New birational invariants

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Outline

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Chen-Ruan Cohomology and (equivariant) rationality, joint with Leonardo Cavenaghi, Lino Grama, and Maxim Kontsevich

What is next?

Birational Geometry

Recall

Recall that an algebraic variety X is said to be *rational* $\mathbb{C}$ if $\mathbb{C}(X) = \mathbb{C}(x_1, \ldots, n)$ where $n = \dim_{\mathbb{C}} X$.

Example

- $\dim_{\mathbb{C}} X = 1$. X a cubic in $\mathbb{P}^2$. $h^{1,0}(X) = 1$ implies that X is not rational.
- $\dim_{\mathbb{C}} X = 2$. X a cubic in $\mathbb{P}^3$. X is rational.
- $\dim_{\mathbb{C}} X = 3$. X smooth cubic in $\mathbb{P}^4$.

$$\mathrm{H}^{2,1}(X)/\mathrm{H}_3^*(X) = Jac(X) \neq Jac(C).$$

Hodge Diamond

$$\begin{array}{ccc} & 1 & \\ & 1 & \\ 5 & & 5 \\ & 1 & \\ & 1 & \end{array}$$

Example

- $\dim_{\mathbb{C}} X = 5$. X a cubic in $\mathbb{P}^5$ and with a plane. Hodge Diamond:

$$\begin{array}{ccc} & 1 & \\ & 1 & \\ 1 & 21 & 1 \\ & 1 & \\ & 1 & \end{array}$$

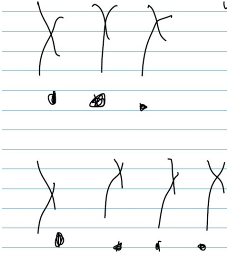
Homological Mirror Symmetry

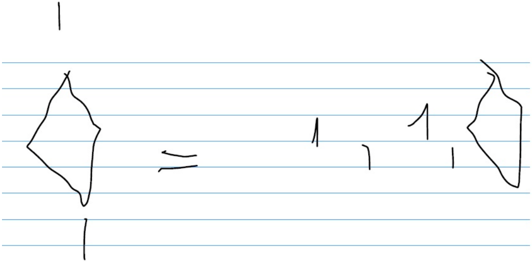
Homological Mirror Symmetry

$$D^b(\mathbb{P}^2) \longrightarrow FS(\mathbb{C}^2, W = x+y+\frac{1}{xy}).$$

1
1
1

1
2 1+1
1





F - bundles

- In general, the convergence of series in the definition of quantum product is not known. One possible fix is to work in an algebraically closed non-archimedean field $K := \bigcup_{N\geq 1} \mathbb{Q}((y^{1/N}))$.
- Let us consider the K-analytic super manifold F_X with coordinates $q_1, \dots, q_r$ and t_i for $i \notin \{1, \dots, r\}$ where $0 < |q_i| < 1$, $0 \leq |t_i| < 1$ for j such that Δ_j is an even class. Quantum multiplication gives an associative commutative product $\star$ on the tangent bundle TF_X identified with $H^\bullet(X)$ via $\Delta_i \mapsto (q_i\partial_{q_i})$ if $i \in \{1, \dots, r\}$ and ∂_{t_i} otherwise.
- Another important structure is the Euler vector field given by the cohomology class:

$$Eu := c_1(TX) + \sum_{i:\deg\Delta_i\neq 2} \frac{\deg\Delta_i - 2}{2} t_i\Delta_i$$

Decomposition theorem

- Denote $M := F_X$. The multiplication $\star \in \Gamma(M, (T^*M) \otimes 2 \otimes TM)$ and the Euler field $Eu \in \Gamma(M, TM)$ are related by $\mathrm{Lie}_{Eu}(\star) = \star$.
- Let us consider a point $p \in M_{\text{even}}$ and a finite collection of disjoint open discs $(D_\alpha) \in K$ such that the spectrum of the operator $Eu \star \cdot$ acting on T_pM is contained in the union $\bigcup_\alpha D_\alpha$. Then, locally near p, the same is true, and we get a decomposition of TM in the vicinity of p into a direct sum of subspaces. The general result is that this decomposition comes from a canonical decomposition $(M, \star, Eu) = \bigoplus_\alpha (M_\alpha, \star_\alpha, Eu_\alpha)$ near p of (quotient) varieties endowed with products and Euler fields.

Blow up formulae

- Let $Z \subset X$ be a smooth closed subvariety of codimension $m \geq 2$.
- By making a blowup with center at Z, we obtain a new smooth projective variety $X_e = \mathrm{Bl}_Z X$.
- It is well-known that there is a canonical identification of cohomology spaces (breaking Z-grading and cup-product):
$$H^\bullet(X_e) \simeq H^\bullet(X) \oplus M(m-1) \cdot H^\bullet(Z)$$
- If we consider the spectrum of $(Eu \star \cdot)|_{T_p F_{\mathbf{even}} X_e}$ for a point $p \in F_{\mathbf{even}} X_e$, corresponding to an ample class on $\widetilde{X}$ sufficiently close to the semi-ample class $[\widetilde{X} \to X]$, where $*\omega_X$ is an ample class, we obtain a picture like this:

Blow up formulae 1

- Eigenvalues close to 0 correspond to classes in $H^\bullet(X)$.
- Eigenvalues close to the rescaled $(m-1)$-st roots of 1 correspond to classes in $H^\bullet(Z)$.
- The calculation is very easy, similar to the calculation of the quantum product for $\mathbb{CP}^n$ at the beginning of this lecture.
- The only relevant curves are constant maps and lines in the projectivization of the normal bundle to $Z \subset X$.

By the general decomposition theorem, we conclude that $M(X_e)$ is locally isomorphic to the product of m different F-manifolds with Euler fields, which have the same dimensions as $M(X)$ and $(m-1)$ copies of $M(Z)$.

Atoms

Atoms

- Let X be a complex projective variety. Consider the subspace of its even cohomology $H^{2\bullet}(X, \mathbb{Q})$ spanned by the Hodge classes:
$$\mathcal{H}_{\text{Hodge}}(X) := \bigoplus_i H^{i,i}(X) \cap H^{2i}(X, \mathbb{Q})$$
- This subspace gives a purely even submanifold $M_{X,\text{Hodge}} \subset M_X$ over K, of dimension equal to the rank of $\mathcal{H}_{\text{Hodge}}(X)$.
- The spectrum of the operator $E_{up} \star \cdot$ where $p \in M_{X,\text{Hodge}}$ achieves a certain maximal value μ at a dense open nonempty connected subset $M_o^{\text{Hodge}} \subset M_{X,\text{Hodge}}$. Eigenvalues of $E_{up} \star \cdot$ give a μ-fold spectral cover of M_o^{Hodge}, possibly disconnected.
- Definition: the set of local atoms Atoms_X is the set of connected components of the spectral cover described above.
- Important example: if $K_X = \det T^*X$ is numerically effective (has non-zero intersection with any curve), then Atoms_X consists just of one point. Reason: quantum product preserves filtration $H^{\geq \bullet}(X)$.

Atom 1

- Now consider the following huge set:
$$\mathcal{G} := \text{iso classes of } X/C$$
$$\text{Atoms}_X/\text{Aut}X$$
- Iritani's theorem implies that one can relate certain elements of M_{X_e} with some elements of M_X or M_Z. This generates a certain equivalence relation on the set above, and we denote by Atoms_C the set of equivalence classes. This set is naturally filtered by the minimal dimension of a variety in which an atom can appear.
- Well-known fact: birational equivalences between smooth projective varieties are generated by blowups with smooth centers of codimension ≥ 2. Hence, the non-rationality criterion: If for an N-dimensional variety X (here $N \geq 2$) at least one of the atoms of X does not appear in varieties of dimension $\leq N-2$, then X is not rational.

Invariants of Atoms

- Our goal is to prove the non-rationality of certain 4-dimensional varieties. Hence, we have to study atoms coming from all ≤ 2-dimensional varieties, i.e., from points, curves, and surfaces. Moreover, it is sufficient to consider only one representative in each birational class of surfaces.
- For every atom α (in general), we have the following invariants:
  - the rank ρ_α of the space of Hodge classes $\mathcal{H}_{\mathbf{Hodge}}(X) \otimes_{\mathbb{Q}} K$ in the corresponding generalized eigenspace of $E_u \star \cdot$,
  - the Hodge polynomial $P_\alpha \in \mathbb{Z}[t, t^{-1}]$ whose coefficient at t^k is equal to the rank of the generalized α-eigenspace in $\oplus_{p,q:p-q=k} H^{p,q}(X)$.
- Using these two types of invariants, we can distinguish certain atoms of the generic cubic 4-fold from those coming from points, curves, and surfaces.

Atom < 2

- For any atom α coming from points or curves, we obviously have $\mathrm{Coeff}_t(2P_\alpha) = 0$.
- For minimal models X of all surfaces, except surfaces of general type and K3 surfaces, we have $\mathrm{Coeff}_t(2P_\alpha) = 0$ for any atom coming from X, because $H^{2,0}(X) = 0$.
- For the minimal resolution X of ADE singularities of the minimal model of a K3 surface or a surface of general type, we have $K_X \geq 0$, hence only one atom α, and then $\rho_\alpha \geq 3$, as X has two non-trivial algebraic cycles of dimensions 0 and 2 and at least one non-trivial algebraic cycle of dimension 1.

Generic 4 dim cubic

- Generic cubic 4-fold $X \subset \mathbb{CP}^5$ has the following Hodge diamond (and the decomposition into the sum of Hodge classes and the transcendental part):

$$
\begin{array}{cccccc}
 & 1 & & & & \\
1 & 20 & 1 & 1 & 1 & 1 \\
 & 1 & & & &
\end{array}
$$

- Classical Givental's calculation: at a special (maybe non-generic) point of $M_{X,\text{Hodge}}$ the spectrum of E_u is:

$$
\begin{array}{ccc}
 & 1 & \\
1 & 20 & 1 \\
 & 1 &
\end{array}
$$

Hence, the middle part has $\rho = 2$, $\text{Coeff}_t(2P_\alpha) = 1 \Rightarrow$ it cannot come from ≤ 2 dimensions.

Algebraically nonclosed fields

- By Y. André's theory of motivated cycles (1996), for any field k of characteristic 0, we have a pro-reductive algebraic group $G/\mathbb{Q}$ and a universal Weil cohomology theory for k-varieties with values in representations of G. One has $\pi_0(G) = \mathrm{Gal}(k/\bar{k})$.
- Action of G on $H^2(\mathbb{P}^1)$ gives an epimorphism $G \twoheadrightarrow \mathbb{G}_m = \mathrm{GL}(1)$, denote by $G_{\mathbf{nc}}$ the kernel of this map. Then the image of algebraic cycles in $H^\bullet(X)$ is contained in $(H^{\mathrm{even}}(X))^{G_{\mathbf{nc}}}$.
- Gromov-Witten invariants are given by algebraic cycles $\Rightarrow$ algebraic group $G_{\mathbf{nc}}$ acts on M_X. We define local atoms, as well as Atoms k by replacing $M_{X,\mathbf{Hodge}} \rightsquigarrow (M_X)^{G_{\mathbf{nc}}}$ (the fixed locus).
- A basic invariant of an atom α: an isomorphism class of a representation $[R_\alpha]$ of $G_{\mathbf{nc}}$ over K (typically reducible). In the special case $k = \mathbb{C}$, one can recover invariants $\rho_\alpha = \dim(R_\alpha^{G_{\mathbf{nc}}})$ (number of Hodge classes) and the Hodge polynomial P_α.
- The total representation of $G_{\mathbf{nc}}$ in $H^\bullet(X)$ splits into a direct sum of "atomic" ones.

Examples

Examples

Example

- Consider a smooth hypersurface X_{geom} of degree $(1,1,1,1)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, defined over an algebraically closed field k. It is the blowup of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ at an elliptic curve E, and hence it has 8 point-like atoms and one more complicated atom α_E associated with E.
- Now, consider a model X of X_{geom} defined over a non-closed field k such that the Galois group $\text{Gal}(k/\bar{k})$ acts by a transitive group of permutations of 4 factors in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then at the most naive point of M_X with coordinates $q_i = 1, t_j = 0$, there are 3 different eigenvalues of $E_u \star \cdot$, with multiplicities $1, 4, 7$. The last piece has Hodge polynomial $5 + t + t^{-1}$ and only 2 algebraic classes defined over k. This implies that this representation of G_{nc} cannot split furthermore into atomic representations coming from 0- and 1-dimensional varieties over k, $\Rightarrow$ nonrationality of X/k.

Example

- 1 1 1 1 1

  Hypersurface in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with a S_5-action. The atom over zero has only two algebraic cycles.

- $X = \widehat{\mathbb{P}^3_C}$ $\mathbb{Z}_2$ 2 : 1

1

2

2 2

2

1

2

2 2 1 1

2

. . .

Asymptotic of the quantum differential equation

The asymptotic $\sigma_1, \ldots, \sigma_N$ of the solutions of the quantum differential equation

$$\left(\frac{\partial}{\partial u} - \frac{K}{u^2} + \frac{G}{u}\right)\psi = 0$$

are birational invariants.

Theorem

- *For a Fano hypersurface of degree d*

$$\delta := \dim X - 2\frac{N-d}{d}$$

- *Assume that $\delta > \dim X - 2$. Then X is not rational.*

Example

X a 4-dimensional quadric. We have $\delta = 4 - 2\frac{6-4}{4} = 3 > 1$.

Chen-Ruan Cohomology and (equivariant) rationality, joint with Leonardo Cavenaghi, Lino Grama, and Maxim Kontsevich

Chen-Ruan cohomology and the Burnside group

- Chen-Ruan cohomology of an orbifold X is the orbifold cohomology (with real or complex coefficients) of the inertial orbifold IX. It is motivated by the role that orbifolds play as target spaces in perturbative string theory, as in the algebraic operation of orbifolds 2d CFTs.
- Inertia orbifold is a particular model for the free loop space object of an orbifold X
- M. Kontsevich, V. Pestun, and Y. Tschinkel introduced (2019) new invariants in equivariant birational geometry.
  Assume that a finite group G acts (birationally and generically free) on a projective variety Y (of dimension d) such that $X = Y/G$ is an orbifold. Let Y^G be the fixed point for this G-action. We decompose Y^G into irreducible subvarieties components'
  $$Y^G = \sqcup_l F_l.$$
  The G-action induces, for each l, a G-action in $T_y Y$, $y \in F_\alpha$ with characters $\{a_{j,l}\}$, $j = 1, \ldots, d$. The symbols $\{[a_{1,l}, \ldots, a_{d,l}]\}_l$ under some relations define elements in a group $\mathcal{B}_d(G)$ whose classes $\beta(Y) := \sum_l [a_{1,l}, \ldots, a_{d,l}]$ are equivariant birational invariants. An enhancement of $B_d(G)$ leads to the *Burnside group*.

A criterium

We connect the group $\mathcal{B}_d(G)$ with Chen-Ruan cohomology via the following

Theorem

Assume that G is abelian. Consider a generically free birational action of G in $\mathbb{P}^d$. Write

$$Y^G = \sqcup_l F_l, \ {\mathbb{P}^d}^G = \sqcup_m G_m.$$

Let $\mathrm{H}^*(\cdot)$ stand to the cohomology ring of $(\cdot)$ and consider the respective induced G-actions on it. If

$$\prod_l \mathrm{H}^*(F_l)^G \ncong \prod_m \mathrm{H}^*(G_m)^G$$

then

$$Y \not\sim_G \mathbb{P}^d,$$

i.e., $\beta(Y) \neq \beta(\mathbb{P}^d)$.

Idea of the proof.

Translate the statement into eigenvalues for the action on the fixed point set. Collect this information from *twisted sectors*. Realize the relation of this with β. □

Example (Fixed point set for linear finite group actions on $\mathbb{P}^n$)

Lemma

Let G be a group with the following G-irreducible complex representation $\psi = \sum_j n_j \psi_j$, i.e., $\psi : G \to \mathrm{GL}(\mathbb{C}^{d_j})$. Then,

$$\mathbb{P}^{\sum_j n_j - 1}{}^G = \sqcup_{j:d_j=1} \mathbb{P}^{n_j - 1}.$$

Proposition

If G is abelian and ψ is as former, then

$$\mathbb{P}^{\sum_j n_j - 1}{}^G = \sqcup_{j=1}^{\sum_j n_j - 1} \{[0 : \ldots : 1 : 0 : \ldots : 0]\}$$

Thus, for any commutative ring R the singular cohomology ring of $\mathbb{P}^{\sum_j n_j - 1}{}^G$ is given by

$$\mathrm{H}^*(\mathbb{P}^{\sum_j n_j - 1}{}^G; R) \cong R[x]/(x^2) \oplus 0 \oplus \ldots \oplus 0$$

where x is a generator of degree 2.

Take $R = \mathbb{C}(\mathbb{P}^n)$. The G-action in $\mathbb{P}^n$ induces a G-action in $\mathbb{C}(\mathbb{P}^n)$. Consider the quantity

$$C(\mathbb{P}^n)^G[x]/(x^2) \oplus 0 \oplus \ldots \oplus 0.$$

An Application

Example (Fixed point set for (cyclic permutation) actions on $\mathbb{P}^1 \times \ldots \times \mathbb{P}^1$)

Write for each $\mathbb{P}^1$ in the factor its coordinates as $[u_0 : u_1]$. Suppose we are given d-factors. Let $\mathbb{C}_d$ act on $\mathbb{P}^1 \times \ldots \mathbb{P}^1$ via cyclic permuting the coordinates of each factor. We have that

$$\mathbb{P}^1 \times \ldots \mathbb{P}^{1\, C_d} \cong \mathbb{P}^1.$$

Indeed, if $[u_0^j : u_1^j]$ stands to the general coordinates in the j-th factor, we readily check that $\mathbb{P}^1 \times \ldots \mathbb{P}^{1\, C_d} = \{([u_0^1 : u_1^1], \ldots, [u_0^1 : u_1^1])\} \cong \{[u_0 : u_1] \in \mathbb{P}^1\}$.

Consequently, for a chosen commutative ring R, we have that

$$\mathrm{H}^*(\mathbb{P}^1 \times \ldots \times \mathbb{P}^{1\, C_d}) \cong \frac{R[x]}{(x^2)}$$

where x is a generator of degree 2. Pick $R = \mathbb{C}(\mathbb{P}^1)$ and let $G = C_d$ act on it accordingly. Consider the quantity

$$\frac{\mathbb{C}(\mathbb{P}^1)^{C_d}[x]}{(x^2)}.$$

An application

Theorem

For any linear action of $G = C_4$ *in* $\mathbb{P}^4$ *and the permuting coordinates* C_4 *action on* $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ *we have*

$$\mathbb{P}^4 \not\sim_{C_4} \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

Let $X(1,1,1,1)$ *be given by the blowup of* $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ *on an elliptic curve. Assume that* C_4 *acts in it via restricting the cyclic coordinate permutation in* $\mathbb{P}^1 \times \ldots \times \mathbb{P}^1$*. We have that for any* C_4*-linear action in* $\mathbb{P}^3$

$$X(1,1,1,1) \not\sim \mathbb{P}^3.$$

Proof.

$$\frac{\mathbb{C}(\mathbb{P}^1)^{C_4}[x]}{(x^2)} \not\cong \frac{C(\mathbb{P}^1)^{C_4}[x]}{(x^2)} \oplus 0 \oplus 0 \oplus 0.$$

□

Cubic with a plane

Theorem

- *Assume that the* Γ *conjecture holds for* X *and for any* $Z \subset X$*. Then the* Γ*-conjecture holds for* $\widehat{X}_Z$*.*
- *Let* X *be a 4-dimensional quadric bundle. If there exists a class* B *such that* $\langle B, B \rangle = \frac{1}{2}$ *then* X *is not rational.*

Example

The former applies to, for instance,

- 4-dimensional cubic with a plane
- Intersection of three quadrics in $\mathbb{P}^7$.

What is next?

What is next?

- Seek finer birational invariants combining group actions with atoms theory.
- The concept of *gerbes* is related to Chen-Ruan cohomology via the following. Let X be an orbifold. The group of gerbes with connection over X are classified by the Deligne cohomology group $\mathrm{H}^3(X, \mathbb{Z}(3)^\infty_D)$. Discrete torsion of B-fields (as in the works of Vafa and Witten) are the curvature of these gerbe connections.
- Gerbes with connections correspond with twisted bundles $\mathcal{L}$ over X. The Grothendieck group generated by the isomorphism classes of $\mathcal{L}$ is the $\mathcal{L}$ twisted K-theory ${}^{\mathcal{L}}K_{\mathrm{grp}}(X)$. Under mild hypothesis on X, it holds that
$$^{\mathcal{L}}K_{\mathrm{grp}}(X) \otimes \mathbb{C} \cong \mathrm{H}^*_{CR}(X; \mathbb{C}).$$
- The ring tmf can be used to recover "global information" for orbifolds X. This means that we can use it combined with the theory of Chen-Ruan cohomology to classify all the possible T^2-fibratios whose base is X. This may allow us to relate equivariant birational invariants with smooth invariants.

Thank you!