A Multiscale Inverse Problem approached via Homogenization

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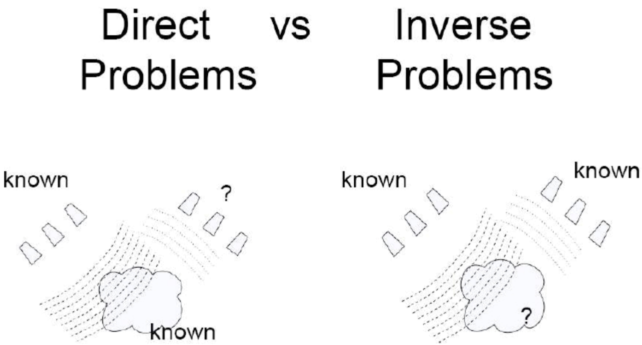

A general inverse problem

- A heat-conducting material in $\Omega \subset \mathbb{R}^d$ that varies on a fine scale $\varepsilon > 0$ is represented by its conductivity tensor, say $A^\varepsilon(x), x \in \Omega$.
- Stimulated by f, it reaches a temperature $u^\varepsilon(t,x)$, solution to

$$(P)\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} - \mathrm{div}(A^\varepsilon(x)\nabla u^\varepsilon) = f \quad \text{in} \quad (0,T)\times\Omega \\ u^\varepsilon = 0 \quad \text{on} \quad (0,T)\times\partial\Omega \\ u^\varepsilon(0,\cdot) = u_0 \in L^2(\Omega) \end{cases}$$

- Inverse Problem
  To determine $A^\varepsilon = A^\varepsilon(x)$ from measurements of the flux of u^ε on $\partial\Omega$ and the knowledge of $u^\varepsilon(\theta,\cdot)$ at a given time $\theta \in (0,T)$.
- "Source to flux" map

$$\begin{cases} \Lambda_{A^\varepsilon} : L^2(0,T;H^{-1}(\Omega)) \longrightarrow L^2(0,T;H^{-1/2}(\partial\Omega)) \\ f \mapsto \Lambda_{A^\varepsilon} f \ = \ A^\varepsilon \nabla u^\varepsilon \cdot \hat{n}\big|_{\partial\Omega \text{ or } \Gamma_0 \subseteq \partial\Omega} \end{cases}$$

Practical constraints

- Only a subregion $\omega \subseteq \Omega$ can be accessed to locate heat sources, where ω is a given open subset of Ω, and hence

  f will be assumed to be compactly supported in ω

- Fluxes $A^{\varepsilon}\nabla u^{\varepsilon} \cdot \hat{n}$ can only be measured over a subset of the boundary, say Γ_0; Γ_0 being an open subset of $\partial\Omega$, $|\Gamma_0| > 0$.
- What will be measured or known is actually

$$A^{\varepsilon}\nabla u^{\varepsilon} \cdot \hat{n}\big|_{\Gamma_0}$$

All of our analysis of the IP will be carried out with (RHS)'s

$$f \in \mathcal{C}_0^{\infty}((0, T) \times \omega)$$

Existence & regularity of the solution of (P)

A. Pazy (1983), D. Gilbarg & N.S. Trudinger (2^{nd} ed. 2001)

By semi-group theory, (P) has a unique solution

$$u^\varepsilon \in \mathcal{C}([0,T];L^2(\Omega)) \cap \mathcal{C}((0,T);H^2(\Omega)\cap H_0^1(\Omega)) \cap \mathcal{C}^1((0,T),L^2(\Omega))$$

Let $N \in \mathbb{N}$ and τ_1, τ_2 be fixed s.t.

$$N > \frac{d}{2} + 3; \qquad 0 < \tau_1 < \tau_2 < T.$$

Regularity for coefficients in $\mathcal{C}^{N-1,1}(\overline{\Omega})^{d\times d}$

$$||u^\varepsilon||_{\mathcal{C}^\ell((\tau_1,\tau_2);H^{m_0}(\Omega))} \leq C_\ell \left(||u^\varepsilon(0,\cdot)||_{L^2(\Omega)} + ||f||_{W^{1,1}(0,T;H^{m_0}(\omega))} \right) \ \forall \ell \geq 0$$

where $m_0 = 2\left[\frac{N+1}{2}\right]$.

Local stability & identifiability (I)

For $0 < \alpha < \beta$, define

$$\mathcal{M}(\alpha,\beta,\Omega) \overset{\text{(def)}}{=} \left\{ A \in \mathcal{C}^{N-1,1}(\overline{\Omega}; Sym_d) \,\middle|\, \alpha|\xi|^2 \leq A(x)\xi\cdot\xi \leq \beta|\xi|^2 \;\forall\, \xi, x \right\}$$

Sym_d: class of $d \times d$ real-valued symmetric matrices

Admissible tensors around a given set $\eta = (\eta_{ij})_{i,j=1}^{d}$

Let $M_0, r_0 > 0$ be given (r_0 small). We define

$$\omega_1 \overset{\text{(def)}}{=} \{x \in \Omega \mid dist(x, \partial\Omega) < r_0\}.$$

For any set of smooth functions $\eta = (\eta_{ij})_{i,j=1}^{d}$ defined on $\overline{\Omega}$, we set

$$\mathcal{U}[M_0,\eta] \overset{\text{(def)}}{=} \left\{ A \in \mathcal{M}(\alpha,\beta,\Omega) \,\middle|\, ||A||_{\mathcal{C}^{N-1,1}(\overline{\Omega})^{d\times d}} \leq M_0,\, A\big|_{\omega_1} = \eta \right\}$$

Stability & Identifiability (II)

Theorem Let $A^{(2)} \in \mathcal{U}$ be fixed. Then we can choose $\ell_0 = d(d+3)/2$ functions $(f_\ell)_{\ell=1}^{\ell_0}$ in $\mathcal{C}_0^\infty((0,T)\times\omega)$ such that, for every $A^{(1)} \in \mathcal{U}$,

$$||A^{(1)} - A^{(2)}||_{H^1(\Omega)^{d\times d}} \leq C \sum_{\ell=1}^{\ell_0} ||u[A^{(1)}, f_\ell](\theta,\cdot) - u[A^{(2)}, f_\ell](\theta,\cdot)||_{H^3(\Omega)} +$$

$$+ C \sum_{\ell=1}^{\ell_0} ||A^{(1)}\nabla u[A^{(1)}, f_\ell]\cdot\hat{n} - A^{(2)}\nabla u[A^{(2)}, f_\ell]\cdot\hat{n}]||_{H^2((\tau_1,\tau_2);L^2(\Gamma_0))},$$

where C depends continuously on $\mathcal{U}$, $(f_\ell)_{\ell=1}^{\ell_0}$ and $||A^{(2)}||_{\mathcal{C}^1(\overline{\Omega})^{d\times d}}$.

Good and not-so-good consequences

- (Uniqueness) Same solutions and fluxes $\Rightarrow$ tensors coincide
- (Reconstruction) It suggests that it would be enough to do it using a finite number of sources
- (Multiscale case) The constant C scales to an $O(\varepsilon^{-1})$

Reconstruction

Two considerations before reconstruction

- Reconstruction of multiple coefficients is currently outside the scope of inverse problems theory

  The number of unknowns should be reduced ($\frac{d(d+1)}{2} \to 1$)

- A numerical reconstruction departing from the fine-scale model requires a resolution (i.e. a mesh size) resolving the scale ε (which, although fixed, is very small), often representing a prohibited cost.

Going down to the fine-scale should be avoided,
which we'll do via homogenization

Why a homogenization strategy?

$$(A^\varepsilon)_{\varepsilon\to 0} \xrightarrow{\mathrm{G}} A^\circ \Rightarrow \forall f \in L^2(0,T;H^{-1}(\Omega)), \begin{cases} \Lambda_{A^\varepsilon} f \overset{\varepsilon\to 0}{\rightharpoonup} \Lambda_{A^\circ} f \quad \text{in} \\ \quad L^2(0,T;H^{-1/2}(\partial\Omega)) \end{cases}$$

Homogenization strategy

A new class of admissible highly oscillating tensors

Admissible tensors

A^ε is sought in the subclass of parametrizable, locally periodic tensors of the form

$$A^\varepsilon(x) = A^\varepsilon_\sigma(x) = F(\sigma(x), \frac{x}{\varepsilon}),$$

where $\left\{ \begin{array}{rcl} F: \mathbb{R} \times \mathbb{R}^d & \to & Sym_d \subset \mathbb{R}^{d\times d} \\ (s,y) & \mapsto & F(s,y) \end{array} \right.$ is $Y = [0,1)^d$-periodic in y,

and $\sigma : \Omega \to \mathbb{R}$ is the parametrizing function. F, σ sufficiently smooth.

Our IP revisted (restricted to less admissible tensors)

- The map $F(\cdot,\cdot)$ is assumed to be known, we look for $\sigma(\cdot)$, say

$$\sigma \in \mathcal{A}_{\text{ad}} \overset{\text{(def)}}{=} \left\{ m \in \mathcal{C}^{N-1,1}(\overline{\Omega}) \;\middle|\; A^\varepsilon_m \in \mathcal{M}(\alpha,\beta,\Omega) \right\}$$

Parametrizable, locally $[0,1)^d$-periodic homogenization

Explicit formulae for the homogenized coefficients

- Any sequence $(A^\varepsilon_\sigma)_\varepsilon$ is G-convergent, say $\text{G-}\lim\limits_{\varepsilon\to 0} A^\varepsilon_\sigma = A^0_\sigma \in \mathcal{A}_{\text{ad}}$
- Explicit formulae for the coefficients $(A^0_\sigma)_{ij}(x)$ of A^0_σ are available in terms of $(F(\cdot,\cdot))_{ij}$ and σ: $\forall i,j = 1,\ldots d$,

$$(A^0_\sigma)_{ij}(x) = \int_Y F_{ij}(\sigma(x),y)dy - \sum_{k=1}^{d} \int_Y (F(\sigma(x),y))_{ik} \frac{\partial \hat{\chi}_j}{\partial y_k} dy,$$

where $\hat{\chi}_j$ is the unique solution to finding $\hat{\chi}_j \in H^1_\#(Y)$ s.t.

$$\int_Y F(\sigma(x),y)\nabla_y \hat{\chi}_j \cdot \nabla_y v\, dy = \int_Y F(\sigma(x),y)\, \mathbf{e}_j \cdot \nabla_y v\, dy$$

$\forall v \in H^1_\#(Y)$

The effective inverse problem (I)

- As said in the first slide, our IP's input data are the maps

$$f \mapsto (\Lambda_{A^\varepsilon_\sigma} f)|_{(\tau_1,\tau_2)\times\Gamma_0} \quad \& \quad f \mapsto u[A^\varepsilon_\sigma, f](\theta,\cdot),$$

corresponding to boundary measurements of the fluxes and the solution of (P) at time $t = \theta$.

This is a considerable amount of information

- The stability theorem suggests that a finite number of source terms would be enough for reconstruction

The effective inverse problem (II)

- The stability theorem suggests that a finite number of source terms would be enough for reconstruction
- and leads naturally to consider the optimisation problem of finding $\sigma \in \mathcal{A}_{\mathrm{ad}}$ s.t.

$$\Phi^{\varepsilon}(\sigma) \;=\; \inf_{m \in \mathcal{A}_{\mathrm{ad}}} \Phi^{\varepsilon}(m),$$

$$\Phi^{\varepsilon}(m) \overset{(\mathrm{def})}{=} \sum_{\ell=1}^{\ell_0} ||(\Lambda_{A^{\varepsilon}_{\sigma}} - \Lambda_{A^{0}_{m}}) f_{\ell}||^2_{H^2(\tau_1,\tau_2;L^2(\Gamma_0))} + ||(u[A^{\varepsilon}_{\sigma}, f_{\ell}] - u[A^{0}_{m}, f_{\ell}])(\theta,\cdot)||^2_{H^3(\Omega)}$$

Here, $u[A^0_m, f_\ell] = u^0_m$ is the unique solution for $A^0_m(x) = \underset{\varepsilon \to 0}{\mathrm{G\text{-}lim}}\, A^{\varepsilon}_m$, i.e.,

$$(HP)_\ell \begin{cases} \dfrac{\partial u^0_m}{\partial t} - \mathrm{div}(A^0_m \nabla u^0_m) = f_\ell & \text{in} \quad (0,T) \times \Omega \\ u^0_m = 0 & \text{on} \quad (0,T) \times \partial\Omega \\ u^0_m(0,\cdot) = u_0 & \text{in} \quad \Omega \end{cases}$$

Reconstruction Algorithm

for σ or A^ε_σ

Input data: The maps $f \mapsto (\Lambda_{A^\varepsilon_\sigma} f)|_{(\tau_1,\tau_2)\times\Gamma_0}$, $f \mapsto u[A^\varepsilon_\sigma, f](\theta,\cdot)$, for a fixed $\theta \in (\tau_1,\tau_2)$. Take a first $m \in \mathcal{A}_{\text{ad}}$.

1. Compute the homogenized tensor A^0_m
2. Set $A^{(2)} = A^0_m$ in the stability theorem. Obtain the sources $(f_\ell)_{\ell=1}^{\ell_0}$
3. Recover data: $(\Lambda_{A^\varepsilon_\sigma} f_\ell)|_{(\tau_1,\tau_2)\times\Gamma_0}$, $u[A^\varepsilon_\sigma, f_\ell](\theta,\cdot)$
4. For $\ell = 1,\ldots,\ell_0$, solve the homogenized problem $(\text{HP})_\ell$
5. Compute the corresponding fluxes $(\Lambda_{A^0_m} f_\ell)|_{(\tau_1,\tau_2)\times\Gamma_0}$, and evaluate $u^0_m(\theta,\cdot)$ for $\ell = 1,\ldots,\ell_0$
6. Evaluate $\Psi^\varepsilon(m) = \Phi^\varepsilon(m) + \gamma||m - m_0||^2_{H^1(\Omega) \text{ or } L^2(\Omega)}$ (Tikhonof's regularization, parameter γ, guess m_0 for σ, given)
7. Depending on the optimization method chosen for minimizing Ψ, upgrade m, go back to the first step
8. End when a tolerance criterion is achieved

Mathematical justification

Theorem(c) Let $\sigma \in \mathcal{A}_{\text{ad}}$ be fixed. For each ε, we consider the minimization problem

$$\inf_{m \in \mathcal{A}_{\text{ad}}} \Phi^{\varepsilon}(m),$$

Then for every minimizing sequence $(m_n^{\varepsilon})_n$, there is a subsequence $(m_{n_k}^{\varepsilon})_k$ such that

$$\mathrm{G}\text{-}\lim_{k\to\infty} A^{\varepsilon}_{m^{\varepsilon}_{n_k}}(x) = A^{\varepsilon}_{\sigma}(x).$$

Moreover,

$$\mathrm{G}\text{-}\lim_{\varepsilon\to 0}\left(\mathrm{G}\text{-}\lim_{k\to\infty} A^{\varepsilon}_{m^{\varepsilon}_{n_k}}(x)\right) = A^{0}_{\sigma}(x).$$

Thank You for your Attention !