Unifying stability theory for stochastic equations

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Based on previous and ongoing works with

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Outline

1. Modern integration theory
2. Generalized ODEs
3. Stochastic Calculus
4. Generalized Stochastic Equations
5. Stability Outcomes

Riemman Integral

- It is limited not only to bounded functions, but also to only those with elementary anti-derivatives (F. T. C).

- the Riemann integral does not yield “good” convergence theorems.

$$f_1(x) = \begin{cases} 1, & x = r_1 \\ 0, & \text{otherwise} \end{cases} \quad f_2(x) = \begin{cases} 1, & x = r_1, r_2 \\ 0, & \text{otherwise} \end{cases}$$

$$f_n(x) = \begin{cases} 1, & x = r_1, r_2, \ldots, r_n \\ 0, & \text{otherwise} \end{cases}$$

$\{r_n\}_{n\in\mathbb{N}} \subset [0, 1]$ sequence of rational numbers.

Lebesgue Integral

- It adds additional constraints on the F. T. C.
- It is a complex method.

- There are still a large number of functions which can not be integrated.

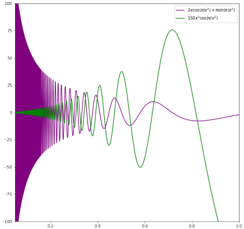
- It does not guarantee that every derivative is integrable.

Example:

Consider $F : [0,1] \to \mathbb{R}$ given by

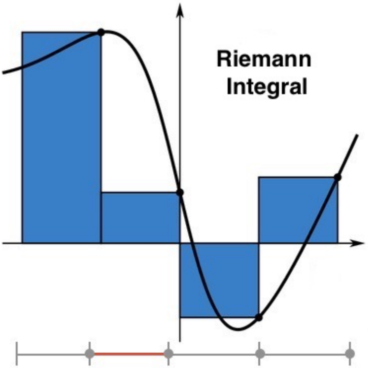
$$F(t) = \begin{cases} t^2 \sin \dfrac{1}{t^2}, & 0 < t \leq 1, \\ 0, & t = 0. \end{cases}$$

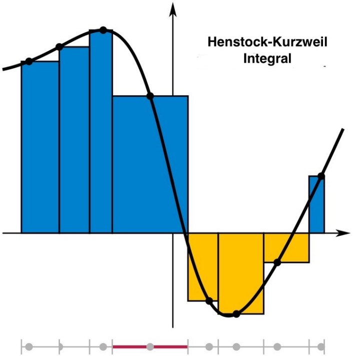
Let $g = F'$. Then,



g is not Lebesgue integrable

Henstock-Kurzweil non-absolute integration


Riemann
Integral


Henstock-Kurzweil
Integral

A **tagged division** of $[a, b] \subset \mathbb{R}$ is a finite collection of point-interval pairs $(\tau_i, [s_{i-1}, s_i])$, where

1. $a = s_0 \leq s_1 \leq \ldots \leq s_k = b$
2. $\tau_i \in [s_{i-1}, s_i], \quad \forall\, i.$

Given a function $\delta : [a, b] \to (0, +\infty)$ (called **gauge** of $[a, b]$), a tagged division $d = (\tau_i, [s_{i-1}, s_i])$ is δ-**fine**, whenever

1. $[s_{i-1}, s_i] \subset (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)), \quad \forall\, i.$

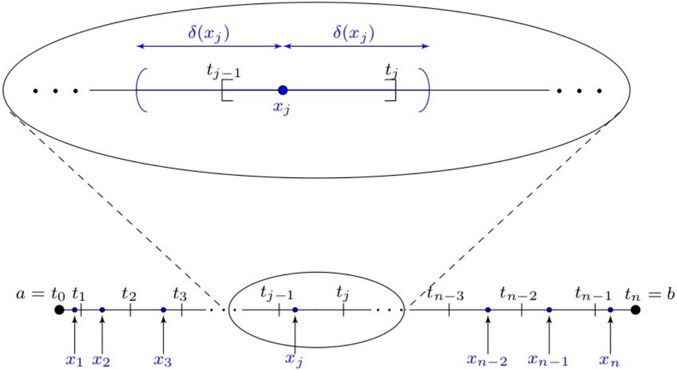


Figure: A geometric representation of δ-fineness

Lemma (Cousin)

Given a gauge δ on $[a, b]$, there is a $\delta-$fine tagged division of $[a, b]$.

Let X be a Banach space. A function $U : [a, b] \times [a, b] \to X$ is *Henstock-Kurzweil integrable* over $[a, b]$, if there is an element $K \in X$ s.t given $\varepsilon > 0$, there is a gauge δ of $[a, b]$ s.t for every $\delta-$fine tagged division $d = (\tau_i, [s_{i-1}, s_i])$ of $[a, b]$, we have

$$\| \sum_i [U(\tau_i, s_i) - U(\tau_i, s_{i-1}) - K\| < \varepsilon.$$

In this case, we write $K = \int_a^b DU(\tau, t)$ and use the convention $\int_a^b DU(\tau, t) = -\int_b^a DU(\tau, t)$, whenever $a < b$.

The Perron-Stieltjes integral

Given functions $f, g\colon [a,b] \to \mathbb{R}$, consider $U : [a,b] \times [a,b] \to \mathbb{R}$ as

$$\boxed{U(\tau, t) = f(\tau)g(t).}$$

Then,

$$\sum_j U(\tau_j, s_j) - U(\tau_j, s_{j-1}) = \sum_j f(\tau_j)[g(t_j) - g(t_{j-1})]$$

and hence

$$\boxed{\int_a^b DU(\tau, t) = \int_a^b f(s)dg(s).}$$

In particular, when $g(s) = s$ for all $s \in [a,b]$, we say that f is Perron integrable and

$$\boxed{\int_a^b f(s)ds = \int_a^b DU(\tau, t)}$$

Using δ-fine division instead traditional, we have more variation in how we can choose the length of subintervals. Because of this fact, we can integral functions of unbounded variation, highly oscillating and having many discontinuities points.

Another advantage of using a non-constant gauge is that we can take intervals of a set of points that is finite or countable whose union has small length. Because the total length of these intervals are arbitrarily small, their lengths end up not contributing to the Riemann sum in any significant way. An example of this would be the Dirichlet's function, where we can choose δ being arbitrarily small at the rational points and, for the irrational points, it does not matter how we define δ, since the function value at the irrational points is zero. For example, we can define δ as

$$\delta(t) = \begin{cases} \frac{\varepsilon}{2^{i+1}}, & \text{if } t = r_i \in \mathbb{Q} \\ 1, & \text{if } t \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Using a non constant gauge, we also can choose a certain gauge and "force" a division to have a specific tag. By doing this, we can control certain "bad" points that would normally make the function non-integrable. An example of this is $f : [0,1] \to \mathbb{R}$ given by

$$f(t) = \begin{cases} \frac{1}{\sqrt{t}}, & t \in (0,1] \\ 0, & t = 0. \end{cases}$$

Thus, f is not Reimann integrable. However, if we choose a gauge δ correctly, we can force the function value at the first tag to be zero. This will make any terms of the Riemann sum with this tag equal to zero. On the remaining part of the intervals the function f will be bounded and continuous and thus the function is now integrable.

Moreover, a non-constant gauge improves some classical results, as the Fundamental Theorem of Calculus.

The Fundamental Theorem of Calculus

Theorem

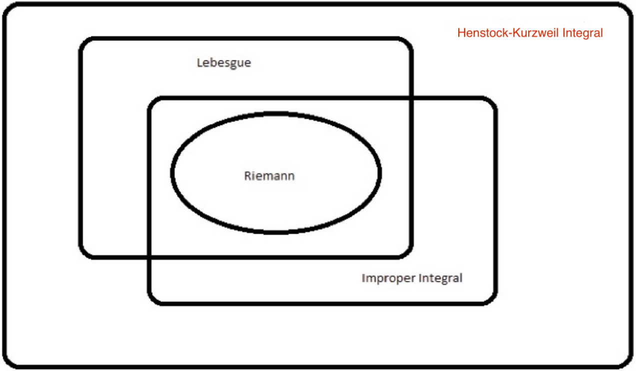
Let $F : [a,b] \to \mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ be functions. Suppose F is differentiable on $[a,b]$, s.t, $F'(t) = f(t)$ for all $t \in [a,b]$. Then,

$$\int_a^b f(s)ds = F(b) - F(a).$$

Remark

Notice that the Fundamental Theorem of Calculus for Henstock-Kurzweil integrals does NOT require any additional assumptions on f, unlike the Riemann integral version which requires f to be Riemann integrable and even the Lebesgue version, which requires f to be absolutely continuous.

- Every derivative is integrable (F. T. C).
- It deals well with highly oscillating (unbounded variation) functions.
- It has theorems which generalize
  - the Monotone Convergence Theorem
  - the Dominated Convergence Theorems of measure theory


Henstock-Kurzweil Integral
Lebesgue
Riemann
Improper Integral

Generalized ODEs

A function $x : [\alpha, \beta] \to X$ is called a **solution** of the generalized ODE

$$\frac{dx}{d\tau} = DF(x, t)$$

if

$$x(s) - x(\tau_0) = \int_{\tau_0}^{s} DF(x(\tau), t), \quad \forall\ s \in [\alpha, \beta].$$

In the sense of Kurzweil integral $U(\tau, t) = F(x(\tau), t)$

Generalized ODEs encompass:

- ordinary and functional differential equations;
- impulse and measure differential equations;
- dynamic equations on time scales;
- integral equations (e.g. Volterra type);
- a class of partial differential equation.

The theory of generalized ODEs has been shown to act as a unifying theory for many equations.

ODEs x Generalized ODEs

Consider the following ODE

$$x' = f(x, t),$$

where $f : \Omega \subset C([t_0, T], \mathbb{R}^n) \times [t_0, T]$ and Ω is open.

Its corresponding integral form is

$$\boxed{x(t) = x(t_0) + \int_{t_0}^{t} f(x(s), s)ds, \quad t \in [t_0, T],}$$

whenever the integral exists in some sense.

The integral $\int_{t_0}^{t} f(x(s),s)ds$ can be approximated by

$$\int_{t_0}^{t} f(x(s),s)ds \cong \begin{cases} \sum_{i=1}^{m} f(x(\tau_i),\tau_i)(t_i - t_{i-1}) \\ \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} f(x(\tau_i),s)ds. \end{cases}$$

where

$$t_0 < t_1 < t_2 < \ldots < t_m = t$$

is a sufficiently fine division of $[t_0, t]$ and $\tau_i \in [t_{i-1}, t_i]$, $\forall\ i$.

Let

$$F(x,t) = \int_{t_0}^{t} f(x,s)ds, \quad (x,t) \in \Omega.$$

Then

$$F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1}) = \int_{s_{i-1}}^{s_i} f(x(\tau_i), s)ds \cong \int_{s_{i-1}}^{s_i} f(x(s), s)ds.$$

Hence

$$\int_{t_0}^{t} DF(x(\tau), s) \cong \sum_i [F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})] \cong$$

$$\cong \sum_i \int_{s_{i-1}}^{s_i} f(x(s), s)ds = \int_{t_0}^{t} f(x(s), s)ds.$$

Therefore there exists a one-to-one relation between the integrals

$$\int_{t_0}^{t} f(x(s), s)ds \quad \text{and} \quad \int_{t_0}^{t} DF(x(\tau), t)$$

and, hence, between the integral forms

- $x(t) = x(t_0) + \int_{t_0}^{t} f(x(s), s)ds, \; t \in [t_0, T],$
- $x(t) = x(t_0) + \int_{t_0}^{t} DF(x(\tau), t), \; t \in [t_0, T],$

or, equivalently, between the "*differential equations*"

$$\dot{x} = f(x, t) \quad \text{and} \quad \frac{dx}{d\tau} = DF(x, t).$$

MFDEs x Generalized ODEs

Coming to initial value problems for measure functional differential equations given by the system

$$\begin{cases} y(t) = y(t_0) + \int_{t_0}^{t} f(y_s, s)\, dg(s), & t \in [t_0, t_0 + \sigma] \text{ with } \sigma > 0 \\ y_{t_0} = \phi, & \end{cases}$$

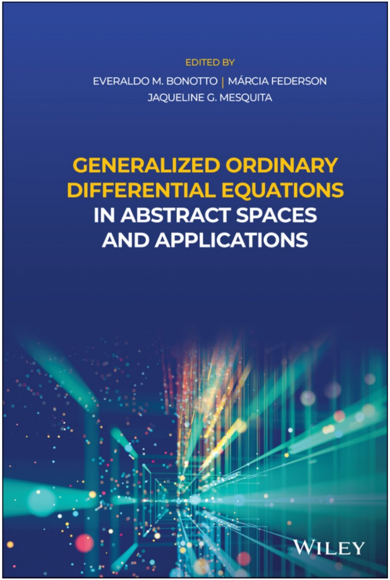
where ϕ is some initial history.

There exists a one-to-one relation between the solution of the MFDE and the Generalized ODE

$$\frac{dx}{d\tau} = DF(x(\tau), t)$$

where the function $F : \Omega \times [t_0, t_0 + \sigma] \to G^{-}([t_0, t_0 + \sigma], \mathbb{R}^n)$, $\Omega \subset G^{-}([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$, is given by

$$F(x,t)(\vartheta) = \begin{cases} 0, & t_0 - r \leq \vartheta \leq t_0, \\ \int_{t_0}^{\vartheta} f(x_s, s)\, \mathrm{d}g(s), & t_0 \leq \vartheta \leq t \leq t_0 + \sigma, \\ \int_{t_0}^{t} f(x_s, s)\, \mathrm{d}g(s), & t \leq \vartheta \leq t_0 + \sigma \end{cases}$$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GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS IN ABSTRACT SPACES AND APPLICATIONS
WILEY

Stochastic Calculus: Motivation

Price of financial assets that varies over time $\rightarrow$ Investment portfolio

Density integral over a curve

$$\int_\gamma \delta(\gamma(s))ds$$

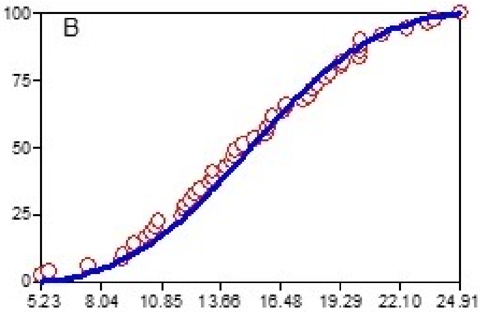


Figure: $\lim \delta(\gamma(t_i))\widehat{arco}$

Itô-Integral: Motivation

Riemann Integral

$$\int_a^b f(x)dx$$

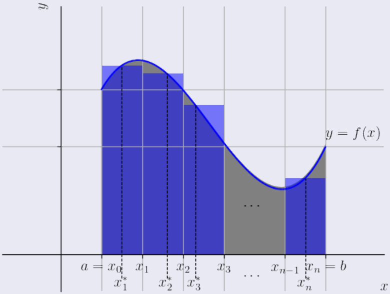


Figure: $\lim \sum_{i=1}^{n} f(x_i)\Delta x_i$

Itô-Integral

$$Y_t = \int_0^t H_s dB_s \quad \text{or}$$

$$I(f)(\omega) = \int_0^t f(\omega, s) dB_s$$

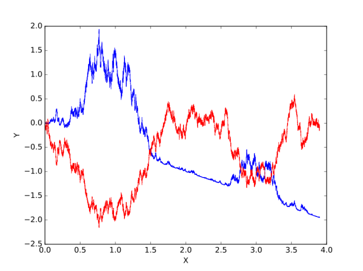


Figure: Itô integral Y_t of a Brownian motion B_t

Construction of Itô-Integral

Consider $f(\omega, t) = \chi_{[a,b]}$. We hope

$$I(f)(\omega) = \int_{\mathbb{R}} f(\omega, s) dB_s = \int_a^b dB_s = (B_b - B_a)(w).$$

Now, if $\{t_i\}$ is a division of $[a, b]$ and consider

$$f(\omega, s) = \sum_{i=0}^{n-1} a_i(\omega) \chi_{[t_i, t_{i+1}]}$$

Then,

$$I(f)(\omega) = \sum_{i=0}^{n-1} a_i(\omega) \left[(B_{t_{i+1}} - B_{t_i})(\omega) \right]$$

If $\{f_n\}$ is a sequence of functions in $\mathcal{H}_0^2$ and $f_n \to f$, then

$$\int f dB_s = \lim_{n\to\infty} \int f_n dB_s$$

Let π_n is a sequence of division of $[0, t]$ s.t.

$$diam(\pi_n) \to 0$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} H_{t_{i-1}}(B_{t_i} - B_{t_{i-1}}) = \int_0^t H_s dB_s$$

$$\mathbb{E}\left[\int_0^t f^2(\omega, s)ds\right] < \infty$$

Figure: University of São Paulo

Figure: Federal University of Uberlândia / Howard University

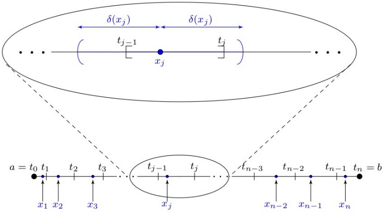
A new research field began

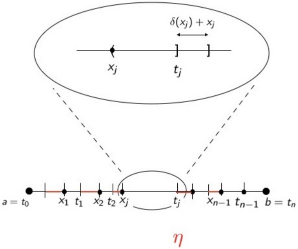
called

Generalized Stochastic Equations

E. M. Bonotto, R. Collegari, M. Federson, T. Gill, Operator-valued stochastic differential equations in the context of Kurzweil-like equations, *JMAA* 527, (2023), 1–27.

Kurzweil-belated integrable





Notations

- $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in I}, \mathbb{P})$ denotes a filtering probability space
- $L^p(\Omega, V)$, $1 \leq p < \infty$ is the space of all $\mathcal{F}$-measurable random variables $Z \colon \Omega \to V$ endowed with the norm
$$\|Z\|_{L^p} = \left(\int_\Omega \|Z(\omega)\|_V^p d\mathbb{P}\right)^{\frac{1}{p}}$$
- $\mathfrak{F}(\Omega, V)$ the space of all operators from Ω to V

Belated partial division

Let $\delta\colon [a,b] \to [0,+\infty)$ be a non-negative function (called gauge on $[a,b]$). A δ-belated partial division of $[a,b]$

- point-interval pairs $D = \{(x_{i-1},(x_{i-1},x_i]) : i = 1,2,\ldots,|D|\}$
- $(x_{i-1},x_i]$, $i = 1,2\ldots,|D|$, are disjoint
- $(x_{i-1},x_i] \subset (x_{i-1},x_{i-1}+\delta(x_{i-1}))$, $\quad \forall\ i = 1,2\ldots,|D|$.

Belated partial division

Let $\delta\colon [a,b] \to [0,+\infty)$ be a non-negative function (called gauge on $[a,b]$). A δ-belated partial division of $[a,b]$

- point-interval pairs $D = \{(x_{i-1}, (x_{i-1}, x_i]) : i = 1, 2, \ldots, |D|\}$
- $(x_{i-1}, x_i]$, $i = 1, 2 \ldots, |D|$, are disjoint
- $(x_{i-1}, x_i] \subset (x_{i-1}, x_{i-1} + \delta(x_{i-1})), \quad \forall\ i = 1, 2 \ldots, |D|.$

In addition, D is called a (δ, η)-belated partial division of $[a,b]$ if

$$\left| b - a - \sum_{i-1}^{|D|} (x_i - x_{i-1}) \right| \leq \eta$$

Kurzweil-belated integral

Let $G\colon [a,b]\times[a,b]\to\mathfrak{F}(\Omega,V)$ be a $\{\mathcal{F}_t\}$-adapted process on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in I},\mathbb{P})$. We say that G is Kurzweil-belated integrable over $[a,b]$, if $\forall\ \epsilon>0$, $\exists\ K\in L^p(\Omega,V)$, a gauge δ on $[a,b]$ and $\eta>0$ s.t

$$\mathbb{E}=\left[\left\|\sum_{i=1}^{|D|} G(s_{i-1},s_i)-G(s_{i-1},s_{i-1})-K\right\|_V^p\right]<\epsilon$$

$\forall$ (δ,η)-fine belated partial division

$D=\{(s_{i-1},(s_{i-1},s_i]):i=1,2\ldots,|D|\}$ of $[a,b]$.

In this case, we write

$$K=\int_a^b G(\tau,s)$$

Kurzweil-belated integral × Itô-Henstock integral

For $f : [a, b] \to L(U, V)$ and $B = \{B_t : t \in I\}$ a Q-Brownian motion, we define

$$G(\tau, s) = f_\tau B_s.$$

Then,

$$\sum_{i=1}^{|D|} G(s_{i-1}, s_i) - G(s_{i-1}, s_{i-1}) = \sum_{i=1}^{|D|} f_{s_{i-1}}(B_{s_i} - B_{s_{i-1}})$$

and, hence,

$$\boxed{\int_a^b G(\tau, s) = (IH)\int_a^b f dB}$$

Generalized Stochastic Equations

$F\colon L^p(\Omega, V) \times I \to \mathfrak{F}(\Omega, V)$ be an operator. A $\{\mathcal{F}_t\}$-adapted process $X = \{X_t : t \in J\}$, $J \subset I$, on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in I}, \mathbb{P})$, $X_t \in L^p(\Omega, V)$, $\forall\ t \in J$, is a solution of the GSE

$$X_t = X_s + \int_s^t F(X_r, \tau), \quad t, s \in J, \tag{1}$$

on J, whenever $X_t(\omega) \in V\ \forall\ t \in J$ and $\mathbb{P}$-almost every $\omega \in \Omega$ and the integral equation (1) holds, where the integral is in the sense of the Kurzweil-belated integral with $G(r, \tau) = F(X_r, \tau)$.

- I is unbounded and $J = [s_0, +\infty) \subset I$, we say that X is a global forward solution with i.c X_{s_0}
- $X_t \equiv 0\ \forall\ t \in J$, X is called the trivial solution.

Cauchy problem

$$\begin{cases} X_t &= X_{t_0} + \int_{t_0}^{t} F(X_r, \tau), \quad t, t_0 \in [a, b], \\ X_{t_0} &= \widetilde{X} \in L^p(\Omega, V). \end{cases}$$

$\mathcal{L}^p([a,b],\Omega,V)$, $1 \leq p < \infty$, is the space of all $\{\mathcal{F}_t\}$-adapted process $X = \{X_t : a \leq t \leq b\}$ s.t

$$\|X\|_{\mathcal{L}^p}^p = \int_a^b \left(\int_\Omega \|X_t(\omega)\|_V^p d\mathbb{P} \right) dt = \int_a^b \mathbb{E}\left[\|X_t\|_V^p\right] dt < \infty,$$

where $X_t \in L^p(\Omega, V)$ $\forall$ $t \in [a,b]$.

- $h\colon [a,b] \to \mathbb{R}$ be nondecreasing
- $L \subseteq L^p(\Omega, V)$
- $F\colon L^p(\Omega, V) \times [a,b] \to \mathfrak{F}(\Omega, V)$ be an operator

We say that F belongs to the class $\mathcal{G}(L \times [a,b], h)$ if, $\forall$ $\{\mathcal{F}_t\}$-adapted process $Z = \{Z_t : a \leq t \leq b\}$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in I}, \mathbb{P})$, with $Z_t \in L$ $\forall$ $t \in [a,b]$,

- the Kurzweil-belated integral $\int_a^b F(Z_t, s)$ exists
- $\forall$ $s_1, s_2 \in [a,b]$

$$\mathbb{E}\left[\left\|\int_{s_1}^{s_2} F(Z_t, s)\right\|_V^p\right] \leq |h(s_2) - h(s_1)| \quad \text{and}$$

$$\mathbb{E}\left[\left\|\int_{s_1}^{s_2} [F(Z_t, s) - F(Y_t, s)]\right\|_V^p\right] \leq \|Z - Y\|_{\mathcal{L}^p}^p |h(s_2) - h(s_1)|$$

If F belongs to the class $\mathcal{G}(L \times [a,b], h)$, where $h\colon [a,b] \to \mathbb{R}$ is continuous, then $\exists\ \Delta > 0\ \exists!$ $\{\mathcal{F}_t\}$-adapted process $X = \{X_t : t_0 \leq t \leq t_0 + \Delta\}$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in I}, \mathbb{P})$ which is a solution of the Cauchy Problem

$$\begin{cases} X_t &= X_{t_0} + \displaystyle\int_{t_0}^{t} F(X_r, \tau), \\ X_{t_0} &= \widetilde{X} \in L^p(\Omega, V). \end{cases}$$

New results concerning

existence and uniqueness of solution

F. Andrade da Silva, E. M. Bonotto, M. Federson, T. Gill, Stability for generalized stochastic equations, *Stochastic Processes and their Applications* 173, (2024), 1-214.

Proposition

If F belongs to the class $\mathcal{G}(L \times [a,b], h)$, then every solution $X = \{X_t : t \in [a,b]\}$ of the GSE is continuous in the expectation, that is,

$$\lim_{t \to s} \mathbb{E}[\|X_t - X_s\|_V^p] = 0,$$

$\forall \ t \in [a,b]$.

Theorem

- $I = [t_0, +\infty)$
- $h\colon I \to \mathbb{R}$ *a nondecreasing left-continuous function*
- $F\colon L^p(\Omega, V) \times I \to \mathfrak{F}(\Omega, V) \in \mathcal{G}(L^p(\Omega, V) \times I, h)$
- $X = \{X_t : t \in [a, b)\}$ *and* $Y = \{Y_t : t \in I_Y\}$ *are solutions of the GSE*
  - $[a, b) \subset I$
  - $I_Y \in \{[b, \nu], [b, \nu), [b, +\infty) : \nu \in (b, +\infty)\}$

If $\lim_{t \to b^-} \mathbb{E}[\|X_t - Y_b\|_V^p] = 0$, *then* $Z = \{Z_t : t \in [a, b) \cup I_Y\}$, *defined by*

$$Z_t = \begin{cases} X_t, & t \in [a, b), \\ Y_t, & t \in I_Y, \end{cases}$$

is a solution of the GSE.

Lemma

- $I = [t_0, +\infty)$
- $h\colon I \to \mathbb{R}$ *a nondecreasing continuous function*
- $F\colon L^p(\Omega, V) \times I \to \mathfrak{F}(\Omega, V) \in \mathcal{G}(L^p(\Omega, V) \times I, h)$
- $X = \{X_t : t \in I_X\}, Y = \{Y_t : t \in I_Y\} \in S_{s_0, \widetilde{X}}$, $s_0 \geq t_0$

Then,

$$X_t = Y_t \quad \forall t \in I_X \cap I_Y$$

Theorem

- $I = [t_0, +\infty)$
- $h\colon I \to \mathbb{R}$ a *nondecreasing continuous function*
- $F\colon L^p(\Omega, V) \times I \to \mathfrak{F}(\Omega, V) \in \mathcal{G}(L^p(\Omega, V) \times I, h)$

$\forall\ s_0 \geq t_0\ \widetilde{X} \in L^p(\Omega, V)$, $\exists!$ *maximal solution* $X = \{X_t : t \in I_X\}$ *of the GSE, with* $X_{s_0} = \widetilde{X}$.

Theorem

- $I = [t_0, +\infty)$
- $h\colon I \to \mathbb{R}$ a *nondecreasing continuous function*
- $F\colon L \times I \to \mathfrak{F}(\Omega, V) \in \mathcal{G}(L \times I, h)$ $L \subset L^p(\Omega, V)$

If L *is compact, then* $\exists!$ *global forward solution.*

$$dX_t = g(X_t, t)dt + f(X_t, t)dB_t \longleftrightarrow X_t = X_{t_0} + \int_{t_0}^{t} F(X_r, s)$$

$$F(\nu, s) = \int_{t_0}^{s} g(\nu, r)dr + \int_{t_0}^{s} f(\nu, r)dB_r$$

$$\dot{x}(t) + ax(t) - b[\dot{x}(t-h) + ax(t-h)] = 0, \quad a > 0.$$

- $h = 0$, $x(t) = x(0)e^{-at}$ is asymptotically stable for $a > 0$ and arbitrary b.
- $h > 0$, the appropriate characteristic equation $(z+a)(1-be^{-hz})$ has all roots
$$z_k = \frac{1}{h}(\ln|b| + 12k\pi), \quad k = 0, \pm 1, \pm 2, \ldots$$
with real parts $\mathrm{Re}(z_k) = \frac{1}{h}\ln|b|$. So, if $|b| > 1$, then the trivial solution is unstable for each $h > 0$.

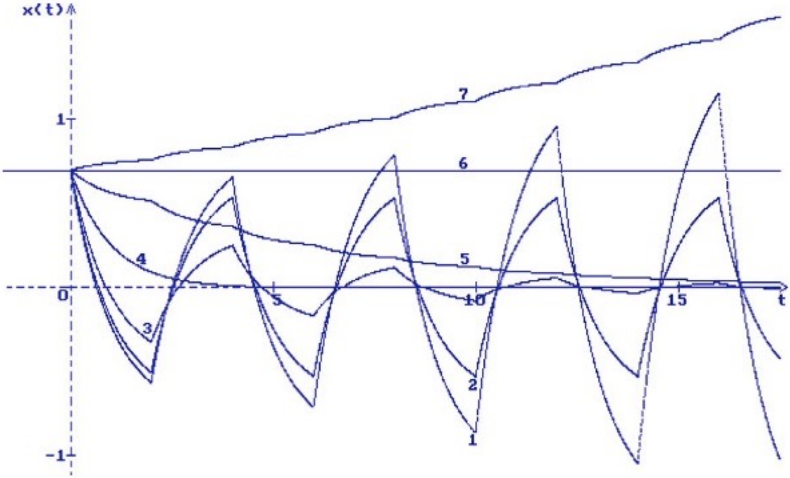
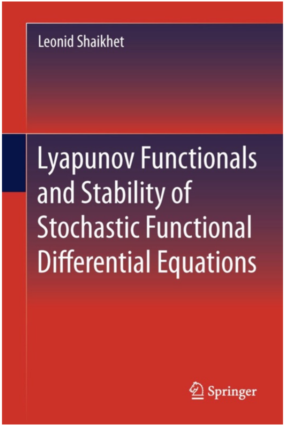


Figure: $a = 1$, $h = 2$, $x(0) = 0.7$, $s \in [-h, 0]$, (1): $b = -1.1$, (2): $b = -1$, (3): $b = -0.7$, (4): $b = 0$, (5): $b = 0.7$, (6): $b = 1$, (7): $b = 1.1$


Leonid Shaikhet
Lyapunov Functionals and Stability of Stochastic Functional Differential Equations
Springer

$$dx(t) = a(t, x_t)dt + b(t, x_t)dw(t) \longleftrightarrow X_t = X_{t_0} + \int_{t_0}^{t} F(X_r, s)$$

Let $1 \leq p < \infty$. The trivial solution of the GSE

$$X_t = X_{t_0} + \int_{t_0}^{t} F(X_r, \tau), \quad t, t_0 \in [a, b], \tag{2}$$

is said to be *p-stable*, if $\forall\ \epsilon > 0$, $\exists\ \delta = \delta(\epsilon) > 0$ s.t

$$\mathbb{E}[\|X_t\|_V^p] < \epsilon, \quad \text{for all } t \geq s_0,$$

whenever $\|\widetilde{X}\|_{L^p}^p < \delta$.

is *exponentially p-stable*, if it is p-stable and $\exists\ \lambda > 0$ and $C > 0$ (which may depend on $\widetilde{X}$) s.t

$$\mathbb{E}[\|X_t\|_V^p] \leq Ce^{-\lambda(t-s_0)}, \quad \text{for all } t \geq s_0,$$

Let $\mathcal{K}$ be the family of all nondecreasing continuous functions $\mu\colon \mathbb{R}^+ \to \mathbb{R}^+$ s.t $\mu(0) = 0$ and $\mu(t) > 0$, $t > 0$.

A functional $\mathcal{V}\colon [t_0, +\infty) \times L^p(\Omega, V) \to \mathbb{R}^+$ is said to be *positive definite* (in the sense of Lyapunov) if $\mathcal{V}(t, 0) \equiv 0$ and, for some $\mu \in \mathcal{K}$,

$$\mathcal{V}(t, Z) \geq \mu(\|Z\|_{L^p}), \quad \text{for all } (t, Z) \in [t_0, +\infty) \times L^p(\Omega, V).$$

Let $\mathcal{V}\colon [t_0, +\infty) \times L^p(\Omega, V) \to \mathbb{R}^+$ be a positive definite functional s.t, $\forall$ $X = \{X_t : t \geq s_0\}$ of the GSE (2), with $s_0 \geq t_0$, we have

- $\exists\ \alpha > 0$ f.w
$$\mathbb{E}[\mathcal{V}(t, X_t) - \mathcal{V}(s, X_s)] \leq -\alpha \int_s^t \mathbb{E}[\mathcal{V}(\tau, X_\tau)] d\tau, \quad \forall s_0 \leq s < t$$
- $\exists\ c > 0$ s.t $\mathbb{E}[\mathcal{V}(t, X_t)] \geq c\mathbb{E}[\|X_t\|_V^p] \quad \forall t \geq s_0$
- $\exists\ \sigma \in \mathcal{K}$ s.t $\mathbb{E}[\mathcal{V}(s_0, X_{s_0})] \leq \sigma\left(\|X_{s_0}\|_{L^p}^p\right).$

Then, the trivial solution of the GSE (2) is exponentially p-stable.

Consider

$$F(\nu, s) = -\int_0^s a(\nu, r)dr + \int_0^s f_r dB_r,$$

where $a(Z, t) = Z$, $f : [0, +\infty) \times \Omega \to L(U, V)$ be a process Kurzweil-belated integrable and $\{B_t : t \geq 0\}$ is a Brownian motion. Then,

$$X_t = X_s + \int_s^t F(X_r, \tau), \quad t, s \in [0, +\infty)$$

is a solution of the SDE

$$dX_t = -a(X_t, t)dt + f_t dB_t, \quad t \geq 0. \tag{3}$$

Define

$$\mathcal{V}(t, X_t) = X_t^2(\omega).$$

Thus,

$$\frac{1}{2}X_t^2(\omega) \leq \mathcal{V}(t, X_t) \leq \frac{3}{2}X_t^2(\omega), \quad \text{for all } t \geq 0.$$

Now,

$$d\mathcal{V}(t, X_t) = 2X_t dX_t(\omega) = 2[-X_t a(X_t, t)d_t + X_t f_t dB_t] = -2[X_t^2 - X_t f_t dB_t]$$

which implies

$$\mathcal{V}(t, X_t) - \mathcal{V}(s, X_s) = -2\left[\int_s^t X_\tau^2 d\tau - \int_s^t X_\tau f_\tau dB_\tau\right].$$

Taking the expectation and applying the zero mean property of the Kurzweil-belated integral, we get

$$\mathbb{E}\left[\int_s^t X_\tau f_\tau dB_\tau\right] = 0$$

and, consequently,

$$\mathbb{E}[\mathcal{V}(t, X_t) - \mathcal{V}(s, X_s)] = -2\mathbb{E}\left[\int_s^t X_\tau^2 d\tau\right].$$

By Itô's Isometric

$$\mathbb{E}[\mathcal{V}(t, X_t) - \mathcal{V}(s, X_s)] \leq -c \int_s^t \mathbb{E}[X_\tau^2] d\tau = -c \int_s^t \mathbb{E}[\mathcal{V}(\tau, X_\tau)] d\tau, \quad c > 2$$

and, therefore, the trivial solution of (3) is mean-square exponentially stable.

Assume that $g\colon \Omega \to L(U,V)$ is a random variable bounded in $L^2(U,V)$, that is, there $\exists\ M > 0$ s.t $\|g(\omega)\|_{L^2} \leq M\ \forall\ \omega \in \Omega$ and $\theta\colon \Omega \to L(U,V)$ is a random variable s.t, $\forall\ \omega \in \Omega$, $\theta(\omega)$ is the null operator in $L(U,V)$.

Then, the process $f\colon [0,+\infty) \times \Omega \to L(U,V)$ given by

$$f_t = \begin{cases} g, & \text{if} \quad t = s \\ \theta, & \text{if} \quad t \neq s \end{cases}$$

Kurzweil-belated integrable, but it is not Itô integrable.

The trivial solution of a GSE is said to be *p-stable in probability*, if $\forall$ $\epsilon \in (0,1)$ and $r > 0$, $\exists\ \delta = \delta(\epsilon, r, s_0) > 0$ s.t

$$\mathbb{P}\left(\sup_{t \geq s_0} \|X_t\|_V > r\right) < \epsilon,$$

whenever $\|\widetilde{X}\|_{L^p} < \delta$ (i.e., $\mathbb{P}(\|\widetilde{X}\|_V < \delta) = 1$

If $\exists\ \mathcal{V}\colon [t_0,+\infty)\times L^p(\Omega,V)\to\mathbb{R}^+$, $1\leq p<\infty$, s.t

1. $\exists\ c>0$ f.w. $c\|Z\|_{L^p}\leq \mathcal{V}(t,Z)$, $\forall\ (t,Z)\in[t_0,+\infty)\times L^p(\Omega,V)$

Moreover, for every solution $X=\{X_t : t\geq s_0\}$ of the GSE (2), we have

2. $\exists\ \sigma\in\mathcal{K}$ s.t $\mathbb{E}[\mathcal{V}(s_0,X_{s_0})]\leq\sigma\left(\|X_{s_0}\|_{L^p}\right)$;
3. the function $[s_0,+\infty)\ni t\mapsto \mathcal{V}(t,X_t)$ is nonincreasing;
4. the functional $L\mathcal{V}(t,X_t)$, defined by

$$L\mathcal{V}(t,X_t)=\lim_{h\to 0^+}\frac{1}{h}\mathbb{E}\left[\mathcal{V}(t+h,X_{t+h})-\mathcal{V}(t,X_t)|\mathcal{F}_t\right],$$

   is non-positive $\forall\ t\geq s_0$.

Then, the trivial solution of the GSE (2) is p-stable in probability.

Let

$$F(\nu, s) = \int_0^s [ax(\nu) + bx(\nu - h)]dr + \int_0^s \sigma x(\nu - \tau)dB_r,$$

where $h > 0$, $\tau > 0$ $B : [0, +\infty) \to \mathbb{R}$ is a Brownian motion and

$$a + |b| + \frac{1}{2}\sigma^2 < 0. \tag{4}$$

Consider

$$\begin{cases} \dot{x} &= ax(t) + bx(t-h) + \sigma x(t-\tau)dB(t) \\ x_0 &= \phi, \end{cases}$$

where ϕ is a $\mathcal{F}_t$-measurable random function defined for each $t \in [0, T]$ fulfilling

$$\int_0^t \mathbb{E}[\phi^2(s)]ds < \infty.$$

Let $X = \{X_t : t \geq 0\}$ be a solution of the GSE (2) and consider $\mathcal{V}\colon [t_0, +\infty) \times L^2(\Omega, V) \to \mathbb{R}^+$ given by

$$\mathcal{V}(t, X_t) = x^2(t) + |b| \int_{t-h}^{t} x^2(s)ds + \sigma^2 \int_{t-\tau}^{t} x^2(s)ds.$$

By Itô's Lemma, we have

$$\begin{aligned} L\mathcal{V}(t, X_t) &= 2x(t)(ax(t) + bx(t-h)) \\ &\quad +\sigma^2 x^2(t-\tau) + |b|(x^2(t) - x^2(t-h)) \\ &\quad +\sigma^2(x^2(t) - x^2(t-\tau)) \\ &\leq (2(a+|b|) + \sigma^2)x^2(t). \end{aligned}$$

Together with (4), the trivial solution of the GSE (2) is mean-square stable in probability.

Stability for GSEs

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Thanks for your attention!