Neutral functional differential equations with state-dependent delays and applications

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Trends in Mathematical Sciences

Emmy Noether

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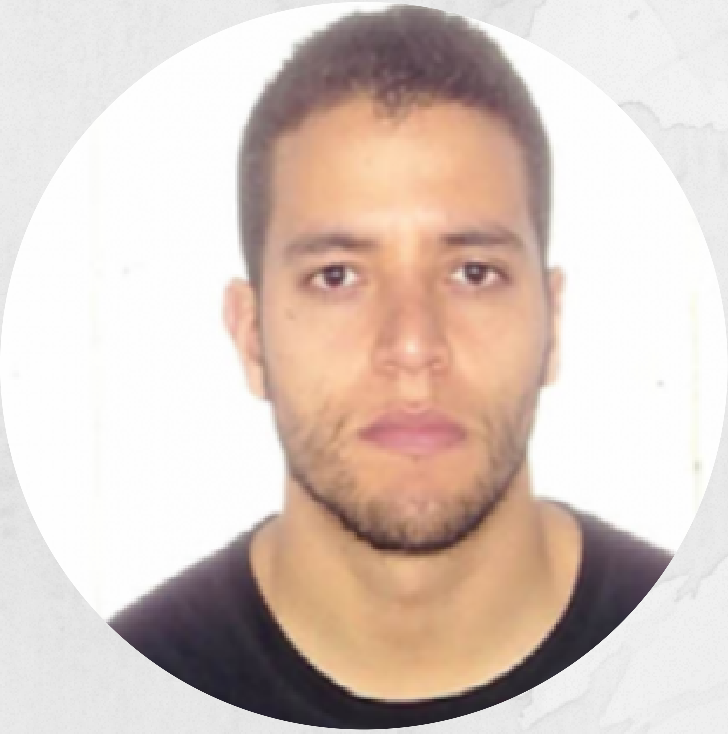
OUTLINE

B. Lani-Wayda, J. G. M.,
Linearized instability for differential equations with dependence on the past derivative
Electron. J. Qual. Theory Differ. Equ. 2023, No. 52, 1-52.

BERNHARD LANI-WAYDA
JUSTUS-LIEBIG UNIVERSITÄT
GERMANY

H. Henríquez, J.G. M., H. C. dos Reis
Existence results for abstract functional differential equations with infinite state-dependent delay and applications
Mathematische Annalen, 88, pages 1817–1840, 2024



HENRIQUE
C. DOS REIS
UNB

HERNÁN
HENRÍQUEZ
USACH
IN MEMORIAM


timeline

Rodney Driver

Two-body problems in eletrodynamics

"It was observed that for Antarctic whale and seal populations, the length of time to maturity is a function of the amount of food (mostly krill) available. Prior to World War II, it was observed that individual seals took five years to mature, small whales took seven to ten years, and large whale species took twelve to fifteen years to reach maturity.
Subsequent to the introduction of factory ships after the war, and with it a depletion of the large whale populations, there was an increase in the krill available for the seals and the remaining whales. It was then noted that seals took three to four years to mature and small whales now only took five years."

Aiello, Freedman & Wu 1992

$$\dot{x}(t) = \frac{\tilde{b}_1(x(t-\tau(x(t))))e^{-\mu_0\tau(x(t))} - \tilde{\mu}_1(x(t))x(t)}{1+\dot{\tau}(x(t))\tilde{b}_1(x(t-\tau(x(t))))e^{-\mu_0\tau(x(t))}}$$

Aiello, W. G., Freedman, H. I., and Wu, J. ***Analysis of a model representing stage-structured population growth with state-dependent time delay*****, SIAM Journal on Applied Mathematics 52(3), pp. 855–869, 1992.**

Balázs and Krisztin

Processing data

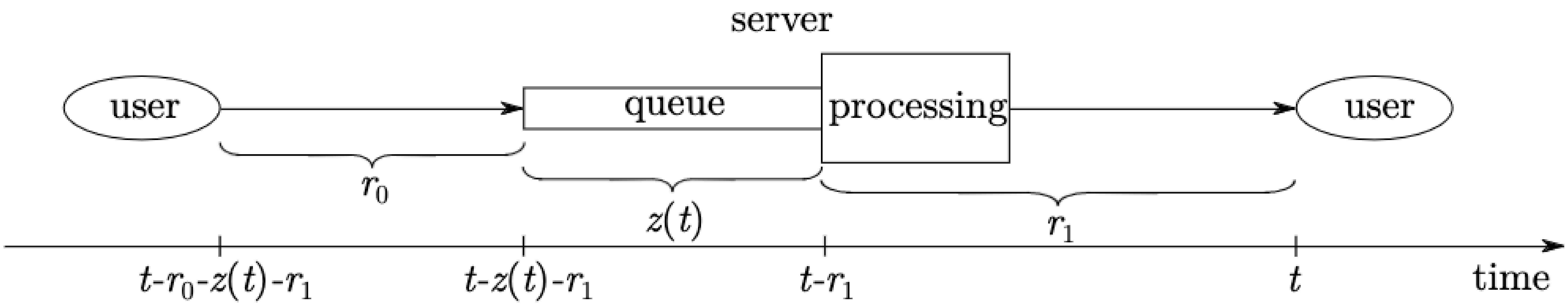


FIG. 1. *The process in time.*

SIAM J. MATH. ANAL.
Vol. 52, No. 4, pp. 3697–3737

A DIFFERENTIAL EQUATION WITH A STATE-DEPENDENT QUEUEING DELAY*

ISTVÁN BALÁZS† AND TIBOR KRISZTIN‡

Differential Equations with state-dependent delays

Problem

Consider the following DDE with state–dependent delays

$$\dot{x}(t) = g(x(t - \tau(x_t))), \tag{1}$$

where $g : \mathbb{R}^n \to \mathbb{R}^n$ and a delay functional $\tau : U \to [0, h]$ on a subset $U_0 \subset C([-h, 0], \mathbb{R}^n)$.

Problem

For initial data in the space $C = C([-h, 0], \mathbb{R}^n)$ of solutions to equations with state–dependent delay are in general NOT unique in cases where for similar equations with constant delay the IVP is well-posed.

Example

The functions

$$x(t) = t + 1 \text{ and } \; x(t) = t + 1 - t^{3/2}$$

for a small $t > 0$ are both solutions of the equation

$$x'(t) = -x(t - |x(t)|)$$

with initial values

$$x(t) = \begin{cases} -1, & \text{if } \; t \le -1; \\ \frac{3}{2}(t+1)^{1/3} - 1, & \text{if } \; -1 < t \le -\frac{7}{8}; \\ \frac{10}{7}t + 1, & \text{if } \; -\frac{7}{8} < t \le 0. \end{cases}$$

Consider

$$\dot{x}(t) = f_0(x_t), \tag{2}$$

where $f_0 : C([-h,0],\mathbb{R}^n) \to \mathbb{R}^n$.

Define $f_0 : C([-h,0],\mathbb{R}^n) \to \mathbb{R}^n$ by

$$f_0 := g \circ ev \circ (\mathrm{id} \times (-\tau))$$

with the evaluation

$$ev : C([-h,0],\mathbb{R}^n) \times [-h,0] \to \mathbb{R}^n$$

given by $ev(\phi, s) = \phi(s)$.

Then, we get the following DDE with state–dependent delays

$$\dot{x}(t) = g(x(t - \tau(x_t))), \tag{3}$$

where $g : \mathbb{R}^n \to \mathbb{R}^n$.

But the evaluation map *ev* is in general NOT continuously differentiable, NOR even locally Lipschitz continuous.

- BRUNOVSKÝ, ERDÉLYI AND WALTHER (2004)

$$\dot{x}(t) = ax(t) - ax(t-1) + f(x(t))$$

- H-O WALTHER (2005) AND (2006)
- GARAB, KOVÁCS AND KRISZTIN (2016)

STUMPF (2012)

$$\dot{x}(t) = a(x(t) - x(t - r(x(t)))) + f(x(t))$$

H-O. WALTHER (2013)

$$\dot{x}(t) = a\dot{x}(t + d(x(t))) + f(x(t))$$

Problem

$$x'(t) = ax'(t + d(x(t))) + f(x(t)) \tag{1}$$

with $a \in \mathbb{R}$ and given function $d : \mathbb{R} \to (-h, 0)$ and $f : \mathbb{R} \to \mathbb{R}$.

Goal 1: We want to write the equation (1) in the form

$$x'(t) = g(\partial x_t, x_t)$$

for a functional $g : C \times C^1 \supset W \to \mathbb{R}^n$, W open, which satisfies some conditions (g0)–(g4), (g6) and (g7) which we will explain later

For $t \in [a, b]$, define $x_t : [-h, 0] \to \mathbb{R}^n$ by

$$x_t(\theta) = x(t + \theta).$$

Let $h > 0$, consider the equation

$$x'(t) = ax'(t + d(x(t))) + f(x(t)) \tag{2}$$

with $a \in \mathbb{R}$ and $d : \mathbb{R} \to (-h, 0)$, $f : \mathbb{R} \to \mathbb{R}$, $r : \mathbb{R} \to \mathbb{R}$ all continuously differentiable, $f(0) = 0$, $r(\mathbb{R}) \subset [-h, 0]$. Consider the evaluation map

$$ev : [-h, 0] \times C \ni (t, \phi) \mapsto \phi(t) \in \mathbb{R},$$

where ev is continuous and the induced map

$$ev_1 : (-h, 0) \times C^1 \ni (t, \varphi) \mapsto \varphi(t) \in \mathbb{R}$$

is continuously differentiable.

Define $g : C \times C^1 \to \mathbb{R}$ by

$$g = a(ev \circ ((d \circ ev_1(0,\cdot) \circ pr_2) \times pr_1)) + f \circ ev_1(0,\cdot) \circ pr_2,$$

with the projections pr_1, pr_2 onto the first and second factor.

Assumptions

Let $U_1 \subset C^1$ denote the open set of all $\psi \in C^1$ with $(\partial\psi, \psi) \in W$.

(g0) Continuity: The function g is continuous;

(g1) The delay in the neutral term "never vanishes":

For every $\psi \in U_1 \subset C^1$, there exists $\Delta \in (0, h)$ and a neighbourhood $N \subset W$ of $(\partial\psi, \psi)$ in $C \times C^1$ such that for all (ϕ_1, χ), (ϕ_2, χ) in N with

$$\phi_1(t) = \phi_2(t), \quad \forall t \in [-h, -\Delta]$$

we have

$$g(\phi_1, \chi) = g(\phi_2, \chi).$$

Assumptions

(g2) Local estimates for g

For every $\psi \in U_1 \subset C^1$, there exists $L_2 \geq 0$ and a neighbourhood $N \subset W$ of $(\partial\psi, \psi)$ in $C \times C^1)$ such that for all $(\phi_1, \psi_1), (\phi_2, \psi_2)$ in N, we have:

$$|g(\phi_2, \psi_2) - g(\phi_1, \psi_1)| \leq L_2(|\phi_2 - \phi_1|_C + (\mathrm{Lip}(\phi_2) + 1)|\psi_2 - \psi_1|_C).$$

Assumptions

(g3) Linear extension of the derivative Dg

The restriction g_1 of g to the open subset $W_1 = W \cap (C^1 \times C^1)$ of the space $C^1 \times C^1$ is continuously differentiable, every derivative $Dg_1(\phi, \psi) : C^1 \times C^1 \to \mathbb{R}^n$, $(\phi, \psi) \in W_1$, has a linear extension:

$$D_e g_1(\phi, \psi) : C \times C \to \mathbb{R}^n$$

and the map

$$W_1 \times C \times C \ni (\phi, \psi, \chi, \rho) \mapsto D_e g_1(\phi, \psi)(\chi, \rho) \in \mathbb{R}^n.$$

Assumptions

A condition like (g3) was introduced as **almost Fréchet differentiability** by Mallet-Paret, J., Nussbaum, R. D., and P. Paraskevopoulos.

Assumptions

Property (g3) implies that the set

$$X_{g,2} = \{\psi \in U_1 \cap C^2 : \psi'(0) = g(\partial\psi, \psi)\},$$

if nonempty, is a continuously differential submanifold of the Banach space C^2 of the twice continuously differentiable functions $\phi : [-h, 0] \to \mathbb{R}^n$ with the norm given by

$$|\phi|_2 = |\phi| + |\partial\phi| + |\partial\partial\phi|.$$

Assumptions

(g4) For every $(\phi_0, \psi_0) \in W_1$ there exist $c_4 \geq 0$ and a neighborhood $N \subset W_1$ of (ϕ_0, ψ_0) in $C^1 \times C^1$ such that for all $(\phi, \psi), (\phi_1, \psi_1)$ in N and for all $\chi \in C^1$, we have

$$|(Dg_1(\phi, \psi) - Dg_1(\phi_1, \psi_1))(\chi, 0)| \leq c_4 |\partial\chi| |\psi - \psi_1|.$$

Assumptions

(g6) $(0,0) \in W$, $g(0,0) = 0$, (g3) holds and the map

$$(\phi, \psi) \mapsto \| D_e g_1(\phi, \psi)(\cdot, 0) \|_{L_c(C, \mathbb{R}^n)} \in \mathbb{R}$$

is upper semicontinuous at $(0,0)$.

Assumptions

(g7) $(0,0) \in W$, $g(0,0) = 0$, g_1 is differentiable, and there exist $\eta > 0$, $c_7 \geq 0$ and a function $\xi : [0,\infty) \to [0,\infty)$ with

$$\xi(0) = 0 = \lim_{t \to 0} \xi(t)$$

so that for every $(\phi, \psi) \in W_1$ with $|\phi| + |\psi| \leq \eta$ and for all $\rho \in C^1$, we have

$$|[Dg_1(\phi,\psi) - Dg_1(0,0)](0,\rho)| \leq c_7(\xi(|\phi|_1 + |\psi|_1)|\rho| + |\rho|_1|\psi|).$$

Proposition (H-O Walther, 2016)

The map g satisfies (g0)–(g4), (g6) and (g7).

Consider the following closed subset

$$X_{g,2*} = \{\psi \in X_{g,2} : \psi''(0) = D_e g_1(\partial\psi, \psi)(\partial\partial\psi, \partial\psi)\}$$

on the manifold $X_{g,2}$.

Proposition (Hans-Otto Walther, 2013)

For each $\varphi \in X_{g,2*}$, the solution x^{φ} is twice continuously differentiable, and for all $t \in [0, t_{\varphi})$, $x_t^{\varphi} \in X_{g,2*}$.

Goal

Prove that 0 is unstable.

Strategy

Use invariance of cone

Assumptions

(A1) $(E, |\cdot|_0)$ is a Banach space, $E_2 \subset E$ subspace, $|\cdot|_2$ is a norm on E_2 such that $|\cdot|_0 \leq |\cdot|_2$. The operator $\{S(t)\}_{t\geq 0} \in L_c(E, E)$ form a C_0–semigroup of linear operators. There exist real numbers $\alpha < \beta$ with $\beta > 0$ and a decomposition $E = U \oplus V$ into $S(t)$–invariant closed subspaces, where $U \neq \{0\}$, and a constant $K > 0$ such that

$$\forall u \in U, \forall t \geq 0 : |S(t)u|_0 \geq K^{-1} e^{\beta t} |u|_0 \tag{4}$$

$$\forall v \in V, \forall t \geq 0 : |S(t)v|_0 \leq K e^{\alpha t} |v|_0. \tag{5}$$

Assumptions

(A2) $X \subset E_2$ is a subset, $I_x = [0, t_x^+) \subset [0, \infty)$ open interval for $x \in X$ with $0 \in I_x$, $t_x^+ \in (0, \infty]$ lower semicontinuous as function of $x \in X$,

$$\Omega := \bigcup_{x \in X} I_x \times \{x\},$$

$\phi : \Omega \to X$ is a semiflow on X. In other words,

- If $t \in I_x$, $s \in I_{\phi(t,x)}$, then $s + t \in I_x$;
- $\phi(t + s, x) = \phi(s, \phi(t, x))$;
- $I_0 = [0, \infty)$;

Assumptions

(A3) Define

$$R(t,x) := \phi(t,x) - S(t)x \text{ for } x \in X, t \in I_x.$$

Assume there exists a $t_1 > 0$ such that for every $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$, for all $x \in B_{|\cdot|_2}(0,\delta_\varepsilon) \cap X$ such that $t_1 \in I_x$, and

$$|R(t_1,x)|_0 \leq \varepsilon |x|_0.$$

Assumptions

Under these assumptions, an equivalent norm $||\cdot||_0$ exists on C_0 such that

$$\forall u \in U, \forall t \geq 0 : ||S(t)u||_0 \geq e^{\beta t}||u||_0 \tag{6}$$

$$\forall v \in V, \forall t \geq 0 : ||S(t)v||_0 \leq e^{\alpha t}||v||_0 \tag{7}$$

i.e., the estimates in I above hold with $K = 1$, and

$$||u+v||_0 = \max\{||u||_0, ||v||_0\}. \tag{8}$$

Definition

For $c \in (0,1]$, we define the cone

$$K_c := \{u + v \in E : v \in V, u \in U, \|u\|_0 \geq c\|u+v\|_0\}.$$

Invariance of cone and expansion

Let $c \in (0,1]$ *and* $0 < t_1$ *as in (A3). Set* $q := \frac{e^{\beta t_1 + 1}}{2}$ *(so* $q > 1$*, since* $\beta > 0$*). There exists* $\delta > 0$ *such that*

$$x = u + s \in K_c \cap X \cap B_{|\cdot|_2}(0,\delta)$$

$$\Rightarrow \phi(t_1, x) := \tilde{s} + \tilde{u} \in K_c \textit{ and } \|\tilde{u}\|_0 \geq q\|u\|_0.$$

Abstract *versus* Concrete

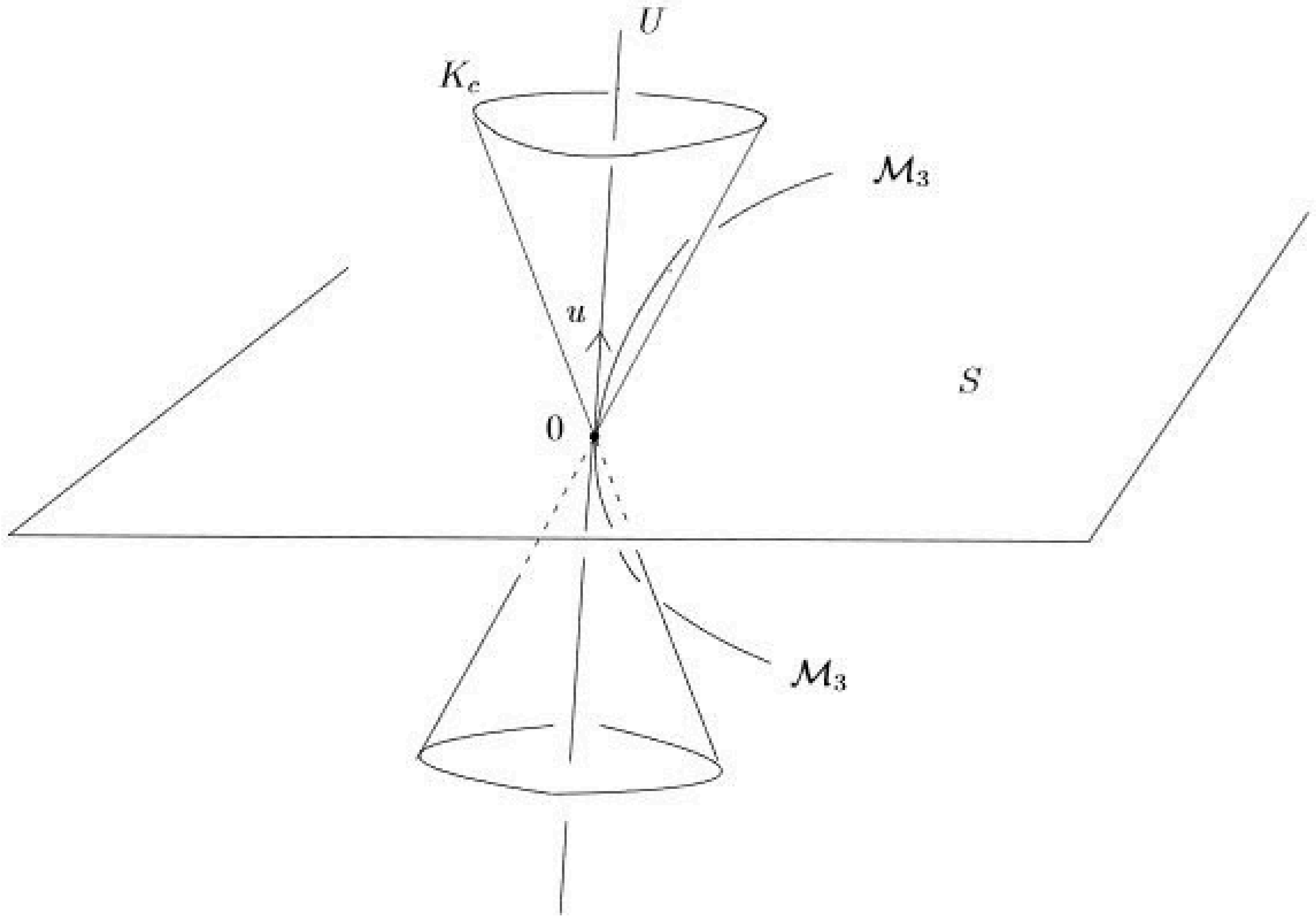
Abstract framework	Concrete framework
E	C_0
E_2	C_2
X	$X_{g,2*}$
$S(t)$	$S(t)$

Assumptions

Consider the following additional hypothesis on g

(A) $g_1 : W_1 \to \mathbb{R}^n$ is C^2 on W_1 and for $(\psi, \phi) \in W_1$, $D^2 g(\psi, \phi)$ has a continuous extension $D_e^2 g_1(\psi, \phi)$ to $C \times C$.

With this condition, we can show that $\mathcal{M}_3 = X_{g,2*} \cap C^3$ is a C^1-submanifold of C^3.


U
K_c
$\mathcal{M}_3$
u
S
0
$\mathcal{M}_3$

Assumptions

Consider the following condition

(A4) $0 \in \overline{X_{g,2*} \setminus \{0\} \cap K_c}^{|\cdot|_2}$.

Theorem (Linearized Instability Principle)

Assume (A1), (A4), (g0)–(g4), (g6) and (g7) are satisfied. Then 0 is unstable for the semiflow ϕ on X. More precisely, for a sequence (u_n) as in (A4) and $n_0 \in \mathbb{N}$ such that $u_n \in B_{|\cdot|_2}(0,\delta)$ for all $n \geq n_0$, one has for every $n \geq n_0$, there exists $t_n > 0$ such that $\phi(t_n, u_n) \notin B_{|\cdot|_2}(0,\delta)$.

Second Part

$$x'(t) = Ax(t) + f\left(t, x_{\rho(t,x_t)}\right), \quad t \in [0,a], \tag{1}$$

$$x_0 = \varphi \tag{2}$$

We consider (1)–(2) as **an abstract retarded functional differential equation with infinite delay**. Because of the infinite delay, the function x_t, which is usually known as the segment of $x(\cdot)$ at t, is defined by

$$x_t : (-\infty, 0] \to X, x_t(\theta) = x(t+\theta)$$

We assume that $x_t \in \mathcal{B}$, where $\mathcal{B}$ is the phase space for the problem (1)–(2).

$$\frac{\partial u(t,\xi)}{\partial t} = \frac{\partial^2 u(t,\xi)}{\partial \xi^2} + f\left(t, u_{\rho(t,u_t)}\right), \quad 0 \le t \le a, \tag{3}$$

$$u(t,0) = u(t,\pi) = 0, \tag{4}$$

$$u(\theta,\xi) = \varphi(\theta,\xi), \quad -\infty < \theta \le 0, \tag{5}$$

for $0 \le \xi \le \pi$ and where $u\colon (-\infty, a] \times [0,\pi] \to \mathbb{R}$ represents the temperature distribution in the bar, and the function $\varphi\colon (-\infty, 0] \times [0,\pi] \to \mathbb{R}$ is the initial temperature distribution.

Open and Developing Problems

Invitation to Brazil

Save the date
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