Spectral Deconvolution of Random Matrices via Free Probability

Octavio Arizmendi
CIMAT

joint work with Pierre Tarrago and Carlos Vargas

Trends in Mathematical Sciences.
Friedrich-Alexander-Universität Erlangen-Nürnberg,
June 10 – 14, 2024,

In probability and more generally in mathematics there are many objects that can be represented as matrices and which have a random component by their nature..

Classical Examples

- **Multivariate Statistics** Random Matrices were considered by John Wishart for the statistical analysis of large samples, (covariance matrices) (1928)
- **Physics:** Random matrices were studied by Eugene Wigner to model the spectra of heavy atoms. (1955)
- **Number Theory** Hugh Montgomery and Freeman J. Dyson discovered that the pair correlation distribution of the zeros of the Riemann's zeta function is modeled by the pair correlation distribution of the eigenvalues of certain random matrices. (GUE´s)(1970's) .

Free Probability

Free Probability is a non-commutative probability theory initiated by Voiculescu to tackle problems in Operator Algebras

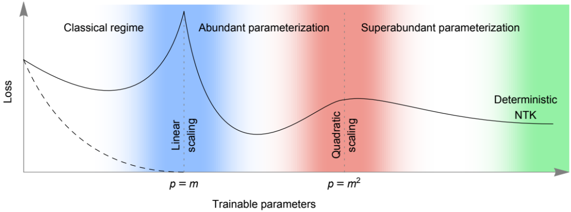
Initial Developments include:

- Limit Theorems, Levy Processes, Infinite Divisibility. (Bercovici & Pata 1999)
- Free Entropy. (Voiculescu 1990's)
- Relation with Random Matrices: Asymptotic Freeness. (Voiculescu 1991)
- Combinatorial Approach based in NC-partitions. (Speicher 1994)

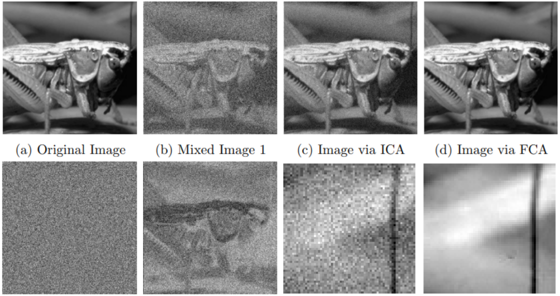
Rcently, random matrices and free have provided insight in connection with:

- **Graphs** :
  Adam Marcus; Daniel Spielman; Nikhil Srivastava (2015). Interlacing families IV: Bipartite Ramanujan graphs of all sizes (PDF). Foundations of Computer Science (FOCS), 2015 IEEE 56th Annual Symposium.
- **Wireless communication.**
  Zheng Z., Wei L., Speicher R., Müller R., Hämäläinen J., Corander J.: Outage capacity of Rayleigh product channels: a free probability approach In: IEEE Transactions on Information Theory 63 (2017), S. 1731-1745
- **Quantum Information:**
  Belinschi, S. T., Collins, B., & Nechita, I. (2016). Almost one bit violation for the additivity of the minimum output entropy. Communications in Mathematical Physics, 341, 885-909.

- **Neural Networks/Machine Learning** Adlam, B., & Pennington, J. (2020, November). The neural tangent kernel in high dimensions: Triple descent and a multi-scale theory of generalization. In International Conference on Machine Learning (pp. 74-84). PMLR.



- **Image Analysis.** Nadakuditi, R.R., Wu, H. Free Component Analysis: Theory, Algorithms and Applications. Found. Comput. Math 23, 973–1042 (2023).



(a) Original Image (b) Mixed Image 1 (c) Image via ICA (d) Image via FCA

(e) Gaussian Noise (f) Mixed Image 2 (g) Image via ICA (zoom in) (h) Image via FCA (zoom in)

Random Matrices

What properties are we interested in Random Matrices?

- Properties which are studied include eigenvalues, eigensapces invertibility, norm, rank, spectral gap, etc.
- **Today we are interested in the global behavior of the eigenvalues** $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$ **from large random matrices A.**
- A is random ,thus $\{\lambda_1, \ldots, \lambda_n\}$ are random.

Random Matrices

- Given a matrix A (not necessarily random) the probabiity measure that assigns $1/n$ to each eigenvalue is called the **spectral distribution** of A:

$$\mu_A := \frac{1}{n}\delta_{\lambda_1} + \cdots + \frac{1}{n}\delta_{\lambda_n}$$

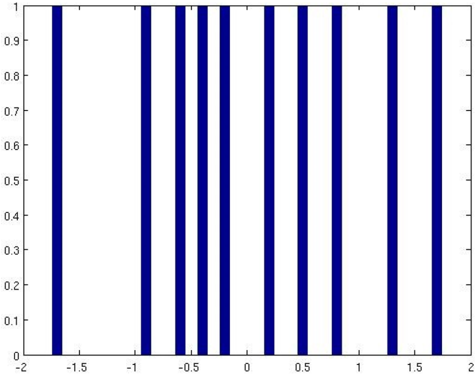
- If A random, the spectral distribution of A is a **random distribution.**
- When $n \gg 1$ for many ensembles $\{A_n\}_{n>1}$ of random matrices, as $n \to \infty$, the behavior their eigenvalues stabilizes.
- In other words, there is convergence to a **unique** spectral distribution.

Wigner Matrices

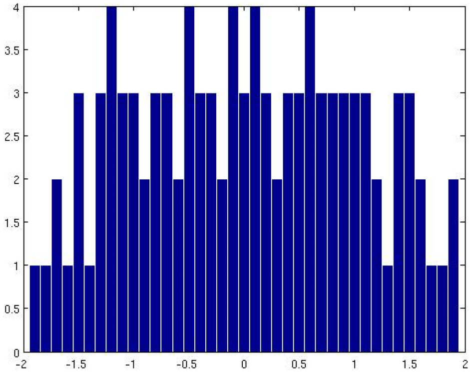
Definition

A Wigner matrix is an $N \times N$ random matrix $W^{(N)} = (W_{ij})_{i,j=1}^{N}$ with $W = W^*$ and such that the entries $(W_{ij})_{i,j=1,i\geq j}^{N}$ are independent and identically distributed with mean zero and variance σ.
The family $\{W^{(N)}\}_{N\in\mathbb{N}}$ is called a Wigner ensemble.

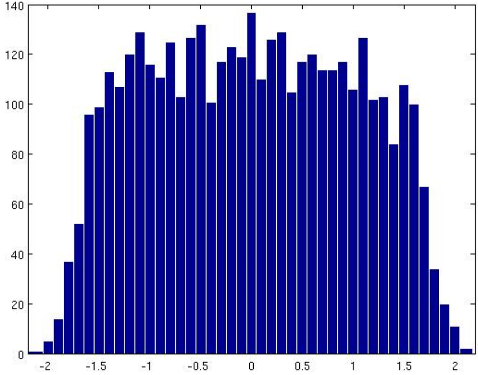
Matrices de Wigner



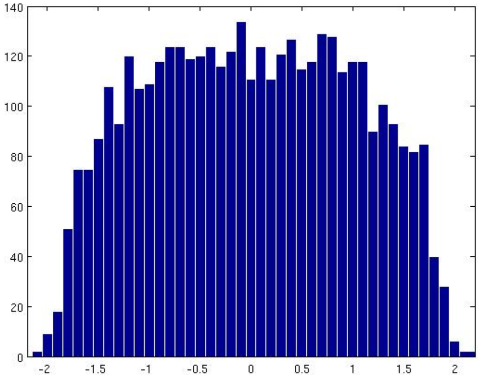
Matrices de Wigner



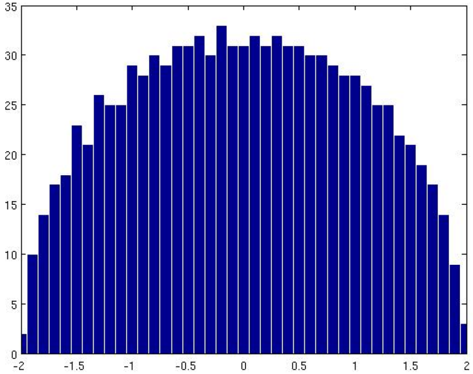
Matrices de Wigner



Matrices de Wigner



Matrices de Wigner

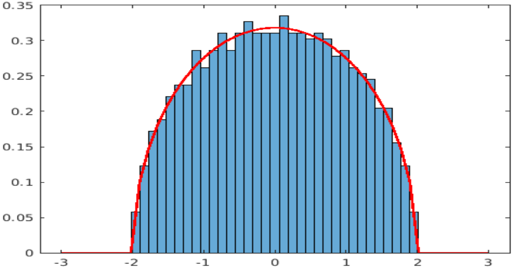


Wigner's Theorem is stated as follows.

Theorem (Wigner's Semicircle Law)

Let $W^{(N)}_{N\in\mathbb{N}}$ be a Wigner ensemble whose entries have variance $1/N$. Then **a.s.** *the spectral distribution of W_N converges weakly as $N \to \infty$, towards a semicircle distribution,*

$$W^{(N)} \to s.$$



(General) spectral deconvolution problem

Three ingredients :

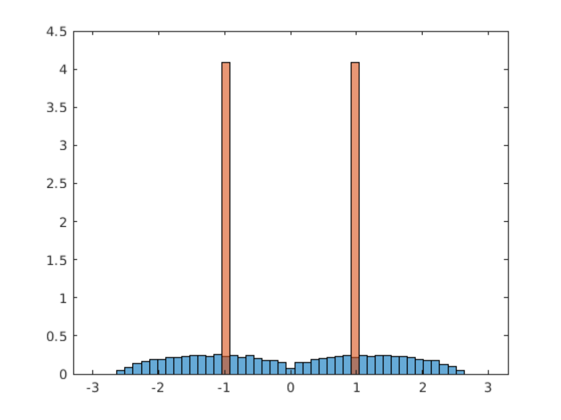
1. $A \in M_N(\mathbb{C})$, $A = A^*$, the *unknown* "signal" matrix,
2. $\mathbf{X} = (X_1, \ldots, X_r) \in M_N(\mathbb{C})^r$, the "noise" matrices, known only statistically in the large N limit.
3. $W = f(A, X_1, X_1^*, \ldots, X_r, X_r^*)$, the *known* "data" matrix, with $f \in \mathbb{C}\langle U, V_1, V_1^*, \ldots, V_r, V_r^*\rangle$ self-adjoint polynomial.

Main goal : recover μ_A from W.

Example

$$W = A + X,$$

with X Wigner, and A diagonal with $\mu_A = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_1$.



Eigenvalues of the data (*blue*) and of the signal (*orange*).

Applications

- Wireless communication :

$$W = (X_1 A + X_2)^*(X_1 A + X_2),$$

with X_1, X_2 Ginibre matrices (A matrix of emission powers, X_1 matrix of random messages, X_2 noise matrix from environment, W signal matrix) $\rightsquigarrow$ General linear model (Ryan, Debbah, 2007).

Applications

- Wireless communication :

$$W = (X_1 A + X_2)^*(X_1 A + X_2),$$

with X_1, X_2 Ginibre matrices (A matrix of emission powers, X_1 matrix of random messages, X_2 noise matrix from environment, W signal matrix) $\rightsquigarrow$ General linear model (Ryan, Debbah, 2007).

- Cleaning of covariance matrix :

$$W = XAX^*$$

with X Ginibre matrix (Ledoit-Péché (2011), Ledoit-Wolf (2013))

Applications

- Wireless communication :

$$W = (X_1 A + X_2)^*(X_1 A + X_2),$$

with X_1, X_2 Ginibre matrices (A matrix of emission powers, X_1 matrix of random messages, X_2 noise matrix from environment, W signal matrix) $\rightsquigarrow$ General linear model (Ryan, Debbah, 2007).

- Cleaning of covariance matrix :

$$W = XAX^*$$

with X Ginibre matrix (Ledoit-Péché (2011), Ledoit-Wolf (2013))

- Generalized/structured covariance matrix :

$$W = XAX^*$$

with $X = BG$ $B \geq 0$, G Ginibre matrix (Buns, Allez, Bouchaud, Potter (2016)). e.g. $(B^2)_{ij} = \exp(|i - j|/\tau)$.

Other approaches:

- Covariance matrices :
  - El Karoui (2008) : first results on the subject.
  - Rao (2008) : atomic case.
  - Bai and al. (2010) : deconvolution using moments.
  - Ledoit and Wolf (2013) : "QuEST" algorithm for estimating covariance matrices, numerical inversion of the Marchenko-Pastur equation.

Other approaches:

- Covariance matrices :
  - El Karoui (2008) : first results on the subject.
  - Rao (2008) : atomic case.
  - Bai and al. (2010) : deconvolution using moments.
  - Ledoit and Wolf (2013) : "QuEST" algorithm for estimating covariance matrices, numerical inversion of the Marchenko-Pastur equation.
- Bun, Allez, Bouchaud, Potters (2016) : Rotationally Invariant Estimator of noisy matrix models, using the spectral deconvolution as an oracle (some spectral deconvolution are achieved using the R/S-transform).
- Hayase (2020) : use of Cauchy distribution.
- Maïda et al. (preprint, 2022) : study of the backward Fokker-Planck equation.
- Chhaibi, et al (2023). Estimation of large covariance matrices using free probability. Using R/S-transform and analytic extension)

Today's problem

Consider a matrix of the form

$$B = A + W$$

or

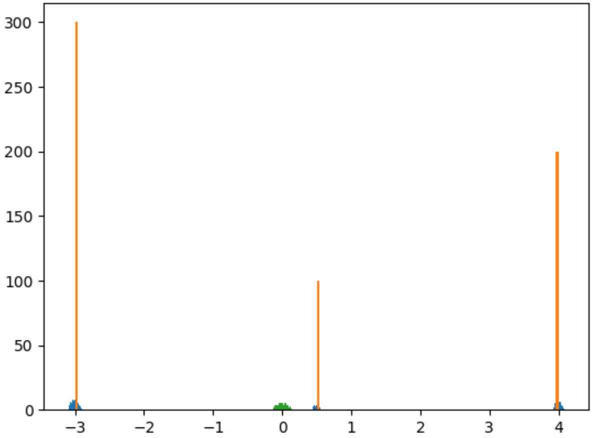
$$B = WAW^T$$

where

- W is some noise. (e.g W is Gaussian), whose spectrum we know statistically.
- We observe the spectrum of B.

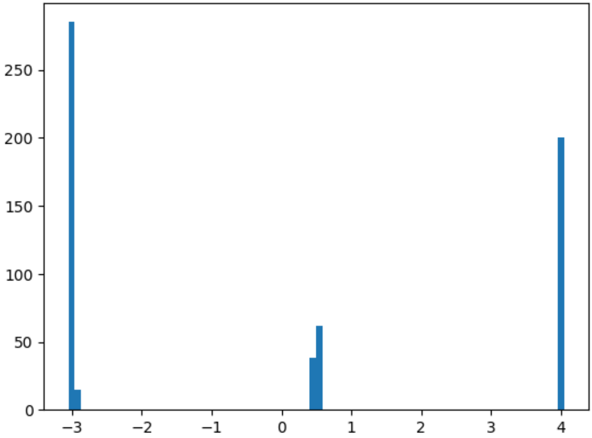
Problem: Estimate the spectrum of A.

Additive noise



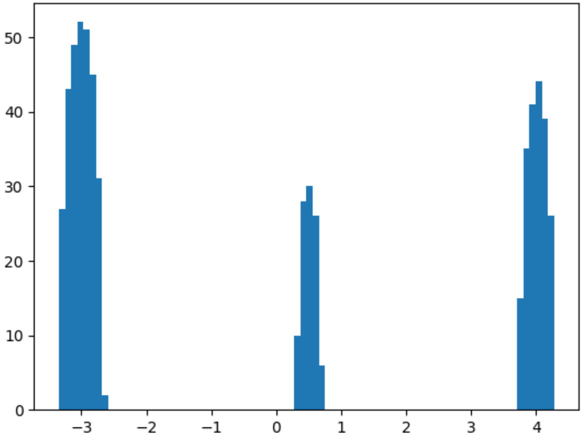
$$B = A + W$$

Additive noise



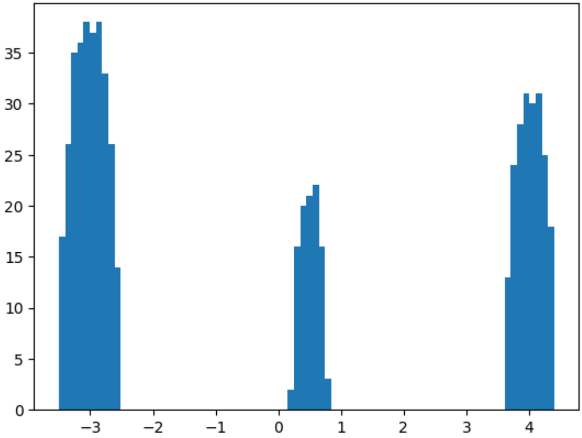
$$B = A + W$$

Additive noise



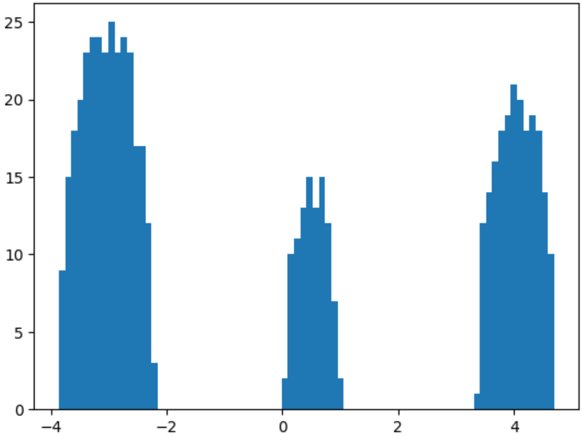
$$B = A + W$$

Additive noise



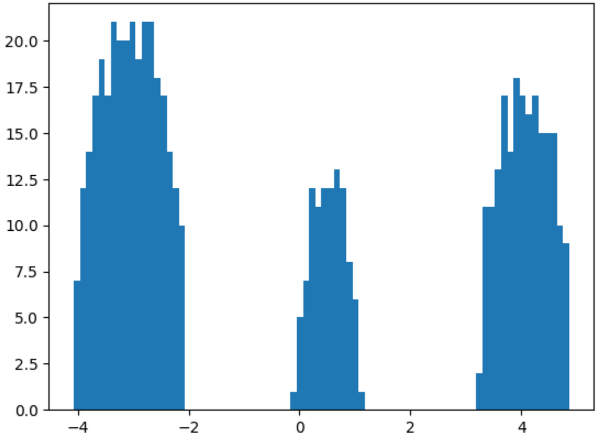
$$B = A + W$$

Additive noise



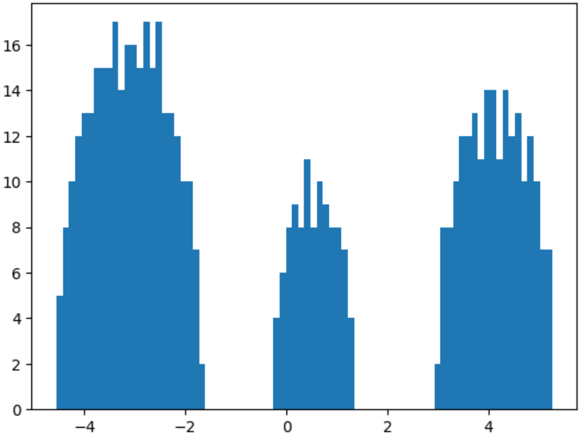
$$B = A + W$$

Additive noise



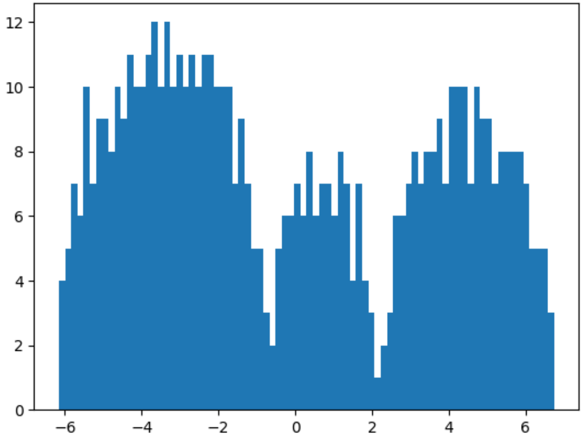
$$B = A + W$$

Additive noise



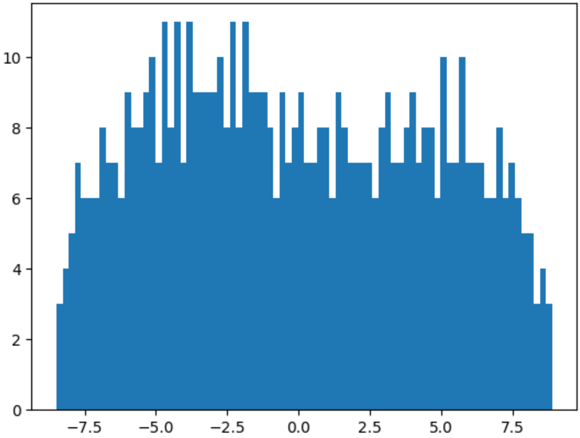
$$B = A + W$$

Additive noise



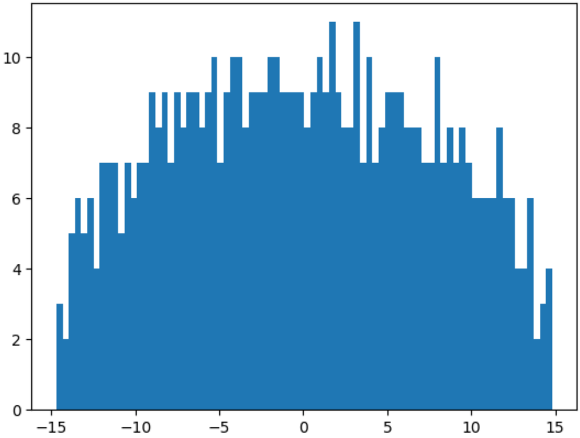
$$B = A + W$$

Additive noise



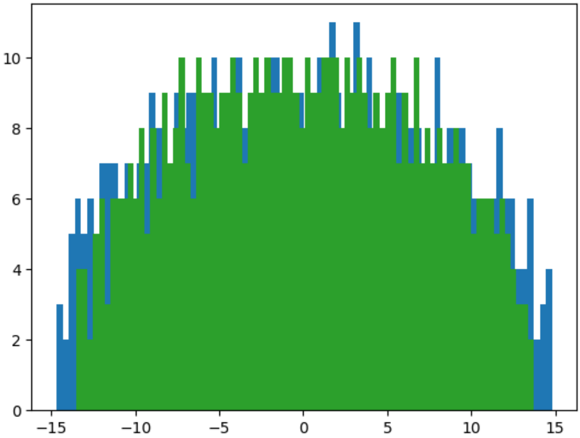
$$B = A + W$$

Additive noise



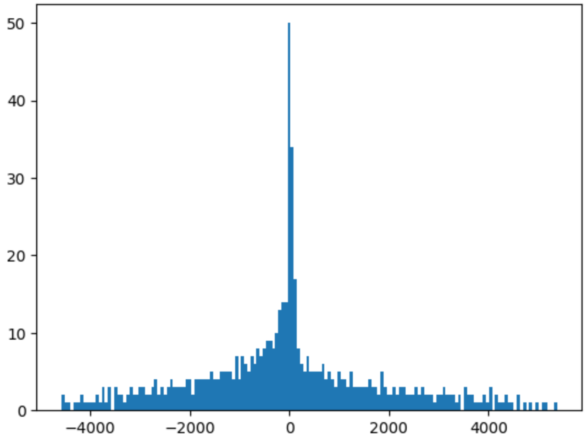
$$B = A + W$$

Additive noise



$$B = A + W(blue) \quad vs \quad W(green)$$

Multiplicative noise



$$B = WAW^T$$

I. Free Probability

Non Commutative Probability

Definition

A non commutative probability space is a pair $(\mathcal{A}, \tau)$, where $\mathcal{A}$ is an algebra (over $\mathbb{C}$) with unit $1_{\mathcal{A}}$ and τ is a linear functional such that $\tau(1_{\mathcal{A}}) = 1$.

Main spirit:
Look at

- Algebras instead of σ- algebras.
- "Expectation" τ, instead of probability.
- Random variables rather than events.

Non Commutative Probability

- **Random Variables**: The elements of $\mathcal{A}$ are called (non-commutative) random variables.
- **Real ones:** An element $a \in \mathcal{A}$ such that $a = a^*$ is called self-adjoint.
- **Positivity:** τ is *positive* if $\tau(a^*a) \geq 0$.
- **Usual Framework;** $\mathcal{A}$ is a unital C^*-algebra and $\tau : \mathcal{A} \to \mathbb{C}$ is a positive unital linear functional. The pair $(\mathcal{A}, \tau)$ is called a *C^*-probability space*.

- **Moments**: For $a \in \mathcal{A}$, we will refer to the value of $\tau(a^k)$, $k \in \mathbb{N}$ as the k-th moment of a.

Non Commutative Probability

- **Moments**: For $a \in \mathcal{A}$, we will refer to the value of $\tau(a^k)$, $k \in \mathbb{N}$ as the k-th moment of a.
- **Distribution:** For any self-adjoint element $a \in \mathcal{A}$ (by the Riesz-Markov-Kakutani representation theorem) there exists a unique probability measure μ_a with the same moments as a, that is,

$$\int_{\mathbb{R}} x^k \mu_a(dx) = \tau(a^k), \quad \forall k \in \mathbb{N}.$$

  We call μ_a the distribution of a with respect to τ.

Tensor/Free Independence

Definition

A family of subalgebras $(\mathcal{A}_i)_{i\in I}$ of $\mathcal{A}$ is said **tensor** independent if they **commute** and for all $a_1 \in A_{i(1)}, ..., a_n \in A_{i(n)}$

$$\phi(a_1 a_2 ... a_n) = \phi(a_1)\phi(a_2)...\phi(a_n)$$

whenever $i(j) \neq i(k)$ for all $j \neq k$.

Definition (Voiculescu)

Let (A, ϕ) be a NCPS and let $\mathcal{A}_1, \mathcal{A}_2$ be subalgebras of A. We say that $\mathcal{A}_1$ is free from $\mathcal{A}_2$ if for all $a_i \in A_1$ $b_j \in A_2$ such that $\phi(a_i) = 0$ and $\phi(b_i) = 0$ for all i then

$$\phi(a_1 b_1 a_2 b_2 \cdots a_n b_n) = 0.$$

First non-trivial mixed moment:

Tensor

$$\phi(abab) = \phi(a^2)\phi(b^2)$$

Free

$$\phi(abab) = \phi(a)^2\phi(b^2) + \phi(a^2)\phi(b)^2 - \phi(a)^2\phi(b)^2$$

Free Convolutions

- Free additive convolution $\boxplus : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, that associates to the couple (μ, ν) the distribution $\mu \boxplus \nu := \mu_{X+Y}$, where $X \sim \mu$, $Y \sim \nu$ and X is free from Y.
- Free multiplicative convolution $\boxtimes : \mathcal{M}^+ \times \mathcal{M} \to \mathcal{M}$, that associates to the couple (μ, ν) the distribution $\mu \boxtimes \nu := \mu_{X^{1/2} Y X^{1/2}}$, where $X \sim \mu$, $Y \sim \nu$ and X is free from Y.

Asymptotic Freeness

Convergence in joint distribution

Definition (Convergence in joint distribution)

Let $(A_n, \tau_n)_{n>0}$ and (A, τ) be NCPS an let $a_n, b_n \in A_n$. We say that the pair (a_n, b_n) converges in joint ditribution to (a, b)

$$\tau_n(a_n^{l_1} b_n^{m_1} \cdots a_n^{l_k} b_n^{m_k}) \to \tau(a^{l_1} b^{m_1} \cdots a^{l_k} b^{m_k}).$$

Convergence in joint distribution

Definition (Convergence in joint distribution)

Let $(A_n, \tau_n)_{n>0}$ and (A, τ) be NCPS an let $a_n, b_n \in A_n$. We say that the pair (a_n, b_n) converges in joint ditribution to (a, b)

$$\tau_n(a_n^{l_1} b_n^{m_1} \cdots a_n^{l_k} b_n^{m_k}) \to \tau(a^{l_1} b^{m_1} \cdots a^{l_k} b^{m_k}).$$

- Two sequences of random variables $\{a_n\}_n$ and $\{b_n\}_n$ are said to be **asymptotically free** if they converge to a pair a, b of free random variables in some NCPS.

Asymptotic Freeness

Theorem (Voiculescu)

- *Let* $\{G_1^{(n)}\}_{n>0}$ *and* $\{G_2^{(n)}\}_{n>0}$ *be two independent sequence of nxn Gaussian matrices, then* $\{G_1^{(n)}\}_{n>0}$ *and* $\{G_2^{(n)}\}_{n>0}$ *are asymptotically free, as* $n \to \infty$.

Asymptotic Freeness

Theorem (Voiculescu)

- *Let* $\{G_1^{(n)}\}_{n>0}$ *and* $\{G_2^{(n)}\}_{n>0}$ *be two independent sequence of nxn Gaussian matrices, then* $\{G_1^{(n)}\}_{n>0}$ *and* $\{G_2^{(n)}\}_{n>0}$ *are asymptotically free, as* $n \to \infty$*.*
- *Let* $\{A_n\}_{n>0}$ *and* $\{B_n\}_{n>0}$ *be two sequences of deterministic random matrices with limiting distributions and let* U_n *be a Haar distributed unitary matrix. Then* $\{U_n A U_n^*\}_{n>0}$ *and* $\{B_n\}_{n>0}$ *are asymptotically free.*

Theorem (Voiculescu)

- *Let* $\{G_1^{(n)}\}_{n>0}$ *and* $\{G_2^{(n)}\}_{n>0}$ *be two independent sequence of nxn Gaussian matrices, then* $\{G_1^{(n)}\}_{n>0}$ *and* $\{G_2^{(n)}\}_{n>0}$ *are asymptotically free, as* $n \to \infty$.
- *Let* $\{A_n\}_{n>0}$ *and* $\{B_n\}_{n>0}$ *be two sequences of deterministic random matrices with limiting distributions and let* U_n *be a Haar distributed unitary matrix. Then* $\{U_n A U_n^*\}_{n>0}$ *and* $\{B_n\}_{n>0}$ *are asymptotically free.*

"Intuition": When in random position all the relations are lost and thus they behave like free.

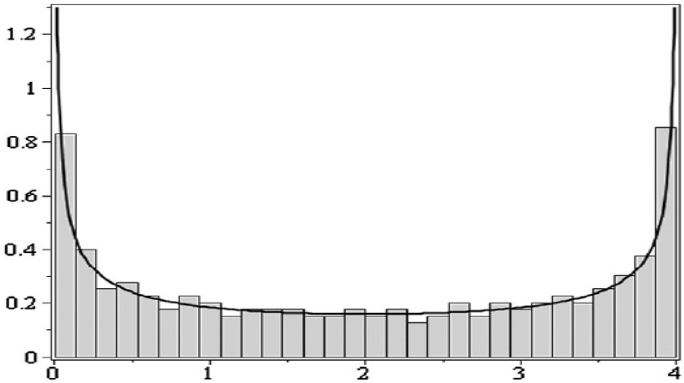
Consquences of Asymtotic Freeness

Asymtotic Freeness implies that for large N.

1. If $B = A + W$ (*additive case*) on $\mu_B \simeq \mu_A \boxplus / \boxtimes \mu_W$
2. If $B = WAW^*$ (*multiplicative case*), and we set $X = WW^*$ and $\mu_B \simeq \mu_A \boxplus \mu_X$.

- as N goes to infinity in law : Voiculescu (1991),
- as N goes to infinity, almost surely : Speicher (1993),
- concentration results for fixed N : Kargin (2015), Bao, Erdösz and Schnelli (2017), Meckes and Meckes (2013).
- Large deviation principle : Belinschi, Guionnet and Huang (2020).

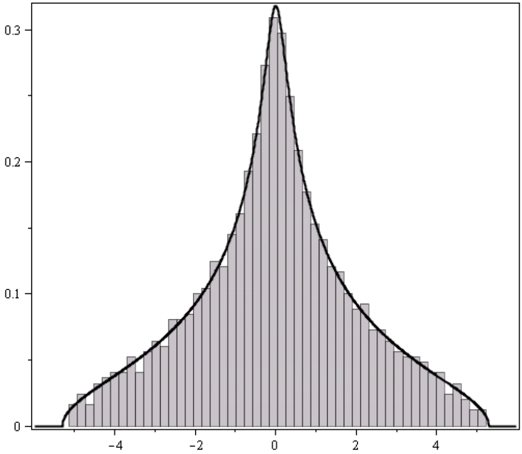
Asymptotic Freeness



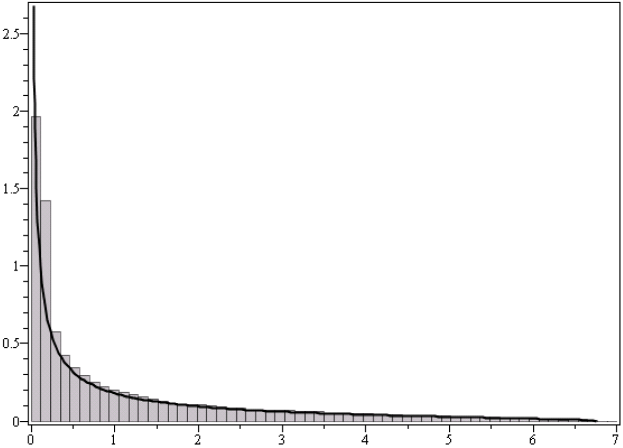
$$\frac{1}{\pi\sqrt{4-t^2}} \quad |t| < 2.$$

$$B_1 + UB_2U^* : \textit{freeaddition}$$

Asymptotic Freeness

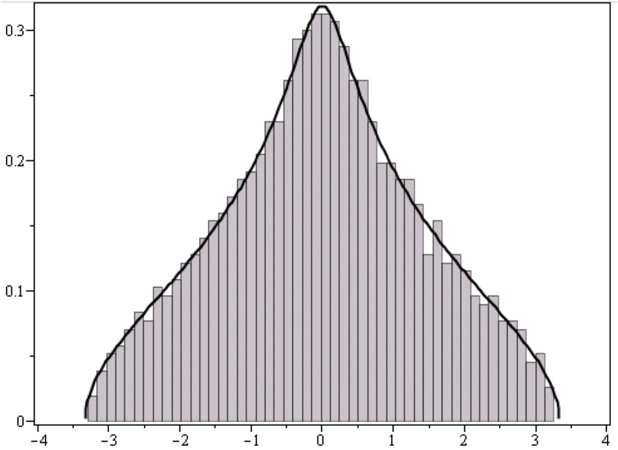


$W_1 W_2^2$: *free multiplication*



$W_1^2 W_2^2$: *free multiplication*

Asymptotic Freeness



$W_1 W_2 - W_2 W_1$: *free commutator*

Convolutions of measures.
(analytic perspective)

The **Cauchy transform** of a probability measure μ is defined as

$$G_\mu(z) := \int_{\mathbb{R}} \frac{1}{z-t} d\mu(t) \quad \text{for } z \in \mathbb{C} \setminus supp(\mu).$$

Importance:

1. $G_{\mu_n}(z) \to G_{\mu_n}(z)$ implies weak convergence.
2. Used to calculate convolutions in free probability.
3. We can recover the density by knowning G close to the real line: Stieljes inversion formula.

$$d\mu(x) = -\frac{1}{\pi} \lim_{y \to 0+} ImG_\mu(x+iy) \text{ for almost all } x. \tag{1}$$

Free Convolutions

The **free convolution** $\mu \boxplus \nu$

$$G_{\mu \boxplus \nu}^{<-1>}(z) = G_{\mu}^{<-1>}(z) + G_{\nu}^{<-1>}(z) - 1/z \quad \text{for } z \in \Gamma.$$

So if we call $G_{\mu}^{-1}(z) - 1/z = R_{\mu}(z)$.
Then

$$R_{\mu \boxplus \nu} = R_{\mu} + R_{\nu}$$

Warning!!! In general, the exact computation is not posible.

Free convolution via Subordination

Suppose that $\mu_3 = \mu_1 \boxplus \mu_2$. How to compute μ_3 from μ_1 and μ_2 ?

There exist $\omega_1, \omega_2 : \mathbb{C}^+ \to \mathbb{C}^+$ such that for $z \in \mathbb{C}^+$,

1. $G_{\mu_3}(z) = G_{\mu_1}(\omega_1(z)) = G_{\mu_2}(\omega_2(z))$ (*subordination property*)
2. $\omega_1(z) + \omega_2(z) = z + \frac{1}{G_{\mu_3}(z)}$ (*free additive relation*)

Free convolution via Subordination

Suppose that $\mu_3 = \mu_1 \boxplus \mu_2$. How to compute μ_3 from μ_1 and μ_2 ?

There exist $\omega_1, \omega_2 : \mathbb{C}^+ \to \mathbb{C}^+$ such that for $z \in \mathbb{C}^+$,

1. $G_{\mu_3}(z) = G_{\mu_1}(\omega_1(z)) = G_{\mu_2}(\omega_2(z))$ (*subordination property*)
2. $\omega_1(z) + \omega_2(z) = z + \frac{1}{G_{\mu_3}(z)}$ (*free additive relation*)

Using 1. and 2., one has access to $\omega_1(z)$ as a fixed point equation/iteration limit

$$\omega_1(z) = K_z(\omega_1(z)) = \lim_{n\to\infty} K_z^{\circ n}(w),$$

for any $w \in \mathbb{C}^+$, with

$$K_z(w) = h_{\mu_2}(h_{\mu_1}(w) + z) + z,$$

where $h_\mu(z) = G_\mu(z)^{-1} - z$.

Set $\mathbb{C}_\sigma = \{z \in \mathbb{C}, \Im z > \sigma\}$, and write $\sigma_1^2 = \mathrm{Var}(\mu_1)$.

Free deconvolution : the subordination method

Set $\mathbb{C}_\sigma = \{z \in \mathbb{C}, \Im z > \sigma\}$, and write $\sigma_1^2 = \mathrm{Var}(\mu_1)$.

Theorem (Arizmendi, Vargas, T.)

There exist two analytic functions $\omega_1, \omega_3 : \mathbb{C}_{2\sqrt{2}\sigma_1} \to \mathbb{C}^+$ *such that for all* $z \in \mathbb{C}_{2\sqrt{2}\sigma_1}$,

1. $G_{\mu_2}(z) = G_{\mu_1}(\omega_1(z)) = G_{\mu_3}(\omega_3(z))$ *(subordination property).*
2. $\omega_1 + z = \omega_3 + F_{\mu_1}(\omega_1(z)) = \omega_3 + F_{\mu_3}(\omega_3(z))$.

Moreover, $\omega_3(z)$ *is the unique fixed point of the function* $K_z(w) = z - h_{\mu_1}(w + F_{\mu_3}(w) - z)$ *in* $\mathbb{C}_{3\Im(z)/4}$ *and we have*

$$\omega_3(z) = \lim K_z^{\circ n}(w),\ w \in \mathbb{C}_{3\Im(z)/4}.$$

Free deconvolution : the subordination method

Suppose that $\mu_3 = \mu_1 \boxplus \mu_2$. How to compute μ_2 from μ_1 and μ_3 ?

Deconvolution method :

1. using the latter theorem, compute ω_3 and $G_{\mu_2} = G_{\mu_3} \circ \omega_3$ on the horizontal line $L = \{x + 2\sqrt{2}\sigma_1 i, x \in \mathbb{R}\}$.

Free deconvolution : the subordination method

Suppose that $\mu_3 = \mu_1 \boxplus \mu_2$. How to compute μ_2 from μ_1 and μ_3 ?

Deconvolution method :

1. using the latter theorem, compute ω_3 and $G_{\mu_2} = G_{\mu_3} \circ \omega_3$ on the horizontal line $L = \{x + 2\sqrt{2}\sigma_1 i, x \in \mathbb{R}\}$.
2. Fact : using the Stieltjes inversion formula on L instead of $\mathbb{R}$, we obtain the density f of the classical convolution $\mu_2 * \mathcal{C}_{2\sqrt{2}\sigma_1}$, where $d\mathcal{C}_\lambda(t) = dCauchy_\lambda(t) = \frac{\lambda}{\pi(t^2+\lambda^2)}$.

Free deconvolution : the subordination method

Suppose that $\mu_3 = \mu_1 \boxplus \mu_2$. How to compute μ_2 from μ_1 and μ_3 ?

Deconvolution method :

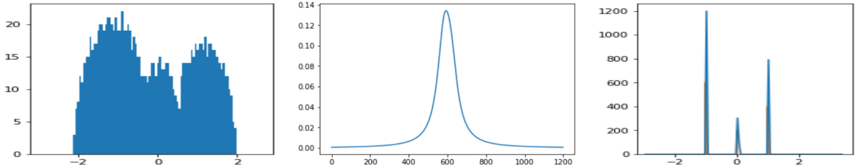
1. using the latter theorem, compute ω_3 and $G_{\mu_2} = G_{\mu_3} \circ \omega_3$ on the horizontal line $L = \{x + 2\sqrt{2}\sigma_1 i, x \in \mathbb{R}\}$.
2. Fact : using the Stieltjes inversion formula on L instead of $\mathbb{R}$, we obtain the density f of the classical convolution $\mu_2 * \mathcal{C}_{2\sqrt{2}\sigma_1}$, where $d\mathcal{C}_\lambda(t) = dCauchy_\lambda(t) = \frac{\lambda}{\pi(t^2+\lambda^2)}$.
3. Using classical deconvolution tools, recover μ_2 by doing the deconvolution of f by $\mathcal{C}_{2\sqrt{2}\sigma_1}$.

Exemple of deconvolution tools (regularization needed !):

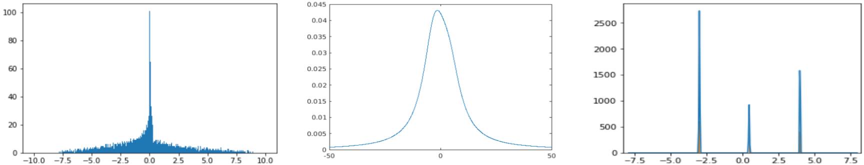
- regularized Fourier transform,
- Tychonov's regularization,
- Total Variation minimization procedure.

Examples

Addition:



Multiplication



Comparison with Ledoit-Wolf

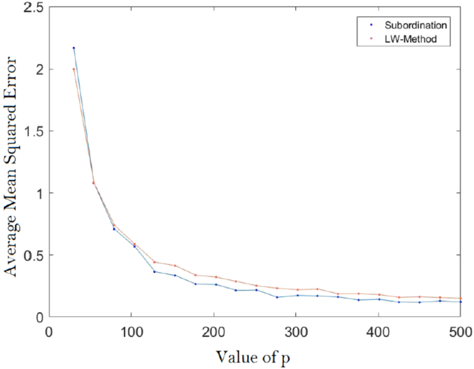


Fig. 7. Accuracy in terms of mean squared error in the subordination method compared to Ledoit–Wolf method.

Comparison with Ledoit-Wolf

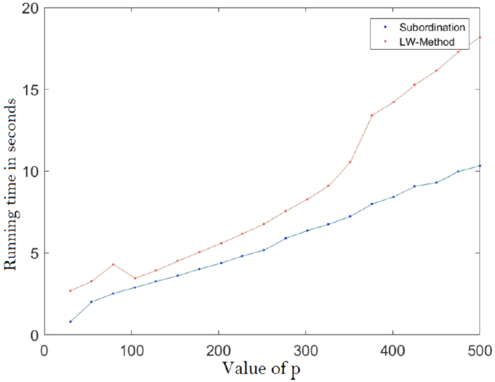


Fig. 8. Speed of the subordination method compared to the speed in the Ledoit–Wolf method.

Comparison with Ledoit-Wolf

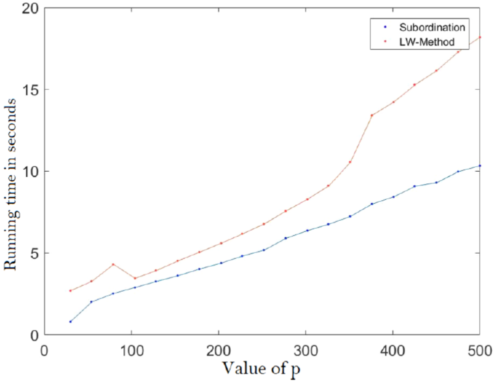


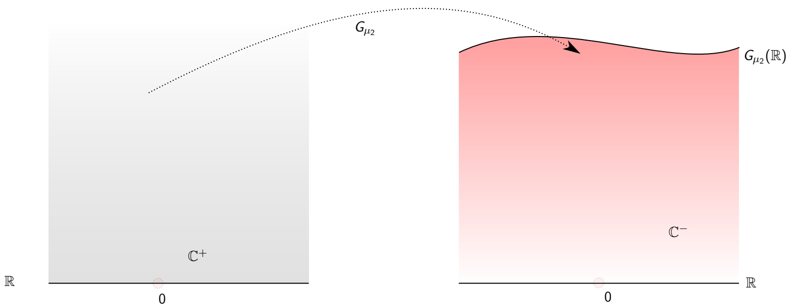
Fig. 8. Speed of the subordination method compared to the speed in the Ledoit–Wolf method.

Futher analysis: Tarrago, P. (2023). Spectral deconvolution of matrix models: the additive case. Information and Inference: A Journal of the IMA, 12(4), 2629-2689.

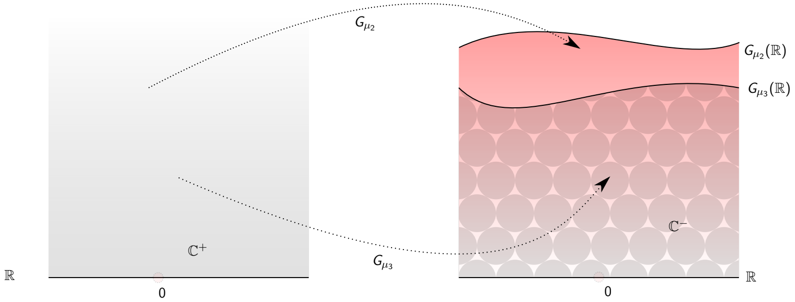
Thanks!

Geometry of subordination property and classical deconvolution

Geometry of subordination property and classical deconvolution

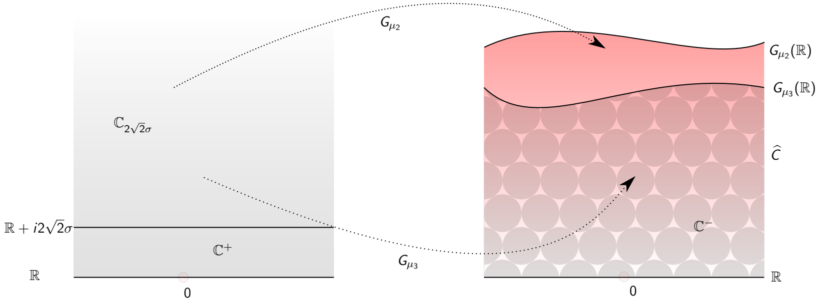


Geometry of subordination property and classical deconvolution

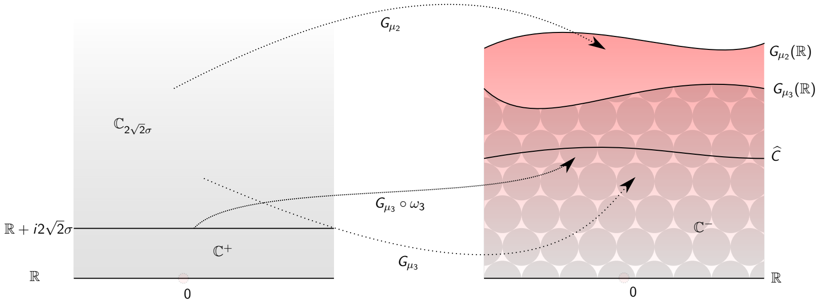


$$G_{\mu_3}(\mathbb{C}^+) \subset G_{\mu_2}(\mathbb{C}^+).$$

Geometry of subordination property and classical deconvolution

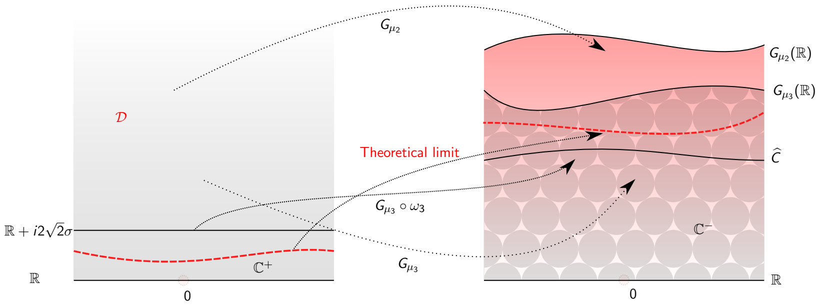


Stability of the free deconvolution : heuristic from the subordination property



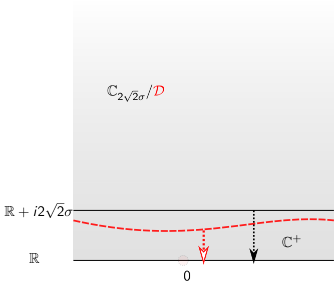
Subordination of the deconvolution.

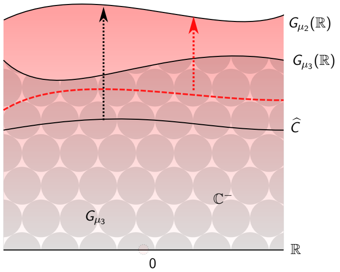
Geometry of subordination property and classical deconvolution



Maximum domain of recovery of G_{μ_2}.

Geometry of subordination property and classical deconvolution





Extension of G_{μ_2} from $\mathbb{C}_{2\sqrt{2Var(\mu_0)}}/\mathcal{D}_{\max}$ to $\mathbb{C}^+$.

In the case $\mathbb{C}_{2\sqrt{2Var(\mu_0)}}$, this equivalent to a classical deconvolution.