A general theory of norm and time optimal control problems for linear PDE systems

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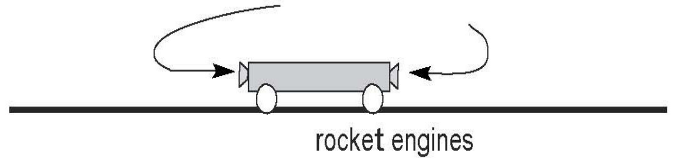
First example of time optimal control



A rocket car on a train track

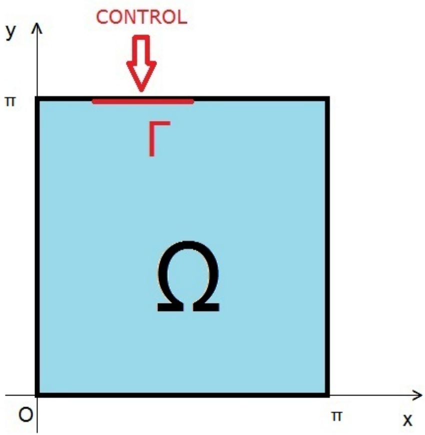
Steer the rocket car from rest to rest, from the initial position z_0 to the final position z_1, with a bounded acceleration:

$$\ddot{z}(t) = u(t), \quad z(0) = z_0, \quad \dot{z}(0) = 0,$$

$$z(\tau) = z_1, \quad \dot{z}(\tau) = 0, \quad -1 \leqslant u(t) \leqslant 1.$$

The solution is to apply max acceleration, i.e., $u^*(t) = 1$ until the unique switching point, and then max braking, i.e., $u^*(t) = -1$. This solution, can be "uncomfortable" for the passengers...

A PDE example: controlling the temperature in a room



$$\frac{\partial z}{\partial t} = \Delta z \text{ in } \mathbb{R}_+ \times \Omega \quad (1)$$

$$\begin{cases} z = u & \text{on } \mathbb{R}_+ \times \Gamma \\ z = 0 & \text{on } \mathbb{R}_+ \times (\partial\Omega \setminus \Gamma) \end{cases} \quad (2)$$

$$z(0, x) = \psi(x) \qquad (x \in \Omega) \quad (3)$$

Statement of the norm optimal control problem

Given $\psi \in L^2(\Omega)$ and $\tau > 0$ the *norm optimal control problem for the system described by* (1)-(3) consists in determining $\hat{u} \in L^\infty([0,\infty) \times \Gamma)$ such that

$$\|\hat{u}\|_{L^\infty([0,\infty)\times\Gamma)} = \min_{\substack{u \in L^\infty([0,\infty)\times\Gamma) \\ z(\tau,\cdot)=0}} \|u\|_{L^\infty([0,\infty)\times\Gamma)}.$$

If the above optimal control problem admits a solution $\hat{u}$ for some $\tau > 0$, we denote by $N^\infty(\tau)$ the optimal cost, i.e., we set

$$N^\infty(\tau) = \|\hat{u}\|_{L^\infty([0,\infty)\times\Gamma)}.$$

Statement of the time optimal control problem

Given $\psi \in L^2(\Omega)$, $M > 0$ and setting

$$\mathcal{U}_M := \left\{u \mid u \in L^\infty((0,\infty)\times\Gamma) \quad \text{s.t. } \|u\|_{L^\infty((0,\infty)\times\Gamma)} \leqslant M\right\}.$$

the *time optimal control problem for the system described by* (1)-(3) consists in determining $(\tau^\infty(M), u_M^\infty) \in (0,\infty)\times\mathcal{U}_M$ such that u_M^∞ drives the initial state ψ to zero in time $\tau^\infty(M)$, i.e., the state trajectory z_M^∞ associated to u_M^∞ satisfies $z_M^\infty(\tau,\cdot) = 0$, and

$$\tau^\infty(M) := \min_{u\in\mathcal{U}_M} \{\tau \mid \exists u \in \mathcal{U}_M \ \text{ with } \ z(\tau,\cdot) = 0\}.$$

First main results

Proposition 1

With the above notation, we have:

1. *For every $\tau > 0$ and $\psi \in L^2(\Omega) \setminus \{0\}$ the norm optimal control problem for (1)-(3) admits at least one solution. Moreover, $\hat{u}$ can be chosen s.t. $|\hat{u}(t,x)| = N^\infty(\tau)\text{a.e.}$.*
2. *For every $\psi \in L^2(\Omega) \setminus \{0\}$, the time optimal control problem for (1)-(3) admits at least one solution $(\tau^\infty(M), u_M^\infty)$. Moreover, u_M^∞ can be selected s.t. $|u_M^\infty(t,x)| = M \text{ a.e.})$.*
3. *N^∞, τ^∞ defined via the two assertions above satisfy*

$$N^\infty(\tau^\infty(M)) = M \qquad (M > 0)$$

$$\tau^\infty(N^\infty(\tau)) = \tau \qquad (\tau > 0).$$

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General framework

- X (the state space) and U are Hilbert spaces.
- $\mathbb{T} = (\mathbb{T}_t)_{t \geqslant 0}$ is a strongly continuous semigroup on X generated by A.
- $B \in \mathcal{L}(U; X_{-1})$ is an **admissible control operator for** $\mathbb{T}$ and $u \in L^2([0, \infty), U)$ is an input function.

Some background

Let $\dot{z}(t) = Az(t) + Bu(t)$, or $z(t) = \mathbb{T}_t z(0) + \Phi_t u$, where

$\Phi_t \in \mathcal{L}(L^2([0,\infty),U);X), \quad \Phi_t u = \int_0^t \mathbb{T}_{t-\sigma} Bu(\sigma)\,\mathrm{d}\sigma.$

$(\mathbb{T},\Phi)$ is said *exactly controllable in time* τ if $\mathrm{Ran}\,\Phi_\tau = X$. This is equivalent to the existence of $K_\tau > 0$ such that for $\varphi \in \mathcal{D}(A^*)$, $K_\tau^2 \int_0^\tau \|B^*\mathbb{T}_t^*\varphi\|^2\,\mathrm{d}t \geqslant \|\varphi\|^2$ (exact observability)

$(\mathbb{T},\Phi)$ is *approximatively controllable in time* τ if $\overline{\mathrm{Ran}\,\Phi_\tau} = X$, or, ($B^*\mathbb{T}_t^*\varphi = 0$ for $t \in [0,\tau] \Rightarrow \varphi = 0$) (approx. observability)

More background

Let $U = L^2(\mathcal{O})$, where $\mathcal{O}$ is a compact Riemannian manifold. The L^∞ reachable space in time τ of Σ, denoted $\mathcal{R}_\tau^\infty$ is

$$\mathcal{R}_\tau^\infty = \Phi_\tau \left(L^\infty([0,\infty) \times \mathcal{O}) \right).$$

$\mathcal{R}_\tau^\infty$, endowed with the norm

$$\|\xi\|_{\mathcal{R}_\tau^\infty} = \inf_{\substack{u \in L^\infty([0,\infty)\times\mathcal{O}) \\ \Phi_\tau u = \xi}} \|u\|_{L^\infty([0,\tau]\times\mathcal{O})} \qquad (\xi \in \mathcal{R}_\tau^\infty),$$

is a normed space.

$\Sigma = (\mathbb{T}, \Phi)$ is said L^∞ null controllable in time τ if $\mathcal{R}_\tau^\infty \supset \operatorname{Ran} \mathbb{T}_\tau$.

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Leading assumptions

[H1] $U = L^2(\mathcal{O})$, where $\mathcal{O}$ is a compact Riemannian manifold.

[H2] $\mathbb{T}$ is analytic and its generator A has compact resolvents.

[H3] For $\tau > 0$ we have $\mathcal{R}_\tau^\infty = X$, i.e., $(\mathbb{T}, \Phi)$ is L^∞ null controllable in any time.

[H4] If $\tau > 0$, $\psi \in X$ are such that $(\Phi_\tau^* \psi)(t, x) = 0$ for (t, x) in a subset of positive measure of $[0, \tau] \times \mathcal{O}$ then $\psi = 0$.

A stronger hypothesis: null controllability from measurable sets

[H5] For every $\tau > 0$ and every set of positive measure $e \subset [0, \tau] \times X$ there exists $K_{\tau,e} > 0$ such that

$$K_{\tau,e} \int_e |(\Phi_\tau^* \psi)(t, x)| \, \mathrm{d}x \, \mathrm{d}t \geqslant \|\mathbb{T}_\tau^* \psi\|_X \qquad (\psi \in X).$$

Norm optimal controls. Problem statement

Given $\psi \in X$, we define $N^\infty : (0,\infty) \to [0,\infty)$ by

$$N^\infty(\tau) := \|\mathbb{T}_\tau \psi\|_{\mathcal{R}_\tau^\infty}.$$

Thanks to [H3], N^∞ is clearly well defined and takes values in $[0,\infty)$. Moreover, N^∞ clearly depends on the choice of $\psi \in X$ but, to avoid notational complexity, this dependence will not appear explicitly in what follows. A norm optimal control at time τ is an input function $\hat{u}$ such that

$$\Phi_\tau \hat{u} + \mathbb{T}_\tau \psi = 0, \qquad \|\hat{u}\|_{L^\infty([0,\tau]\times\mathcal{O})} = N^\infty(\tau).$$

The typical questions to be solved are the existence of such controls and the study of their properties (such as optimality conditions or the bang-bang-property).

Norm optimal controls. Main result

Theorem 1

Let $\tau > 0$ and let $\Sigma = (\mathbb{T}, \Phi)$ a well-posed control LTI system with state space X and input space U satisfying assumptions [H1]-[H4]. Then for every $\psi \in X \setminus \{0\}$ the norm optimal control problem $(NP)^\tau$ admits at least one solution. Moreover, this control $\hat{u}$ can be chosen to have the bang-bang property, i.e.,

$$|\hat{u}(t,x)| = N^\infty(\tau) \qquad ((t,x) \in (0,\tau) \times \mathcal{O} \text{ a.e.}).$$

Finally, if [H5] replaces [H3] and [H4] then the solution of the norm optimal control problem $(NP)^\tau$ is unique.

Time optimal controls. Problem statement

Let $M > 0$ and denote

$$\mathcal{U}_M := \left\{u \mid u \in L^\infty((0,\infty)\times\mathcal{O}) \quad \text{s.t. } \|u\|_{L^\infty((0,\infty)\times\mathcal{O})} \leqslant M\right\}.$$

Solving the time optimal control problem $(TP)^M$ consists in determining $(\tau^\infty(M), u_M^\infty) \in (0,\infty)\times\mathcal{U}_M$ such that u_M^∞ drives the initial state ψ to zero in time $\tau^\infty(M)$, i.e.,

$$\mathbb{T}_{\tau^\infty(M)}\psi + \Phi_{\tau^\infty(M)}u_M^\infty = 0,$$

and

$$(TP)^M : \qquad \tau^\infty(M) := \min_{u\in\mathcal{U}_M}\left\{\tau \mid \exists u \in \mathcal{U}_M \ \text{ with } \ \mathbb{T}_\tau\psi + \Phi_\tau u = 0\right\}.$$

Here the main questions are the existence of optimal pairs, describing their qualitative properties (namely bang-bang) and establishing the relation between time and norm optimal controls.

Time optimal controls. Main result

Theorem 2

Given $\psi \in X \setminus \{0\}$, the time optimal control problem $(TP)^M$ admits at least one solution $(\tau^\infty(M), u_M^\infty)$ if and only if $M > \hat{N} := \inf_{\tau>0} N^\infty$. Moreover,

$$N^\infty(\tau^\infty(M)) = M \qquad (M > \hat{N})$$

$$\tau^\infty(N^\infty(\tau)) = \tau \qquad (\tau \in (0,\infty)).$$

and u_M^∞ can be selected such that it has the bang-bang property

$$|u_M^\infty(t,x)| = M \qquad ((t,x) \in [0, \tau^\infty(M)] \times \mathcal{O} \text{ a.e.}).$$

Finally, if [H5] replaces [H3] and [H4] then this solution is unique.

The multiplier space (I)

For $\tau > 0$, the *multiplier space* Z_τ is the completion of X for

$$\|\eta\|_{Z_\tau} = \int_0^\tau \int_{\mathcal{O}} |(\Phi_t^* \eta)(t,x))| \, \mathrm{d}x \, \mathrm{d}t = \int_0^\tau \int_{\mathcal{O}} |(B^* \mathbb{T}_t^* \eta)(x))| \, \mathrm{d}x \, \mathrm{d}t$$

Proposition 2

Let $\tau > 0$. Then $\mathcal{R}_\tau^\infty$ is the dual of Z_τ with respect to the pivot space X, i.e.,

$$\mathcal{R}_\tau^\infty = \{\xi \in X \;\mid\; \text{s.t.} \sup_{\substack{\eta \in X \\ \|\eta\|_{Z_\tau} \leqslant 1}} |\langle \eta, \xi \rangle_X| < \infty\},$$

$$\|\xi\|_{\mathcal{R}_\tau^\infty} = \sup_{\substack{\eta \in X \\ \|\eta\|_{Z_\tau} \leqslant 1}} |\langle \eta, \xi \rangle_X| \qquad (\xi \in \mathcal{R}_\tau^\infty).$$

The multiplier space (II)

Proposition 3

For every $h > 0$ and every $\eta \in X$ we have

$$\int_0^{\tau+h} \int_{\mathcal{O}} \left| \left(\Phi^*_{\tau+h}\eta\right)(t,x)\right| \,\mathrm{d}x\,\mathrm{d}t \leqslant \left(1+\kappa_h K_\tau^{-1}\right) \int_0^{\tau} \int_{\mathcal{O}} \left|\left(\Phi^*_\tau \eta\right)(t,x)\right| \,\mathrm{d}x\,\mathrm{d}t,$$

where κ_h and K_τ are the constants depending only on h and τ, respectively. Moreover, Z_τ and $\mathcal{R}^\infty_\tau$ do not depend on $\tau > 0$.

Proof of Proposition 3

For every $\eta \in \mathcal{D}(A^*)$ and $\tau,\ h > 0$ we have

$$\int_0^{\tau+h} \int_{\mathcal{O}} |B^* \mathbb{T}_t^* \eta| \, \mathrm{d}x \, \mathrm{d}t$$

$$\leqslant \int_0^{\tau} \int_{\mathcal{O}} |B^* \mathbb{T}_t^* \eta| \, \mathrm{d}x \, \mathrm{d}t + [h\mu(\mathcal{O})]^{\frac{1}{2}} \left[\int_{\tau}^{\tau+h} \int_{\mathcal{O}} |B^* \mathbb{T}_t^* \eta|^2 \, \mathrm{d}x \, \mathrm{d}t \right]^{\frac{1}{2}},$$

$$\int_{\tau}^{\tau+h} \int_{\mathcal{O}} |B^* \mathbb{T}_t^* \eta|^2 \, \mathrm{d}x \, \mathrm{d}t = \int_0^h \int_{\mathcal{O}} |B^* \mathbb{T}_t^* \mathbb{T}_\tau^* \eta|^2 \, \mathrm{d}x \, \mathrm{d}t \leqslant \kappa_h^2 \|\mathbb{T}_\tau^* \eta\|_X^2.$$

The two above inequalities imply our first conclusion. From this we deduce that for every $\tau_1,\ \tau_2 > 0$ the norms $\|\cdot\|_{Z_{\tau_1}}$ and $\|\cdot\|_{Z_{\tau_2}}$ are equivalent on X, thus our second conclusion.

Extending the adjoint semigroup (I)

Proposition 4

Let $\tau > 0$. Then for every $t \in (0, \tau]$ the operator $\mathbb{T}_t^*$ uniquely extends to an operator $\mathbb{S}_t \in \mathcal{L}(Z_\tau, X)$. Moreover, we have

$$\|\mathbb{S}_\tau \eta\|_X \leqslant K_\tau \|\eta\|_{Z_\tau} \qquad\qquad (\eta \in Z_\tau).$$

Moreover, the family $(\mathbb{S}_\sigma)_{\sigma \geqslant 0}$ obtained by setting $\mathbb{S}_0 = \mathbb{I}_{Z_\tau}$ forms a C^0 semigroup on Z_τ.

Extending the adjoint semigroup (II)

Corollary 3

For every $t \in [0,\tau]$ we define the operator Ψ_t by

$$(\Psi_t \eta)(\sigma) = \begin{cases} B^* \mathbb{S}_{t-\sigma}\eta & (\sigma \in [0,t), \eta \in Z_t), \\ 0 & (\sigma > t, z_0 \in Z_t), \end{cases}$$

were $\mathbb{S}$ is the C^0-semigroup introduced in Proposition 4. Then

$$\Psi_t \in \mathcal{L}(Z_\tau, L^1([0,\infty) \times \mathcal{O})), \qquad \Psi_t|_X = \Phi_t^*,$$

$$\left(1 + \kappa_{\tau-t} K_t^{-1}\right) \|\Psi_t \eta\|_{L^1([0,\infty)\times\mathcal{O})} \geqslant \|\eta\|_{Z_\tau},$$

i.e., $(\mathbb{S}, \Psi)$ is L^1 exactly observable on Z_τ.

A dual minimization problem and a maximum principle

For $\tau > 0$ and $\psi \in X$ we set

$$J_\psi^\tau(\eta) = \frac{1}{2}\|\eta\|_{Z_\tau}^2 + \langle \psi, \mathbb{S}_\tau \eta \rangle_X \qquad (\eta \in Z_\tau).$$

Theorem 4

For every $\psi \in X$ and $\tau > 0$ the functional J_ψ^τ admits at least one minimizer $\hat{\eta} \neq 0$ on Z_τ. Moreover, setting

$$\gamma(t,x) = \left(B^* \mathbb{S}_{\tau - t} \widehat{\eta}\right)(x),$$

$$\hat{u}(t,x) = \left(\int_0^\tau \int_{\mathcal{O}} |\gamma(\sigma, y)| \, \mathrm{d}y \, \mathrm{d}\sigma \right) \frac{\gamma(t,x)}{|\gamma(t,x)|},$$

then $\hat{u}$ is a norm optimal control solving $(NP)^\tau$.

Conclusions for the parabolic case

- We obtain the existence of *pointwise* bang-bang norm optimal controls in a very general context, without assuming null controllability from measurable sets.
- Assuming null controllability from measurable sets, we prove also their uniqueness.
- The existence and the properties of time optimal controls are obtained by exploiting their relation with norm optimal controls.

Main result for exactly controllable systems

Theorem 5 (J. Lohéac and M.T.,SICON (2013))

Assume that $B \in \mathcal{L}(U, X)$ *and that* (A, B) *is exactly controllable in any time* $\tau > 0$. *Let* $u^*(t)$ *be the time optimal control, defined on* $[0, \tau^*]$. *Then there exists* $z \in X$, $z \neq 0$ *such that*

$$\mathrm{Re}\,\langle B^*\mathbb{T}^*_{\tau^*-t}z, u^*(t)\rangle = \max_{\|\mathrm{u}\|\leq 1} \mathrm{Re}\,\langle B^*\mathbb{T}^*_{\tau^*-t}z, \mathrm{u}\rangle$$

Moreover, assume that (A, B) *is approximatively controllable from sets of positive measure. Then* $u^*(t)$ *is bang-bang, unique and given by*

$$u^*(t) = \frac{1}{\|B^*\mathbb{T}^*_{\tau^*-t}z\|} B^*\mathbb{T}^*_{\tau^*-t}z \qquad (t \in [0, \tau^*] \quad a.e.),$$

Idea of the proof

- For each $\tau > 0$, we endow X with the norm

$$|||z||| = \inf\{\|u\|_{L^\infty([0,\tau],U)} \ \mid \ \Phi_\tau u = z\}.$$

  Note that $|||\cdot|||$ is equivalent with the original norm $\|\cdot\|$.

- For $\tau > 0$ we set

$$B^\infty(\tau) = \{\Phi_\tau u \ \mid \ \|u\|_{L^\infty([0,\tau],U)} \leqslant 1\},$$

  and we show that if (τ^*, u^*) is an optimal pair then $\Phi_{\tau^*} u^* \in \partial B^\infty(\tau^*)$.

- Using the fact that $B^\infty(\tau^*)$ has a non empty interior, we apply a geometric version of the Hahn-Banach theorem to get the conclusion.

The Schrödinger equation with internal control

Proposition 5 (J. Loheac and M.T., 2013)

Let $\Omega \subset \mathbb{R}^m$ *be a rectangular domain* $\mathcal{O} \subset \Omega$ *be open and non-empty, or let* $\mathcal{O}$ *satisfy the BLR geometric optics condition. Let* $z_0,\ z_1 \in L^2(\Omega)$*. Then the time optimal control problem*

$$\dot{z} = i\Delta z + \chi_{\mathcal{O}} u \qquad \text{in } [0,\infty) \times \Omega,$$

$$z = 0 \quad \textit{on} \ \ (0,\infty) \times \partial\Omega,$$

$$z(0,x) = z_0(x), \quad z(\tau,x) = z_1(x) \qquad (x \in \Omega)$$

$$\|u(\cdot,t)\|_{L^2(\mathcal{O}} \leqslant 1 \qquad a.e. \ .$$

admits a unique solution u^**. Moreover,* u^* *is bang-bang.*

Main steps of the proof of Proposition 5 (I):

We apply Theorem 5 which requires

1. Admissibility: bounded control operator, so obvious here.
2. L^∞ exact controllability in any time: known from works of S. Jaffard (1990) and Lebeau (1994).
3. Approximative controllability with controls supported in any set of positive measure $e \subset [0, \infty)$: this is harder.
   By duality this reduces to a unique continuation problem,which consists in showing that if

$$\dot{z} = i\Delta z \qquad \text{in } [0, \infty) \times \Omega,$$

$$z = 0 \qquad \text{on} \ \ (0, \infty) \times \partial\Omega,$$

$$z = 0 \qquad \text{in} \ \ e \times \mathcal{O},$$

then $z = 0$.

Main steps of the proof of Proposition 5 (II):

Let $(\varphi_n)_{n\in I}$ be an orthonormal basis (in $L^2(\Omega)$) formed of eigenvalues of the Dirichlet Laplacian and let $(\lambda_n)_{n\in I}$ be a corresponding sequence of eigenvalues with $\lambda_n \to \infty$. Then

$$z(t,\cdot) = \sum_{n\in I} a_n e^{i\lambda_n t}\varphi_n \qquad (t\in\mathbb{R})\,,$$

where $a_n = \langle z(0,\cdot),\varphi_n\rangle_X$ for every $n\in I$. We thus know that for all $v\in L^2(\mathcal{O})$

$$g(t) = \sum_{n\in I} a_n\langle\varphi_n, v\rangle_{L^2(\mathcal{O})} e^{i\lambda_n t} \qquad (\ t\in e)\,.$$

Since $g\in\mathcal{H}^\infty(\mathbb{C}_+)\cap C(\overline{\mathbb{C}_+})$ it follows that $g(t)=0$ for $t\in\mathbb{R}$. It follows that $z(t,x)=0$ for $\underline{t\in\mathbb{R}}$ and $x\in\mathcal{O}$ which ends the proof.

1D Schrödinger with space-time constraints

Problem statement. Let $0 \leqslant \alpha < \beta \leqslant \pi$ and $z_0,\ z_1 \in L^2[0,\pi]$. Consider the time optimal control problem

$$\dot{z} = iz_{xx} + \chi_{[\alpha,\beta]}u \qquad \text{in } [0,\infty) \times [0,\pi],$$

$$z(t,0) = z(t,\pi) = 0 \quad \text{on} \quad (0,\infty),$$

$$z(0,x) = z_0(x), \quad z(\tau,x) = z_1(x) \qquad (x \in (0,\pi))$$

$$|u(t,x)| \leqslant 1 \qquad a.e. \ .$$

1. Admissibility: bounded control operator, so obvious here.
2. L^∞ exact controllability in any time? Equivalent to the fact that for every $\tau > 0$ there exists $K_\tau > 0$ such that for $a \in l^2$

$$?\ K_\tau \int_0^\tau \int_\alpha^\beta \sum_{n\geqslant 1} \left| a_n e^{in^2 t} \sin(nx) \right| \, \mathrm{d}x \, \mathrm{d}t \geqslant \|a\|_{l^2}\ ?$$

1D Schrödinger with boundary control

Problem statement. Consider the time optimal control problem

$$\dot{z}(t,x) = i z_{xx}(t,x) \qquad (t \geqslant 0,\ x \in (0,\pi)),$$

$$z_x(t,0) = u(t), \qquad z_x(t,\pi) = 0 \qquad (t > 0).$$

$$z(0,x) = z_0(x), \quad z(\tau,x) = z_1(x) \qquad (x \in (0,\pi))$$

$$|u(t)| \leqslant 1 \qquad a.e.\ .$$

1. Admissibility: Already seen.
2. L^∞ exact controllability in any time? Equivalent to the fact that for every $\tau > 0$ there exists $K_\tau > 0$ such that for $a \in l^2$

$$?\ K_\tau \int_0^\tau \sum_{n\geqslant 1} \left| a_n e^{in^2 t} \right| \mathrm{d}t \geqslant \|a\|_{l^2}\ ?$$

The Kirchhoff plate equation

Proposition 6 (Micu, Roventa and M.T., 2023)

Let $\Omega \subset \mathbb{R}^m$ be a rectangular domain $\mathcal{O} \subset \Omega$ be open and non-empty, or let $\mathcal{O}$ satisfy the BLR geometric optics condition. Let $f_0,\ f_1 \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and $g_0,\ g_1 \in L^2(\Omega)$. Then the time optimal control problem

$$\ddot{z} + \Delta^2 z = \chi_{\mathcal{O}} u \qquad (\text{in } \Omega \times [0, \infty)),$$

$$z(x,t) = \Delta z(x,t) = 0 \quad \text{on} \ \ \partial\Omega \times (0, \infty),$$

$$z(x,0) = f_0(x),\ \dot{z}(x,0) = g_0(x), \quad z(x,\tau) = f_1(x),\ \dot{z}(x,\tau) = g_1(x)$$

$$\|u(\cdot,t)\| \leqslant 1 \qquad a.e.\ .$$

admits a unique solution u^*. Moreover, u^* is bang-bang.

Main steps of the proof of Proposition 6:

1. Denote $\psi = \dot{w}$ and by setting

$$\gamma = \dot{\psi} - iA_0\psi, \tag{4}$$

where A_0 is the "variational" extension of the Dirichlet Laplacian. Then

$$\dot{\gamma}(t) + iA_0\gamma(t) = 0 \qquad (t \geqslant 0),$$

in a weak sense.

2. If $\dot{w}(t,x) = 0$ for $t \in e$ and $x \in \mathcal{O}$ then there exists a set of positive measure $\widetilde{e} \subset e$ such that

$$\langle \gamma(t,\cdot), \varphi \rangle = 0 \qquad (t \in \widetilde{e},\ \varphi \in W_0^{2,2}(\mathcal{O})).$$

3. Adapt the Privalov's theorem based procedure (used for Schrödinger) to prove first that $\gamma = 0$ and then that $w = 0$.

Comments, extensions and open questions

- Clamped boundary conditions ($w = \frac{\partial w}{\partial \nu} = 0$ on $\partial\Omega$): impossible to reduce the unique continuation to the similar question for Schrödinger.
- Boundary control: L^∞ exact controllability is an open question.
- More natural constraint $|u(t,x)| \leqslant 1$ for almost every (t,x): both L^∞ exact controllability and approximate controllability from measurable sets are open questions.