Trends in Mathematical Sciences

Impulsive evolution processes: abstract results and an application to a coupled wave equations

Everaldo de Mello Bonotto

Instituto de Ciências Matemáticas e de Computação
Universidade de São Paulo

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Joint work with

Marcelo José Dias Nascimento
Universidade Federal de São Carlos, UFSCar - Brazil

Eric Busatto Santiago
Universidade Federal de São Carlos, UFSCar - Brazil

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Summary

Evolution Processes

Consider the non-autonomous problem

$$\begin{cases} x' = f(x,t) \\ x(s) = x_0, \end{cases}$$

($f : \mathbb{R}^{n+1} \to \mathbb{R}^n$) such that there exists a unique globally well-defined solution $x(t, s, x_0)$ for each initial condition $(s, x_0) \in \mathbb{R} \times \mathbb{R}^n$.

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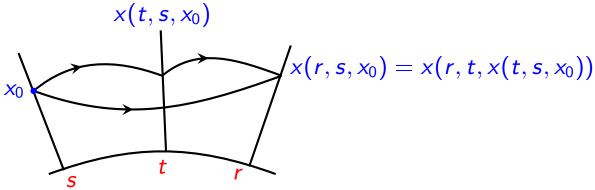
- $x(s,s,x_0) = x_0 \quad \forall s \in \mathbb{R}$ (initial value condition);
- $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \ni (t,s,x_0) \mapsto x(t,s,x_0)$ is continuous;
- $x(r,s,x_0) = x(r,t,x(t,s,x_0)) \quad \forall r,s,t \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$ (cocycle property).



Let (Z, d) be a metric space.

Definition 1

An **evolution process** acting in Z is a two-parameter family $\{S(t,\tau) : t \geq \tau \in \mathbb{R}\}$ of maps from Z into itself such that:

(a) $S(t,t) = I \quad \forall\, t \in \mathbb{R}$,

(b) $S(t,\tau) = S(t,s)S(s,\tau) \quad \forall\, t \geq s \geq \tau$, and

(c) the map $\{(t,\tau) \in \mathbb{R}^2 : t \geq \tau\} \times Z \ni (t,\tau,x) \mapsto S(t,\tau)x \in Z$ is continuous.

Evolution Processes

"Plenty of problems in the real world are non-autonomous. It is nearly impossible for some events to be time independent: growing cells in a life form, the motions of the wind, an external force being applied to some surface, and so on. We can certainly say that everyday problems are mostly non-autonomous (including the case of random influences as these are non-autonomous by their own nature), and we merely approximate many of these phenomena by an autonomous model to simplify our study."

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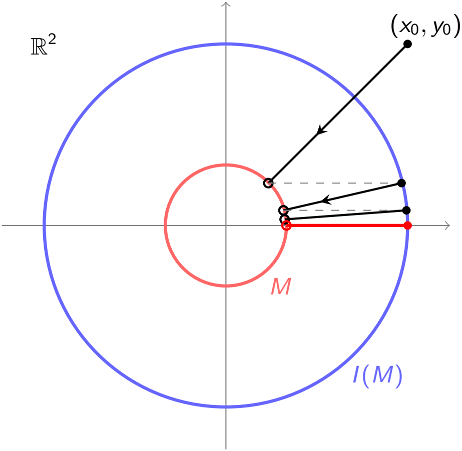
"The theory of impulsive dynamical systems arises to comprehend the structure of systems where the continuity of their evolution is interrupted by abrupt changes of state."

Summary

- $x' = -x$, $y' = -y$;
- $M = \{(x, y) : x^2 + y^2 = 1\}$;



• **impulsive family** is a nonempty collectively closed family $\hat{M} = \{M(t)\}_{t\in\mathbb{R}}$ satisfying the property: for every $\tau \in \mathbb{R}$ and $x \in M(\tau)$ there exists $\epsilon = \epsilon(x,\tau) > 0$ such that

$$\bigcup_{t\in(0,\epsilon)} (\{S(t+\tau,\tau)x\} \cap M(t+\tau)) = \emptyset.$$

The family $\hat{M} = \{M(t)\}_{t\in\mathbb{R}}$ is *collectively closed* if for $t_n \xrightarrow{n\to\infty} t$, $x_n \in M(t_n)$ with $x_n \xrightarrow{n\to\infty} x$, then $x \in M(t)$.

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• **impulse function** is a collectively continuous family $\hat{I} = \{I_t\colon M(t) \to Z\}_{t\in\mathbb{R}}$.

The family $\hat{I} = \{I_t\colon M(t) \to Z\}_{t\in\mathbb{R}}$ is *collectively continuous* if given $t_n \xrightarrow{n\to\infty} t$ and $x_n \in M(t_n)$ for each $n \in \mathbb{N}$ with $x_n \xrightarrow{n\to\infty} x$, then $I_{t_n}(x_n) \xrightarrow{n\to\infty} I_t(x)$.

Definition 2

An **impulsive evolution process** $(Z, S, \hat{M}, \hat{I})$ consists of an evolution process $\{S(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ in Z, an *impulsive family* $\hat{M} = \{M(t)\}_{t\in\mathbb{R}}$ and an *impulse function* $\hat{I} = \{I_t\colon M(t) \to Z\}_{t\in\mathbb{R}}$.

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Definition 3

(Impact Time Map) For $x \in Z$ and $\tau \in \mathbb{R}$,

$$\phi(x,\tau) = \begin{cases} s, & \text{if } S(s+\tau,\tau)x \in M(s+\tau) \text{ and } S(r+\tau,\tau)x \notin M(r+\tau) \text{ for } 0 < r < s, \\ \infty, & \text{if } S(t+\tau,\tau)x \notin M(t+\tau) \text{ for all } t > 0. \end{cases}$$

- If $\phi(x,\tau) < \infty$, then it represents the smallest time for which the trajectory of the point x starting at time τ meets the family $\hat{M}$.

- If $S(r+\tau,\tau)x \in M(r+\tau)$, with $r > 0$, then $\phi(x,\tau) \leq r$.

Given $\tau \in \mathbb{R}$, the **impulsive trajectory** of a point $x \in Z$, starting at time τ, is a map $\tilde{S}(\cdot, \tau)x$ defined on some interval $J_{(x,\tau)} \subseteq [\tau, \infty)$, which contains τ, taking values in Z given inductively by the following rule:

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- if $\phi(x,\tau) < \infty$, by denoting $x = x_0^+$, $\tau_0 = \tau$, and setting $\tau_1 = \phi(x_0^+,\tau_0) + \tau_0$, we define

$$\tilde{S}(t,\tau)x = \begin{cases} S(t,\tau_0)x_0^+, & \text{if } \tau_0 \leq t < \tau_1, \\ I_{\tau_1}(S(\tau_1,\tau_0)x_0^+), & \text{if } t = \tau_1. \end{cases}$$

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where $x_2^+ = I_{\tau_2}(x_2) \quad$ with $\quad x_2 = S(\tau_2,\tau_1)x_1^+.$

Impulsive Evolution Processes

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The impulsive trajectory $\tilde{S}(\cdot, \tau)x$ is defined on the interval $[\tau, T(x, \tau))$, where

$$T(x, \tau) = \tau + \sum_{i=0}^{\infty} \phi(x_i^+, \tau_i).$$

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Lemma 4

The following properties hold:

(i) $\tilde{S}(t,t)x = x$ *for all* $x \in Z$ *and all* $t \in \mathbb{R}$*;*

(ii) $\tilde{S}(t,\tau) = \tilde{S}(t,s)\tilde{S}(s,\tau)$ *for all* $t \geq s \geq \tau \in \mathbb{R}$*.*

Example 5

Consider the evolution process $\{S(t,\tau)\colon t \geq \tau \in \mathbb{R}\}$ in $\mathbb{R}$ given by

$$S(t,s)x = \sqrt[3]{t} - \sqrt[3]{s} + x, \quad t \geq s, \ \ x \in \mathbb{R}.$$

Set $M(t) = \{-t\}$ for each $t \in \mathbb{R}$. Then $\hat{M} = \{M(t)\}_{t\in\mathbb{R}}$ is an impulsive family, since

$$S(r+s,s)(-s) = -r - s \Longleftrightarrow r < 0.$$

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- $\phi(-1,0) = \tau_1 \in (0,1)$ ($\tau_1 \approx 0.317$). If we choose $\hat{I} = \{I_t\colon M(t) \to Z\}_{t\in\mathbb{R}}$ such that

$$x_1^+ = I_{\tau_1}(S(\tau_1,0)(-1)) \geq -\tau_1 \ \text{ then } \ \phi(x_1^+,\tau_1) = \infty,$$

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- $\phi(-1,1) = \infty$ and $\phi(-1-\frac{1}{n},1) \xrightarrow{n\to\infty} 0$.

$$x_n \xrightarrow{n\to\infty} x \text{ and } \tau_n \xrightarrow{n\to\infty} \tau \ \ ?? \Longrightarrow ?? \ \ \phi(x_n,\tau_n) \xrightarrow{n\to\infty} \phi(x,\tau).$$

$$\text{For each } \tau \in \mathbb{R} \text{ fixed, the impact time map } \phi(\cdot,\tau)\colon Z \to (0,\infty] \text{ is continuous on } Z \backslash M(\tau). \tag{H1}$$

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Theorem 6

Let $\tau \in \mathbb{R}$, $x \in Z$, $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ and $\{x_n\}_{n\in\mathbb{N}} \subset Z$ be sequences such that $x_n \xrightarrow{n\to\infty} x$ and $\tau_n \xrightarrow{n\to\infty} \tau$. If $x \notin M(\tau)$, then

$$\phi(x_n, \tau_n) \xrightarrow{n\to\infty} \phi(x, \tau).$$

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Theorem 7

Assume that $I_s(M(s)) \cap M(s) = \emptyset$ for all $s \in \mathbb{R}$. Let $t,\tau \in \mathbb{R}$, $x \in Z\backslash M(\tau)$, $\{x_n\}_{n\in\mathbb{N}} \subset Z$ and $\{\tau_n\}_{n\in\mathbb{N}} \subset (-\infty,t]$ be sequences such that $x_n \xrightarrow{n\to\infty} x$ and $\tau_n \xrightarrow{n\to\infty} \tau$. Then, there exists a sequence $\{\eta_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, with $t+\eta_n \geq \tau_n$ and $\eta_n \xrightarrow{n\to\infty} 0$, such that

$$\tilde{S}(t+\eta_n,\tau_n)x_n \xrightarrow{n\to\infty} \tilde{S}(t,\tau)x.$$

Assume that condition (H1) holds and $I_s(M(s)) \cap M(s) = \emptyset$ for all $s \in \mathbb{R}$.

Lemma 8

Let $\tau \in \mathbb{R}$, $x \in M(\tau)$, $\{\alpha_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, $\{\beta_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, and $\{x_n\}_{n\in\mathbb{N}} \subset Z$ be sequences such that $\alpha_n \geq \beta_n$ for all $n \in \mathbb{N}$, $\alpha_n \xrightarrow{n\to\infty} 0$, $\beta_n \xrightarrow{n\to\infty} 0$, and $x_n \xrightarrow{n\to\infty} x$. Then there exists a subsequence $\{\phi(x_{n_k}, \tau + \beta_{n_k})\}_{k\in\mathbb{N}}$ of $\{\phi(x_n, \tau + \beta_n)\}_{n\in\mathbb{N}}$ such that $\phi(x_{n_k}, \tau + \beta_{n_k}) \xrightarrow{k\to\infty} 0$. Moreover,

- (i) *if $\alpha_{n_k} - \beta_{n_k} < \phi(x_{n_k}, \tau + \beta_{n_k})$ for all $k \in \mathbb{N}$, then $\tilde{S}(\tau + \alpha_{n_k}, \tau + \beta_{n_k})x_{n_k} \xrightarrow{k\to\infty} x$;*
- (ii) *if $\alpha_{n_k} - \beta_{n_k} \geq \phi(x_{n_k}, \tau + \beta_{n_k})$ for all $k \in \mathbb{N}$, then $\tilde{S}(\tau + \alpha_{n_k}, \tau + \beta_{n_k})x_{n_k} \xrightarrow{k\to\infty} I_\tau(x)$.*

Summary

"To understand the asymptotic behavior of evolution equations the notion of attractor plays a fundamental role."

"In general, an attractor means a compact set which satisfies an invariance property and that attracts (in some sense) a class of subsets of the phase space in which the equation is stated."

"In the case of non-autonomous systems, we can find in the literature at least two different approaches to describe their dynamics: the pullback attraction and the forward attraction. Here, we are concerned with the pullback attraction."

Definition 9

The **impulsive pullback ω-limit set** of a subset $B \subset Z$ at time $t \in \mathbb{R}$ is defined by

$$\tilde{\omega}(B,t) = \{x \in Z\colon \text{ there are sequences } \{\tau_n\}_{n\in\mathbb{N}} \subset \mathbb{R}, \{\epsilon_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R} \text{ and } \{x_n\}_{n\in\mathbb{N}} \subseteq B \\ \text{such that } \tau_n \xrightarrow{n\to\infty} -\infty,\ \epsilon_n \xrightarrow{n\to\infty} 0,\ \{\tau_n - \epsilon_n\}_{n\in\mathbb{N}} \subset (-\infty, t] \\ \text{and } \tilde{S}(t+\epsilon_n, \tau_n)x_n \xrightarrow{n\to\infty} x\}.$$

The **impulsive pullback ω- limit set of B** is the family $\tilde{\omega}(B) = \{\tilde{\omega}(B,t)\}_{t\in\mathbb{R}}$. Note that $\tilde{\omega}(B)$ is a family of closed subsets of Z.

Definition 10

Let $\hat{B} = \{B(t)\}_{t\in\mathbb{R}}$ be a family of nonempty subsets of Z. We say that $\hat{B}$ is:

- **positively $\tilde{S}$-invariant** if $\tilde{S}(t,\tau)B(\tau) \subseteq B(t)$ for all $t \geq \tau \in \mathbb{R}$.
- **negatively $\tilde{S}$-invariant** if $\tilde{S}(t,\tau)B(\tau) \supseteq B(t)$ for all $t \geq \tau \in \mathbb{R}$.
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Definition 11

A family $\hat{A} = \{A(t)\}_{t\in\mathbb{R}}$ **pullback $\tilde{S}$-attracts bounded subsets of Z**, if for every bounded set $B \subset Z$, every $t \in \mathbb{R}$ and all sequences $\{\epsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ and $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ with $\{\tau_n - \epsilon_n\}_{n\in\mathbb{N}} \subset (-\infty, t]$, $\epsilon_n \xrightarrow{n\to\infty} 0$ and $\tau_n \xrightarrow{n\to\infty} -\infty$ we have

$$\lim_{n\to\infty} \mathrm{d_H}(\tilde{S}(t+\epsilon_n, \tau_n)B, A(t)) = 0.$$

The *Hausdorff semidistance* between two nonempty subsets A and B of Z is given by

$$\mathrm{d_H}(A, B) = \sup_{a\in A} \inf_{b\in B} d(a, b).$$

Definition 12

A family $\hat{\mathbb{A}} = \{\mathbb{A}(t)\}_{t\in\mathbb{R}}$ of subsets of Z is called an **impulsive pullback attractor** for the impulsive evolution process $(Z, S, \hat{M}, \hat{I})$ if:

(*i*) $\{\mathbb{A}(t)\}_{t\in\mathbb{R}}$ is compact;

(*ii*) $\{\mathbb{A}(t)\backslash M(t)\}_{t\in\mathbb{R}}$ is $\tilde{S}$-invariant;

(*iii*) $\{\mathbb{A}(t)\}_{t\in\mathbb{R}}$ pullback $\tilde{S}$-attracts bounded subsets of Z;

(*iv*) $\{\mathbb{A}(t)\}_{t\in\mathbb{R}}$ is the minimal family of closed sets satisfying property (*iii*).

Condition (*iv*) says that, provided $\hat{\mathbb{A}}_1$ and $\hat{\mathbb{A}}_2$ are two impulsive pullback attractors for an impulsive evolution process $(Z, S, \hat{M}, \hat{I})$, then $\mathbb{A}_1(t) = \mathbb{A}_2(t)$ for every $t \in \mathbb{R}$.

Definition 13

A process $(Z, S, \hat{M}, \hat{I})$ is said to be **pullback $\tilde{S}$-asymptotically compact** if, given $t \in \mathbb{R}$, sequences $\{\epsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ and $\{x_n\}_{n\in\mathbb{N}} \subset Z$ such that $\{\tau_n - \epsilon_n\}_{n\in\mathbb{N}} \subset (-\infty, t]$, $\tau_n \xrightarrow{n\to\infty} -\infty$, $\epsilon_n \xrightarrow{n\to\infty} 0$ and $\{x_n\}_{n\in\mathbb{N}} \subset Z$ is bounded, then the sequence $\{\tilde{S}(t+\epsilon_n, \tau_n)x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence.

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A process $(Z, S, \hat{M}, \hat{I})$ is said to be **pullback $\tilde{S}$-strongly bounded dissipative** if, for each $t \in \mathbb{R}$, there exists a bounded subset $B(t)$ of Z which pullback $\tilde{S}$-absorbs bounded subsets of Z at time t, that is, there exists $\epsilon_0 > 0$ such that, for each bounded subset D of Z, one can find a time $T = T(t, D) \leq t$ such that

$$\tilde{S}(t+\epsilon, \tau)D \subset B(t) \quad \text{for all} \quad \tau \leq T - \epsilon_0 \quad \text{and} \quad |\epsilon| \leq \epsilon_0.$$

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In this case, the family $\hat{B} = \{B(t)\}_{t\in\mathbb{R}}$ is called an **absorbing set**.

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In this case, the family $\hat{B} = \{B(t)\}_{t\in\mathbb{R}}$ is called an **absorbing set**. If the absorbing set $\hat{B} = \{B(t)\}_{t\in\mathbb{R}}$ is compact and there exists $t_0 \in \mathbb{R}$ such that $\bigcup_{t\leq t_0} B(t)$ is bounded in Z, then $(Z, S, \hat{M}, \hat{I})$ is **pullback $\tilde{S}$-strongly compact dissipative**.

The Impulsive Pullback Attractor

Assume that $I_s(M(s)) \cap M(s) = \emptyset$ for every $s \in \mathbb{R}$.

Theorem 15

Let $(Z, S, \hat{M}, \hat{I})$ be pullback $\tilde{S}$-strongly compact dissipative with compact absorbing set $\hat{K} = \{K(t)\}_{t\in\mathbb{R}}$. Then $(Z, S, \hat{M}, \hat{I})$ admits an impulsive pullback attractor $\hat{\mathbb{A}} = \{\mathbb{A}(t)\}_{t\in\mathbb{R}}$ such that

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for some $t_0 \in \mathbb{R}$, that is, $\hat{\mathbb{A}}$ is pullback bounded.

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Proof. For each $t \in \mathbb{R}$, define

$$\mathbb{A}(t) = \bigcup\{\tilde{\omega}(B, t)\colon\ B \subset Z,\ B \text{ bounded}\}.$$

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• the pullback $\tilde{S}$-strongly compact dissipativeness $\Longrightarrow \tilde{\omega}(B) = \{\tilde{\omega}(B, t)\}_{t\in\mathbb{R}}$ is a nonempty collectively compact family in Z which pullback $\tilde{S}$-attracts B.

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Assume that $I_s(M(s)) \cap M(s) = \emptyset$ for every $s \in \mathbb{R}$.

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Let $(Z, S, \hat{M}, \hat{I})$ be pullback $\tilde{S}$-strongly compact dissipative with compact absorbing set $\hat{K} = \{K(t)\}_{t\in\mathbb{R}}$. Then $(Z, S, \hat{M}, \hat{I})$ admits an impulsive pullback attractor $\hat{\mathbb{A}} = \{\mathbb{A}(t)\}_{t\in\mathbb{R}}$ such that

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Proof. For each $t \in \mathbb{R}$, define

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• the pullback $\tilde{S}$-strongly compact dissipativeness + $I_s(M(s)) \cap M(s) = \emptyset \Longrightarrow \tilde{\omega}(B)\backslash\hat{M}$ is negatively $\tilde{S}$-invariant.

- The family $\hat{\mathbb{A}} = \{\mathbb{A}(t)\}_{t\in\mathbb{R}}$ is nonempty, pullback $\tilde{S}$-attracts bounded subsets of Z, and $\{\mathbb{A}(t)\backslash M(t)\}_{t\in\mathbb{R}}$ is $\tilde{S}$-invariant.

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• $\mathbb{A}(t) \subset K(t)$ for all $t \in \mathbb{R}$. Thus, $\hat{\mathbb{A}} = \{\mathbb{A}(t)\}_{t\in\mathbb{R}}$ is compact.

• Let $\hat{C} = \{C(t)\}_{t\in\mathbb{R}}$ be a family of closed sets that pullback $\tilde{S}$-attracts bounded subsets of Z. We can prove that $\tilde{\omega}(B, t) \subset C(t)$ for all bounded set $B \subset Z$ and $t \in \mathbb{R}$, i.e., $\mathbb{A}(t) \subset C(t)$ for all $t \in \mathbb{R}$.

• The family $\hat{\mathbb{A}} = \{\mathbb{A}(t)\}_{t\in\mathbb{R}}$ is nonempty, pullback $\tilde{S}$-attracts bounded subsets of Z, and $\{\mathbb{A}(t)\backslash M(t)\}_{t\in\mathbb{R}}$ is $\tilde{S}$-invariant.

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Therefore, the family $\hat{\mathbb{A}} = \{\mathbb{A}(t)\}_{t\in\mathbb{R}}$ is the impulsive pullback attractor for the impulsive evolution process $(Z, S, \hat{M}, \hat{I})$. ■

Summary

A non-autonomous wave coupled system with impulsive action

Consider the following impulsive non-autonomous problem

$$\begin{cases} u_{tt} - \Delta u + u + \eta(-\Delta)^{\frac{1}{2}} u_t + a(t)(-\Delta)^{\frac{1}{2}} v_t = f(u), & (x,t) \in \Omega \times (\tau, \infty), \\ v_{tt} - \Delta v + \eta(-\Delta)^{\frac{1}{2}} v_t - a(t)(-\Delta)^{\frac{1}{2}} u_t = 0, & (x,t) \in \Omega \times (\tau, \infty), \\ u = v = 0, & (x,t) \in \partial\Omega \times (\tau, \infty), \\ \{I_t \colon M(t) \subset Y_0 \to Y_0\}_{t \in \mathbb{R}}, \end{cases}$$

with initial conditions

$$u(\tau, x) = u_0(x),\ u_t(\tau, x) = u_1(x),\ v(\tau, x) = v_0(x),\ v_t(\tau, x) = v_1(x),\ x \in \Omega,\ \tau \in \mathbb{R},$$

where $Y_0 = H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$ is the phase space.

- Ω is a bounded smooth domain in $\mathbb{R}^n$ ($n \geq 3$) with boundary $\partial\Omega$ assumed to be regular enough.

- $\eta > 0$.

- $f \in C^1(\mathbb{R})$ satisfies the dissipativeness condition

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} \leq 0,$$

and the growth condition

$$|f'(s)| \leq c(1 + |s|^{\rho-1}), \quad \forall s \in \mathbb{R},$$

where $1 < \rho < \frac{n}{n-2}$, with $n \geq 3$, and $c > 0$ is a constant.

- $a \colon \mathbb{R} \to (0, \infty)$ is continuously differentiable in $\mathbb{R}$ and satisfies:

$$0 < a_0 \leq a(t) \leq a_1, \quad \forall t \in \mathbb{R},$$

and there exists $b_0 > 0$ such that

$$|a'(t)| \leq b_0, \qquad \forall\, t \in \mathbb{R},\ \epsilon \in [0, 1].$$

We can rewrite the system

$$\begin{cases} u_{tt} - \Delta u + u + \eta(-\Delta)^{\frac{1}{2}} u_t + a(t)(-\Delta)^{\frac{1}{2}} v_t = f(u), & (x,t) \in \Omega \times (\tau, \infty), \\ v_{tt} - \Delta v + \eta(-\Delta)^{\frac{1}{2}} v_t - a(t)(-\Delta)^{\frac{1}{2}} u_t = 0, & (x,t) \in \Omega \times (\tau, \infty), \\ u = v = 0, \ (x,t) \in \partial\Omega \times (\tau, \infty), \end{cases}$$

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$$u(\tau, x) = u_0(x), \ u_t(\tau, x) = u_1(x), \ v(\tau, x) = v_0(x), \ v_t(\tau, x) = v_1(x), \ x \in \Omega, \ \tau \in \mathbb{R},$$

as an ordinary differential equation in the following abstract form

$$\begin{cases} W_t + \mathcal{A}(t) W = F(W), & t > \tau, \\ W(\tau) = W_0, & \tau \in \mathbb{R}, \end{cases}$$

where $W = W(t)$, for all $t \in \mathbb{R}$, and $W_0 = W(\tau)$ are respectively given by

$$W = [u \ \ u_t \ \ v \ \ v_t] \ \text{ and } \ W_0 = [u_0 \ \ u_1 \ \ v_0 \ \ v_1].$$

A non-autonomous wave coupled system with impulsive action

$$\begin{cases} u_{tt} - \Delta u + u + \eta(-\Delta)^{\frac{1}{2}} u_t + a(t)(-\Delta)^{\frac{1}{2}} v_t = f(u), \qquad (x,t) \in \Omega \times (\tau, \infty), \\ v_{tt} - \Delta v + \eta(-\Delta)^{\frac{1}{2}} v_t - a(t)(-\Delta)^{\frac{1}{2}} u_t = 0, \qquad (x,t) \in \Omega \times (\tau, \infty), \\ u = v = 0, \ (x,t) \in \partial\Omega \times (\tau, \infty), \\ u(\tau, x) = u_0(x), \ u_t(\tau, x) = u_1(x), \ v(\tau, x) = v_0(x), \ v_t(\tau, x) = v_1(x), \ x \in \Omega, \ \tau \in \mathbb{R}. \end{cases}$$

E. M. Bonotto, M. J. D. Nascimento and E. B. Santiago, *Long-time behaviour for a non-autonomous Klein-Gordon-Zakharov system. J. Math. Anal. Appl., v. 506, p. 125670, 2022.*

- Well-Posedness
- Pullback Attractor

We assume the following conditions:

(*H*1) $I_s(M(s)) \cap M(s) = \emptyset$ for every $s \in \mathbb{R}$;

(*H*2) there exists $\mu > 0$ such that $\|I_s(w)\|_{Y_0}^2 \leq \mu$ for all $s \in \mathbb{R}$ and $w \in M(s)$;

(*H*3) there exists $\xi > 0$ such that $\phi(I_s(x), s) \geq 2\xi$ for all $(x, s) \in M(s) \times \mathbb{R}$.

Let $\tilde{W}(t) = \tilde{S}(t,\tau)W_0$, $t \geq \tau$, be the impulsive solution of the impulsive non-autonomous problem

$$\begin{cases} W_t + \mathcal{A}(t)W = F(W), \ t > \tau, \\ W(\tau) = W_0 \in Y_0, \ \tau \in \mathbb{R}, \\ \{I_t \colon M(t) \subset Y_0 \to Y_0\}_{t\in\mathbb{R}}, \end{cases}$$

and $(Y_0, S, \hat{M}, \hat{I})$ be its associated impulsive evolution process.

Lemma 16

There exists $R > 0$ such that for any bounded subset B of Y_0, one can find $t_0(B) > 0$ such that

$$\|\tilde{S}(t,\tau)W_0\|^2_{Y_0} \leq R,$$

for all $W_0 \in B$ and $t \geq \tau + t_0(B)$.

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$(H3)$ there exists $\xi > 0$ such that $\phi(I_s(x), s) \geq 2\xi$ for all $(x, s) \in M(s) \times \mathbb{R}$.

Lemma 17

If G is a precompact subset of Y_0 and $t \geq \tau \in \mathbb{R}$ satisfies $0 \leq t - \tau < \xi$, then $\tilde{S}(t,\tau)G$ is precompact in Y_0.

- The operator $S(t,\tau)\colon Y_0 \to Y_0$ is compact for $t > \tau$.

Theorem 18

The impulsive evolution process $(Y_0, S, \hat{M}, \hat{I})$ *is pullback* $\tilde{S}$*-strongly compact dissipative with compact absorbing set* $\hat{K} = \{K(t)\}_{t\in\mathbb{R}}$*. Furthermore, there exists* $\delta > 0$ *such that* $\bigcup_{t\in\mathbb{R}} K(t) \subset \overline{B_{Y_0}(0,\delta)}$*.*

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Proof. Let $B_0 = \{w \in Y_0 \colon \|w\|^2_{Y_0} \leq R\}$, where $R > 0$ comes from Lemma 16, and let $\tau \in (\xi, 2\xi)$ be fixed (ξ comes from condition $(H3)$).

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$$C_1(t) = \{w \in B_0 \colon \phi(w, t-\tau) > \xi\} \quad \text{and} \quad C_2(t) = \{w \in B_0 \colon \phi(w, t-\tau) \leq \xi\},$$

and we prove that

$$G(t) = \tilde{S}(t, t-\tau+\xi)S(t-\tau+\xi, t-\tau)C_1(t) \bigcup S(t, t-\tau+\xi)\tilde{S}(t-\tau+\xi, t-\tau)C_2(t).$$

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Let $\epsilon_0 \in (0, \frac{\xi}{2})$ and define $K(t) = \overline{\bigcup_{|\epsilon|\leq\epsilon_0} \tilde{S}(t+\epsilon, t-\epsilon_0)G(t-\epsilon_0)}$ which is compact in Y_0.

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The impulsive evolution process $(Y_0, S, \hat{M}, \hat{I})$ is pullback $\tilde{S}$-strongly compact dissipative with compact absorbing set $\hat{K} = \{K(t)\}_{t\in\mathbb{R}}$. Furthermore, there exists $\delta > 0$ such that $\bigcup_{t\in\mathbb{R}} K(t) \subset \overline{B_{Y_0}(0,\delta)}$.

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and we prove that

$$G(t) = \tilde{S}(t, t-\tau+\xi)S(t-\tau+\xi, t-\tau)C_1(t) \bigcup S(t, t-\tau+\xi)\tilde{S}(t-\tau+\xi, t-\tau)C_2(t).$$

Let $\epsilon_0 \in (0, \frac{\xi}{2})$ and define $K(t) = \overline{\bigcup_{|\epsilon|\leq\epsilon_0} \tilde{S}(t+\epsilon, t-\epsilon_0)G(t-\epsilon_0)}$ which is compact in Y_0. We prove that the family $\hat{K} = \{K(t)\}_{t\in\mathbb{R}}$ pullback $\tilde{S}$-absorbs bounded subsets of Y_0. Hence, $(Y_0, S, \hat{M}, \hat{I})$ is pullback $\tilde{S}$-strongly compact dissipative.

Theorem 19

The impulsive evolution process $(Y_0, S, \hat{M}, \hat{I})$ *admits an impulsive pullback attractor* $\hat{\mathbb{A}} = \{\mathbb{A}(t)\}_{t\in\mathbb{R}}$*. Moreover, there exists* $\delta > 0$ *such that* $\bigcup_{t\in\mathbb{R}} \mathbb{A}(t) \subset \overline{B_{Y_0}(0,\delta)}$.

Thank You!!!!!!