جامعة خليفة
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Local Volatility Estimation in the Presence of Jumps

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Trends in Mathematical Sciences
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Thanks Enrique ZUAZUA

Motivation for Modeling Derivative Markets

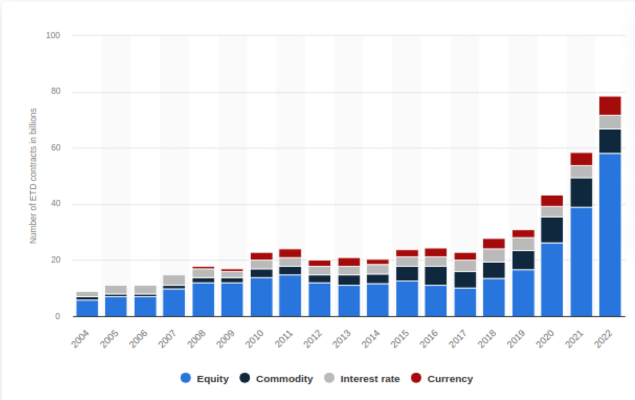


Figure: Number of exchange traded derivatives (ETDs) traded worldwide from 2004 to 2022, by type(in billions) Source: Statista Research Department

Motivation for Modeling Derivative Markets I

Fact

- Top banks in the United States and Europe have tremendous exposure to derivatives: Notional value of $632 trillion at end-june 2022
- The gross market value of outstanding OTC derivatives, summing positive and negative values, rose noticeably in the first half of 2022, to $ 18.3 trillion,
- Deutsche Bank alone had more than 90 $\times 10^9$ Euros Potential future exposure of derivative contracts (2020)

Source: Bank of International Settlements (BIS) and European Bank Association (EBA)

Motivation for Modeling Derivative Markets II

Consequence

Options and Derivatives are a fundamental part of the world economics.

Futures and Energy Markets

- Futures are a type of derivative contract agreement to buy or sell a specific commodity asset or security at a set future date for a set price.
- Futures are crucial in commodities and energy trading
- Imense markets
- Interconnected with fixed income markets and security markets
- Interconnected with currency markets (especially with crypto-currencies)

Motivation for Modeling Derivative Markets I

What are the basic underlying structures?

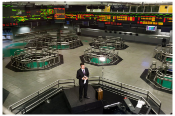
- In Mathy language: DIFFUSIVE PROCESSES, THEIR EXPECTED VALUES, AND CONTROL
- In AI/ML language: Model estimation and selection
- In Financial language: Find the risk premium associated to a derivative contract and the corresponding hedging portfolio.

Motivation for Modeling Derivative Markets II

Main difficulties

Financial Data is highly complex phenomena!

- **Huge noisy data sets**
- **Nongaussian behavior ... in fact: Nonstationary behavior**
- **Volatity is complex and nonstationary phenomena**
- **Design models and understand the complex phenomena**



Financial Models for Options & Derivatives I

1973 - 2023 Golden Jubilee of the Seminal Papers by Black-Scholes and by Merton

- Problem: How to compute the price of a vanila option (call or a put)?
- Method: Introduce a hedging portfolio composed of cash and the underlying asset. Define a model for the underlying and introduce a number of simplifying assumptions.
- Result: A “simple” way of relating the evolution of the option price with its variation with respect to the underlying price. More precisely with the so called “delta” and “gamma” of the option.
- Impact: A fair and robust (albeit very simplistic) way of computing the price of calls and puts.

Financial Models for Options & Derivatives II

Fischer Black

(credit Wikipedia)

Robert Merton - Nobel 1997

(credit Wikipedia)

Myron Scholes - Nobel 1997

(credit Wikipedia)

Financial Models for Options & Derivatives III

1997 Nobel Prize Press release

"Robert C. Merton and Myron S. Scholes have, in collaboration with the late Fischer Black, developed a pioneering formula for the valuation of stock options. Their methodology has paved the way for economic valuations in many areas. It has also generated new types of financial instruments and facilitated more efficient risk management in society "

Challenge

- "Be able to prove that the method works"
- "trustworthness "

Reference: June 2023 *Nature* paper by Blanka Horvath... " Golden jubilee for an iconic financial formula"

Financial Models for Options & Derivatives IV

Black-Scholes-Merton eq.: Price of an option $P(t,x)$

$P(t,x)$ at time t for spot value x and h is the payoff at time T_E.

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 P}{\partial x^2} + r\left(x\frac{\partial P}{\partial x} - P\right) = 0 \qquad P(T_E, \cdot) = h$$

Variables:

x is the price of the underlying (stock)

$P = P(t,x)$ is the price of the option on the underlying x at time t

r is the interest rate and σ is the volatility. The time $t < T_E$.

Note

- **Note the BSM Equation above is a FINAL value problem.**
- **However, in practice we have a Stochastic Behavior of the Volatility**

Classical Black-Scholes-Merton

Under highly simplifying assumptions, the call option price C on an underlying X is given by

$$C^{\mathrm{BS}}(X,t;K,T,r,\sigma) = XN(d_+) - Ke^{-r(T-t)}N(d_-) \tag{1}$$

where N is the cumulative normal distribution.

$$d_\pm = \frac{\log(Xe^{r(T-t)}/K)}{\sigma\sqrt{T-t}} \pm \frac{\sigma\sqrt{T-t}}{2}\,. \tag{2}$$

Some of the Assumptions:

- Non dividend paying (just for simplicity)
- Complete and Frictionless Markets
- Exponential Brownian motion dynamics
- Constant Volatility

Plot of the Black-Scholes Price of a Call

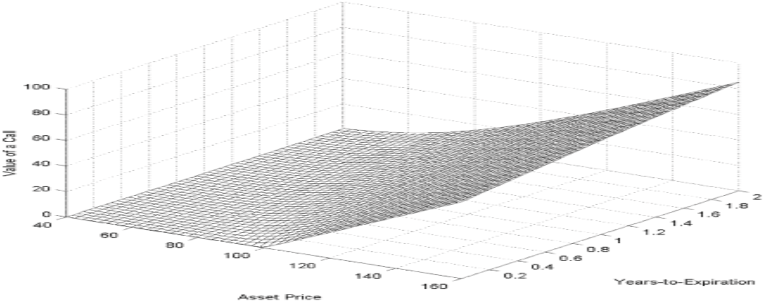


Figure: Typical example of the price of a vanilla option (call).

However... Stochastic Behavior of the Volatility

IBOVESPA Index and its Volatility

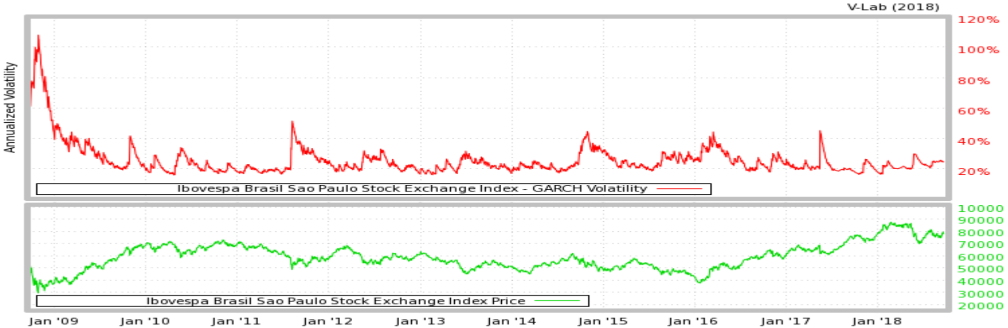


Figure: IBOVESPA Index and its (garch) Volatility (From Vlab at Stern, NYU)

Stochastic Behavior of the Volatility

- Volatility is not deterministic! It is even a multi-scale phenomena!
- It is not true that the underlying undergoes an Exponential Brownian Motion
- Even more so in high frequency contexts...

Implied Volatility The value of the volatility that should be used in the Black-Scholes formula to give the quoted market price of a derivative.

The Concept of Implied Volatility

Recall

$$C^{\mathrm{BS}}(X,t;K,T,r,\sigma_0) = XN(d_+) - Ke^{-r(T-t)}N(d_-) \tag{3}$$

where N is the cumulative normal distribution function and

$$d_\pm = \frac{\log(Xe^{r(T-t)}/K)}{\sigma_0\sqrt{T-t}} \pm \frac{\sigma_0\sqrt{T-t}}{2}\,. \tag{4}$$

Notion of Implied Volatility Fix everything else and consider

$$\sigma \longmapsto C^{\mathrm{BS}}(X,t;K,T,r,\sigma)$$

The implied volatilty is the inverse to this map.
IMPLIED VOL ”wrong number that when plugged into the wrong equation gives the right price”

IMPLIED VOL

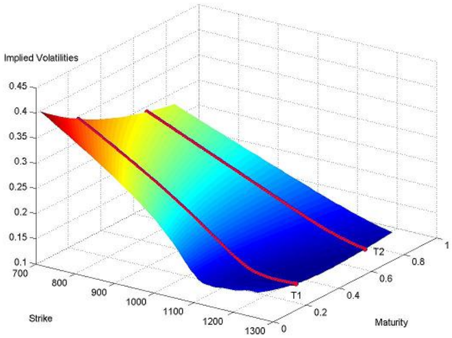


Figure: Implied Volatility Surface- (From Bruno Dupire - IMPA talk)

Volatility

Different Interpretations and Directions

- Econometrics - Historical
- Implied (or Implicit)
- Stochastic Volatility Models
  - fast mean reversion (Papanicolaou, Fouque, et al)
  - for commodities: **jt work Fouque, Saporito, Zubelli; IJTAF2015**
- Local Volatility NON PARAMETRIC (focus of this talk)
- More recent work Stochastic Local Vol (jt work Saporito & Yang)
- More recent work Local vol and jumps (jt work V. Albani)

Central Problems

- Understand volatility behavior.
- Protect portfolios against volatility oscilations.
- Find parsimonious and efficient models (simple but not too simple!)
- Calibrate such models in a robust and effective way.
- Price other derivatives consistently

Dupire's Local Volatility Model

Risk

CUTTING EDGE CLASSIC FROM JANUARY 1994. VOLATILITY

Pricing with a smile

*In the January 1994 issue of Risk, **Bruno Dupire** showed how the Black-Scholes model can be extended to make it compatible with observed market volatility smiles, allowing consistent pricing and hedging of exotic options*

where $r(t)$ is the instantaneous forward rate of maturity t implied from the yield curve.

Some Wall Street houses incorporate this temporal information in their discretisation schemes to price American or path-dependent options.

However, the dependence of implied volatility on the strike, for a given maturity (known as the *smile* effect) is trickier. Researchers have attempted to enrich the Black–Scholes model to compute a theoretical 'smile'. Unfortunately, they have to introduce a non-traded source of risk such as jumps, stochastic volatility or transaction costs, thus losing the completeness (ability to hedge options with the underlying asset) of the model.[3] Completeness is of the highest value: it allows for arbitrage pricing and hedging.

Therefore, we must ask whether it is possible to build a spot process that:

- is compatible with the observed smiles at all maturities, and
- keeps the model complete.

More precisely, given the arbitrage-free prices $C(K, T)$ of European calls of all strikes K and maturities T, is it possible to find a risk-neutral process for the spot in the form of a diffusion:

$$\frac{dS}{S} = r(t)dt + \sigma(S,t)dW$$

where the instantaneous volatility σ is a deterministic function of the spot and of the time?

This would extend the Black-Scholes model to make full use of its diffusion setting without increasing the dimension of the uncertainty. We would have the features of a one-factor model (hence easily discretisable) to explain all European option prices. We could then price and hedge any American or path-dependent options (even for European options, the knowledge of the whole

Figure: Bruno Dupire's seminal contributions. Source: Risk Magazine

Local Volatility Model

B. Dupire

Idea Assume that the volatility is given by

$$\sigma = \sigma(t, X)$$

i.e.: it depends on time and the asset price.
Easy to check that the Black-Scholes eq. holds.

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma(t, X)^2 X^2 \frac{\partial^2 P}{\partial X^2} + r\left(X\frac{\partial P}{\partial X} - P\right) = 0 \quad (5)$$

$$P(T, X) = h(X) \quad (6)$$

From now on $h(X) = (X - K)^+$ **or** $h(X) = (K - X)^+$

Take Home Message

Focus: Dupire local volatility models

Our Achievements/Goals

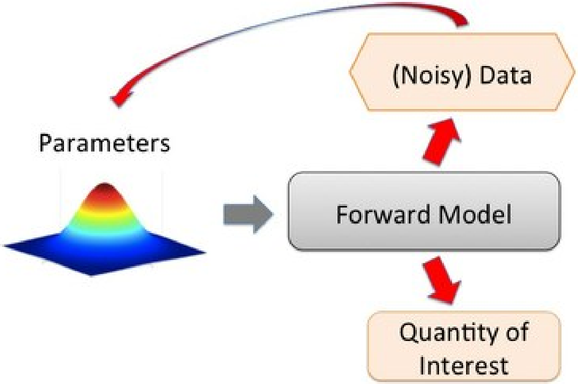
- **Present a unified DATA DRIVEN framework for the calibration of local volatility models**
- **Use recent tools of convex regularization of ill-posed Inverse Problems.**
- **Present convergence results that include convergence rates w.r.t. noise level in fairly general contexts**
- **Go beyond the classical quadratic regularization.**

Applications

- risk management; hedging; evaluation of exotic derivatives

Our approach: Data Driven Inverse Problem Theory

Note: Parameters may live in a very large space



Example of an Inverse Problem

Classical Computerized Tomography

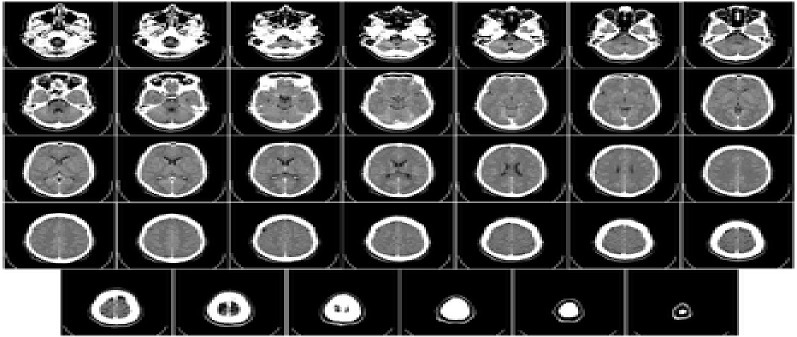


Figure: By Department of Radiology, Uppsala University Hospital (Wikipedia)

Example of an Inverse Problem

Tomography with Diffusion (Discretized Model) published in *Science* Vol. 248 - 1990

Image Reconstruction of the Interior of Bodies That Diffuse Radiation

J. R. SINGER, F. ALBERTO GRÜNBAUM, PHILIP KOHN,
JORGE PASSAMANI ZUBELLI

A method for reconstructing images from projections is described. The unique aspect of the procedure is that the reconstruction of the internal structure can be carried out for objects that diffuse the incident radiation. The method may be used with photons, phonons, neutrons, and many other kinds of radiation. The procedure has applications to medical imaging, industrial imaging, and geophysical imaging.

J. R. Singer, Electrical Engineering and Computer Science, and Biophysics Departments, University of California, Berkeley, CA 94720.
F. A. Grünbaum and J. P. Zubelli, Mathematics Department, University of California, Berkeley, CA 94720.
P. Kohn, International Computer Sciences Institute,

$\cdot f_{ijk}$; the probability of a backward scatter is $w_{ijk} \cdot b_{ijk}$; the probability of a sideways scatter is $w_{ijk} \cdot s_{ijk}$.

4) For each voxel we provide the variables w, f, b, and s. As pointed out above,

Inverse Problems

Connection w/ Neural Networks & Deep Learning... excerpt from our *Science* paper

This problem can also be easily cast as a neural network in which each voxel is represented by six neurons, one neuron for each direction of exit from the voxel. The activity of these cells corresponds to the radiation flux in a given direction and is equal to a weighted sum of the activity coming in from all six neighbors. Learning techniques such as back propagation (*8*) may have applications to these problems.

Message of our work on Calibration of Local Volatility

- We have considered the simultaneous calibration of local volaltility and jump-size dist.
- We have stated the regularity properties of the parameter-to-solution map.
- Tikhonov-type regularization was used to solve the inverse problems separately.
- We have applied a splitting strategy to solve the simultaneous calib. prob.
- We provided numerical examples real and simulated data.
- Paper published in Finance & Stochastics (2020).
  Available also at ArXiV `https://arxiv.org/abs/1811.02028`

Dupire's Equation

$U(T,K) := P(t,X;T,K)$

Fix (t,X) and set

$$U(T,K) := P(t,X;T,K)$$

Assuming that there exists a local volatility function $\sigma = \sigma(t,X)$ for which Eq. (5) holds Dupire(1994) showed that the call price satisfies

$$\begin{cases} \partial_T U - \frac{1}{2}\sigma^2(T,K)K^2\partial_K^2 U + rK\partial_K U = 0 \;, \quad K > 0 \;, \; T \geq 0 \\ U(K,T=0) = (X-K)^+ \;, \end{cases} \tag{7}$$

Problem Statement

The Vol Calibration Problem

Given an observed set

$$\{u = u(t, X, T, K; \sigma)\}_{(T,K)\in\mathcal{X}}$$

find $\sigma = \sigma(t, X)$ that best fits such market data

Noisy data $u = u^\delta$

Parameter-to-solution operator

$F : \mathcal{D}(F) \subset H^{1+\varepsilon}(\Omega) \to L^2(\Omega)$

$$F(a) = u(a) \tag{8}$$

Approach

Convex Tikhonov Regularization

For given convex f minimize the Tikhonov functional

$$\mathbb{F}_{\alpha,u^\delta}(a) := ||F(a) - u^\delta||^2_{L^2(\Omega)} + \alpha f(a) \tag{9}$$

over $\mathcal{D}F$, where, $\alpha > 0$ is the regularization parameter.

Remark that f incorporates the *a priori* info on a.

$$||\bar{u} - u^\delta||_{L^2(\Omega)} \leq \delta\,, \tag{10}$$

where $\bar{u}$ is the data associated to the actual value $\hat{a} \in \mathcal{D}F$.

Questions

Theoretical Questions:

- Does there exist a minimizer of the regularized problem?
- Suppose that the noise level goes to zero... How fast does the regularized go to the true solution?

Results obtained in **joint work with D. Cezaro and O. Scherzer.**
Published in J. Nonlinear Analysis, 2012

Questions:

- Can we devise an iterative algorithm to compute the solution?
- Does this algorithm converge?
- Can we regularize by stopping the iteration judiciously?

We proved:

1. **A tangential cone condition that ensures convergence of the Landwebber iteration. Joint work w/ D. Cezaro.** (IMA J. of Applied Math. 2013)
2. **Obtained a Morozov-type criterion to stop the iteration. Joint work w/ Albani & D. Cezaro** (A.Analysis & Discrete Math. 2014)
3. **Developed a regularization by discretization with a stopping criterion. Joint work w/ Albani & D. Cezaro.** (Inv. Problems in Imaging. 2016)

Furthermore

- Implemented the different algorithms
- Compared with alternatives (such as (ensemble) Kalman filter based iterations)

Impact: Solution of complex derivative problems, efficient hedging; and risk management.

The Splitting Algorithm with DAX Options

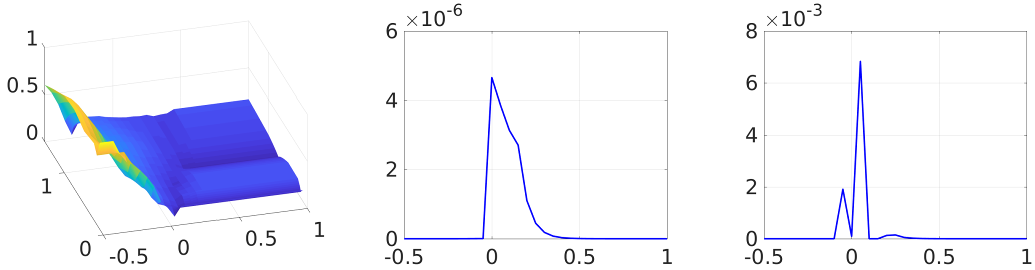


Figure: Reconstructions from Dax options of local volatility surface (left), double exponential tail (center) and jump-size density function (right).

The Splitting Algorithm with DAX Options

Adherence of our models to the implied volaltility at different times to expiration

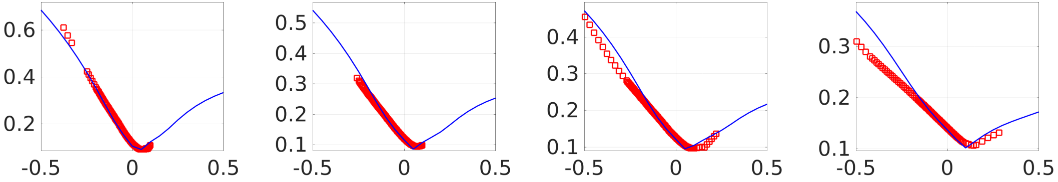


Figure: Market (squares) and model (continuous line) implied volatility of DAX options.

Final considerations

- We have considered the simultaneous calibration of local vol. and jump-size dist.
- We have stated the regularity properties of the parameter-to-solution map.
- Tikhonov-type regularization was used to solve the inverse problems separately.
- We have applied a splitting strategy to solve the simultaneous calib. prob.
- We provided numerical examples.
- We also provided examples with real data.

IMPACT

The data driven methodology provides a fairly accurate way (as measured by the adherence to the implied vol) to model option prices.

Thank You!


mathematics & finance:
R:O
Research in Options
EWG
CFM
COMMODITIES AND
FINANCIAL MODELLING
جامعة خليفة
Khalifa University
Converging Paths in Commodity and Financial Markets Analysis
3-8 December 2023, Abu Dhabi, UAE

Related Work

Very vast!!!

- Avellaneda et al.(1997,2000,2002)
- Bouchouev & Isakov (1997)Isakov (2011), Bouchouev, Isakov & Valdivia (2002)
- Crepey (2003)
- Egger & Engl (2005)
- Hofmann et al. (2005, 2007)
- Achdou & Pironneau (2004)
- Roger Lee (2001,2005)
- Abken et al. (1996)
- Ait Sahalia, Y & Lo, A (1998)
- Berestycki et al. (2000)
- Buchen & Kelly (1996)
- Coleman et al. (1999)
- Cont, Cont & Da Fonseca (2001)
- Jackson et al. (1999)
- Jackwerth & Rubinstein (1998)
- Jourdain & Nguyen (2001)
- Lagnado & Osher (1997)
- Samperi (2001)
- Stutzer (1997)

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Scherzer, O., Grasmair, M., Grossauer, H., Haltmeier, M., and Lenzen, F. *Variational Methods in Imaging*, volume 167 of *Applied Mathematical Sciences*. Springer, New York, 2008.