frac{z_j - p_j(b^{S,i})}{h_j} \right)^2 \right) dz$$
$$= \frac{1}{N_{\mathsf{KDE}} \prod_{j=1}^{n} h_j} \sum_{i=1}^{N_{\mathsf{KDE}}} \prod_{j=1}^{n} \int_{p_j^{\min}}^{p_j^{\max}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} \left(\frac{z_j - p_j(b^{S,i})}{h_j} \right)^2 \right) dz_j$$

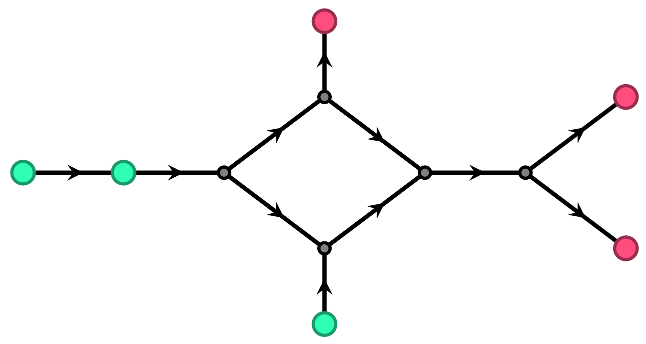
Gauss error function: $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-t^2\right) dt$

probability via KDE

$$\mathbb{P}_{N_{\mathsf{KDE}}}(p \in P_{\min}^{\max}) = \frac{1}{N_{\mathsf{KDE}} 2^n} \sum_{i=1}^{N_{\mathsf{KDE}}} \prod_{j=1}^{n} \left[\operatorname{erf}\left(\frac{p^{\max} - p_j(b^{S,i})}{\sqrt{2}\, h_j} \right) - \operatorname{erf}\left(\frac{p^{\min} - p_j(b^{S,i})}{\sqrt{2}\, h_j} \right) \right]$$

Mathematical Gas Transport

Stationary Gas Transport in Pipeline Networks



p_0	$p^{\min}$	outflow	covariance	α
$\begin{pmatrix} 60 \\ 58 \\ 60 \end{pmatrix}$	$\begin{pmatrix} 40 \\ 40 \\ 40 \end{pmatrix}$	$\begin{pmatrix} 20 \\ 15 \\ 18 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	75%

Deterministic optimization

$$\min_{p^{\max}_{\text{det}}} \sum p^{\max}_{\text{det}}, \quad \text{s.t.} \quad p_i \in [p_i^{\min}, p^{\max}_{\text{det},i}]. \qquad \Rightarrow \qquad p^{\max}_{\text{det}} = \begin{pmatrix} 46.10 \\ 52.04 \\ 51.08 \end{pmatrix}$$

Probabilistic optimization

$$\min_{p^{\max}_{\text{prob}}} \sum p^{\max}_{\text{prob}}, \quad \text{s.t.} \quad \mathbb{P}\left(p_i \in \left[p_i^{\min}, p^{\max}_{\text{prob},i}\right] \right) \geq 0.75. \qquad \Rightarrow \qquad p^{\max}_{\text{prob}} = \begin{pmatrix} 47.52 \\ 53.34 \\ 52.45 \end{pmatrix}$$

How good is the optimal deterministic solution in the probabilistic setting?

$$\mathbb{P}\left(b \in M(p^{\max}_{\text{det}}) \right) \approx 35.4\%$$

https://gaslib.zib.de/

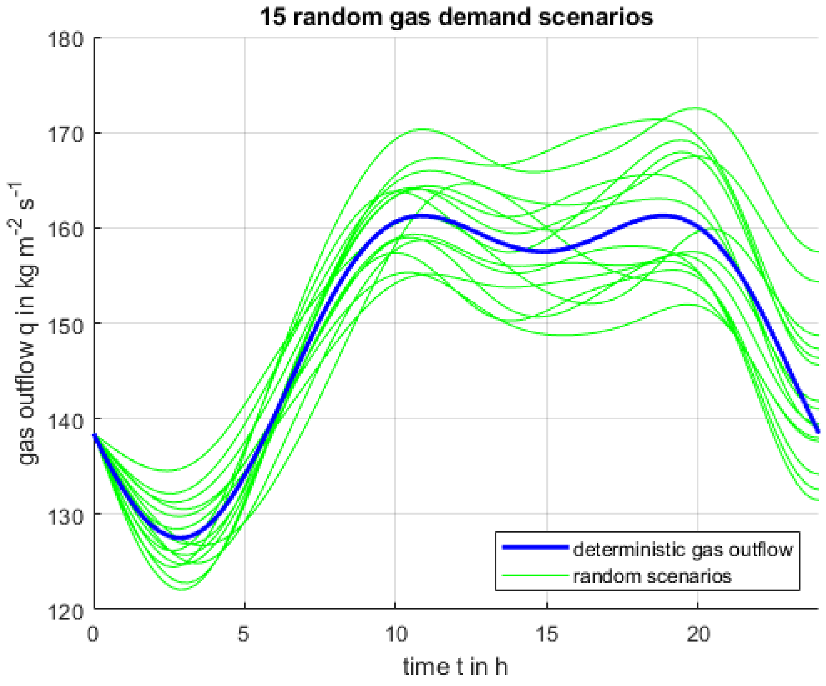
Random Boundary Functions

- Write a function f as Fourier series

$$f(t) = \sum_{m=0}^{\infty} a_m^0(f)\, \psi_m(t)$$

- For random variables $\xi_m \sim \mathcal{N}(1, \sigma)$ define

$$f^\omega(t) = \sum_{m=0}^{\infty} \xi_m(\omega)\, a_m^0(f)\, \psi_m(t)$$



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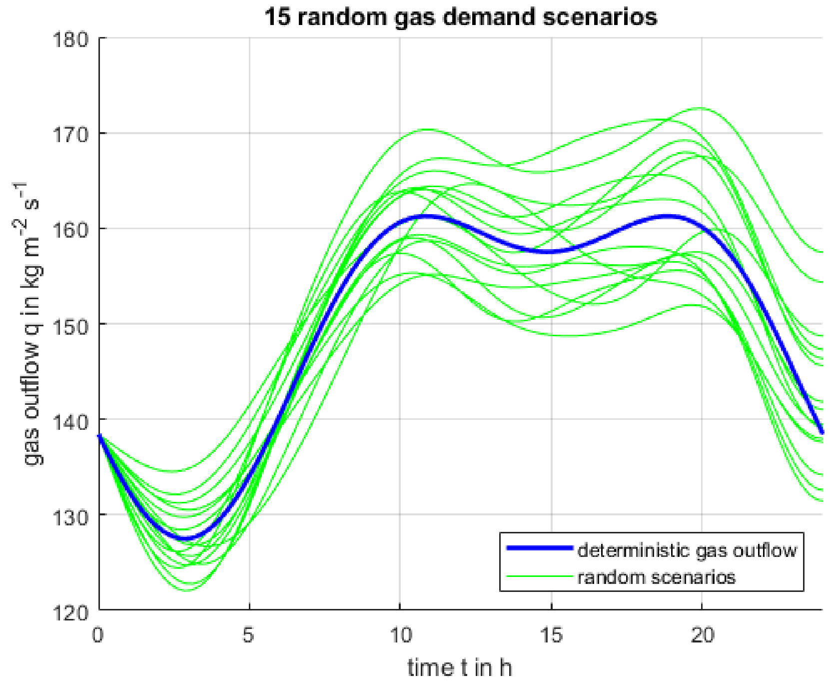


Time dependent probabilistic constraint

$$\mathbb{P}(\ f^\omega \in M(t) \quad \forall\, t \in [0, T]\ )\ \geq\ \alpha$$

„We want to guarantee that a percentage α of all possible random scenarios is feasible in every point in time $t \in [0, T]$.“

Mathematical Gas Transport

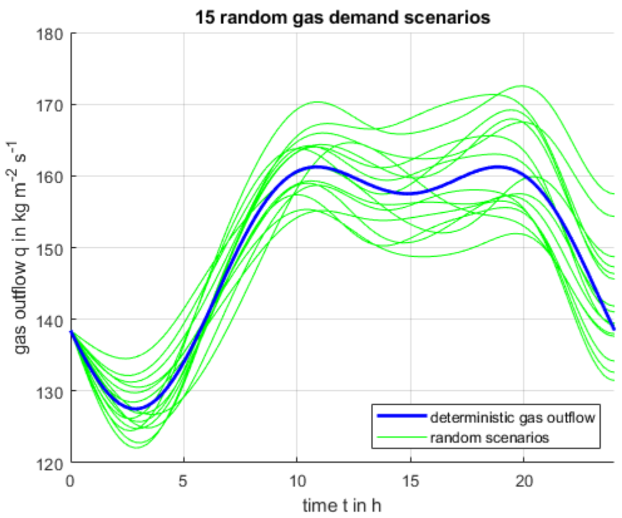
Dynamic Gas Transport in Pipeline Networks

For $(t,x) \in [0,T] \times [0,L]$, $\rho = \rho(t,x)$ and $q = q(t,x)$ consider the isothermal Euler equations for ideal gases:

ρ	density
q	flow
c	sound speed
λ	friction
D	diameter

The isothermal Euler equations

$$\rho_t + q_x = 0,$$
$$q_t + \left(c^2\,\rho + \frac{q^2}{\rho}\right)_x = \frac{\lambda}{2\,D}\frac{q|q|}{\rho},$$
$$\rho(t,0) = \rho_0(t),$$
$$q(t,L) = b(t)$$



- For suitable initial and boundary conditions there exists a unique solution
- We consider random boundary flow by using randomized Fourier series
- We define the set of feasible loads as $M(t) := \{\ \rho(\cdot,L) \mid \rho^{\min} \le \rho(t,L) \le \rho^{max}\ \}$
- We want to compute the probability $\mathbb{P}\left(\ \omega \in \Omega \mid \rho^{\omega}(\cdot,L) \in M(t) \quad \forall\, t \in [0,T]\ \right)$

[Gugat and Ulbrich, 2017]: *Lipschitz solutions of initial boundary value problems for balance laws.* Math. Models Methods Appl. Sci., 28(5): 921–951.

Mathematical Gas Transport

Dynamic Gas Transport in Pipeline Networks

- Let $\mathcal{B} = \{\ b^{S,1}(t), \cdots, b^{S,N_{\text{KDE}}}(t)\ \}$ be independent and identically distributed random boundary functions
- Let $\mathcal{P}_{\mathcal{B}} = \{\ \rho(t; b^{S,1}), \cdots, \rho(t; b^{S,N_{\text{KDE}}})\ \}$ be the corresponding densities at the end of the pipe

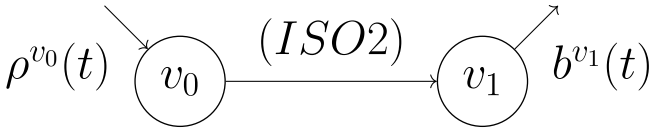
$$\mathbb{P}\left(\ \rho(t,L) \in \left[\rho^{\text{min}}, \rho^{\text{max}}\right] \quad \forall t \in [0,T]\ \right) \quad = \quad \mathbb{P}\begin{pmatrix} \min\limits_{t\in[0,T]} \rho(t,L) \in \left[\rho^{\text{min}}, \rho^{\text{max}}\right] \\ \max\limits_{t\in[0,T]} \rho(t,L) \in \left[\rho^{\text{min}}, \rho^{\text{max}}\right] \end{pmatrix}$$

- Let $\left\{ \begin{bmatrix} \underline{\rho}(b_1) := \min\limits_{t\in[0,T]} \rho(t; b^{S,1}) \\ \overline{\rho}(b_1) := \max\limits_{t\in[0,T]} \rho(t; b^{S,1}) \end{bmatrix}, \cdots, \begin{bmatrix} \underline{\rho}(b_{N_{\text{KDE}}}) := \min\limits_{t\in[0,T]} \rho(t; b^{S,N_{\text{KDE}}}) \\ \overline{\rho}(b_{N_{\text{KDE}}}) := \max\limits_{t\in[0,T]} \rho(t; b^{S,N_{\text{KDE}}}) \end{bmatrix} \right\} \subseteq \mathbb{R}^2$ be a sample of the minimal and maximal densities in $[0,T]$

Kernel density estimator for bandwidths h^{min} and h^{min}

$$\varrho_{p,N_{\text{KDE}}}(z) = \frac{1}{N_{\text{KDE}}\ h^{\text{min}}\ h^{\text{max}}} \sum_{i=1}^{N_{\text{KDE}}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{z-\underline{\rho}(b_i)}{h^{\text{min}}}\right)^2\right) \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{z-\overline{\rho}(b_i)}{h^{\text{max}}}\right)^2\right)$$

[Schuster 2021]: *Nodal Control and Probabilistic Constrained Optimization.* PhD thesis, FAU Erlangen-Nürnberg, Germany, 2021,
https://opus4.kobv.de/opus4-trr154/frontdoor/index/index/searchtype/latest/docId/410/start/2/rows/10



$\rho_0(t)$	$\rho^{\min}$	c	λ^F	D	L	T	α
46.75 kg/m^3	34 kg/m^3	343 m/s	0.1	0.5 m	30 km	24 h	90%

Deterministic Optimization

$$\min_{\rho^{\text{max}}_{\text{det}}} \quad \rho^{\text{max}}_{\text{det}},$$

$$\text{s.t.} \quad \rho(t, L) \in [\rho^{\text{min}}, \rho^{\text{max}}_{\text{det}}].$$

$$\Rightarrow \quad \rho^{*,\text{max}}_{\text{det}} = 42.15\ kg/m^3$$

Probabilistic Optimization

$$\min_{\rho^{\text{max}}_{\text{prob}}} \quad \rho^{\text{max}}_{\text{prob}},$$

$$\text{s.t.} \quad \mathbb{P}\left(\rho(t, L) \in \left[\rho^{\text{min}}, \rho^{\text{max}}_{\text{prob}}\right] \ \forall t \in [0, T] \right) \geq 0.9.$$

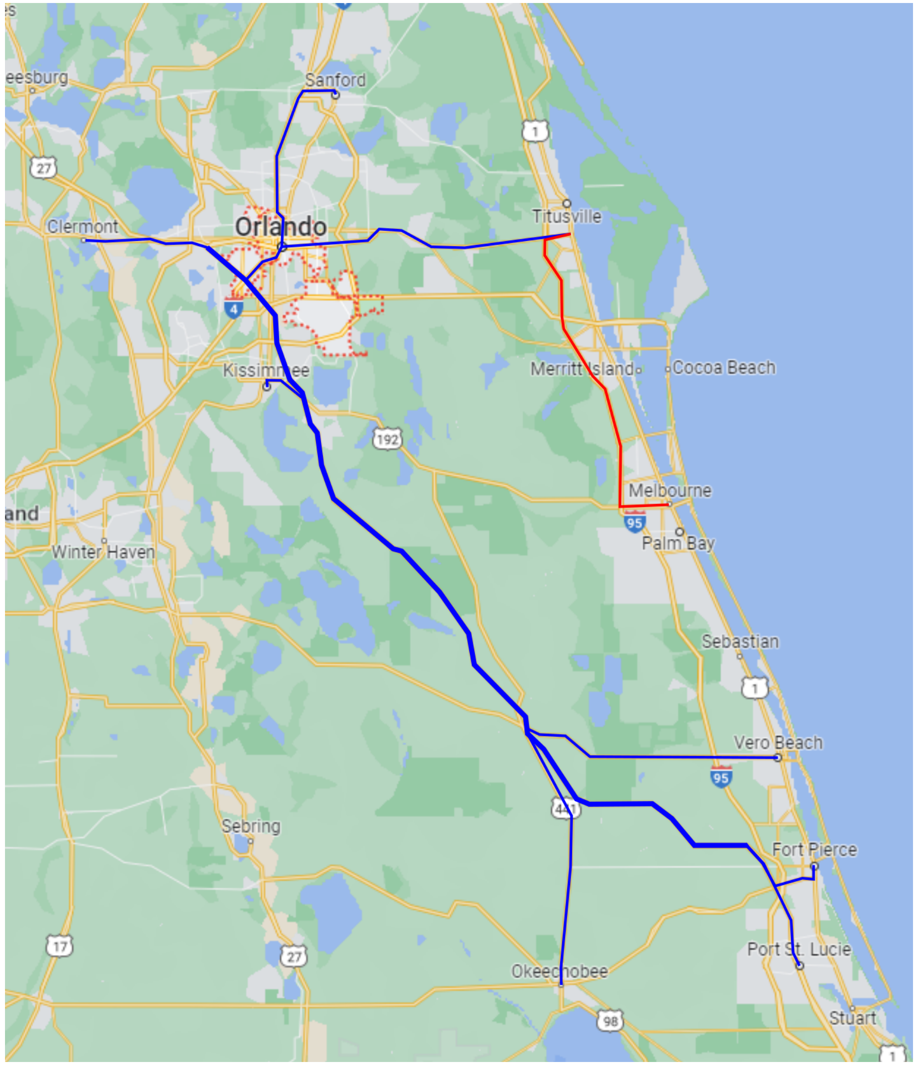
$$\Rightarrow \quad \rho^{*,\text{max}}_{\text{prob}} = 42.49\ kg/m^3$$

How good is the optimal deterministic solution in the probabilistic setting?

$$\mathbb{P}\big(b(t) \in M(t; p^{\max}_{\text{det}}) \quad \forall t \in [0, T] \big) \approx 50\%$$

Nodal Control under Uncertainty

A Turnpike Result for the Transport Equation



- There is a fastest route between any two points and if the origin and destination are close together and far from the turnpike, the best route may not touch the turnpike.
- But if origin and destination are far enough apart, it will always pay to get on to the turnpike and cover distance at the best rate of travel, even if this means adding a little mileage at either end.

[Dorfman, Samuelson, Solow, 1958]: Linear Programming and Economic Analysis. *New York: McGraw-Hill*

Mathematical Gas Transport

Stationary Gas Transport in Pipeline Networks

For $c > 0$ consider the transport equation in one dimension

$$r_t(t,x) + c\, r_x(t,x) = m\, r(t,x),$$

with initial condition and boundary control

$$r(0,x) = r_{\text{ini}}(x) \qquad \text{and} \qquad r(t,0) = u(t).$$

For convex functions f and g consider the optimal control problems

Dynamic Optimal Control Problem

$$\begin{aligned} \min_{u \in L^2(0,T)} \quad & J_T(u) = \int_0^T f\big(u(t)\big) + g\big(r(t,L)\big)\, dt \\ \text{s.t.} \quad & r_t(t,x) + c r_x(t,x) = m\, r(t,x), \\ & r(0,x) = r_{\text{ini}}(x), \\ & r(t,0) = u(t). \end{aligned}$$

Static Optimal Control Problem

$$\begin{aligned} \min_{u \in \mathbb{R}} \quad & J(u) = f\big(u\big) + g\big(r(L)\big) \\ \text{s.t.} \quad & c r_x(x) = m\, r(x), \\ & r(0) = u. \end{aligned}$$

(A1) For $\varepsilon > 0$ let functions f and g satisfy $\big(f'(x_1) - f'(x_2)\big)\big(x_1 - x_2\big) + \big(g'(y_1) - g'(y_2)\big)\big(y_1 - y_2\big) \ \geq\ \varepsilon\, \|x_1 - x_2\|_2^2$.

(A2) Let the derivative of g be Lipschitz continuous with Lipschitz constant L_k, i.e., $\|g'(y_1) - g'(y_2)\|_2 \ \leq\ L_k\, \|y_1 - y_2\|_2$.

Nodal Control under Uncertainty

A Turnpike Result for the Transport Equation

We consider the deterministic case first.

Deterministic Turnpike

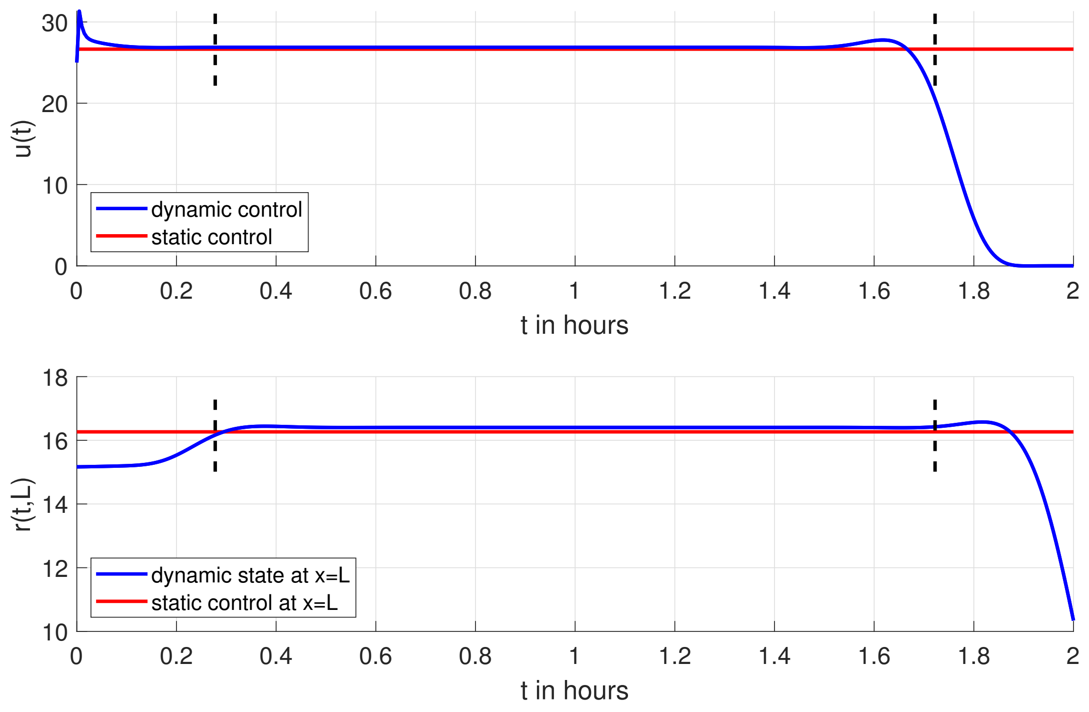
The optimal solution $u^\delta(t)$ of the dynamic optimal control problem

$$\min_{u \in L^2(0,T)} \quad J_T(u) = \int_0^T f\big(u(t)\big) + g\big(r(t,L)\big)\,dt$$

$$\text{s.t.} \quad r_t(t,x) + c r_x(t,x) = m\,r(t,x),$$
$$r(0,x) = r_{\text{ini}}(x),$$
$$r(t,0) = u(t).$$

and the optimal solution u^σ of the corresponding static problem satisfy the integral turnpike property

$$\int_0^T \|u^\delta(t) - u^\sigma\|_2^2\,dt \;\leq\; \mathcal{C}.$$



[Sakamoto and Schuster, 2023]: *A Turnpike Result for Optimal Boundary Control Problems with the Transport Equation under Uncertainty.* Preprint

Nodal Control under Uncertainty

A Turnpike Result for the Transport Equation

We randomize the initial state using a Wiener process: $r_{\text{ini}}^{\omega}(x) = r_{\text{ini}}(x) + W_x$

Deterministic Turnpike

The optimal solution $u^{\delta}(t)$ of the dynamic optimal control problem

$$\min_{u \in L^2(0,T)} \quad J_T(u) = \int_0^T f\big(u(t)\big) + g\big(\mathbb{E}[r(t,L)]\big)\, dt$$

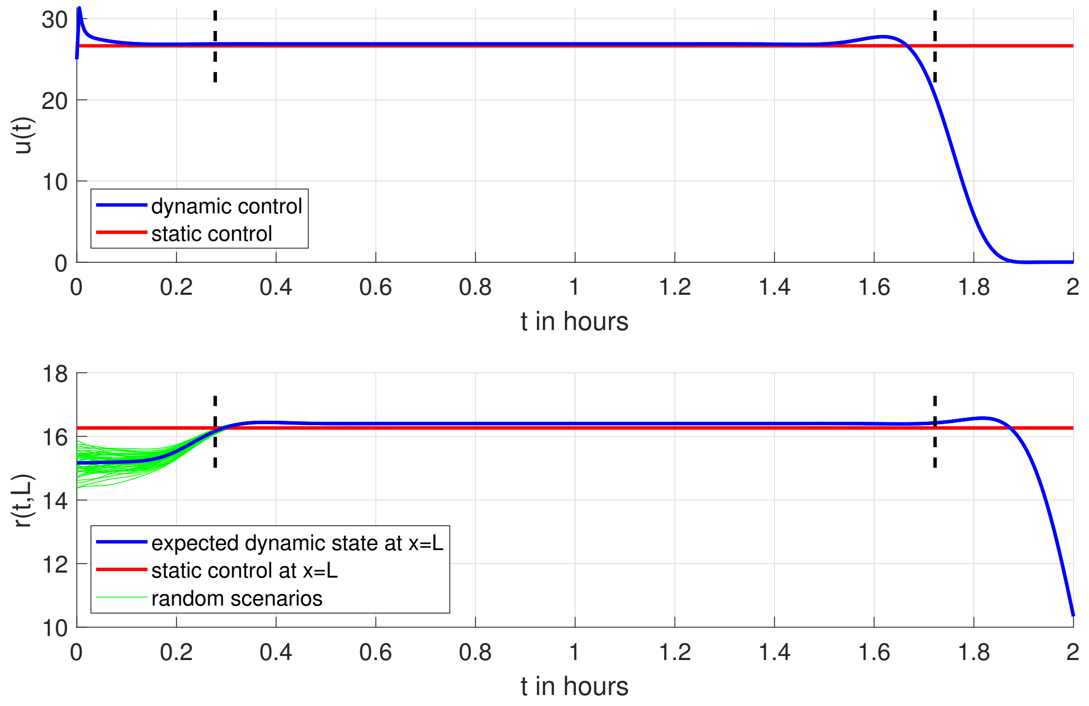
$$\text{s.t.} \quad r_t(t,x) + c r_x(t,x) = m\, r(t,x),$$

$$r(0,x) = r_{\text{ini}}^{\omega}(x),$$

$$r(t,0) = u(t).$$

and the optimal solution u^{σ} of the corresponding static problem satisfy the integral turnpike property

$$\int_0^T \|u^{\delta}(t) - u^{\sigma}\|_2^2 \, dt \;\leq\; \mathcal{C}.$$



[Sakamoto and Schuster, 2023]: *A Turnpike Result for Optimal Boundary Control Problems with the Transport Equation under Uncertainty.* Preprint

Nodal Control under Uncertainty

A Turnpike Result for the Transport Equation

We randomize the source term by a Gaussian distribution: $m^\omega = m(\omega) \sim \mathcal{N}(\mu, \sigma)$

Deterministic Turnpike

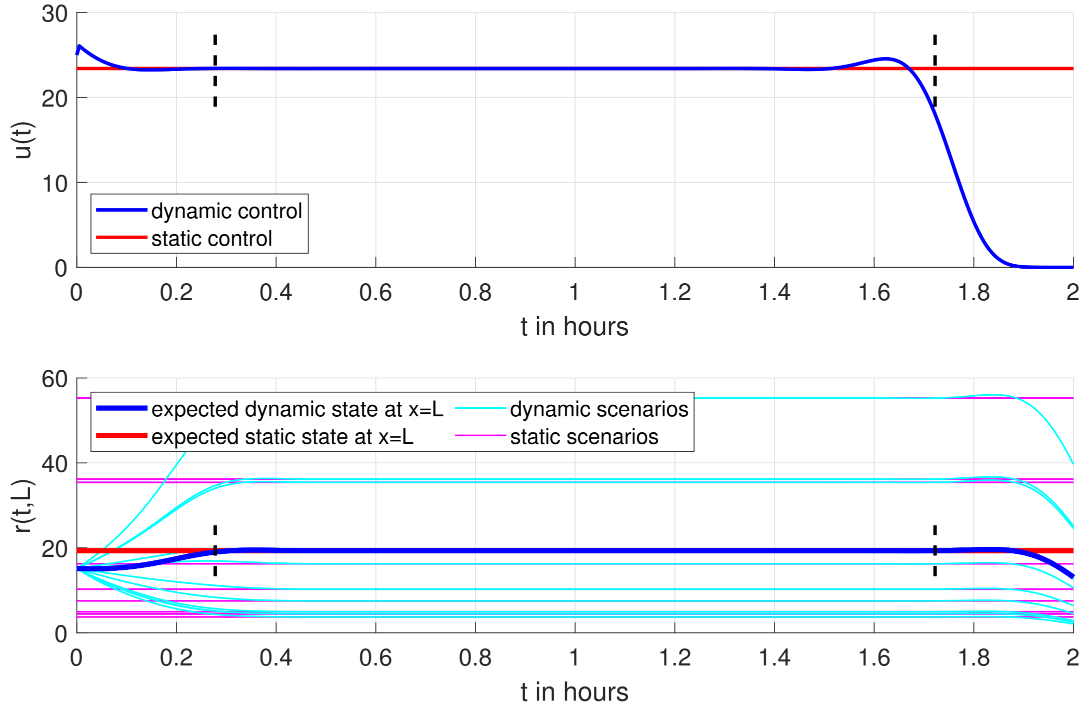
The optimal solution $u^\delta(t)$ of the dynamic optimal control problem

$$\min_{u \in L^2(0,T)} \quad J_T(u) = \int_0^T f\big(u(t)\big) + g\big(\mathbb{E}[r(t,L)]\big)\, dt$$

$$\text{s.t.} \quad r_t(t,x) + c r_x(t,x) = m^\omega\, r(t,x),$$
$$r(0,x) = r_{\text{ini}}(x),$$
$$r(t,0) = u(t).$$

and the optimal solution u^σ of the corresponding static problem satisfy the integral turnpike property

$$\int_0^T \|u^\delta(t) - u^\sigma\|_2^2\, dt \;\leq\; \mathcal{C}.$$



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Nodal Control and Probabilistic Constrained Optimization

Friedrich-Alexander-Universität
DYNAMICS, CONTROL, MACHINE LEARNING AND NUMERICS

Department MATHEMATIK

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