Local null controllability of the complete N-dimensional Ladyzhenskaya-Boussinesq model

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Problem Formulation

We will consider $\Omega \subset \mathbb{R}^N$ ($N = 2$ or $N = 3$) be a non-empty bounded connected open set, with regular boundary $\partial\Omega$ and let $T > 0$ be given. We will us denote by Q the cylinder $\Omega \times (0, T)$ with side boundary $\Sigma = \partial\Omega \times (0, T)$.

Let $\omega \subset \Omega$ be a (small) non-empty open set. We denote by $(.,.)$ and $\|.\|$ respectively the L^2 scalar product and norm in Ω. We will use C to denote a generic positive constant. Thus, we will study the null controllability for the nonlinear systems:

$$\begin{cases} y_t - \nabla \cdot (\nu(\nabla y) Dy) + (y \cdot \nabla) y + \nabla P = v\tilde{1}_\omega + \nu_0 \theta e_N & \text{in} \quad Q, \\ \nabla \cdot y = 0 & \text{in} \quad Q, \\ \theta_t - \nabla \cdot (\nu(\nabla y) \nabla\theta) + y \cdot \nabla\theta = v_0 \tilde{1}_\omega + \nu(\nabla y) Dy : \nabla y & \text{in} \quad Q, \\ y(x,t) = 0, \theta(x,t) = 0 & \text{on} \quad \Sigma, \\ y(x,0) = y^0(x),\ \theta(x,0) = \theta^0(x) & \text{in} \quad \Omega, \end{cases} \tag{1}$$

where

$$\nu(\nabla y) := \nu_0 + \nu_1 \int_\Omega |\nabla y|^2 dx \tag{2}$$

and

$$
\begin{cases}
y_t - \nabla \cdot (\bar{\nu}(\nabla y) D y) + (y \cdot \nabla) y + \nabla P = v \tilde{1}_\omega + \nu_0 \theta e_N & \text{in} \quad Q, \\
\nabla \cdot y = 0 & \text{in} \quad Q, \\
\theta_t - \nabla \cdot (\bar{\nu}(\nabla \theta) \nabla \theta) + y \cdot \nabla \theta = v_0 \tilde{1}_\omega + \bar{\nu}(\nabla y) D y : \nabla y & \text{in} \quad Q, \\
y(x,t) = 0, \theta(x,t) = 0 & \text{on} \quad \Sigma, \\
y(x,0) = y^0(x), \ \theta(x,0) = \theta^0(x) & \text{in} \quad \Omega,
\end{cases}
\tag{3}
$$

where $\bar{\nu}(\nabla \varsigma) := \nu_0 + \nu_1 \|\nabla \varsigma\|^2_{L^p}$, for $3 < p \leqslant 6$, and in both systems

$$
e_N = \begin{cases} (0,1) \text{ if } N = 2, \\ (0,0,1) \text{ if } N = 3. \end{cases}
$$

In (1) and (3), $y = y(x,t)$ stands the "averaged" velocity field, $\theta = \theta(x,t)$ and $p = p(x,t)$ represent, respectively, temperature and pressure of a fluid whose particles are in Ω during the time interval $(0,T)$; ν_0 and ν_1 are positive constants representing the kinematic viscosity and turbulent viscosity, respectively. (y^0, θ^0) are the initial states, that is to say, the states at time $t = 0$; $\tilde{1}_\omega \in C_0^\infty(\Omega)$ such that $0 < \tilde{1}_\omega \leqslant 1$ in ω and $\tilde{1}_\omega = 0$ outside ω; Dy stands for the symmetrized gradient of y: $Dy = \frac{1}{2}(\nabla y + \nabla^T y)$ and

$$Dy : \nabla y := \sum_{i,j=1}^{N} \frac{1}{2}\left(\frac{\partial y_j}{\partial x_i} + \frac{\partial y_i}{\partial x_j}\right)\frac{\partial y_i}{\partial x_j}. \tag{4}$$

Furthermore, $\omega \times (0,T)$ is the control domain and v (force) and v_0 (heat sources) represent the controls acting on the system.

The following vector spaces, frequently used in the context of incompressible fluids, which will be used throughout the article are:

$$H := \{u \in L^2(\Omega)^N : \nabla \cdot u = 0 \,\mathrm{in}\, \Omega, u \cdot \eta = 0 \,\mathrm{on}\, \partial\Omega\}$$

and

$$V^p := \{u \in W_0^{1,p}(\Omega)^N : \nabla \cdot u = 0 \,\mathrm{in}\, \Omega\},$$

where η is the normal vector exterior to $\partial\Omega$ and $W_0^{1,p}(\Omega)$ is the closure of the space of test functions in Ω, $\mathcal{D}(\Omega)$, in $W^{1,p}(\Omega)$(the standard Sobolev space). In particular, when $p = 2$ we will denote $V = V^p$.

For $N = 2$, $y^0 \in V$, $\theta^0 \in W_0^{1,3/2}(\Omega)$, and any $v \in L^2(\omega \times (0,T))^N$, $v_0 \in L^2(\omega \times (0,T))$ sufficiently small in their respective spaces, (1) possesses exactly a strong solution (y, p, θ) with

$$\begin{cases} y \in L^2(0,T; H^2(\Omega)^N \cap V) \cap C^0([0,T]; V), \;\; y_t \in L^2(0,T; H) \\ \theta \in L^2(0,T; W^{2,3/2}(\Omega)), \;\; \theta_t \in L^2(0,T; L^{3/2}(\Omega)). \end{cases} \tag{5}$$

For $N = 3$, this is true if y^0, θ^0, v and v_0 are sufficiently small in their respective spaces.

Main Results

Definition 1

Let any non-empty open set $\omega \subset \Omega$. It will be said that (1) (respectively (3)) is locally null-controllable at time $T > 0$ if there exists $\delta > 0$ such that, for every $(y^0, \theta^0) \in V \times W_0^{1,3/2}(\Omega)$ (respectively $(y^0, \theta^0) \in V^p \times W_0^{1,p}(\Omega)$) with

$$\|(y^0, \theta^0)\|_{V \times W_0^{1,3/2}(\Omega)} < \delta \,(\text{respectively } \|(y^0, \theta^0)\|_{V^p \times W_0^{1,p}(\Omega)} < \delta),$$

there exists controls $v \in L^2(\omega \times (0, T))^N$, $v_0 \in L^2(\omega \times (0, T))$ and associated solutions (y, p, θ) satisfying

$$y(x, T) = 0 \;\text{ and }\; \theta(x, T) = 0 \;\text{ in }\; \Omega. \tag{6}$$

Thus, the main results are given by the following:

Theorem 2

The nonlinear system (1) *is locally null-controllable at any* $T > 0$.

Theorem 3

The nonlinear system (3) *is locally null-controllable at any* $T > 0$.

In order to prove Theorems 2 and 3, we will first see a result of null controllability for the linear system associated with (1) and (3)

$$\begin{cases} \mathcal{L}_1 y + \nabla P = v\tilde{1}_\omega + \nu_0 \theta e_N + F_1, \;\; \nabla \cdot y = 0 & \text{in} \quad Q, \\ \mathcal{L}_2 \theta = v_0 \tilde{1}_\omega + F_2 & \text{in} \quad Q, \\ y(x,t) = 0, \;\; \theta(x,t) = 0 & \text{on} \quad \Sigma, \\ y(x,0) = y^0(x), \;\; \theta(x,0) = \theta^0(x) & \text{in} \quad \Omega, \end{cases} \tag{7}$$

where, $\mathcal{L}_1 y := y_t - \nu_0 \Delta y$ and $\mathcal{L}_2 \theta := \theta_t - \nu_0 \Delta \theta$.
Furthermore, when $N = 2$ we also show a result of null controllability in a large time for the solutions of the system (1).

Theorem 4 (Large time Null-Controllability)

For $N = 2$*, let* $(y^0, \theta^0) \in V \times H_0^1(\Omega)$ *and* $r > 0$ *a positive constant such that* $\|(y^0, \theta^0)\|_{V \times H_0^1(\Omega)} < r$*, then there exists a sufficiently large time* $T > 0$ *such that the nonlinear system* (1) *is null-controllable at* T*.*

Well-posedness results

The results here will be applied when we study the null controllability of system (7), since once we have the appropriate regularity for θ^0 and y^0 the results described here can be applied to equation formed by $(7)_1$ and $(7)_2$.
The first lemma we mention here is applied to parabolic equations in $L^p - L^q$ spaces and its verification can be based on [3]:

Lemma 5

Let $1 < r, s < \infty$ *and suppose that* $\phi^0 \in W^{1,s}(\Omega)$ *and* $h \in L^r(0, T; L^s(\Omega))$. *Then the problem*

$$\begin{cases} \phi_t - \Delta\phi = h & \text{in} \quad Q, \\ \phi = 0 & \text{on} \quad \Sigma, \\ \phi(0) = \phi^0 & \text{in} \quad \Omega \end{cases}$$

admits a unique solution

$$\phi \in W^{1,r}(0, T; L^s(\Omega)) \cap L^r(0, T; W^{2,s}(\Omega)),$$

Furthermore, there exist a constant $C > 0$ *such that*

$$\|\phi_t\|_{L^r(0,T;L^s(\Omega))} + \|\Delta\phi\|_{L^r(0,T;L^s(\Omega))} \leqslant C(\|\phi^0\|_{W^{1,s}(\Omega)} + \|h\|_{L^r(0,T;L^s(\Omega))}). \tag{8}$$

The second result is valid for Stokes systems with homogeneous Dirichlet boundary conditions and can be found in [11]:

Lemma 6

For every $T > 0$, $u^0 \in V$ *and* $f \in L^2(Q)^N$, *there exists a unique solution* $(u, q) \in \left(L^2(0, T; H^2(\Omega)^N \cap V) \cap L^\infty(0, T; V) \times L^2(0, T; H^1(\Omega)\right)$ *to the Stokes system*

$$\begin{cases} u_t - \Delta u + \nabla q = f, \;\; \nabla \cdot u = 0 & in \quad Q, \\ u = 0 & on \quad \Sigma, \\ u(0) = u^0 & in \quad \Omega. \end{cases}$$

The next result, proven in [6], concerns the regularity of the solutions of the Stokes system in $L^p - L^q$ spaces (see also [7] for additional comments):

Lemma 7

Let $1 < p_1, p_2 < \infty$ *and suppose that* $u^0 \in W^{1,p_2}(\Omega)^N$ *and* $f \in L^{p_1}(0,T;L^{p_2}(\Omega))$*. Then, the weak solution* $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ *of system*

$$\begin{cases} u_t - \Delta u + \nabla q = f, \;\; \nabla \cdot u = 0 & in \quad Q, \\ u = 0 & on \quad \Sigma, \\ u(0) = u^0 & in \quad \Omega \end{cases}$$

actually verifies, together with a pressure q*, that*

$$(u, \nabla q) \in \left(L^{p_1}(0,T;W^{2,p_2}(\Omega)^N) \cap W^{1,p_1}(0,T;L^{p_2}(\Omega)^N)\right) \times L^{p_1}(0,T;L^{p_2}(\Omega)$$

Moreover, there exists a positive constant C *just depending on* Ω *such that*

$$\|u\|_{L^{p_1}(0,T;W^{2,p_2}(\Omega)^N) \cap W^{1,p_1}(0,T;L^{p_2}(\Omega)^N)} + \|\nabla q\|_{L^{p_1}(0,T;L^{p_2}(\Omega)^N)} \leqslant C(\|f\|_{L^{p_1}(0,T;L^{p_2}(\Omega)^N)} + \|u^0\|_{W^{1,p_2}(\Omega)^N}).$$

Carleman estimate

Let's introduce a new non-empty open set $\omega_0 \Subset \omega$. Due to Fursikov and Imanuvilov [5] we have the following result:

Lemma 8

There exists a function $\eta^0 \in C^2(\overline{\Omega})$ *satisfying*

$$\begin{cases} \eta^0(x) > 0, & \forall x \in \Omega, \\ \eta^0(x) = 0, & \forall x \in \partial\Omega, \\ |\nabla \eta^0(x)| > 0, & \forall x \in \overline{\Omega} \setminus \omega_0. \end{cases}$$

Let us introduce the function $\ell \in C^\infty([0, T])$ such that

$$\ell(t) = \begin{cases} \dfrac{T^2}{4}, & 0 \leqslant t \leqslant T/2, \\ t(T - t), & T/2 < t \leqslant T. \end{cases}$$

Thus, for all $\lambda > 0$ and $m > 4$, we consider the following weight functions:

$$\alpha(x,t) = \frac{e^{5/4\lambda m\|\eta^0\|_\infty} - e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))}}{\ell(t)^4}, \quad \xi(x,t) = \frac{e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))}}{\ell(t)^4},$$
$$\alpha^*(t) = \max_{x\in\overline{\Omega}} \alpha(x,t), \quad \xi^*(t) = \min_{x\in\overline{\Omega}} \xi(x,t),$$
$$\hat{\alpha}(t) = \min_{x\in\overline{\Omega}} \alpha(x,t), \quad \hat{\xi}(t) = \max_{x\in\overline{\Omega}} \xi(x,t).$$

The constant m will be chosen large enough, in particular such that

$$36\hat{\alpha} > 33\alpha^* \text{ in } (0,T). \tag{9}$$

Consider the adjoint system of (7) which is given by

$$\begin{cases} \mathcal{L}_1^*\varphi + \nabla\pi = G_1, \;\; \nabla\cdot\varphi = 0 & \text{in} \quad Q, \\ \mathcal{L}_2^*\psi = \varphi e_N + G_2 & \text{in} \quad Q, \\ \varphi(x,t) = 0, \;\; \psi(x,t) = 0 & \text{on} \quad \Sigma, \\ \varphi(x,T) = \varphi^T(x), \;\; \psi(x,T) = \psi^T(x) & \text{in} \quad \Omega, \end{cases} \tag{10}$$

where $\mathcal{L}_1^*\varphi := -\varphi_t - \nu_0\Delta\varphi$, $\mathcal{L}_2^*\psi := -\psi_t - \nu_0\Delta\psi$, $\varphi^T \in H$, $\psi^T \in L^2(\Omega)$, $G_1 \in L^2(Q)^N$ and $G_2 \in L^2(Q)$.

Thus, we will present a Carleman estimate given by the following lemma:

Lemma 9 (Lemma 2, [7])

For any sufficiently large s and λ, there exists a positive constant C (depending on T, s and λ) such that, for all $\varphi^T \in H$ and $\psi^T \in L^2(\Omega)$ and any $G_1 \in L^2(Q)^N$ and $G_2 \in L^2(Q)$, the solution to (10) verifies

$$\|\varphi(.,0)\|^2 + \|\psi(.,0)\|^2 + \iint_Q e^{-2s\alpha}[\xi^3(|\varphi|^2 + |\psi|^2) + \xi(|\nabla\varphi|^2 + |\nabla\psi|^2)]dx$$

$$\leqslant C\left(\iint_{\omega\times(0,T)} e^{-8s\hat{\alpha}+6s\alpha^*}\hat{\xi}^{16}(|\varphi|^2 + |\psi|^2)dx\,dt + \iint_Q e^{-4s\hat{\alpha}+2s\alpha^*}\hat{\xi}^{15/2}(|G_1|^2 + |G_2|^2)dx\,dt\right). \quad (11)$$

We emphasize that two null controllability results will be obtained, since we will consider different cases for the initial data y^0, θ^0 and the functions F_1, F_2. More precisely, in the first case we will consider more common spaces in control theory, such as $H_0^1(\Omega)$ and $L^2(Q)$ while in the second case we will work with spaces less usual ones, like $W_0^{1,p}(\Omega)$ and $L^q(0,T;L^p(\Omega))$, for $3 < p \leqslant 6$ and $\frac{7}{3} \leqslant q < \infty$.
Let us set the following weights

$$\begin{cases} \rho = e^{s\alpha}\xi^{-3/2}, \ \rho_1 = e^{2s\hat{\alpha}-s\alpha*}\hat{\xi}^{-15/4}, \ \ \rho_2 = e^{4s\hat{\alpha}-3s\alpha^*}\hat{\xi}^{-8}, \\ \rho_3 = e^{s\alpha^*}(\xi^*)^{-1/2}, \ \mu_1 = e^{8s\hat{\alpha}-7s\alpha^*}\hat{\xi}^{-15}, \ \ \mu_2 = e^{8s\hat{\alpha}-7s\alpha^*}\hat{\xi}^{-16}, \\ \mu_3 = e^{8s\hat{\alpha}-7s\alpha^*}\hat{\xi}^{-17}, \ \kappa = e^{9s\hat{\alpha}-8s\alpha^*}\hat{\xi}^{-17}, \end{cases} \tag{12}$$

So that the values of s and λ satisfy the Lemma 9. By inequality (9), we can see that

$$\begin{cases} \kappa \leqslant C\mu_3 \leqslant C\mu_2 \leqslant C\rho_2 \leqslant C\rho_3 \leqslant C\mu_2^2, \\ |\mu_{2,t}| \leqslant C\rho_1, |\mu_3\mu_{3,t}| \leqslant C\mu_2^2 \text{ and } \kappa_t \leqslant C\mu_3 \text{ in } (0,T). \end{cases} \tag{13}$$

With Lemma 9 we will be able to obtain a null controllability result for (7), in which the right-hand side F_1 and F_2 decay sufficiently fast to zero as $t \to T$. In other words, the following propositions are valid:

Proposition 6.1

Let us assume that

- *if* $N = 2$*:* $y^0 \in H$, $\theta^0 \in L^2(\Omega)$, $\rho_3 F_1 \in L^2(Q)^2$ *and* $\rho_3 F_2 \in L^2(0, T; L^{3/2}(\Omega))$.
- *if* $N = 3$*:* $y^0 \in H \cap L^4(\Omega)^3$, $\theta^0 \in L^2(\Omega)$, $\rho_3 F_1 \in L^2(Q)^3$ *and* $\rho_3 F_2 \in L^2(0, T; L^{3/2}(\Omega))$.

Then, we can find state-controls (y, P, θ, v, v_0) *for (7) such that*

$$\iint\limits_{Q} \rho_1^2(|y|^2 + |\theta|^2)dx\,dt + \iint\limits_{\omega\times(0,T)} \rho_2^2(|v|^2 + |v_0|^2)dx\,dt \leqslant C\left(\|y^0\|_H^2 + \|\theta^0\|^2 + \|\rho_3 F_1\|^2_{L^2(Q)^N} + \|\rho_3 F_2\|^2_{L^2(0,T;L^{3/2}(\Omega))}\right). \tag{14}$$

In particular, one has $y(x, T) = 0$ *and* $\theta(x, T) = 0$*. Moreover, if* $(y^0, \theta^0) \in V \times W_0^{1,3/2}(\Omega)$ *then* $y \in L^2(0, T; V) \cap C^0([0, T]; H)$ *and* $\theta \in L^2(0, T; W^{2,3/2}(\Omega)) \cap C^0([0, T]; L^{3/2}(\Omega))$.

Proposition 6.2

Consider $3 < p \leqslant 6$ *and* $\frac{7}{3} \leqslant q < \infty$. *Let us assume that the functions* F_1, F_2 *in* (7) *satisfy* $\rho_3 F_1 \in L^q(0,T;L^p(\Omega)^N)$, $\rho_3 F_2 \in L^q(0,T;L^p(\Omega))$ *and* $(y^0,\theta^0) \in V^p \times W_0^{1,p}(\Omega)$. *Then* (7) *is null-controllable, and its control-state satisfy* $(v, v_0) \in L^2(\omega \times (0,T))^{N+1}$, $y \in L^q(0,T;W^{2,p}(\Omega)^N) \cap C^0([0,T];L^p(\Omega)^N)$ *and* $\theta \in L^q(0,T;W^{2,p}(\Omega)) \cap C^0([0,T];L^p(\Omega))$.

Estimates for the states solutions

Here we will show estimates for the solutions associated with (7), that is, for both the velocity variable and the temperature variable. We will obtain estimates not only for y and θ, but also for $\nabla y, \Delta y, \nabla\theta, \Delta\theta$ and the controls v and v_0. The results obtained in this subsection will be fundamental to obtain the null controllability of the nonlinear systems (1) and (3).

Proposition 6.3

Let the assumptions in Proposition 6.1 be satisfied. Let the state-control (y, P, θ, v, v_0) satisfy (7) and (14). Then, the following estimate holds:

$$\sup_{t\in[0,T]} \int_\Omega \mu_1^2 |y|^2 \, dx + \iint_Q \mu_1^2 |\nabla y|^2 \, dx \, dt$$
$$\leqslant C \left(\|y^0\|_H^2 + \iint_Q [\rho_3^2 |F_1|^2 + \rho_1^2 (|y|^2 + |\theta|^2)] \, dx \, dt + \iint_{\omega\times(0,T)} \rho_2^2 |v|^2 \, dx \, dt \right) \tag{15}$$

Furthermore, if $(y^0, \theta^0) \in V \times W_0^{1,3/2}(\Omega)$, one also has

$$\sup_{t\in[0,T]} \int_\Omega \mu_2^2 |\nabla y|^2 \, dx + \iint_Q \mu_2^2 (|y_t|^2 + |\Delta y|^2) \, dx \, dt$$
$$\leqslant C \left(\|y^0\|_V^2 + \iint [\rho_3^2 |F_1|^2 + \rho_1^2 (|y|^2 + |\theta|^2)] \, dx \, dt + \iint \rho_2^2 |v|^2 \, dx \, dt \right.$$

Continuation Proposition 6.3
and

$$\int_0^T \mu_2^2 \|\theta_t\|^2_{L^{3/2}(\Omega)}\, dt + \int_0^T \mu_2^2 \|\Delta\theta\|^2_{L^{3/2}(\Omega)}\, dt$$
$$\leqslant C \left(\|\theta^0\|^2_{W_0^{1,3/2}(\Omega)} + \iint_Q \rho_1^2 |\theta|^2\, dx\, dt + \iint_{\omega\times(0,T)} \rho_2^2 |v_0|^2\, dx\, dt \right. \tag{17}$$
$$\left. + \int_0^T \|\rho_3 F_2\|^2_{L^{3/2}(\Omega)}\, dt \right).$$

Proposition 6.4

Let the assumptions in Proposition 6.2 be satisfied. Then, the controls verifies

$$\kappa v \in L^2(0,T;H^2(\omega)^N) \cap C^0([0,T];H^1(\omega)^N), \ (\kappa v)_t \in L^2(\omega \times (0,T))^N. \quad (18)$$

$$\kappa v_0 \in L^2(0,T;H^2(\omega)) \cap C^0([0,T];H^1(\omega)), \ (\kappa v_0)_t \in L^2(\omega \times (0,T)). \quad (19)$$

with the estimate

$$\int_0^T \int_\omega \left[|(\kappa v)_t|^2 + |(\kappa v_0)_t|^2 + |\kappa \Delta v|^2 + |\kappa \Delta v_0|^2 \right] dxdt + \sup_{[0,T]} \|\kappa v\|^2_{H^1(\omega)}$$
$$+ \sup_{[0,T]} \|\kappa v_0\|^2_{H^1(\omega)} \leqslant C \left(\|y^0\|^2_{V^p} + \|\theta^0\|^2_{W_0^{1,p}(\Omega)} + \|\rho_3 F_1\|^2_{L^q(0,T;L^p(\Omega)^N)} \right.$$
$$\left. + \|\rho_3 F_2\|^2_{L^q(0,T;L^p(\Omega))} \right).$$

Continuation Proposition 6.4
Furthermore, the associated states satisfy

$$\iint\limits_{Q} \mu_3^2 |y_t|^2 dx\,dt + \sup_{[0,T]} \int_{\Omega} \mu_3^2 |\nabla y|^2 dx + \iint\limits_{Q} \mu_3^2 |\Delta y|^2 dx\,dt$$
$$+ \sup_{[0,T]} \int_{\Omega} \mu_2^2 |y|^2 dx + \iint\limits_{Q} \mu_2^2 |\nabla y|^2 dx\,dt \leqslant C \Big(\|y^0\|_{V^p}^2$$
$$+ \iint\limits_{Q} \rho_1^2 (|\theta|^2 + |y|^2)\, dx\,dt + \iint\limits_{\omega \times (0,T)} \rho_2^2 |v|^2\, dx\,dt + \|\rho_3 F_1\|^2_{L^q(0,T;L^p(\Omega)^N)} \Big) \tag{20}$$

Continuation Proposition 6.4
and

$$\iint\limits_{Q} \mu_3^2|\theta_t|^2 dx\,dt + \sup_{[0,T]}\int_\Omega \mu_3^2|\nabla\theta|^2 dx + \iint\limits_{Q} \mu_3^2|\Delta\theta|^2 dx\,dt + \sup_{[0,T]}\int_\Omega \mu_2^2|\theta|^2 c$$

$$+ \iint\limits_{Q} \mu_2^2|\nabla\theta|^2 dx\,dt \leqslant C \left(\|\theta^0\|^2_{W_0^{1,p}(\Omega)} + \iint\limits_{Q} \rho_1^2|\theta|^2\, dx\,dt \right.$$

$$\left. + \iint\limits_{\omega\times(0,T)} \rho_2^2|v_0|^2\, dx\,dt + \|\rho_3 F_2\|^2_{L^q(0,T;L^p(\Omega))} \right). \tag{21}$$

The next result is a proposition from [8] and will be of great importance for us to conclude our main theorems.

Proposition 6.5

If $u \in L^q(0,T;W^{2,p}(\Omega))$, $u_t \in L^q(0,T;L^p(\Omega))$ *then* $u \in C^0([0,T];W^{1,p}(\Omega))$, $p > 2$ *and* $q > \max\{2, \frac{3p-2}{p}\}$.

The following proposition will be fundamental to guarantee the null controllability of the system (3) and its proof is acquired from the previous results of this section.

Proposition 6.6

Let the assumptions in Proposition 6.4 be satisfied. Then, the following estimates are valid

$$\|(\kappa\theta)_t\|_{L^q(0,T;L^p(\Omega))} + \|\kappa\theta\|_{L^q(0,T;W^{2,p}(\Omega))} + \|\kappa\theta\|_{C^0(0,T;W^{1,p}(\Omega))}$$
$$\leqslant C\left(\|y^0\|_{V^p} + \|\theta^0\|_{W_0^{1,p}(\Omega)} + \|\rho_3 F_1\|_{L^q(0,T;L^p(\Omega)^N)} + \|\rho_3 F_2\|_{L^q(0,T;L^p(\Omega))}\right) \quad (22)$$

and

$$\|(\kappa y)_t\|_{L^q(0,T;L^p(\Omega)^N)} + \|\kappa y\|_{L^q(0,T;W^{2,p}(\Omega)^N)} + \|\kappa y\|_{C^0(0,T;W^{1,p}(\Omega)^N)}$$
$$\leqslant C\left(\|y^0\|_{V^p} + \|\theta^0\|_{W_0^{1,p}(\Omega)} + \|\rho_3 F_1\|_{L^q(0,T;L^p(\Omega)^N)} + \|\rho_3 F_2\|_{L^q(0,T;L^p(\Omega))}\right) \quad (23)$$

Proof of Theorem 2

We will proved the local null controllability for the system (1). Let us consider the Stokes operator $A : D(A) \to H$, where $D(A) := V \cap H^2(\Omega)^N$, $Aw = P(-\Delta w)$ for all $w \in D(A)$ and $P : L^2(\Omega)^N \to H$ is the orthogonal projector.
Let $\mathcal{E}_N$ be (for $N = 2$ or $N = 3$) the following space:

$$\begin{aligned}
&\mathcal{E}_N = \{(y, P, \theta, v, v_0) : \rho_1 y, \rho_2 v\tilde{1}_\omega \in L^2(Q)^N, y \in L^2(0, T; D(A)),\\
&\nabla y \in L^2(Q)^{N\times N}, P \in L^2(0, T; H^1(\Omega)), \rho_1\theta, \rho_2 v_0\tilde{1}_\omega \in L^2(Q),\\
&\theta \in L^2(0, T; W^{2,3/2}(\Omega)), \text{for } F_1 := \mathcal{L}_1 y + \nabla P - \nu_0\theta e_N - v\tilde{1}_\omega \text{ and}\\
&F_2 := \mathcal{L}_2\theta - v_0\tilde{1}_\omega, \rho_3 F_1 \in L^2(Q)^N, \rho_3 F_2 \in L^2(0, T; L^{3/2}(\Omega)), \nabla \cdot y \equiv 0,\\
&y(.,0) \in V, \theta(.,0) \in W_0^{1,3/2}(\Omega), \theta\,|_\Sigma = 0\},
\end{aligned} \tag{24}$$

emphasizing that $\mathcal{L}_1 y = y_t - \nu_0\Delta y$ and $\mathcal{L}_2\theta = \theta_t - \nu_0\Delta\theta$. Thus, it's clear that $\mathcal{E}_N$ is a Banach space for the norm $\|.\|_{\mathcal{E}_N}$, where

$$
\begin{aligned}
\|(y,P,\theta,v,v_0)\|^2_{\mathcal{E}_N} &:= \|y\|^2_{L^2(0,T;D(A))} + \|\theta\|^2_{L^2(0,T;W^{2,3/2}(\Omega))} + \|\rho_1 y\|^2_{L^2(Q)^N} \\
&+\|\rho_1\theta\|^2_{L^2(Q)} + \|P\|^2_{L^2(0,T;H^1(\Omega))} + \|\rho_2 v\|^2_{L^2(\omega\times(0,T))^N} + \|\rho_2 v_0\|^2_{L^2(\omega\times(0,T))} \\
&+\|\rho_3 F_1\|^2_{L^2(Q)^N} + \|\rho_3 F_2\|^2_{L^2(0,T;L^{3/2}(\Omega))} + \|\theta(.,0)\|^2_{W_0^{1,3/2}(\Omega)}.
\end{aligned}
$$

Due to Proposition 6.3 we get:

$$
\begin{aligned}
&\|\mu_1 y\|_{L^\infty(0,T;H)} + \|\mu_1 y\|_{L^2(0,T;V)} + \|\mu_2 y\|_{L^\infty(0,T;V)} + \|\mu_2 y\|_{L^2(0,T;D(A))} \\
&+ \|\mu_2 y_t\|_{L^2(0,T;L^2(\Omega)^N)} + \|\mu_2\theta_t\|_{L^2(0,T;L^{3/2}(\Omega))} + \|\mu_2\theta\|_{L^2(0,T;W^{2,3/2}(\Omega))} \\
&\leqslant C\|(y,P,\theta,v,v_0)\|_{\mathcal{E}_N}.
\end{aligned}
\tag{25}
$$

Furthermore, if $(y,P,\theta,v,v_0)\in\mathcal{E}_N$, then $y_t\in L^2(Q)^N$, whence $y:[0,T]\to V$ is continuous (see, [4]) and we have $y(.,0)\in V$, with

$$
\|y(.,0)\|_V \leqslant C\|(y,P,\theta,v,v_0)\|_{\mathcal{E}_N}, \tag{26}
$$

Now, let us introduce the Banach space

$$\mathcal{Z}_N = L^2(\rho_3^2; Q)^N \times V \times L^2(\rho_3^2(0,T); L^{3/2}(\Omega)) \times W_0^{1,3/2}(\Omega), \tag{27}$$

where $L^2(\rho_3^2(0,T); L^{3/2}(\Omega))$ be the Banach space formed by the measurable functions $u = u(x;t)$ such that $\rho_3 u \in L^2(0,T; L^{3/2}(\Omega))$. Finally, consider also the mapping $\mathcal{F} : \mathcal{E}_N \to \mathcal{Z}_N$, such that

$$\mathcal{F}(y, P, \theta, v, v_0) = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4)(y, P, \theta, v, v_0) \tag{28}$$

where

$$\begin{cases} \mathcal{F}_1(y, P, \theta, v, v_0) := y_t - \nu(\nabla y)\Delta y + (y \cdot \nabla)y + \nabla P - \nu_0 \theta e_N - v\tilde{1}_\omega, \\ \mathcal{F}_2(y, P, \theta, v, v_0) := y(.,0), \\ \mathcal{F}_3(y, P, \theta, v, v_0) := \theta_t - \nu(\nabla y)\Delta\theta + y \cdot \nabla\theta - \nu(\nabla y)Dy : \nabla y - v_0\tilde{1}_\omega, \\ \mathcal{F}_4(y, P, \theta, v, v_0) := \theta(.,0). \end{cases} \tag{29}$$

Note that, in $(29)_1$ we used the definition of $\nabla \cdot (\nu(\nabla y)Dy)$ to rewrite in the form $\nu(\nabla y)\Delta y$, since $\nabla \cdot y = 0$.

Theorem 2 will be proven if we show that there is $\delta > 0$ such that, $(F_1, y^0, F_2, \theta^0) \in \mathcal{Z}_N$ and $||(F_1, y^0, F_2, \theta^0)||_{\mathcal{Z}N} < \delta$ then the equation,

$$\mathcal{F}(y, P, \theta, v, v_0) = (F_1, y^0, F_2, \theta^0) \ , \ (y, P, \theta, v, v_0) \in \mathcal{E}_N$$

possesses at least one solution.

We are interested in apply the Mapping Inverse Theorem in infinite dimensional spaces, that can be found in [1], and is given below, where $B_r(0)$ and $B_\delta(\zeta_0)$ are open ball, respectively of radius r and δ.

Theorem 10 (Mapping Inverse Theorem)

Let $\mathcal{E}$ and $\mathcal{Z}$ be Banach spaces and let $\mathcal{F} : B_r(0) \subset \mathcal{E} \to \mathcal{Z}$ be a $\mathcal{C}^1$ mapping. Let as assume that $\mathcal{F}'(0)$ is onto and let us set $\mathcal{F}(0) = \zeta_0$. Then, there exist $\delta > 0$, a mapping $W : B_\delta(\zeta_0) \subset \mathcal{Z} \to \mathcal{E}$ and a constant $K > 0$ such that

$$W(z) \in B_r(0), \ \mathcal{F}(W(z)) = z \text{ and } \|W(z)\|_{\mathcal{E}} \leqslant K\|z - \mathcal{F}(0)\|_{\mathcal{Z}} \ \forall\, z \in B_\delta(\zeta_0).$$

In particular, W is a local inverse-to-the-right of $\mathcal{F}$.

Thus, we will prove that we can apply this Theorem 10 to the mapping $\mathcal{F}$ in (28)-(29), through the following three lemmas:

Lemma 11

Let $\mathcal{F} : \mathcal{E}_N \to \mathcal{Z}_N$ be given by (28)-(29). Then, $\mathcal{F}$ is well defined and continuous.

proof. We using that elements $(y, P, \theta, v, v_0) \in \mathcal{E}_{\mathcal{N}}$ have regularity of the linear problem.

Lemma 12

The mapping $\mathcal{F} : \mathcal{E}_N \longrightarrow \mathcal{Z}_N$ is continuously differentiable.

proof. We will the proof for $N = 3$ (the case $N = 2$ is similar). Let us first prove that $\mathcal{F}$ is Gâteaux-differentiable at any $(y, P, \theta, v, v_0) \in \mathcal{E}_3$ and let us compute the *G-derivative* $\mathcal{F}'(y, P, \theta, v, v_0)$.

Let us fix $(y, P, \theta, v, v_0) \in \mathcal{E}_3$ and let us take $(y', P', \theta', v', v_0') \in \mathcal{E}_3$ and $\sigma > 0$. Also, by the decomposition made in (29), we introduce the linear mapping $\mathcal{DF} : \mathcal{E}_3 \longrightarrow \mathcal{Z}_3$ with

$\mathcal{DF}(y, P, \theta, v, v_0) = \mathcal{DF} = (\mathcal{DF}_1, \mathcal{DF}_2, \mathcal{DF}_3, \mathcal{DF}_4)$ where

$$\begin{cases} \mathcal{DF}_1(y', P', \theta', v', v_0') := y_t' - \nu(\nabla y)\Delta y' - 2\nu_1(\nabla y, \nabla y')\Delta y + \nabla P' \\ \qquad\qquad + (y' \cdot \nabla)y + (y \cdot \nabla)y' - \theta' e_3 - v' \tilde{1}_\omega, \\ \mathcal{DF}_2(y', P', \theta', v', v_0') := y'(., 0), \\ \mathcal{DF}_3(y', P', \theta', v', v_0') := \theta_t' - \nu(\nabla y)\Delta\theta' - 2\nu_1(\nabla y, \nabla y')\Delta\theta \\ \qquad\qquad + y' \cdot \nabla\theta + y \cdot \nabla\theta' - v_0' \tilde{1}_\omega - \nu(\nabla y) Dy : \nabla y' \\ \qquad\qquad - [\nu(\nabla y) Dy' + 2\nu_1(\nabla y, \nabla y') Dy] : \nabla y, \\ \mathcal{DF}_4(y', P', \theta', v', v_0') := \theta'(., 0). \end{cases}$$

From the definition of the spaces $\mathcal{E}_3, \mathcal{Z}_3$ and (30), it becomes clear that $\mathcal{D}\mathcal{F} \in \mathcal{L}(\mathcal{E}_3, \mathcal{Z}_3)$. Furthermore, for each $j = \{1, 2, 3, 4\}$ we have

$$\frac{1}{\sigma}[\mathcal{F}_j\left((y, P, \theta, v, v_0) + \sigma(y', P', \theta', v', v_0')\right) - \mathcal{F}_j(y, P, \theta, v, v_0)] \quad \text{converges to } \mathcal{D}\mathcal{F}_j(y', P', \theta', v', v_0') \text{ strong in } \mathcal{Z}_3, \text{ as } \sigma \longrightarrow 0. \tag{31}$$

By the same arguments as the previous lemma it is possible to show that (31) is true.

It is also easily proven that $(y, P, \theta, v, v_0) \longmapsto \mathcal{F}'(y, P, \theta, v, v_0)$ is a continuous mapping. Thus, it is shown that $\mathcal{F}$ is not only Gâteaux-differentiable, but also Fréchet-differentiable. For that, suppose that

$$(y_m, P_m, \theta_m, v_m, v_{0m}) \longrightarrow (y, P, \theta, v, v_0) \text{ in } \mathcal{E}_3$$

and there is existence of $\varepsilon_m(y, P, \theta, v, v_0)$ such that

$$\begin{aligned} &\| \left(\mathcal{F}'(y_m, P_m, \theta_m, v_m, v_{0m}) - \mathcal{F}'(y, P, \theta, v, v_0)\right)(y', P', \theta', v', v_0')\|^2_{\mathcal{Z}_3} \\ &\qquad \leqslant \varepsilon_m \|(y', P', \theta', v', v_0')\|^2_{\mathcal{E}_3}, \end{aligned} \tag{32}$$

for all $(y', P', \theta', v', v_0') \in \mathcal{E}_3$ and $\lim_{m\to\infty} \varepsilon_m = 0$. $\square$

Lemma 13

Let $\mathcal{F}$ be the mapping in (28)-(29). Then, $\mathcal{F}'(0,0,0,0,0)$ is onto.

proof. Let $(F_1, y^0, F_2, \theta^0) \in \mathcal{Z}_N$. From Proposition 6.1 we know there exists (y, P, θ, v, v_0) satisfying (7) and (14). Furthermore, from the usual regularity results for the Stokes system we have $(y, P) \in (L^2(0, T; D(A)) \times L^2(0, T; H^1(\Omega)))$. Consequently, $(y, P, \theta, v, v_0) \in \mathcal{E}_N$ and

$$\mathcal{F}'(0,0,0,0,0)(y, P, \theta, v, v_0) = (F_1, y^0, F_2, \theta^0).$$

□

Proof of Theorem 2. We conclude from Lemmas 11-13 that the Inverse Mapping Theorem (Theorem 10) can be applied to the spaces $\mathcal{E}_N$ and $\mathcal{Z}_N$ together with the mapping $\mathcal{F}$ introduced at the beginning of this Section. Thus, there exists $\delta > 0$ such that, for every $(y^0, \theta^0) \in V \times W_0^{1,3/2}(\Omega)$ satisfying $\|(y^0, \theta^0)\|_{V \times W_0^{1,3/2}} < \delta$.

We have that $(y, P, \theta, v, v_0) = W(0, y^0, 0, \theta^0)$ is the sought solution, as it satisfies: $\mathcal{F}(y, P, \theta, v, v_0) = (0, y^0, 0, \theta^0)$.

This proves that, the nonlinear system (1) is locally null-controllable at time $T > 0$.

Proof of the Theorem 3

Let

$$\begin{aligned}
\mathcal{U}_N = & (y, P, \theta, v, v_0) : \rho_1 y \in L^2(Q)^N, \rho_2 v, (\kappa v)_t, \kappa\Delta v \in L^2(\omega \times (0,T))^N, \\
& y \in L^q(0,T; W^{2,p}(\Omega)^N), \nabla y \in L^2(Q)^{N\times N}, P \in L^q(0,T; L^p(\Omega)), \\
& \rho_1\theta \in L^2(Q), \rho_2 v_0, (\kappa v_0)_t, \kappa\Delta v_0 \in L^2(\omega \times (0,T)), \theta \in L^q(0,T; W^{2,p}(\Omega)), \\
& \text{for } F_1 := \mathcal{L}_1 y + \nabla P - \nu_0 \theta e_N - v\tilde{1}_\omega \text{ and } F_2 := \mathcal{L}_2\theta - v_0\tilde{1}_\omega, \\
& \rho_3 F_1 \in L^q(0,T; L^p(\Omega)^N), \rho_3 F_2 \in L^q(0,T; L^p(\Omega)), \nabla\cdot y \equiv 0, y(.,0) \in V^p, \\
& \theta(.,0) \in W_0^{1,p}(\Omega), y\,|_\Sigma = 0, \theta\,|_\Sigma = 0, \text{where } 3 < p \leqslant 6 \text{ and } \tfrac{7}{3} \leqslant q < \infty\},
\end{aligned} \tag{33}$$

It's clear that $\mathcal{U}_N$ is a Banach space for the norm $\|.\|_{\mathcal{U}_N}$, with

$$\begin{aligned}
\|(y, P, \theta, v, v_0)\|^q_{\mathcal{U}_N} = & \|y\|^q_{L^2(0,T;W^{2,p}(\Omega)^N)} + \|\theta\|^q_{L^q(0,T;W^{2,p}(\Omega))} + \|\rho_1 y\|^q_{L^2(Q)^N} \\
& + \|\rho_1\theta\|^q_{L^2(Q)} + \|P\|^q_{L^q(0,T;L^p(\Omega))} + \|\rho_2 v\|^q_{L^2(\omega\times(0,T))^N} + \|\rho_2 v_0\|^q_{L^2(\omega\times(0,T))} \\
& + \|(\kappa v)_t\|^q_{L^2(\omega\times(0,T))^N} + \|\kappa\Delta v\|^q_{L^2(\omega\times(0,T))^N} + \|(\kappa v_0)_t\|^q_{L^2(\omega\times(0,T))} \\
& + \|\kappa\Delta v_0\|^q_{L^2(\omega\times(0,T))} + \|\rho_3 F_1\|^q_{L^q(0,T;L^p(\Omega)^N)} + \|\rho_3 F_2\|^q_{L^q(0,T;L^p(\Omega))} \\
& + \|y(.,0)\|^q_{V^p} + \|\theta(.,0)\|^q_{W_0^{1,p}(\Omega)}.
\end{aligned}$$

Now, let us introduce the Banach space

$$\mathcal{R}_N := L^q(\rho_3^q(0,T); L^p(\Omega)^N) \times V^p \times L^q(\rho_3^q(0,T); L^p(\Omega)) \times W_0^{1,p}(\Omega), \quad (34)$$

and the mapping $\mathcal{I} : \mathcal{U}_N \to \mathcal{R}_N$, such that

$$\mathcal{I}(y, P, \theta, v, v_0) = (\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4)(y, P, \theta, v, v_0) \quad (35)$$

where

$$\begin{cases} \mathcal{I}_1(y, P, \theta, v, v_0) := y_t - \bar{\nu}(\nabla y)\Delta y + (y \cdot \nabla)y + \nabla P - \nu_0 \theta e_N - v\tilde{1}_\omega, \\ \mathcal{I}_2(y, P, \theta, v, v_0) := y(.,0), \\ \mathcal{I}_3(y, P, \theta, v, v_0) := \theta_t - \bar{\nu}(\nabla\theta)\Delta\theta + y \cdot \nabla\theta - \bar{\nu}(\nabla y)Dy : \nabla y - v_0\tilde{1}_\omega, \\ \mathcal{I}_4(y, P, \theta, v, v_0) := \theta(.,0). \end{cases} \quad (36)$$

Theorem 3 will be proven if we show that there is $\delta > 0$ such that, $(F_1, y^0, F_2, \theta^0) \in \mathcal{R}_N$ and $||(F_1, y^0, F_2, \theta^0)||_{\mathcal{R}N} < \delta$ then the equation,

$$\mathcal{I}(y, P, \theta, v, v_0) = (F_1, y^0, F_2, \theta^0) \ , \ (y, P, \theta, v, v_0) \in \mathcal{U}_N$$

possesses at least one solution.

To simplify the notation, in the norms of $L^p(\Omega)^N$ we will just write $L^p(\Omega)$. That said, we have the following results:

Lemma 14

Let $\mathcal{I} : \mathcal{U}_N \to \mathcal{R}_N$ be given by (35)-(36). Then, $\mathcal{I}$ is well defined and continuous.

proof. Let's prove that, for each $(y, P, \theta, v, v_0) \in \mathcal{U}_N$ we have $\mathcal{I}(y, P, \theta, v, v_0) \in \mathcal{R}_N$.
That $\mathcal{I}_2$ and $\mathcal{I}_4$ are well defined follows immediately from the definition of $\mathcal{U}_N$. So let's find out $\mathcal{I}_1$ and $\mathcal{I}_3$.
Analysis of $\mathcal{I}_1$:

- $$\|\rho_3 F_1\|^q_{L^q(0,T;L^p(\Omega))} \leqslant C\|(y, P, \theta, v, v_0)\|^q_{\mathcal{U}_N}.$$

Taking into account (9) we have $\rho_3\kappa^{-2} \leqslant C$. Moreover, using the fact that $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ (since $p > N$) and the estimate (23) from the Proposition 6.6, we obtain

$$\bullet \|\rho_3(y\cdot\nabla)y\|^q_{L^q(0,T;L^p(\Omega))} \leqslant \int_0^T \left(\int_\Omega \rho_3^p |y|^p|\nabla y|^p dx\right)^{q/p} dt$$

$$= \int_0^T \left(\int_\Omega \rho_3^p \kappa^{-2p}\kappa^{2p}|y|^p|\nabla y|^p dx\right)^{q/p} dt$$

$$\leqslant C\int_0^T \|\kappa y\|^q_{L^\infty(\Omega)} \left(\int_\Omega |\kappa\nabla y|^p dx\right)^{q/p} dt$$

$$\leqslant C\|\kappa y\|^q_{L^\infty(0,T;W^{1,p}(\Omega))}\|\kappa y\|^q_{L^q(0,T;W^{1,p}(\Omega))} \leqslant C\|(y,P,\theta,v,v_0)\|^{2q}_{\mathcal{U}_N}.$$

In a similar way

$$\bullet \|\rho_3\nu_1\|\nabla y\|^2_{L^p}\Delta y\|^q_{L^q(0,T;L^p(\Omega))} \leqslant C\int_0^T \rho_3^q\kappa^{-3q}\|\kappa\nabla y\|^{2q}_{L^p(\Omega)}\|\kappa\Delta y\|^q_{L^p(\Omega)} dt$$

$$\leqslant C\|\kappa y\|^{2q}_{L^\infty(0,T;W^{1,p}(\Omega))}\|\kappa y\|^q_{L^q(0,T;W^{2,p}(\Omega))} \leqslant C\|(y,P,\theta,v,v_0)\|^{3q}_{\mathcal{U}_N}.$$

Hence, $\mathcal{I}_1(y,P,\theta,v,v_0) \in L^q(\rho_3^q(0,T);L^p(\Omega)^N)$.

Analysis of $\mathcal{I}_3$:

$$\bullet \, \|\rho_3 F_2\|^q_{L^q(0,T;L^p(\Omega))} \leqslant \|(y,P,\theta,v,v_0)\|^q_{\mathcal{U}_N}.$$

Using the same previous arguments together with the estimates (22) and (23) from the Proposition 6.6, we get

$$\bullet \, \|\rho_3 y \cdot \nabla\theta\|^q_{L^q(0,T;L^p(\Omega))} \leqslant C \int_0^T \|\kappa\nabla\theta\|^q_{L^\infty(\Omega)} \left(\int_\Omega |\kappa y|^p dx \right)^{q/p} dt$$
$$\leqslant C \int_0^T \|\kappa\nabla\theta\|^q_{W^{1,p}(\Omega)} \|\kappa y\|^q_{L^p(\Omega)} dt$$
$$\leqslant C \|\kappa y\|^q_{L^\infty(0,T;W^{1,p}(\Omega))} \|\kappa\theta\|^q_{L^q(0,T;W^{2,p}(\Omega))}$$
$$\leqslant C \|(y,P,\theta,v,v_0)\|^{2q}_{\mathcal{U}_N};$$

$$\bullet \, \|\rho_3 \nu_1 \|\nabla\theta\|^2_{L^p} \Delta\theta\|^q_{L^q(0,T;L^p(\Omega))} \leqslant C \int_0^T \rho_3^q \kappa^{-3q} \|\kappa\nabla\theta\|^{2q}_{L^p(\Omega)} \|\kappa\Delta\theta\|^q_{L^p(\Omega)} dt$$
$$\leqslant C \|\kappa\theta\|^{2q}_{L^\infty(0,T;W^{1,p}(\Omega))} \|\kappa\theta\|^q_{L^q(0,T;W^{2,p}(\Omega))} \leqslant C \|(y,P,\theta,v,v_0)\|^{2q}_{\mathcal{U}_N};$$

and

- $\|\rho_3\bar{\nu}(\nabla y)Dy : \nabla y\|^q_{L^q(0,T;L^p(\Omega))} \leqslant C\|\kappa y\|^q_{L^\infty(0,T;W^{1,p}(\Omega))}\|\kappa y\|^q_{L^q(0,T;W^{1,p}(\Omega))}$
$+ C\|\kappa y\|^{3q}_{L^\infty(0,T;W^{1,p}(\Omega))}\|\kappa y\|^q_{L^q(0,T;W^{1,p}(\Omega))} \leqslant C\|(y, P, \theta, v, v_0)\|^{4q}_{\mathcal{U}_N}$.

Consequently we have $\mathcal{I}_3(y, P, \theta, v, v_0) \in L^q(\rho_3^q(0, T); L^p(\Omega)^N)$.
This proves the Lemma 14.

Based on the arguments applied in the Lemma 14, the next two results follow in a similar way to the Lemmas 12, 13, respectively.

Lemma 15

The mapping $\mathcal{I} : \mathcal{U}_N \longrightarrow \mathcal{R}_N$ is continuously differentiable.

Lemma 16

Let $\mathcal{I}$ be the mapping in (35)-(36). Then, $\mathcal{I}'(0,0,0,0,0)$ is onto.

According to Lemmas 14-16, we can apply the Inverse Mapping Theorem (Theorem 10), then, there exists $\delta > 0$ and a mapping $W : B_\delta(0) \subset \mathcal{R}_N \to \mathcal{U}_N$ such that

$$W(z) \in B_r(0) \text{ and } \mathcal{I}(W(z)) = z, \ \ \forall z \in B_\delta(0).$$

Taking $(0, y^0, 0, \theta^0) \in B_\delta(0)$ and $(y, P, \theta, v, v_0) = W(0, y^0, 0, \theta^0) \in \mathcal{U}_N$, we have

$$\mathcal{I}(y, P, \theta, v, v_0) = (0, y^0, 0, \theta^0).$$

Thus, we conclude that (3) is locally null controllable at time $T > 0$.

Large time Null-Controllability

Following the ideas of [2, 9], we will make the system (1) evolve without control and certify an asymptotic behavior according to $t \to \infty$ of its solutions, when $N = 2$. That is, we will deal with the energy decay of the solutions of the system complete Ladyzhenskaya-Boussinesq. Having verified this analysis, we will take a time $T^* > 0$ such that the solutions $y(T^*, .)$ and $\theta(T^*, .)$ related to the null local controllability of (1) (Theorem 2). Thus, by setting $y(T^*, .)$ and $\theta(T^*, .)$ as the initial data in (1), Theorem 2 gives us the v and v_0 controls that drive the solutions to zero in some sufficiently large time.

Accordingly we state the following lemma, which will be fundamental for the demonstration of Theorem 4.

Lemma 17

For $N = 2$, any $T > 0$ and $(y^0, \theta^0) \in V \times H_0^1(\Omega)$, if there is positive constant $r > 0$ such that

$$\|(y^0, \theta^0)\|_{V \times H_0^1(\Omega)} < r$$

and (y, p, θ) is a solution of (1) with $v \equiv v_0 \equiv 0$, so this solution has asymptotic behavior as $t \to \infty$. More precisely, for

$$E(t) := \|\nabla y(t,.)\|^2 + \|\theta(t,.)\|^2 + \|\nabla \theta(t.,)\|^2$$

there are positive constants $\mathrm{C}_1, \mathrm{C}_2$ such that

$$E(t) \leqslant \mathrm{C}_2\, e^{-\mathrm{C}_1 t} E(0) \text{ a.e in } (0, T). \tag{37}$$

Proof of Theorem 4 First, let's fix $T_0 > 0$. Applying the Theorem 2 there exists $\delta > 0$ such that the system (1), with any initial data $(\bar{y}^0, \bar{\theta}^0) \in V \times W_0^{1,3/2}(\Omega)$ satisfying $\|(\bar{y}^0, \bar{\theta}^0)\|_{V \times W_0^{1,3/2}(\Omega)} < \delta$, is locally null controllable at T_0.
Determine $(y^0, \theta^0) \in V \times H_0^1(\Omega)$ and consider $r > 0$ as defined in the statement of Lemma 17. Let then T^* be a positive time satisfying

$$T^* > \frac{-1}{C_1} \ln \left(\frac{\delta}{\mathrm{C}_2(\|\nabla y^0\|^2 + \|\theta^0\|^2 + \|\nabla\theta^0\|^2)} \right) \tag{38}$$

and consider a solution (y, p, θ) of the system (1), with $T = T^* + T_0$, $v \equiv v_0 \equiv 0$ and (y^0, θ^0) as the initial data.
From (37) and (38), $y(T^*, .)$, $\theta(T^*, .)$ are such that

$$\begin{aligned} \|\,(y(T^*, .), \theta(T^*, .))\,\|_{V \times W_0^{1,3/2}(\Omega)} &\leqslant \mathrm{C}_2\, e^{-\mathrm{C}_1 T^*}(\|\nabla y^0\|^2 + \|\theta^0\|^2 + \|\nabla\theta^0\|^2) \\ &< \delta. \end{aligned}$$

Consequently, by Theorem 2, (1) is null controllable at $T^* + T_0$.

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