Non-overlapping domain decomposition for time-fractional optimal control problems on networks

Günter Leugering
FAU Erlangen-Nürnberg

Partly joint with V. Mehandiratta (KAUST), M. Mehra (IITD)

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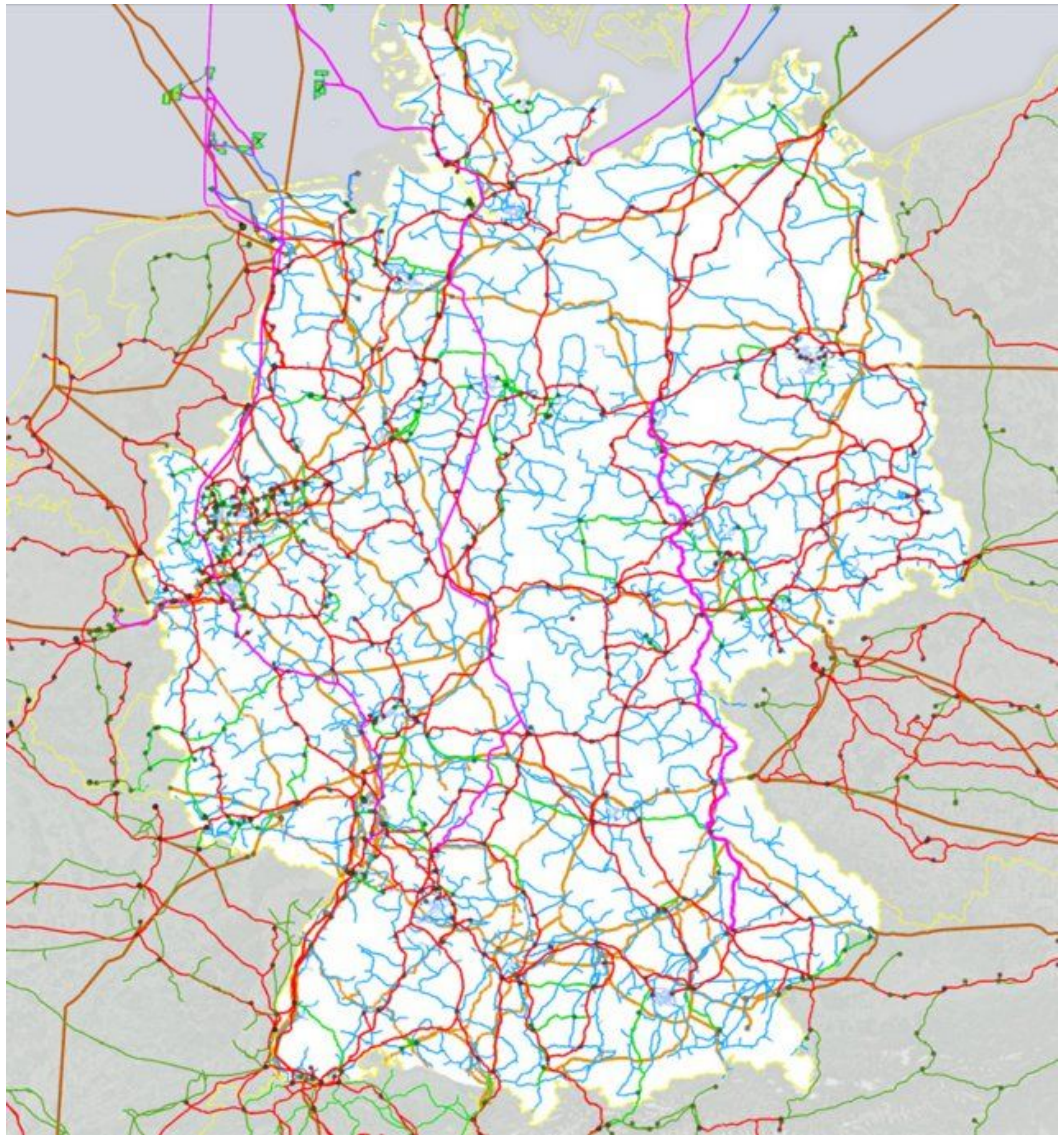
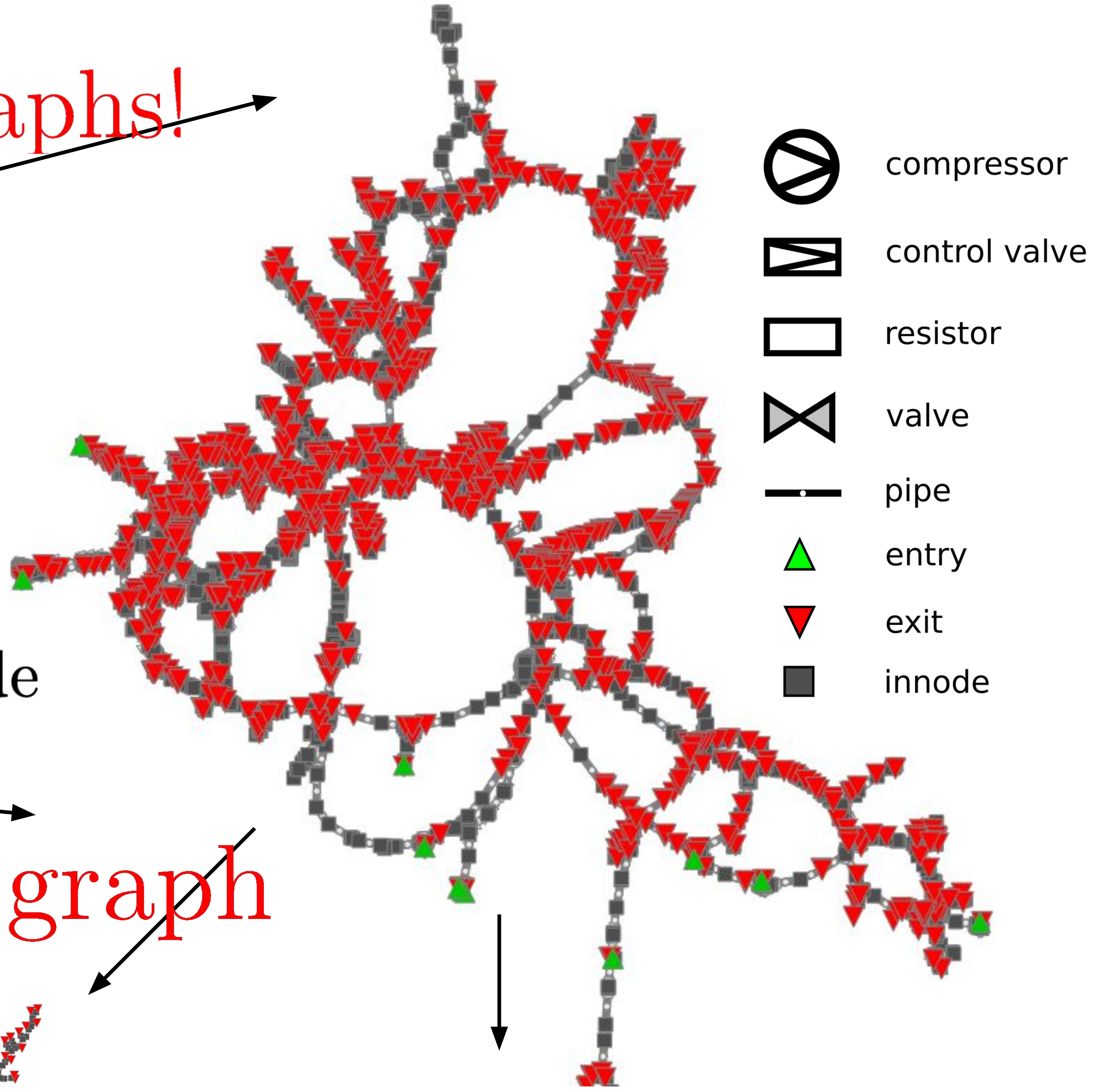
Motivation for optimal control problems on metric graphs: Gas networks and the need for DDMs

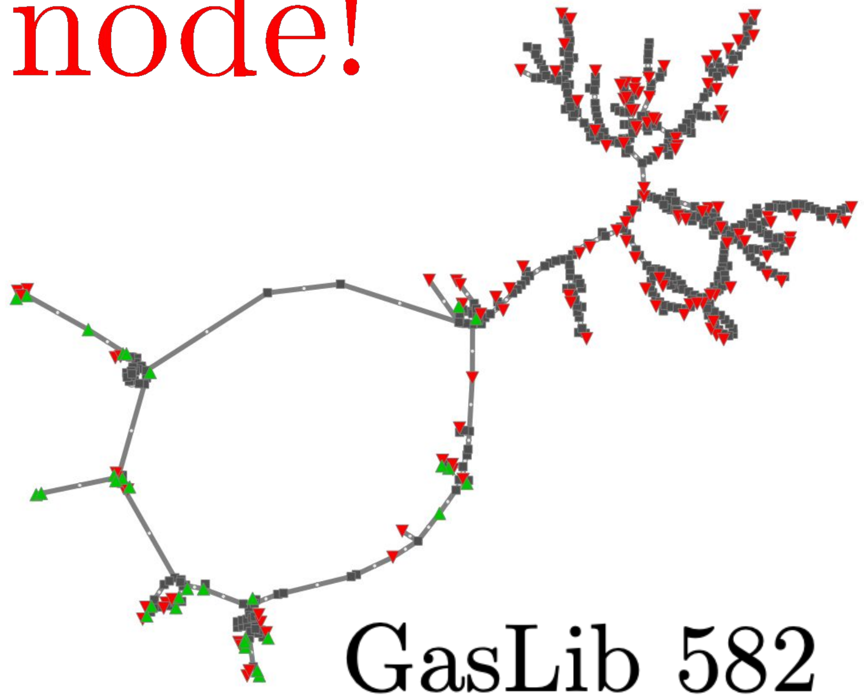
Decompose the full graph into sub-graphs!



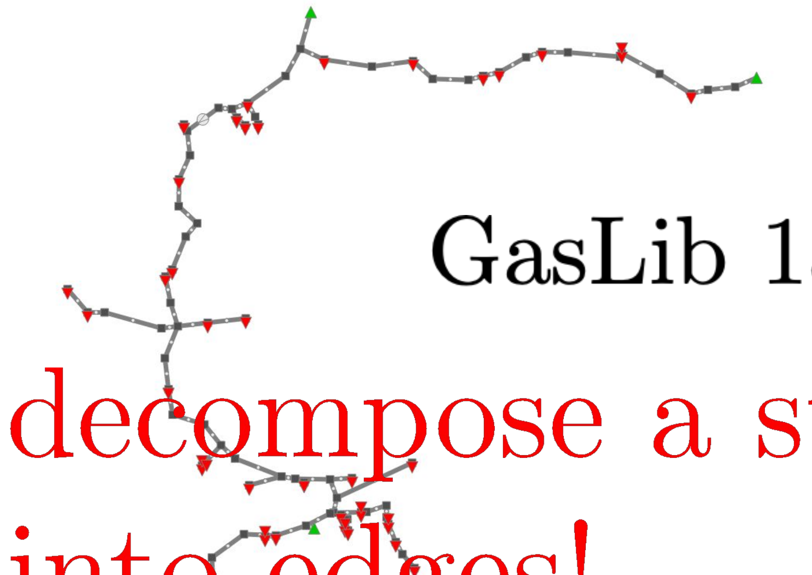
technical zoom
out of a subnetwork
GasLib 4197
see: http://gaslib.zib.de



contract a sub-graph
to a node!



GasLib 582



GasLib 134

decompose a subgraph
into edges!

Motivation: space-time fractional diffusion (STFDE) network model with controls – friction dominated flow

$$
\begin{aligned}
&{}_C\mathcal{D}^{\alpha}_{0,t}\beta(y_i(x,t)) + {}_C\mathcal{D}^{\gamma}_{x,\ell_i}\left(\beta({}_{RL}\mathcal{D}^{\gamma}_{0,x}y_i(x,t))\right) = u_i(x,t), && i\in\mathcal{I},\ x\in(0,\ell_i),\ t\in(0,T),\\
&\mathcal{I}^{1-\gamma}_{0,x}y_i(n_j,t) = \mathcal{I}^{1-\gamma}_{0,x}y_k(n_j,t), && \forall i,k\in\mathcal{I}_j,\ j\in\mathcal{J}^M,\ t\in(0,T),\\
&\sum_{i\in\mathcal{I}_j} d_{ij}\beta({}_{RL}D^{\gamma}_{0,x}y_i(n_j,t)) = 0, && j\in\mathcal{J}^M,\ t\in(0,T)\\
&y_i(n_j,t) = 0, && i\in\mathcal{I}_j, j\in\mathcal{J}^S_D,\ t\in(0,T),\\
&d_{ij}\beta({}_{RL}D^{\gamma}_{0,x}y_i)(n_j,t) = u_j(t), && i\in\mathcal{I}_j, j\in\mathcal{J}^S_N,\ t\in(0,T),\\
&y_i(x,0) = y_i^0(x), && x\in(0,\ell_i),
\end{aligned}
\tag{Net}
$$

transmission conditions

distributed and boundary controls

where the functions $u_i, i\in\mathcal{I}$, $u_j, j\in\mathcal{I}_j, j\in\mathcal{J}^S_N$ serve as distributed and boundary controls, respectively, ${}_C\mathcal{D}^{\mu}_{0,\cdot}, {}_{RL}\mathcal{D}^{\gamma}_{0,\cdot}$ denotes the left-sided Caputo and Riemann-Liouville fractional derivative and $\beta(s) := s|s|^{p-2}$. In the gas context $p = 3/2$. $\mathcal{I}^{\nu}_{0,x}$ is the R-L-fractional integral (see below).

Fractional integrals and derivatives

For $0 < \alpha \leq 1$, the left and right time Caputo fractional derivatives for the interval are given by

$$
{}_{C}D_{0,t}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\int_0^t (t-s)^{-\alpha} f'(s) ds \right),
$$

$$
{}_{C}D_{t,T}^{\alpha} f(t) = -\frac{1}{\Gamma(1-\alpha)} \left(\int_t^T (s-t)^{-\alpha} f'(s) ds \right),
$$

respectively. Accordingly, the spatial left and right Riemann-Liouville fractional derivative are given by

$$
{}_{RL}D_{0,x}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left(\int_0^x (x-s)^{-\alpha} f(s) ds \right),
$$

$$
{}_{RL}D_{x,1}^{\alpha} f(x) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \left(\int_x^1 (s-x)^{-\alpha} f(s) ds \right).
$$

Morevoer, Riemann-Liouville fractional integral is given as follows

$$
\mathcal{I}_{0,x}^{1-\gamma} f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\int_0^x (x-s)^{-\alpha} f(s) ds \right)
$$

Integration by parts....

$$\int_a^b {}_C\mathcal{D}^{\gamma}_{x,b}((a(x)_{RL}D^{\gamma}_{0,x}y(x))w(x)dx$$

$$= \int_a^b a(x)_{RL}D^{\gamma}_{0,x}y(x))_{RL}D^{\gamma}_{0,x}w(x)dx - \left[a(x)_{RL}D^{\gamma}_{0,x}y(x)\mathcal{I}^{1-\gamma}_{a,x}w(x)\right]_a^b$$

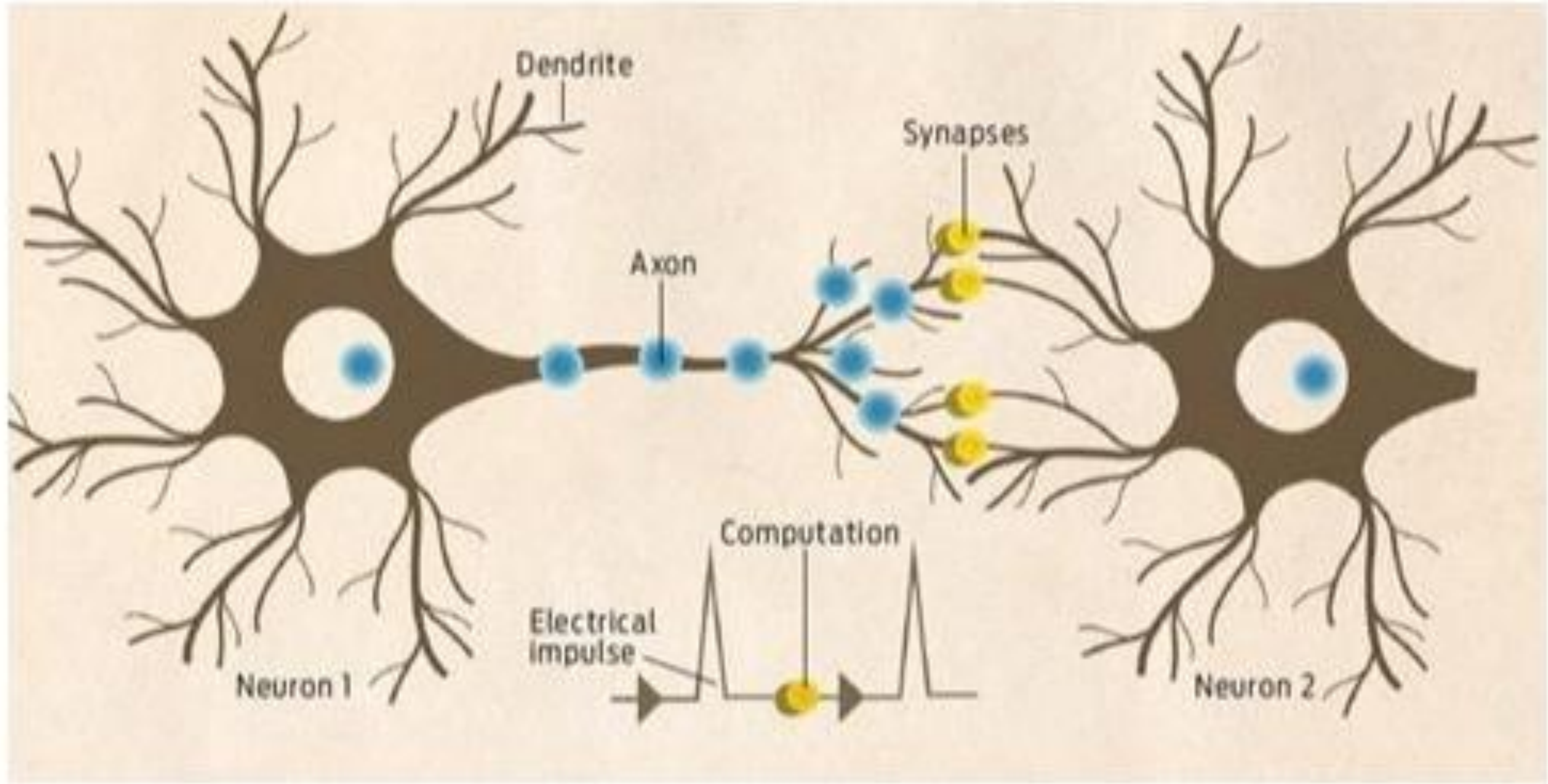
$$= \int_a^b y(x)_C\mathcal{D}^{\gamma}_{x,b}((a(x)_{RL}D^{\gamma}_{0,x}w(x))dx$$

$$- \left[a(x)_{RL}D^{\gamma}_{0,x}y(x)\mathcal{I}^{1-\gamma}_{a,x}w(x)\right]_a^b + \left[a(x)\mathcal{I}^{1-\gamma}_{a,x}y(x)_{RL}D^{\gamma}_{0,x}w(x)\right]_a^b$$

this Sturm-Louville-type identity justifies the choice of the frational operators

Further

- For integer ion holds for open channel flo on a network (still fast diffusi
- Again, for ction term can be added to i neuron with den- drites



- Linear syst nomalous diffusion problems v ineering, like sub- diffusion, s ending on the ker- nels.
- The full sy orting to non-local memristors e Ansgar Jüngel's talk

The optimal control problem

$$I_y(y) := \sum_{i\in\mathcal{I}} \int_0^T \int_0^{\ell_i} \frac{\kappa_i}{2} |y_i(x,t) - y_i^d(x,t)|^2 dx dt, \ I_T(y(x,T)) := \sum_{i\in\mathcal{I}} \int_0^{\ell_i} \frac{\kappa_{i,T}}{2} |y_i(x,T) - y_{i,T}|^2 dx$$

for the state, while the norms of the controls are penalized as follows

$$I_u(u) := \sum_{i\in\mathcal{I}} \frac{\nu_{i,d}}{2} \int_0^T \int_0^{\ell_i} |u_i(x,t)|^2 dx dt + \sum_{j\in\mathcal{J}_N^S} \frac{\nu_{i,b}}{2} \int_0^T |u_j(t)|^2 dt,$$

where $\kappa_i, \kappa_{i,T} \geq 0, \nu_{i,d}, \nu_{i,b} \geq 0$ serve as penalty parameters. We pose the following optimal control problem for (1)

$$\begin{aligned} &\min_{(y,u)} I(y,u) := I_y(y) + I_T(y(\cdot,T)) + I_u(u) \\ &\qquad\qquad s.t. \\ &(y,u) \text{ satisfies } (NET). \end{aligned} \tag{OCP}$$

Linear system for this talk

For $\alpha = \gamma = 1$, we have discussed the corresponding problem in DD27 in Prague 2022. Here, for the sake of simplicity, we treat the linear case with distributed controls.

$$
\begin{aligned}
&{}_C D^{\alpha}_{0,t} y_i + {}_C D^{\gamma}_{x,\ell_i}(a_i\, {}_{RL} D^{\gamma}_{0,x} y_i) + q_i y_i = f_i + u_i^d, \; (x,t) \in (0,\ell_i) \times (0,T), \\
&\mathcal{I}^{1-\gamma}_{0,x} y_i(0,t) = \mathcal{I}^{1-\gamma}_{0,x} y_i^0(0,t), \; t \in (0,T), \quad i = 1,\dots,m \\
&\sum_{i=1}^{m} a_i(0)\, {}_{RL} D^{\gamma}_{0,x} y_i(0,t) = 0, \; t \in (0,T) \\
&\mathcal{I}^{1-\gamma}_{0,x} y_i(\ell_i,t) = 0, \quad i = 1,\dots,m \quad t \in (0,T), \\
&y_i(x,0) = y_{i,0}(x), \quad i = 1,\dots,m \quad x \in (0,\ell_i) \qquad \text{(STFDE)}
\end{aligned}
$$

transmission conditions (the second and third lines)

Augmented Lagrange approach

We define the augmented Lagrangian as follows

$$\mathcal{L}(y,\eta;q) := \sum_i \int_{\Omega_i} (\frac{1}{2}|a_i\,{}_{RL}\mathcal{D}_{0,x}{}^{\gamma} y_i|^2 - u_i y_i)dx$$

$$+ \sum_i (\mathcal{I}_{0,x}^{1-\gamma} y_i(0) - \eta) q_i + \frac{\sigma}{2} \sum_i (\mathcal{I}_{0,x}^{1-\gamma} y_i(0) - \eta)^2.$$

We then formulate the saddle-point problem

$$\inf_{y,\eta} \sup_q \mathcal{L}(y,\eta;q)$$

This problem is now solved using a Uzawa-type saddle-point iteration according to Glowinski and Le Tallec (1989).

Note that this can be formulated both for higher dimensional problems, non-lnear and parabolic problems, as below

AG à la Glowinski‘s Algortihm 3

0) Given η^{l-1}, q^k,

1) solve for $y_i^k, i = 1, 2$:

$$\partial_{y_i} \mathcal{L}(y^k, \eta^{k-1}; q^k) = 0,$$

provides optimality system on each subdomain

2) update $q_i^k, i = 1, 2$

$$q_i^{k+\frac{1}{2}} = q_i^k + \sigma(\mathcal{I}_{0,x}^{1-\gamma} y_i(0) - \eta^{k-1}),$$

fractional ascent step for the Lagrange multiplier

3) solve for η^k

$$\partial_\eta \mathcal{L}(y^k, \eta^k, q^{k+\frac{1}{2}}) = 0,$$

optimality condition for the interface variable

4) update $q^{k+\frac{1}{2}}$

$$q_i^{k+1} = q_i^{k+\frac{1}{2}} + \sigma(\mathcal{I}_{0,x}^{1-\gamma} y_i(0) - \eta^k),$$

complete ascent step for the Lagrange multiplier

4) increase k to $k+1$ check tolerances and stop or and return to 1).

DDM á la Lions for STFDE for 2 domains

We first consider a non-overlapping DD for the full (STFDE). To this end, we introduce the Robin-Robin-type DDM

$$
\begin{aligned}
-a_1(0)\,{}_{RL}D^{\gamma}_{0,x}y_1(0,t)^{n+1} + \sigma\mathcal{I}^{1-\gamma}_{0,x}y_1(0,t)^{n+1} &= \\
& a_2(0)\,{}_{RL}D^{\gamma}_{0,x}y_2(0,t)^{n} + \sigma\mathcal{I}^{1-\gamma}_{0,x}y_2(0,t)^{n} =: g_{12}^{n+1} \\
-a_2(0)\,{}_{RL}D^{\gamma}_{0,x}y_2(0,t)^{n+1} + \sigma\mathcal{I}^{1-\gamma}_{0,x}y_2(0,t)^{n+1} &= \\
& a_1(0)\,{}_{RL}D^{\gamma}_{0,x}y_1(0,t)^{n} + \sigma\mathcal{I}^{1-\gamma}_{0,x}y_1(0,t)^{n} =: g_{21}^{n+1}
\end{aligned}
$$

We have

$$
\begin{aligned}
g_{12}^{n+1} + g_{21}^{n} &= 2\sigma\mathcal{I}^{1-\gamma}_{0,x}y_2(0,t)^{n}, \\
g_{21}^{n+1} + g_{12}^{n} &= 2\sigma\mathcal{I}^{1-\gamma}_{0,x}y_1(0,t)^{n}, \quad t \in (0,T).
\end{aligned}
$$

DDM á la Lions for 2 domains: Steklov-Poincaré

We introduce the Steklov-Poincaré mappings

$$S_i(\eta_i) := -{}_{RL}D^{\gamma}_{0,x} y_i(t,0),\ i = 1,2,$$

where y_i are the solutions of the corresponding initial boundary value problems on $(-1,0)$, $(0,1)$ respectively with boundary data η_i. Then the transmission conditions

$${}_{RL}D^{\gamma}_{0,x} y_1(t,0) + {}_{RL}D^{\gamma}_{0,x} y_2(t,0)) = 0,\ \mathcal{I}^{1-\gamma}_{0,x} y_1(t,0) = \mathcal{I}^{1-\gamma}_{0,x} y_2(t,0)$$

are equivalent to

$$S_1(\eta_1) + S_2(\eta_2) = 0,\ \eta_1 = \eta_2.$$

which, in turn is equivalent to

$$(\sigma \mathcal{I}^{1-\gamma}_{0,x} + S_1)\eta_1 = (\sigma \mathcal{I}^{1-\gamma}_{0,x} - S_2)\eta_2,\ (\sigma \mathcal{I}^{1-\gamma}_{0,x} + S_2)\eta_2 = (\sigma \mathcal{I}^{1-\gamma}_{0,x} - S_1)\eta_1$$

DDM á la Lions for 2 domains

There are now two ways to solve this system iteratively.

- Jacobi-type method

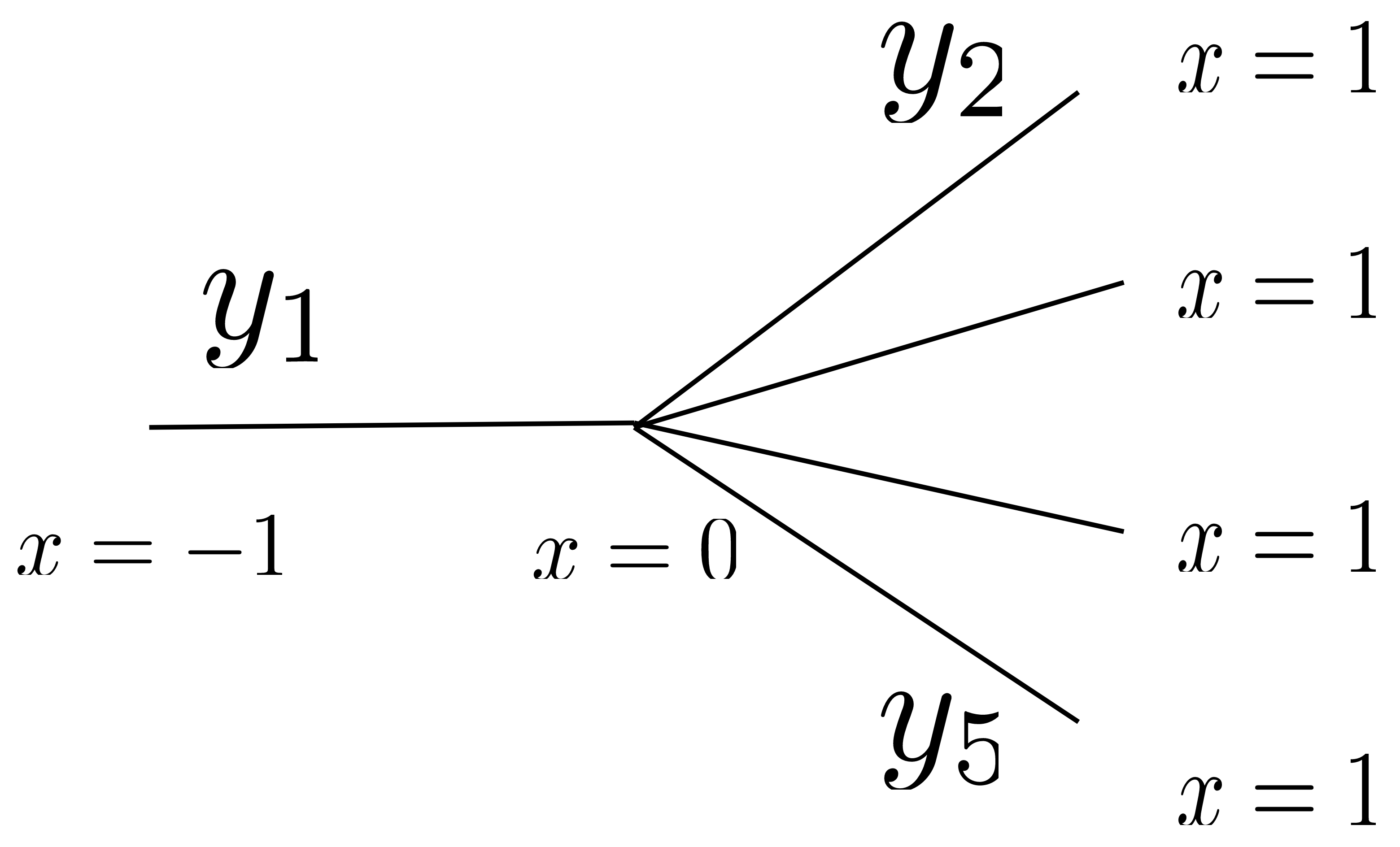
$$(\sigma \mathcal{I}_{0,x}^{1-\gamma} + S_1)\eta_i^{k+1} = (\sigma \mathcal{I}_{0,x}^{1-\gamma} - S_2)\eta_2^{k}$$
$$(\sigma \mathcal{I}_{0,x}^{1-\gamma} + S_2)\eta_2^{k+1} = (\sigma \mathcal{I}_{0,x}^{1-\gamma} - S_1)\eta_1^{k}$$

- Gauß -Seidel type

$$(\sigma \mathcal{I}_{0,x}^{1-\gamma} + S_1)\eta_i^{k+1} = (\sigma \mathcal{I}_{0,x}^{1-\gamma} - S_2)\eta_2^{k}$$
$$(\sigma \mathcal{I}_{0,x}^{1-\gamma} + S_2)\eta_2^{k+1} = (\sigma \mathcal{I}_{0,x}^{1-\gamma} - S_1)\eta_1^{k+1}$$

Notice that the first method is completely parallel, while the second is not. The first iteration is the one, we propose for networks. See Engström, Hansen 2022 for the p-Laplace.

Generalization to a multiple node



$$S_1(\bar{y})(t) := -{}_{RL}D^{\gamma}_{0,x} y_1(t, 0; \bar{y}))$$

$$S_2(\bar{y})(t) := -\sum_{i=2}^{5} {}_{RL}D^{\gamma}_{0,x} y_i(t, 0; \bar{y})$$

Steklov-Poincaré equation at the multiple node

$$S_1(\bar{y}) + S_2(\bar{y}) = 0.$$

Realizing the Steklov-Poincaré setting

We first proceed formally (and then reflect on the Robin-trace operators). We have

$$\eta_1^{k+1} = (\sigma\mathcal{I}_{0,x}^{1-\gamma} + S_1)^{-1}(\sigma\mathcal{I}_{0,x}^{1-\gamma} - S_2)\eta_2^k$$
$$\eta_2^{k+1} = (\sigma\mathcal{I}_{0,x}^{1-\gamma} + S_2)^{-1}(\sigma\mathcal{I}_{0,x}^{1-\gamma} - S_1)\eta_1^{k+1}$$

and introduce

$$\mu^k = (\sigma\mathcal{I}_{0,x}^{1-\gamma} + S_2)\eta_2^k,\ \mu := (\sigma\mathcal{I}_{0,x}^{1-\gamma} + S_2)\eta_2$$
$$\lambda^k = (\sigma\mathcal{I}_{0,x}^{1-\gamma} - S_2)\eta_2^k,\ \lambda := (\sigma\mathcal{I}_{0,x}^{1-\gamma} - S_2)\eta_2$$

Thus

$$\frac{1}{2\sigma}(\mu^k + \lambda^k) = \eta_2^k,\ \frac{1}{2\sigma}(\mu + \lambda) = \eta_2,\ \frac{1}{2\sigma}(\mu^{k+1} + \lambda^k) = \eta_1^{k+1}$$
$$\frac{1}{2}(\mu^k - \lambda^k) = S_2\eta_2^k,\ \frac{1}{2}(\mu - \lambda) = \eta_2,\ \frac{1}{2}(\lambda^k - \mu^{k+1}) = S_1\eta_1^{k+1}.$$

Realizing the Steklov-Poincaré setting

Solve the Dirichlet problem

$$
\begin{aligned}
&{}_C D^{\alpha}_{0,x} \bar{y}_i + {}_C D^{\gamma}_{x,\ell_i} (a_i {}_{RL} D^{\gamma}_{0,x} \bar{y}_i) = f_i, \ (x,t) \in (0,\ell_i,) \times (0,T), \\
&\mathcal{I}^{1-\gamma}_{0,x} \bar{y}_i(0,t) = \bar{g}_i(t), \ \mathcal{I}^{1-\gamma}_{0,x} \bar{y}_i(\ell_i,t) = 0, t \in (0,T), \\
&\bar{y}_i(x,0) = y_i(x), \ x \in (0,\ell_i)
\end{aligned}
$$

for $\bar{y}_i$ and for $\tilde{y}_i$ with $\tilde{g}_i$, respectively. Evaluate $\mathcal{S}_i$. Then

$$
\begin{aligned}
&(\mathcal{S}_i(\bar{g}_i) - \mathcal{S}_i(\tilde{g}_i), \bar{g}_i - \tilde{g}_i) = \\
&= \int_0^T \int_0^{\ell_i} {}_C D^{\alpha}_{0,x} (\bar{y}_i - \tilde{y}_i)(\bar{y}_i - \tilde{y}_i) + a_i ({}_{RL} D^{\gamma}_{0,x} (\bar{y}_i - \tilde{y}_i))^2 dx dt \\
&\geq \frac{1}{2\Gamma(2-\alpha)} \int_0^T \int_0^{\ell_i} \frac{1}{(T-s)^{\alpha}} (\bar{y}_i - \tilde{y}_i)^2 + a_i ({}_{RL} D^{\gamma}_{0,x} (\bar{y}_i - \tilde{y}_i))^2 dx dt
\end{aligned}
$$

Convergence

This implies

$$(S_2\eta_2^k - S_2\eta)(\eta_2^k - \eta) = \frac{1}{4\sigma}\left((\mu^k - \mu)^2 - (\lambda^k - \lambda)^2\right) \geq 0$$
$$(S_1\eta_1^{k+1} - S_1\eta)(\eta_1^{k+1} - \eta) = \frac{1}{4\sigma}\left((\lambda^k - \lambda)^2 - (\mu^{k+1} - \mu)^2\right) \geq 0,$$

where the inequalities follow from the monotonicity of $\beta(\cdot)$ (see below). This implies

$$|\mu^{k+1} - \mu|^2 \leq |\lambda^k - \lambda|^2, \ |\lambda^k - \lambda|^2 \leq |\mu^k - \mu|^2.$$

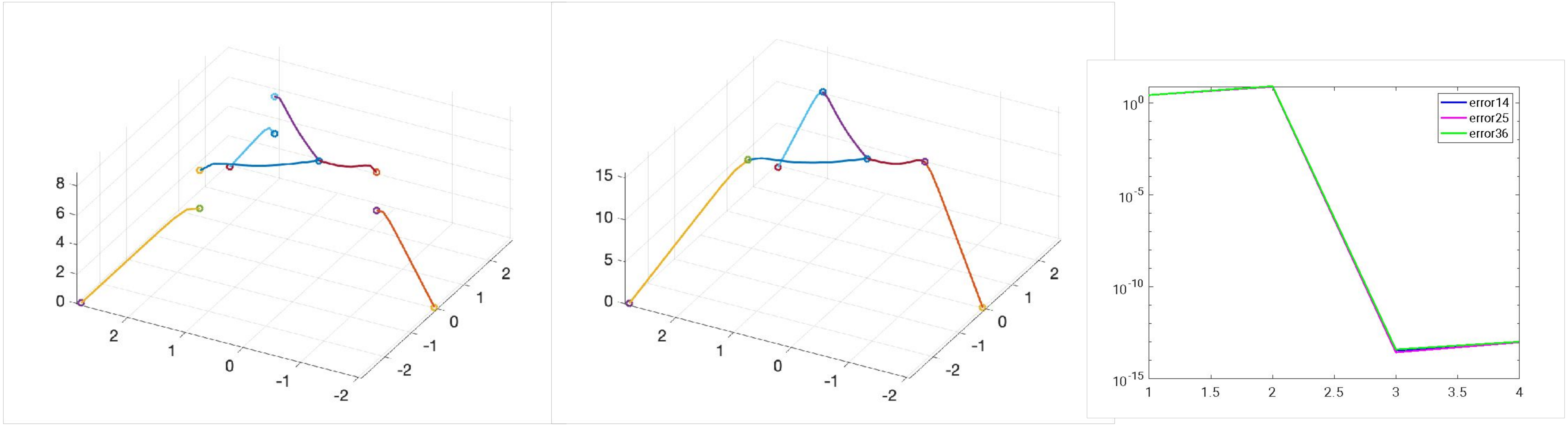
and, hence

$$0 \leq \sum_{k=0}^{K} \left(|\mu^k - \mu|^2 - |\mu^{k+1} - \mu|^2\right) \leq |\mu^0 - \mu|^2, \ \forall K$$

Thus $|\mu^k - \mu|^2 - |\mu^{k+1} - \mu|^2 \to 0$ as $k \to \infty$, and, therefore,

$$(S_2\eta_2^k - S_2\eta)(\eta_2^k - \eta) \to 0, \ (S_1\eta_1^{k+1} - S_1\eta)(\eta_1^{k+1} - \eta) \to 0.$$

Subsrtructuring: cutting out a star



Cutting out a star graph using the Steklov Poncaré map for the star graph.

Back to control

We now recall the optimal control problem. To fix ideas, we confine ourselves to the distributed problem on just 2 domains.

$$\min_{y,u} \mathcal{J}(y,u) := \sum_i \left(\frac{\kappa}{2} \int_0^T \int_0^{\ell_i} |y_i - z_i^d|^2 dx dt + \frac{\nu}{2} \int_0^T \int_0^{\ell_i} |u_i|^2 dx dt \right)$$

subject to

$${}_C D_{0,t}^{\alpha} y_i + {}_C D_{x,\ell_i}^{\gamma} (a_i \, {}_{RL} D_{0,x}^{\gamma} y_i) + q_i y_i = f_i + u_i^d, \; (x,t) \in (0,\ell_i) \times (0,T),$$

$$\mathcal{I}_{0,x}^{1-\gamma} y_i(0,t) = \mathcal{I}_{0,x}^{1-\gamma} y_i^0(0,t), \; t \in (0,T), \quad i = 1,\dots,m$$

$$\sum_{i=1}^{m} a_i(0) \, {}_{RL} D_{0,x}^{\gamma} y_i(0,t) = 0, \; t \in (0,T)$$

$$y_i(\ell_i,t) = 0, \quad i = 1,\dots,m \quad t \in (0,T),$$

$$y_i(x,0) = y_{i,0}(x), \quad i = 1,\dots,m \quad x \in (0,\ell_i)$$

Two strategies: First decompose then optimize or first optimize then decompose

1. **First decompose then optimize**:
   Here we relax the transmission conditions with in the PDE problem using constrained virtual controls in the Neumann condition and augmented Lagrange relaxation of the continuity condition. Then we use a saddle point procedure as above to finally achieve a non-overlapping domain decomposition.

2. **First optimize then decompose**: Here we first derive the optimality condition for the entire optimal control problem and derive a non-overlapping domain decomposition scheme in the sense of P.L. Lions

This parallels the principle of 'First optimize then discretize' versus 'first discretize then optimize.
And, indeed, this, in turn, can be combined. M. Gander has addressed the first issue at the DD28 (KAUST 2024), while we have been mostly working on the second method. Here, we state a connection of the two.

First decompose then optimize: The Lagrangian

$$\min_{y,u,g}\max_{q,z} \mathcal{J}(y,u) := \sum_i \int_0^T \left(\frac{\kappa}{2} \int_0^T \int_0^{\ell_i} |y_i - z_i^d|^2 dx + \frac{\nu}{2} \int_0^T \int_0^{\ell_i} |u_i|^2 dx \right) dt$$

$$+ \int_0^T \left(\sum_i \eta_i((-1)^i g_i(t) - z(t)) + \frac{1}{2\rho} \sum_i |((-1)^i g_i(t) - z(t))|^2 \right) dt$$

$$+ \int_0^T \left(\sum_i \lambda_i(\mathcal{I}_{0,x}^{1-\gamma} y_i(0,t) - q(t)) + \frac{1}{2\rho} \sum_i |((\mathcal{I}_{0,x}^{1-\gamma} y_i(0,t) - q(t))|^2 \right) dt$$

$g_i + g_j = 0$

$\mathcal{I}_{0,x}^{1-\gamma} y_i(0,t) = \mathcal{I}_{0,x}^{1-\gamma} y_j(0,t)$

subject to

$${}_C D_{0,t}^{\alpha} y_i + {}_C D_{x,\ell_i}^{\gamma} (a_i \, {}_{RL} D_{0,x}^{\gamma} y_i) + q_i y_i = f_i + u_i^d, \ (x,t) \in (0,\ell_i) \times (0,T),$$

$$-a_i(0) \, {}_{RL} D_{0,x}^{\gamma} y_i(0,t) = g_i, \ t \in (0,T)$$

virtual control

$$y_i(\ell_i,t) = 0, \quad i = 1,\dots,m \quad t \in (0,T),$$

$$y_i(x,0) = y_{i,0}(x), \quad i = 1,\dots,m \quad x \in (0,\ell_i)$$

First optimize then decompose: The global optimality system

$$
\begin{aligned}
&{}_C D^{\alpha}_{0,t}\bar{y}_i(x,t) + {}_C D^{\gamma}_{x,\ell_i}(a_i(x)\,{}_{RL}D_{0,x}{}^{\gamma}\bar{y}_i(x,t)) = f_i(x,t) + \frac{1}{\nu}p_i(x,t),\\
&{}_C D^{\alpha}_{t,T}p_i(x,t) + {}_C D^{\gamma}_{x,\ell_i}(a_i(x)\,{}_{RL}D_{0,x}{}^{\gamma}p_i(x,t)) = -\kappa(\bar{y}_i(x,t) - z_i^d(x,t)),\\
&\quad \mathcal{I}^{1-\gamma}_{0,x}\bar{y}_i(0,t) = \mathcal{I}^{1-\gamma}_{0,x}\bar{y}_j(0,t),\\
&\quad \mathcal{I}^{1-\gamma}_{0,x}p_i(0,t) = \mathcal{I}^{1-\gamma}_{0,x}p_j(0,t),\ i \neq j,\ i,j = 1,2,\\
&\quad a_1(0)\,{}_{RL}D^{\gamma}_{0,x}\bar{y}_1(0,t) + a_2(0)\,{}_{RL}D^{\gamma}_{0,x}\bar{y}_2(0,t) = 0,\\
&\quad a_1(0)\,{}_{RL}D^{\gamma}_{0,x}\partial p_1(0,t) + a_2(0)\,{}_{RL}D^{\gamma}_{0,x}\partial p_2(0,t) = 0,\\
&\quad \bar{y}_i(\ell_i,t) = 0,\ p_i(\ell_i,t) = 0,\quad ;\\
&\quad \bar{y}_i(x,0) = y_i^0(x),\ p_i(x,T) = 0,\quad x \in (0,\ell_i),\quad i = 1,2,,
\end{aligned}
\qquad \text{(GOS)}
$$

control coupling

transmission conditions

The non-overlapping DDM for the OS

$$ {}_C D^{\alpha}_{0,t} \bar{y_i}^{n+1}(x,t) + {}_C D^{\gamma}_{x,\ell_i}(a_i(x)\, {}_{RL}D_{0,x}{}^{\gamma} \bar{y_i}^{n+1}(x,t)) = f_i(x,t) + \frac{1}{\nu} p_i^{n+1}(x,t), $$

$$ {}_C D^{\alpha}_{t,T} p_i^{n+1}(x,t) + {}_C D^{\gamma}_{x,\ell_i}(a_i(x)\, {}_{RL}D_{0,x}{}^{\gamma} p_i^{n+1}(x,t)) = -\kappa(\bar{y_i}^{n+1}(x,t) - z_i^d(x,t)), $$

$$ (-a_1(0)\, {}_{RL}D^{\gamma}_{0,x} \bar{y_i}^{n+1}(0,t) + \sigma \mathcal{I}^{1-\gamma}_{0,x} \bar{y_i}^{n+1}(0,t) - \mu \mathcal{I}^{1-\gamma}_{0,x} p_i^{n+1}(0,t) $$

$$ = -a_1(0)\, {}_{RL}D^{\gamma}_{0,x} \bar{y_i}^{n}(0,t) + \sigma \mathcal{I}^{1-\gamma}_{0,x} \bar{y_i}^{n}(0,t) - \mu \mathcal{I}^{1-\gamma}_{0,x} p_i^{n}(0,t) $$

$$ =: \lambda_i^{n+1}(t) $$

$$ (-a_1(0)\, {}_{RL}D^{\gamma}_{0,x} p_i^{n+1}(0,t) + \sigma \mathcal{I}^{1-\gamma}_{0,x} p_i(0,t) + \mu \mathcal{I}^{1-\gamma}_{0,x} \bar{y_i}^{n+1}(0,t) $$

$$ = -a_1(0)\, {}_{RL}D^{\gamma}_{0,x} p_i^{n}(0,t) + \sigma \mathcal{I}^{1-\gamma}_{0,x} p_i^{n}(0,t) + \mu \mathcal{I}^{1-\gamma}_{0,x} \bar{y_i}^{n}(0,t) $$

$$ =: \rho_i^{n+1}(t) $$

$$ \bar{y_i}^{n+1}(\ell_i,t) = 0, \; p_i^{n+1}(\ell_i,t) = 0, \quad , $$

$$ \bar{y_i}^{n+1}(x,0) = y_i^0(x), \; p_i^{n+1}(x,T) = 0, \quad x \in (0,\ell_i), \quad i = 1,2,, $$

DDM-update revisited

The history of the coupling is encoded in $\lambda_{ij}^n, \rho_{ij}^n$ and the corresponding update at each iteration step is according to

$$\lambda_i^{n+1}(t) = 2(\sigma \mathcal{I}_{0,x}^{1-\gamma} y_j^n(0,t) - \mu \mathcal{I}_{0,x}^{1-\gamma} p_j^n(0,t)) - \lambda_j^n(t), \qquad \text{(DDM-update)}$$
$$\rho_i^{n+1}(t) = 2(\sigma \mathcal{I}_{0,x}^{1-\gamma} p_j^n(0,t) + \mu \mathcal{I}_{0,x}^{1-\gamma} y_j^n(0,t)) - \rho_j^n(t),\ t \in (0,T).$$

We note that this update looks exactly like the classical one.

Previous work

- **General domains (manifolds, continuous level, no controls; very selective list)**: Early work by P. L. Lions'1989 and O. Pironneau & J.L. Lions'1999 pursued later by J.-D. Benamou'1992-99 for elliptic and parabolic problems, A. Quarteroni'1988-16, F. Nataf' 91-', M. Gander'00-, G. Ciaramella'17-,L. Halpern'00-, J. Haslinger'00-14, J. Kucera,T. Sassi (Signorini-type contact problems), E. Engström, E. Hansen'22 (Robin-type p-Laplace)…M. Dryia, W. Hackbusch'97 (general finite dimensional(!) nonlinear problems)
- **Time domain decomposition (continuous level; again very selective list):** J.L. Lions, Y. Maday, G. Turinici'01, J. Salomon'07-, M. Gander'07-, F. Kwok'18-,G. Ciaramella'21 (semi-linear elliptic) ….(parareal/multiple shooting)…space-time…
- **Optimal control problems:** M. Heinkenschloss'00-11, M. Herty'07, S. Ulbrich'07, M. Gander,'00- F. Kwok'17-, V. Agoshkov'85-, P. Gervasio'04-16, A. Quarteroni'05/06, B. Delourme, L. Halpern, B. Nguyen'06, W. Gong, F. Kwok, Z. Tan'22 (overlapping domains) many others, for linear elliptic and parabolic problems (in almost all cases). **M. Gander et al. 2024**

First decompose then optimize: (VOCP) yet another appproach

Given the iterates λ_{ij}^{n+1}, $\rho_{ij}^{n+1}, i = 1, 2$, we are looking for y_i^{n+1} and the *actual control* u_i^{n+1} as well as the virtual control(s) h_{ij} such that

$$\min J_i(y_i, u_i, h_{ij}) = \frac{\kappa}{2}\int_0^T \int_0^{\ell_i} |y_i(x,t) - z_i^d(x,t)|^2 dxdt$$
$$+\frac{\nu}{2}\int_0^T \int_0^{\ell_i} |u_i^d(x,t)|^2 dxdt + \frac{1}{2\mu}\int_0^T \left(|h_{ij}(t)|^2 + |\mu y_i(0,t) - \rho_{ij}^{n+1}|^2\right) dt, \tag{VOCP}$$

subject to

$$\begin{aligned}
&{}_C D_{0,t}^{\alpha} y_i + {}_C D_{x,\ell_i}^{\gamma}(a_i\, {}_{RL}D_{0,x}^{\gamma} y_i) + q_i y_i = f_i + u_i^d, \ (x,t) \in (0,\ell_i)\times(0,T),\\
&-a_i(0)\,{}_{RL}D_{0,x}^{\gamma} y_i(0,t) + \sigma \mathcal{I}_{0,x}^{1-\gamma} y_i(0,t) = \lambda_{ij}^{n+1} + h_{ij}, \quad t\in(0,T),\\
&y_i(\ell_i,t) = 0, \quad t \in (0,T),\\
&y_i(x,0) = y_i^0(x), \quad x \in (0,\ell_i).
\end{aligned}$$

Equivalence of (DDM-OS) and (VOCP)

This 'first decompose then optimize' approach results in the decomposition of the entire problem into optimization problems on the individual domains! Thus, we can use standard optimization software to iteratively solve on the individual domains, e.g. IPOPT, casADI, SNOOPT, MATLAB's fmincon ....

Theorem Let the optimization penalties $\kappa \geq 0, \mu > 0$ and the initial condition $y^0 \in L^2(\Omega)$ be given. Assume we have the initial transmission data $\lambda_{ij}^0, \rho_{ij}^0 \in L^2(0,T), i,j = 1,2$. Then the domain decomposition procedure for the optimality system (DDM-OS) is equivalent to the sequence of virtual optimal control problems (VOCP).

Theorem

Let the optimization penalties $\kappa \geq 0, \mu > 0$ and the initial condition $y^0 \in L^2(\Omega)$ be given. Assume we have the initial transmission data $\lambda_{ij}^0, \rho_{ij}^0 \in L^2(0,T)$, $i,j = 1,2$. We consider the errors $\tilde{y}_i = y_i^n - y_i$ and $\tilde{p}_i = p_i^n - p_i$, where y_i, p_i solve the original optimality system (GOS) and the iterates y_i^{n+1}, p_i^{n+1} solve (DDM-OS). Then the iteration (DDM-OS) converges under the following conditions.

Case i.) $\sigma = 0, \mu > 0.$

$$\|\tilde{y}_i^n\|_{L^2(0,T;L^2(0,\ell_i))} \to 0, \ \|\tilde{p}_i^n\|_{L^2(0,T;L^2(0,\ell_i))} \to 0.$$

Case ii.) $\sigma > 0, \mu > 0.$
For $\frac{\mu}{\sigma}$ sufficiently large, depending on κ and ν, we have

$$\|\tilde{y}_i^n\|_{L^2(0,T;H^1_{0,1}(0,\ell_i))} \to 0, \ \|\tilde{p}_i^n\|_{L^2(0,T;H^1_{0,1}(0,\ell_i))} \to 0,$$

where $H^1_{0,1}(0,\ell_i)$ denotes the space $H^1(0,\ell_i)$ with elements having zero trace at $x = \ell_i, i = 1,2.$

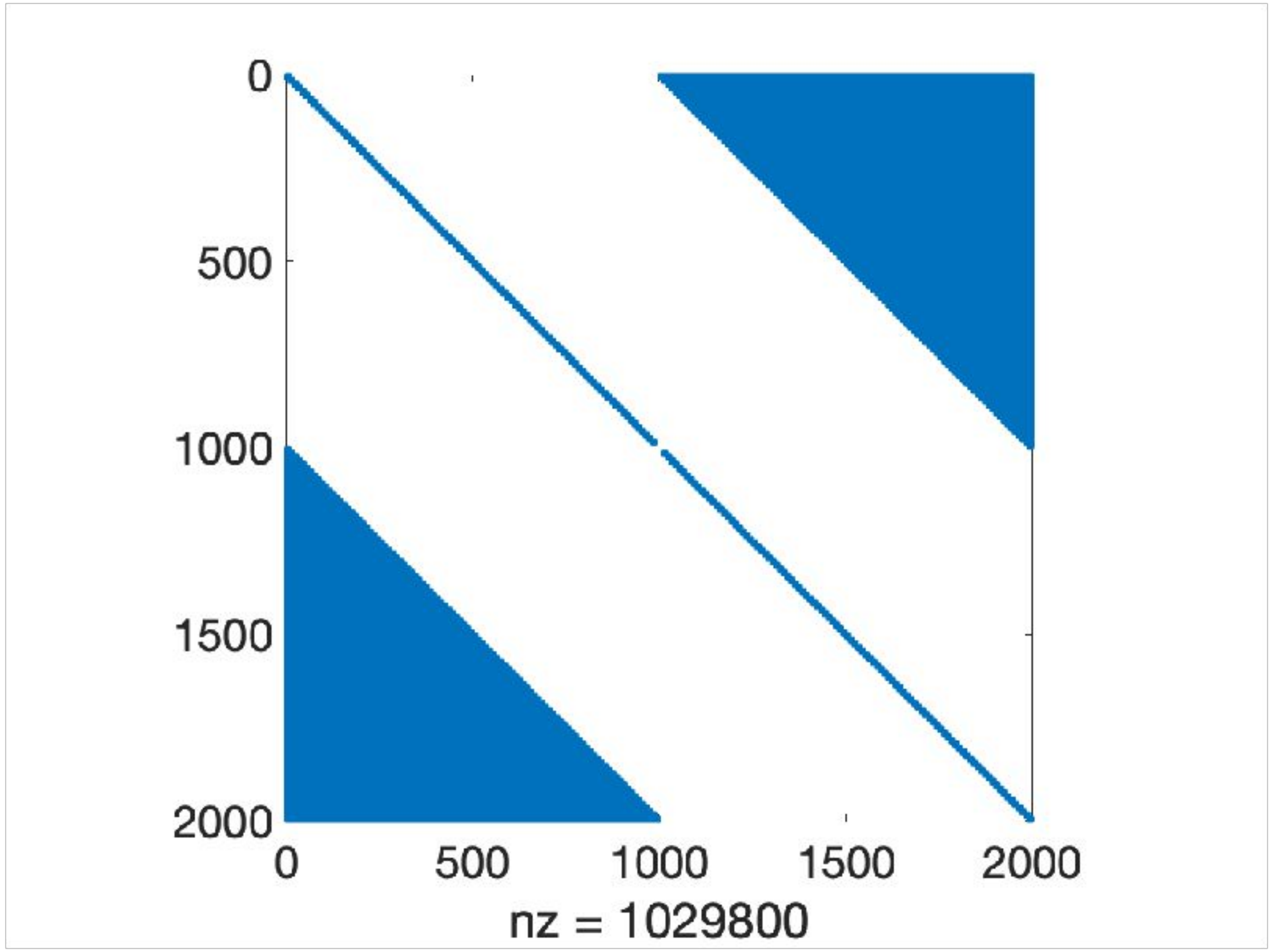
FD approximation

We provide a finite difference (FD) scheme for the state equation. We discretize each the interval $(0, \ell_i)$ as $x_{i,r} = r\Delta x_i$, where Δx_i denotes the spatial discretization step for the edge $e_i i$ given by $\Delta x_i = \ell_i / R$, $r = 0, 1, \ldots, R$, $x_{i,0} = 0$, $x_{i,R} = \ell_i$, $i = 1, 2$. The time grid is given by $t_m = m\Delta t$, $m = 0, 1, \ldots, M$, $t_0 = 0$, $t_M = T$ with $\Delta t = T/M$. We apply the $L1$ method for the discrete approximation of the Caputo derivative,

$$
\begin{aligned}
{}_C D^{\alpha}_{0,t} y_i(x_r, t_{m+1}) &= \frac{1}{\Gamma(1-\alpha)} \sum_{s=0}^{m} \left(\int_{t_s}^{t_{s+1}} (t_{m+1} - \xi)^{-\alpha} \frac{\partial y_i}{\partial \xi}(x_r, \xi) d\xi \right) \\
&\approx \frac{1}{\Gamma(1-\alpha)} \sum_{s=0}^{m} \frac{y_i(x_r, t_{s+1}) - y_i(x_r, t_s)}{\Delta t} \left(\int_{t_s}^{t_{s+1}} (t_{m+1} - \xi)^{-\alpha} d\xi \right) \\
&= \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)} \sum_{s=0}^{n} b_s \left[y_i(x_r, t_{m+1-s}) - y_i(x_r, t_{m-s}) \right],
\end{aligned}
$$

where $b_s = (s+1)^{1-\alpha} - s^{1-\alpha}$, $s = 0, 1, \ldots, m$, $0 < \alpha < 1$ and $i = 1, 2, \ldots, k$.

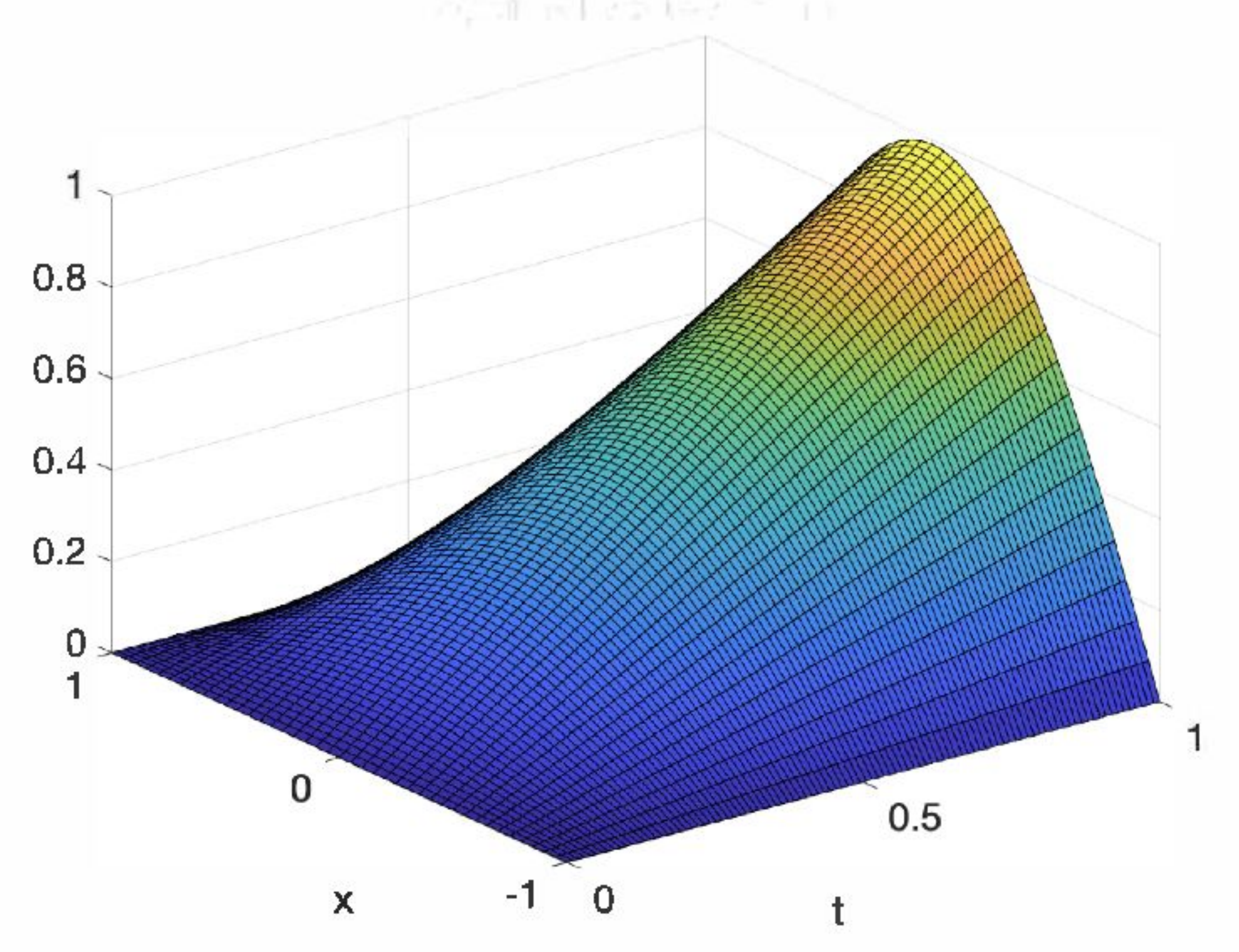
Structure of the all-at-once system matrix

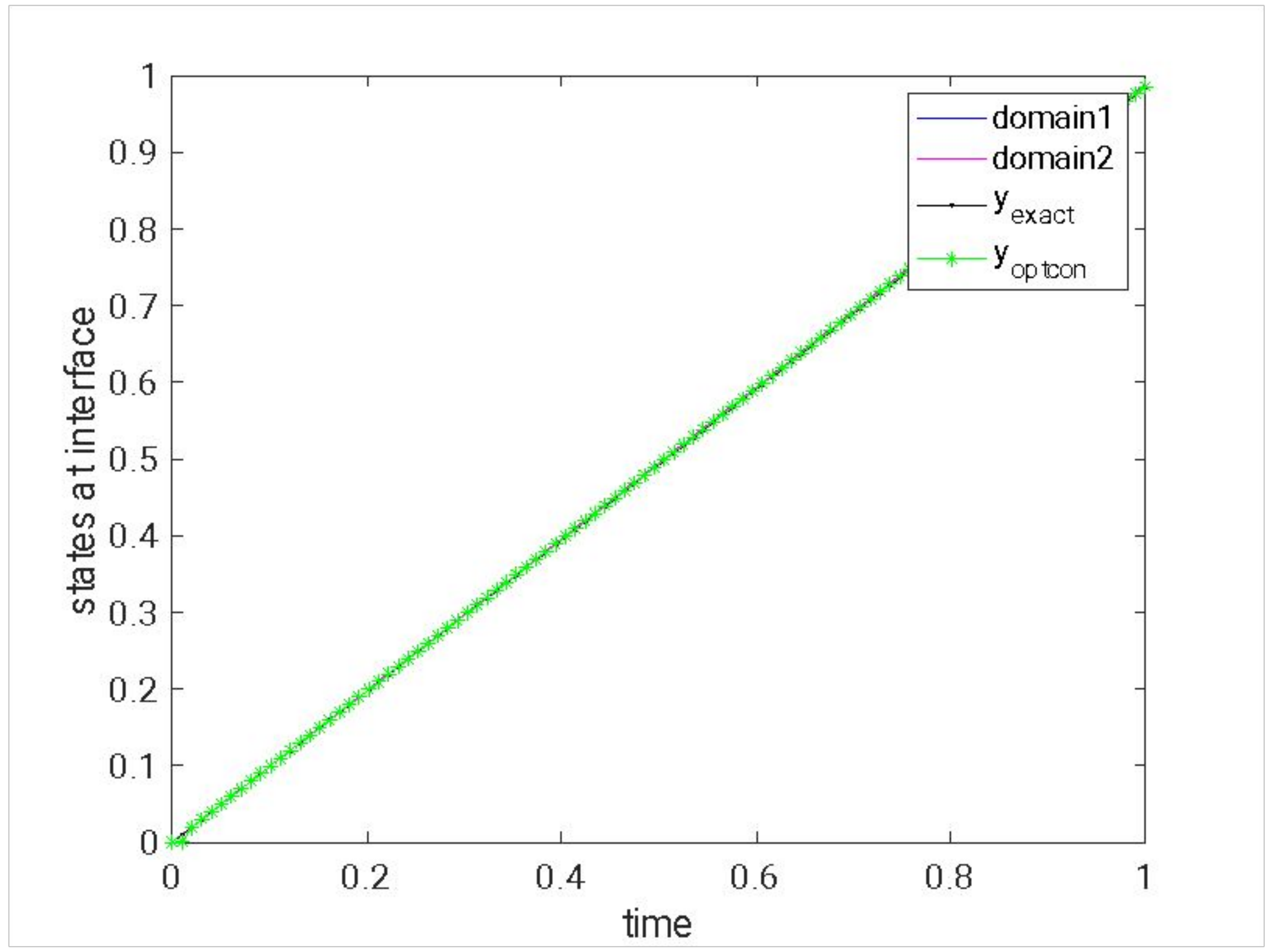


The full diagonal blocks are due to the Volterra-type structure of the operators

Numerical experiment #1: compare with exact solution

For RHS $f(t,x) = (\frac{\Gamma(2)}{\Gamma(2-\alpha}t^{1-\alpha} + t\pi^2)\sin(\pi x)$ we have the exact solution $y_{exact}(t,x) = tsin(\pi x)$. We thus take the target $zd = y_{exaxct}$. On the continous level, the control and hence adjoint are zero.

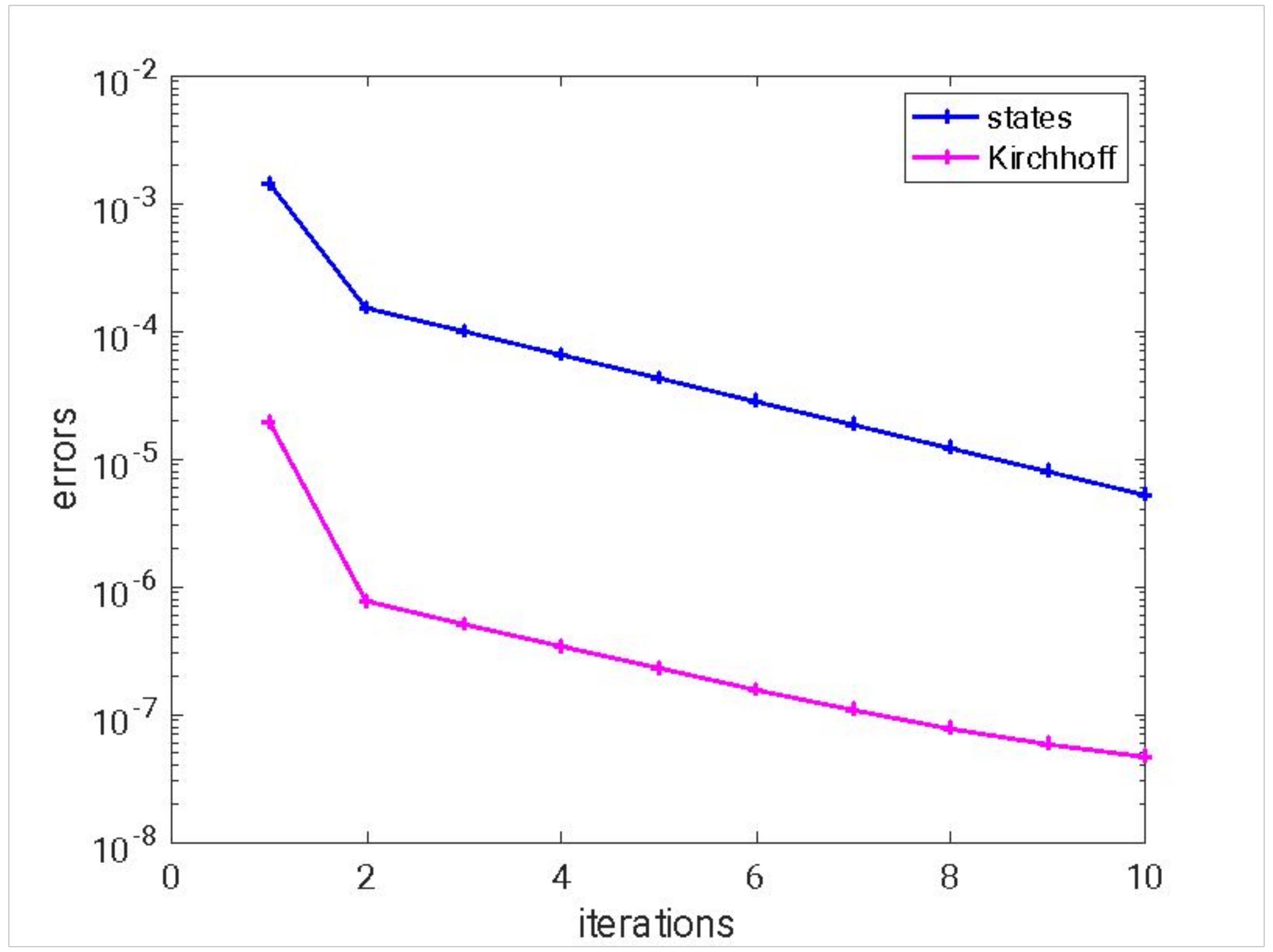




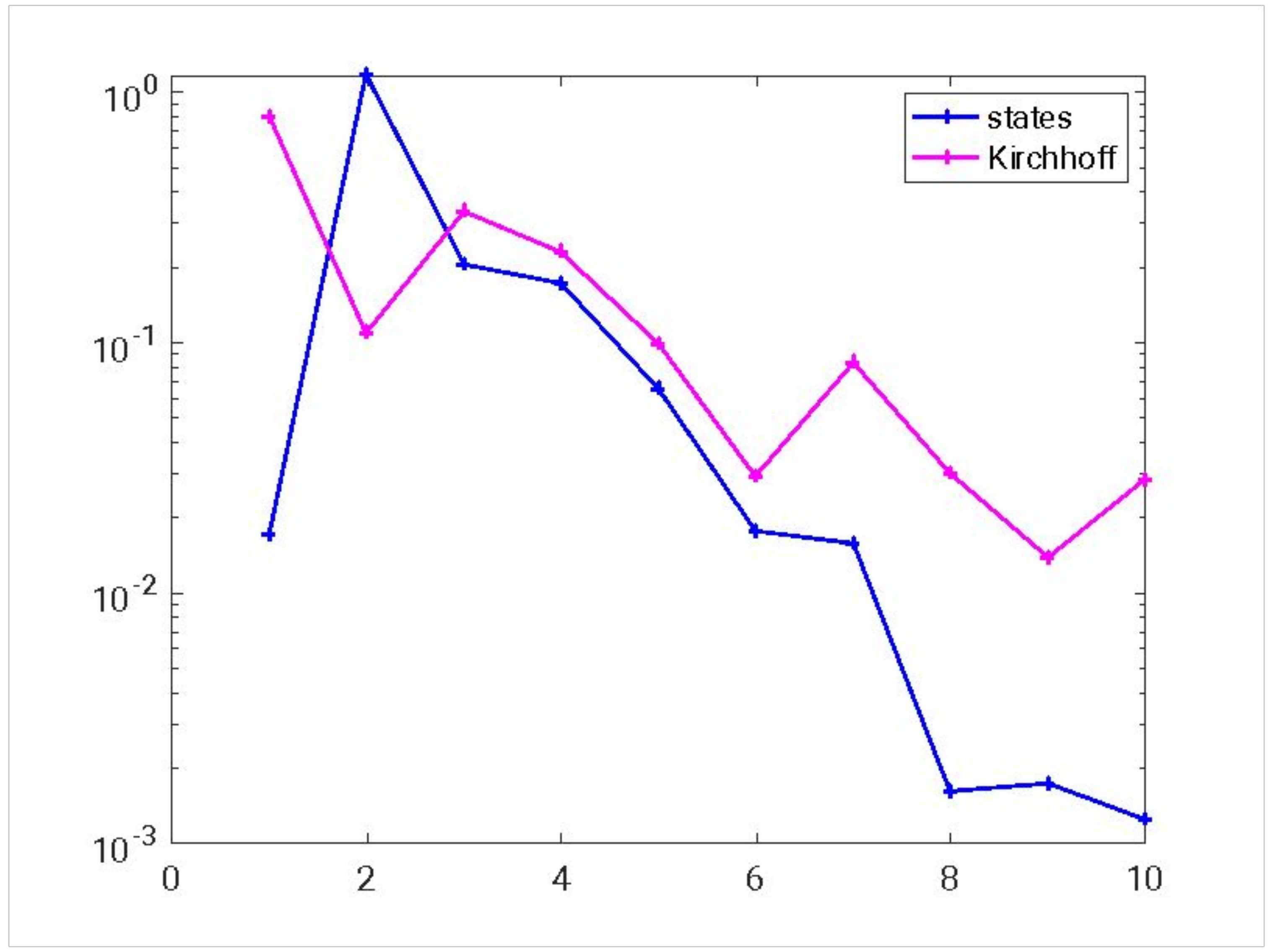
At the interface, the y_{exact} of the global problem and the local solutions of the DDM coincide

Example #1: errors

On the the left:
errors of state and derivatives for DDM-OS



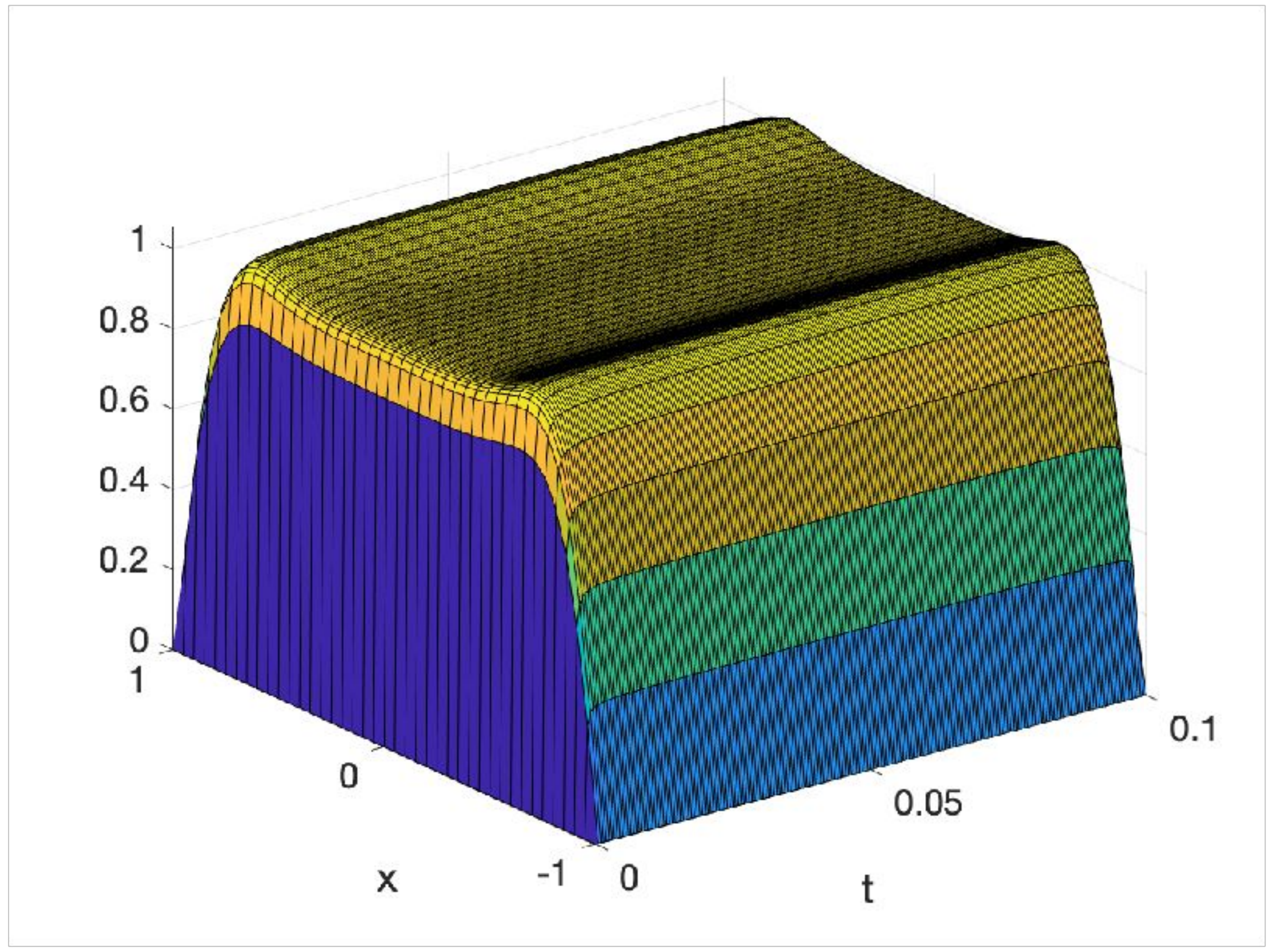
On the the right:
errors of state and derivatives for VOCP

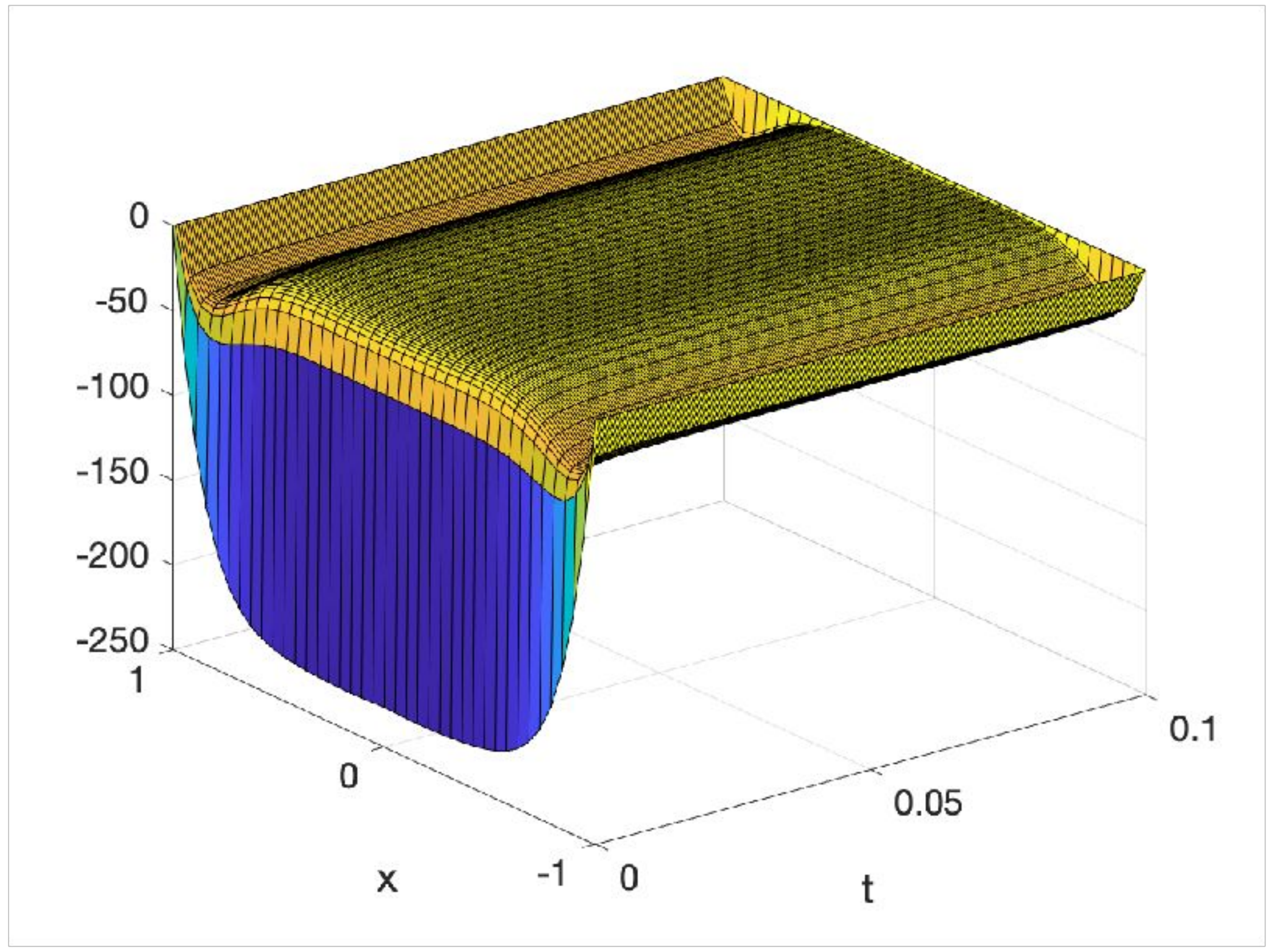


Example #2: final state constant

On the the left:
optimal state

On the the right:
adjoint state



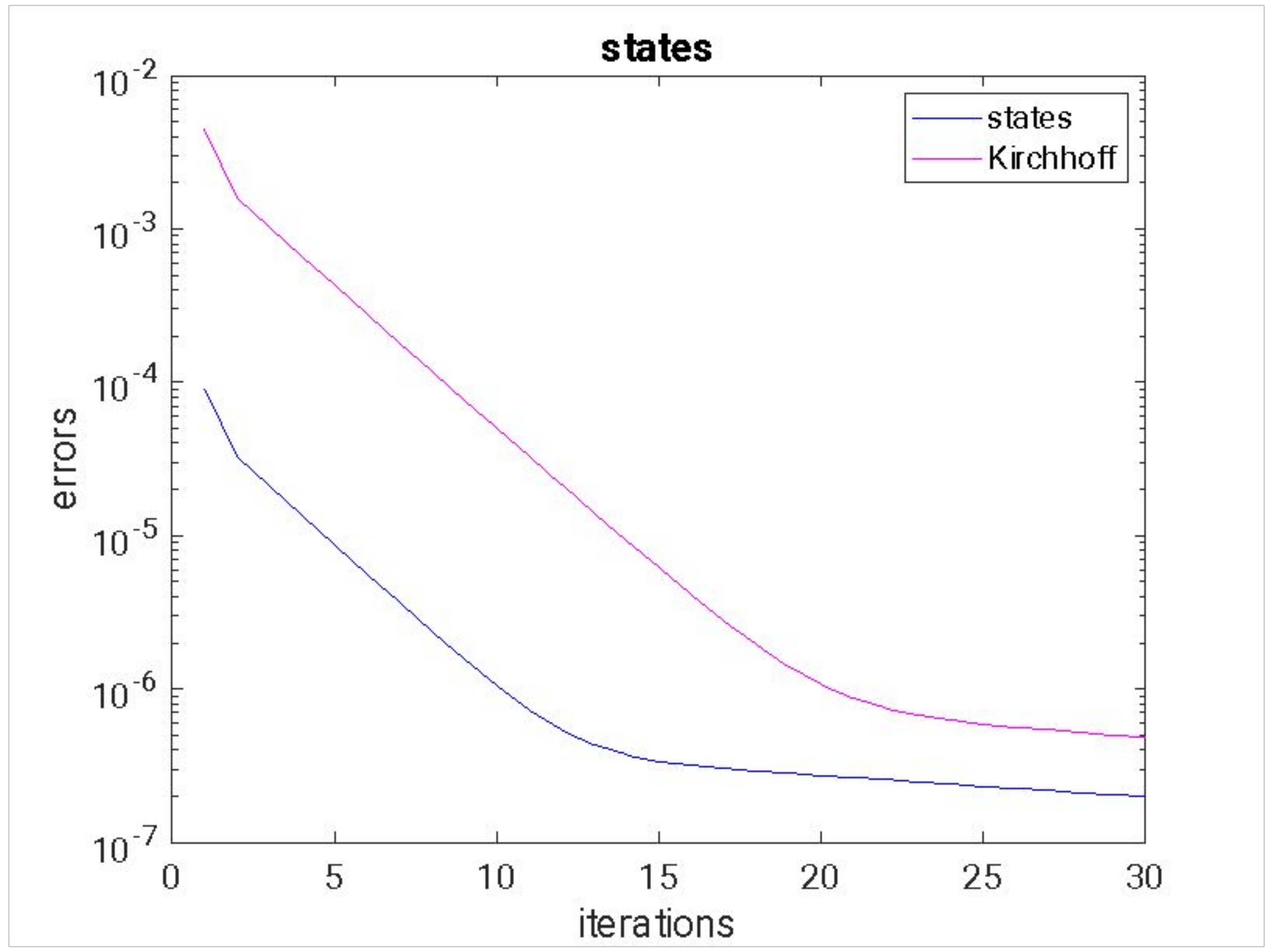


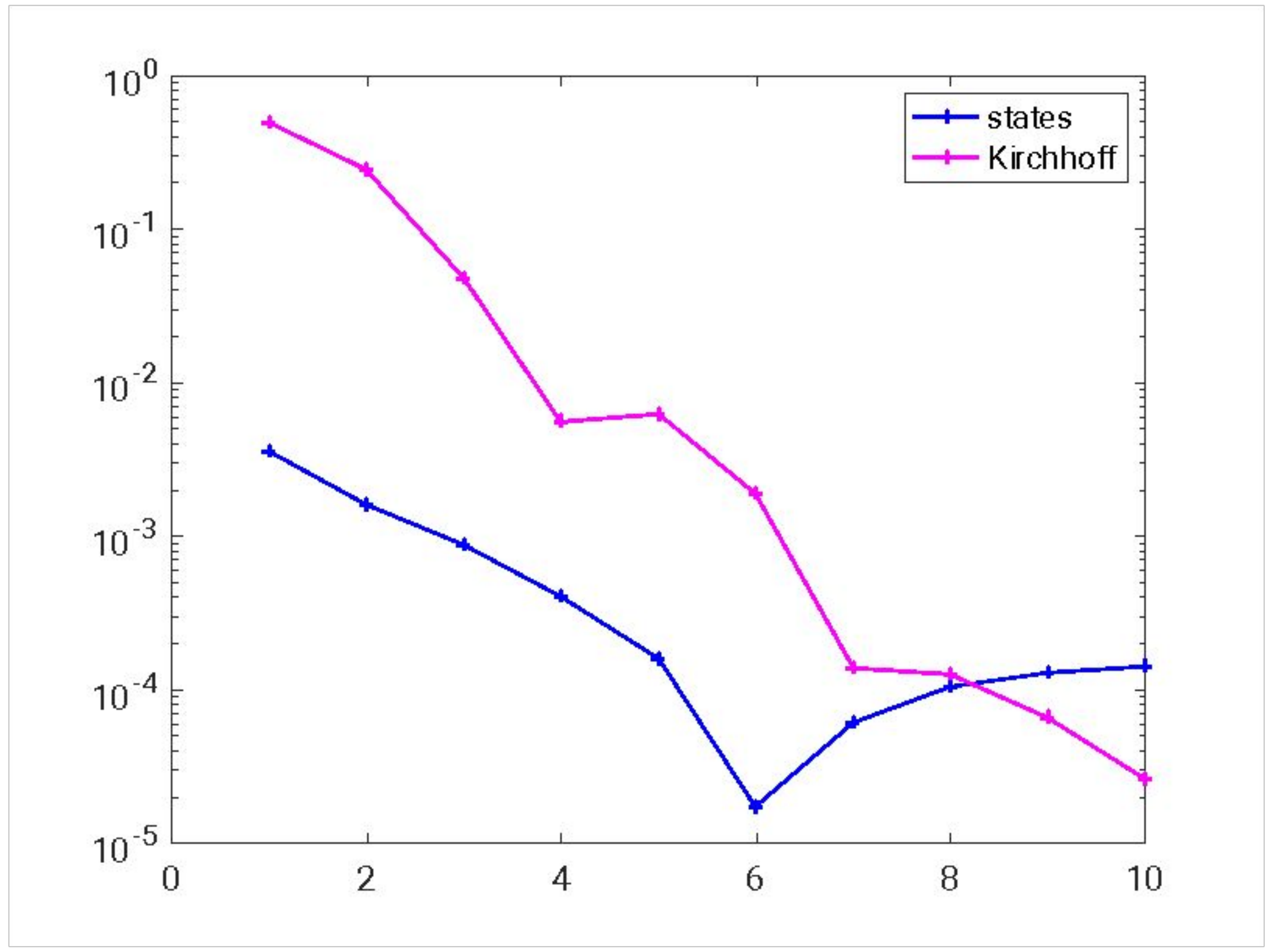
Target $zd = 1$, Dirichlet conditions at both ends, $\kappa = 1.e3, \nu = 1, \sigma = .08, \mu = .08, \alpha = .5$

Example #2: errors

On the left:
Errors of state for DDM-OS

On the right:
Errors of state for VOCP





Thank you for your attention!