The Rayleigh–Taylor Condition for the Muskat Problem

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D. H. Sharp, *An Overview of Rayleigh–Taylor Instability*, Physica D, 1984:

Thus we can infer a simple criterion for the onset of Taylor instability at the interface between two fluids of different densities: *If the heavy fluid pushes the light fluid, the interface is stable. If the light fluid pushes the heavy fluid, the interface is unstable.* A

(A)

(B)

(C)

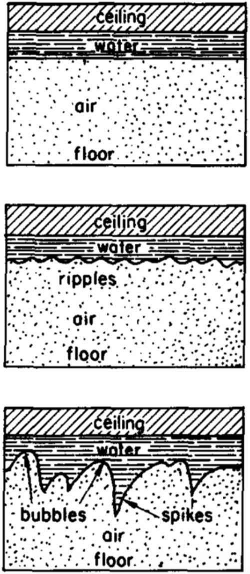
ceiling
water
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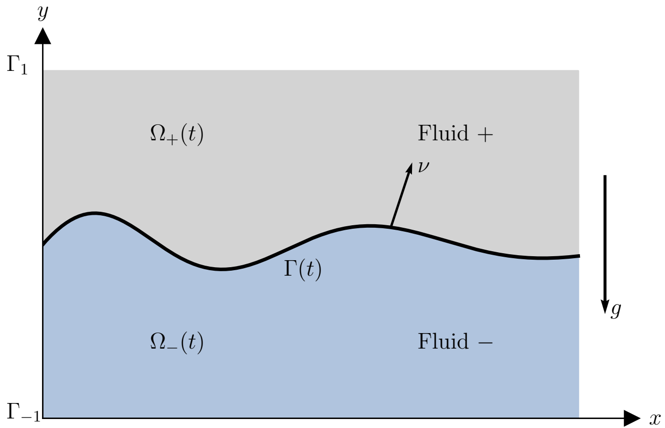
Table I
Some factors influencing the development of Rayleigh–Taylor instability

Factor	Relative size of effect (dimensionless parameter)	Effect on growth of instability
Density ratio	ρ_H/ρ_L or $A=(\rho_H-\rho_L)/(\rho_H+\rho_L)$	A key factor governing the growth rate of Rayleigh–Taylor or Kelvin–Helmholtz instability for small amplitude perturbations of wavelength λ.
Surface tension	Weber number $=2\sigma/(\rho_H-\rho_L)g\lambda^2$	In linear theory, stabilizes wavelengths shorter than a critical wavelength $\lambda=\sqrt{\sigma/g(\rho_H-\rho_L)}$. Establishes a most unstable wavelength, hence probably makes problem well posed mathematically.
Viscosity	$R=\nu t/\lambda^2$	Reduces growth rate; regularizes fluid flow.
Compressibility	$G=g/kc^2=\dfrac{(\text{phase velocity of gravity waves})^2}{(\text{sound speed})^2}$	Reduces growth rate of long wavelength perturbations; decreases active volume of fluid.
Heterogeneity	$\Delta L/\lambda,\ \Delta v/v\ldots$	Can excite secondary, tertiary, ... instabilities of various wavelengths.

The governing equations

Consider the following two-phase flow



The flow in the bulk is governed by Darcy's law:

$$\mu_\pm \vec{v}_\pm = -K \cdot \nabla u_\pm, \qquad K \in \mathbb{R}^{n\times n}_{\mathrm{sym}},\ K > 0.$$

→ common model for laminar flows in porous media and flows in Hele-Shaw cells.

Kinematic boundary condition on the upper free interface:

$$\mu_+ \partial_t h + \partial_\nu u_+ = 0 \quad \text{on} \quad \Gamma(h),$$

→ particles on $\Gamma(h)$ stay there.

Additional *dynamic* boundary condition:

$$u_+ = g\rho_+ h \quad \text{on} \quad \Gamma(h),$$

→ balance of forces on Γ (zero air pressure, no surface tension effects).

On the interface between the fluids:

$$\mu_- \partial_t f + \partial_\nu u_- = 0 \quad \text{on} \quad \Gamma(f),$$

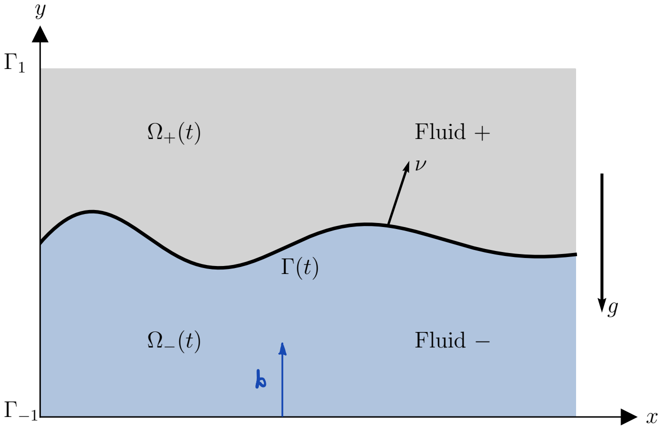
$\rightarrow$ again kinematic boundary condition and continuity of pressure:

$$u_+ - u_- = g(\rho_+ - \rho_-) f \quad \text{on} \quad \Gamma(f).$$

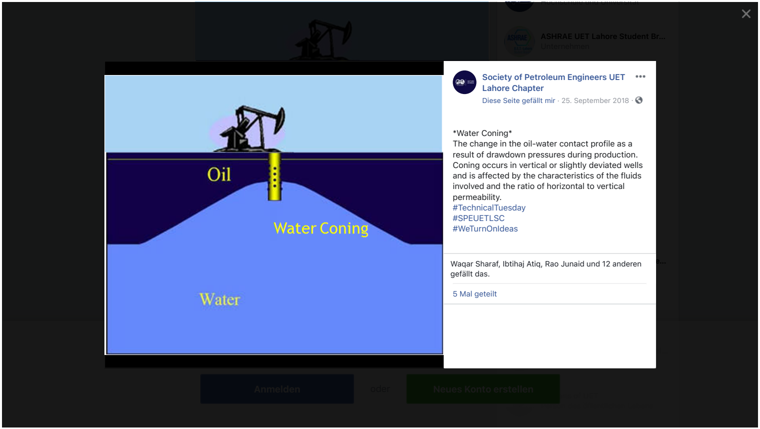
Summarising ($K = k \cdot \text{id}$), we get the following *Muskat* problem:

$$
\begin{aligned}
\Delta u_+ &= 0 && \text{in } \Omega(f,h) \\
\Delta u_- &= 0 && \text{in } \Omega(f) \\
\mu_+ \partial_t h + k \partial_\nu u_+ &= 0 && \text{on } \Gamma(h) \\
u_+ &= g\rho_+ h && \text{on } \Gamma(h) \\
u_+ - u_- &= g(\rho_+ - \rho_-) f && \text{on } \Gamma(f) \\
\mu_- \partial_t f + k \partial_\nu u_- &= 0 && \text{on } \Gamma(f) \\
\partial_y u_- &= b && \text{on } \Gamma_{-1}
\end{aligned}
$$

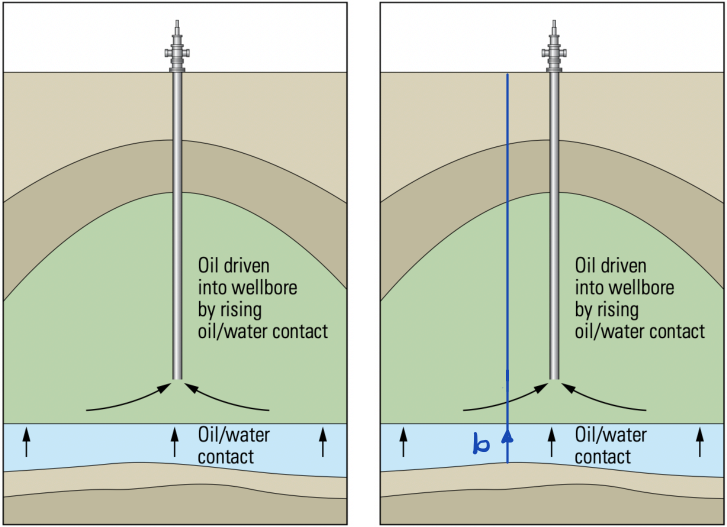
where $b(x)$ is a given injection rate on Γ_{-1}.


y
Γ_1
$\Omega_+(t)$
Fluid +
ν
$\Gamma(t)$
g
$\Omega_-(t)$
Fluid −
b
Γ_{-1}
x

Application in pertoleum engineering[1]:



[1] M Muskat: Two fluid systems in porous media. The encroachment of water into an oil sand, J. Appl. Physics, 5 (1934), 250 – 264


Oil driven
into wellbore
by rising
oil/water contact
Oil/water
contact
Oil driven
into wellbore
by rising
oil/water contact
b
Oil/water
contact

Proposition 1 (A. Matioc, B. Matioc & JE, *Nonlinearity, 2012*)

- The Muskat problem is locally well-posed in the classical sense, provided that

$$\begin{aligned} b\mu_+ + g\rho_+\mu_- &> 0 \\ \frac{\mu_+-\mu_-}{\mu_++\mu_-}(b - g\rho_+) + g(\rho_+ - \rho_-) &< 0 \end{aligned}$$

(generalised Rayleigh–Taylor conditions).

- If $\rho_- > \rho_+$ then the flat solution is asymptotically stable.
- If $\rho_+ > \rho_-$, $b = g\rho_+$, and surface tension is included (and used as a bifurcation parameter) on $\Gamma(f)$ then there are finger shaped unstable steady states of the form:

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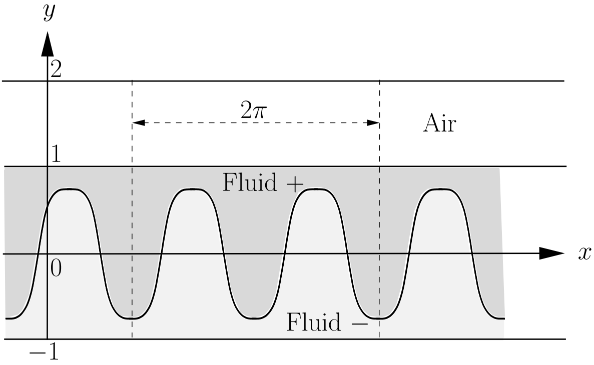
Finger shaped steady states for the Muskat problem:



Related papers (not complete):

- Friedman and Y. Tao., *Nonlinear Analysis TMA*, 2003
- Siegel, Caflish, Howison, *CPAM*, 2004
- Cordoba, Cordoba, Gancedo, *Annals Math.*, 2011
- Prüss, Simmonett, *Evol. Equ. Control Theory*, 2016
- B. Matioc, Ch. Walker, JE, *Indiana Univ. Math. J.*, 2018

- Assume
  - $b \equiv 0$
  - oil as fluid + with: $\rho_+ = 900\,kg/m^3, \quad \mu_+ = 100\,mPas$
  - water as fluid - with: $\rho_- = 1000\,kg/m^3, \quad \mu_- = 1\,mPas$

  $\rightarrow$ Muskat problem is well-posed.
- In application to oil production:
  - $b > 0$
  - oil as fluid + with: $\rho_+ = 900\,kg/m^3, \quad \mu_+ = 100\,mPas$
  - water as fluid - with: $\rho_- = 1000\,kg/m^3, \quad \mu_- = 1\,mPas$

  $\rightarrow$ Then the Muskat problem is no longer well-posed, in cases where b becomes too large.
- Friedman and Y. Tao., 2003: equal densities, $b = 0$. $\rightarrow$ well-posed if $\mu_+ > \mu_-$.
- Siegel, Caflish, Howison, 2004: equal densities, $b = 0$. $\rightarrow$ well-posed if $\mu_+ > \mu_-$.
- Cordoba et al. 2011: equal viscosities, $\rho_- > \rho_+$.

$$(RT) \qquad \begin{array}{lcl} b\mu_+ + g\rho_+\mu_- & > & 0 \\ \frac{\mu_+ - \mu_-}{\mu_+ + \mu_-}(b - g\rho_+) + g(\rho_+ - \rho_-) & < & 0 \end{array}$$

are the Rayleigh-Taylor (RT) condition for the flat equlibrium.

With the associated pressures

$$p_\pm = u_\pm - g\rho_\pm y$$

(RT) are equivalent to

$$\begin{array}{l} \partial_\nu p_+ < 0 \quad \text{on} \quad [y = 2] \\ \partial_\nu p_- - \partial_\nu p_+ < 0 \quad \text{on} \quad [y = 1] \end{array}$$

Proposition 2 (B. Matioc, Ch. Walker & JE, *Indiana Univ. Math. J., 2018*)

- Given (f_0, h_0) of class $C^{2+\alpha}$, the Muskat problem is locally well-posed in the classical sense, provided that

$$\partial_\nu p_+ < 0 \quad \text{on} \quad \Gamma(h_0)$$
$$\partial_\nu p_- - \partial_\nu p_+ < 0 \quad \text{on} \quad \Gamma(f_0)$$

- The Muskat problem is backward parabolic when the Rayleigh-Taylor conditions hold with reversed inequalities.

Steps in the proof

- Reduction of the system by solving for the potentials $u_{\pm}$.

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- Reduction of the system by solving for the potentials $u_\pm$.
- The linearised operator equation then involves resolvents for $u_\pm$ but can nevertheless be represented as a system of Fourier multiplication operators of first order (which is by no means obvious).
- The Rayleigh–Taylor conditions ensure that this system is parabolic in the sense that its propagator is a sectorial operator.
- Maximal regularity then yields the well-posedness result.
- The finger shaped steady states are obtained as bifurcation branches, where the surface tension coefficient serves as a bifurcation parameter.

The symbols of the linearisation:

$$\lambda_1^f(m) = \left[A_\mu(b - g\rho_+) + g(\rho_+ - \rho_-) - \gamma_f m^2\right] \frac{k|m|}{\Sigma_\mu \tanh(|m|)},$$

$$\lambda_2^f(m) = \left[\frac{b[\mu]_\pm + 2g\rho_+\mu_-}{\Sigma_\mu} - g\rho_- - \gamma_f m^2\right] \frac{k|m|}{\Sigma_\mu \sinh(|m|)},$$

$$\lambda_1^h(m) = -\left[\frac{g\rho_+\mu_- + b\mu_+}{\Sigma_\mu} + \gamma_h m^2\right] \frac{k|m|}{\Sigma_\mu \sinh(|m|)},$$

$$\lambda_2^h(m) = -\left[\frac{b\mu_+ + g\rho_+\mu_-}{\Sigma_\mu} + \gamma_h m^2\right] \frac{k|m|}{\mu_+ \tanh(|m|)} - \frac{\mu_-}{\mu_+}\frac{\lambda_1^h(m)}{\cosh(m)}.$$

Here, A_μ is the Atwood number

$$A_\mu := (\mu_+ - \mu_-)/(\mu_+ + \mu_-)$$

and $\Sigma_\mu := \mu_+ + \mu_-$.

Thin film approximation

Using the scaling

$$x = \tilde{x}, \quad y = \varepsilon \tilde{y}, \quad t = \varepsilon \tilde{t} \quad f = \varepsilon \tilde{f}, \quad h = \varepsilon \tilde{h},$$

and expanding

$$\tilde{u}_+ = \sum_{k=0}^{\infty} v_+^k \varepsilon^k, \qquad \tilde{u}_- = \sum_{k=0}^{\infty} v_-^k \varepsilon^k$$

the limit $\varepsilon \searrow 0$ yields:

A system of degenerated parabolic equations for the film heights f and h:

$$\begin{cases} \partial_t f = \partial_x(f\partial_x f) + R\partial_x(f\partial_x h), \\ \partial_t h = \partial_x(f\partial_x f) + R_\mu\partial_x\left[(h-f)\partial_x h)\right] + R\partial_x(f\partial_x h) \end{cases} \quad (1)$$

for $(t,x) \in (0,\infty) \times (0,L)$ and subject to homogeneous Neumann boundary conditions. Moreover:

$$R := \frac{\rho_+}{\rho_- - \rho_+}, \qquad R_\mu := \frac{\mu_-}{\mu_+}R.$$

Proposition 3 (A. Matioc, B. Matioc & JE)

The system (1) is classically well-posed for positive initial conditions and the flat steady state is asymptotically stable in H^2, provided that $\rho_- > \rho_+$.

Observe that in Proposition 3 the full Rayleigh–Taylor conditions have been replaced by $\rho_- > \rho_+$.

Fundamental idea of this approach: Realise the system on a phase space of positive functions as a regular quasilinear parabolic system.

In what follows:

- Construction of global weak solutions to (1) emerging from *nonnegative* initial conditions,
- evidence of global L_2-stability of weak solutions.

For $g := h - f$ the system becomes more symmetric:

$$\begin{cases} \partial_t f = (1+R)\partial_x(f\partial_x f) + R\partial_x(f\partial_x g), \\ \partial_t g = R_\mu\partial_x(g\partial_x f) + R_\mu\partial_x(g\partial_x g). \end{cases} \tag{2}$$

Furthermore, there are two energy functionals:

$$\mathcal{E}_1(f,g) := \int_0^L \left[(f \ln f - f + 1) + \frac{\mu_-}{\mu_+}(g \ln g - g + 1) \right] dx$$

of entropy-type and

$$\mathcal{E}_2(f,g) := \int_0^L \left[f^2 + R(f+g)^2 \right] dx$$

of L_2-type.

Proposition 4 (Ph. Laurençot, B. Matioc & JE)

Assume that $R > 0$ and $R_\mu > 0$. Given f_0, $g_0 \in L_2((0, L))$ with $f_0 \geq 0$ and $g_0 \geq 0$, there exist a global weak solution (f, g) of (2) satisfying

- $f \geq 0$, $g \geq 0$ in $(0, T) \times (0, \infty)$,
- $f, g \in L_\infty((0, T), L_2((0, L))) \cap L_2((0, T), H^1((0, L)))$,
- $\|f(T)\|_1 = \|f_0\|_1, \qquad \|g(T)\|_1 = \|g_0\|_1$,
- $\mathcal{E}_j(f(T), g(T)) \leq \mathcal{E}_j(f_0, g_0)$ for $j = 1, 2$

for all $T > 0$.

In addition, the rate of dissipation of both energies can be estimated from above.

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In particular we have:

$$\mathcal{E}_1(f(T), g(T)) + c_R \int_0^T \int_0^L \left(|\partial_x f|^2 + |\partial_x g|^2\right) \, dx \, dt \leq \mathcal{E}_1(f_0, g_0)$$

for all $T \geq 0$.

This estimates is obviously most helpful in deriving H^1-estimates of solutions.

Proposition 5 (Global exponential stability)

Under the assumptions of Proposition 3 there exist positive constants M and ω such that

$$\left\| f(t) - \frac{1}{L}\int_0^L f_0\,dx \right\|_2^2 + \left\| g(t) - \frac{1}{L}\int_0^L g_0\,dx \right\|_2^2 \leq Me^{-\omega t}$$

for a.e. $t \geq 0$.

To get a regularised system, let

$$F_\varepsilon := (1 - \varepsilon^2 \partial_x^2)^{-1} f, \qquad G_\varepsilon := (1 - \varepsilon^2 \partial_x^2)^{-1} g$$

for $\varepsilon > 0$ and $f, g \in L_2((0, L))$. Then we consider:

The regularised system

$$\begin{cases} \partial_t f_\varepsilon = (1+R)\partial_x(f_\varepsilon \partial_x f_\varepsilon) + R\partial_x(f_\varepsilon \partial_x G_\varepsilon), \\ \partial_t g_\varepsilon = R_\mu \partial_x(g_\varepsilon \partial_x F_\varepsilon) + R_\mu \partial_x(g_\varepsilon \partial_x g_\varepsilon), \end{cases} \tag{3}$$

with regularised initial data

$$f_{0\varepsilon} := (1-\varepsilon^2\partial_x^2)^{-1} f_0 + \varepsilon, \quad g_{0\varepsilon} := (1-\varepsilon^2\partial_x^2)^{-1} g_0 + \varepsilon.$$

The off-diagonal terms of (3) then are of lower order.

$\rightarrow$ system (3) is parabolic (at least on positve solutions)

$\rightarrow$ existence of local classical solutions and a criterion for global solutions:

Let $T_+(\varepsilon)$ denote the positive exit time of the solution. If for every $T < T_+(\varepsilon)$ there exists a $C(\varepsilon, T) > 0$ such that

$$f_\varepsilon \geq \varepsilon/2 + C(\varepsilon, T)^{-1}, \quad g_\varepsilon \geq \varepsilon/2 + C(\varepsilon, T)^{-1}$$

and

$$\max_{t\in[0,T]} \|(f_\varepsilon(t), g_\varepsilon(t))\|_{H^1} \leq C(\varepsilon, T),$$

then the solution $(f_\varepsilon(t), g_\varepsilon(t))$ exists globally.

Furthermore:

The energy estimate for $\mathcal{E}_1$ carries over to the regularised system!

$\rightarrow$ uniform H^1-estimates for $(f_\varepsilon, g_\varepsilon)$

$\rightarrow$ global existence of $(f_\varepsilon, g_\varepsilon)$

Using again the energy estimate for $\mathcal{E}_1$, we get uniform estimates of the form

$$\int_0^T \left(\|h_\varepsilon(t)\|_{H^1}^2 + \|h_\varepsilon(t)\|_{L_3}^3 + \|\partial_t h_\varepsilon(t)\|_{(W_6^1)'}^{6/5} \right) dt \leq C(T)$$

for $h \in \{f, g, F, G\}$ and all $(\varepsilon, T) \in (0,1) \times (0, \infty)$.

These estimates form the core of the construction of weak solutions.

Thank you!