The Hele-Shaw free boundary limit of Buckley-Leverett System

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Outline

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The Buckley-Leverett System

The Buckley-Leverett System has been studied by many authors because, it plays an important role in a wide range of the planning and operation of oil wells, filtration problems, dam study, and also in a large class of interesting similar mathematical models:

1. G. Chavent, J. Jaffre, Mathematical Models and Finite Elements for Reservoir Simulation, North Holland, Amsterdam, 1986.

2. M.C. Bustos, F. Concha, R. Bürger, E.M. Tory, Sedimentation and Thickening: Phenomenological Foundation and Mathematical Theory, Kluwer Academic, Dordrecht, 1999.

The Buckley-Leverett system

The most prototypical example is the encroachment of water into an oil sand, that is to say, the flow-mixture of two phases (oil and water) in a porous medium.

The Buckley-Leverett System consists in a conservation law that expresses the mass balance for the density $u(t, x)$ of the fluid-mixture evolving according to the nonlinear continuity equation

$$\partial_t u + \operatorname{div}(g(u)\,\mathbf{v}) = 0,$$

where $g(u)$ represents an additional volume filling term in the density flux, and $\mathbf{v}(t, x)$ is the (seepage) velocity field given by

$$h(u)\mathbf{v} = -\nabla p,$$

obtained from the empirical Darcy's law, (applied for each component of the mixture), which describes the dynamics of the flow in relation with the scalar function $p(t, x)$, usually called pressure.

Indeed, we observe that here $p(t, x)$ is not given by a law of state, but it is related to the incompressibility condition for the velocity field of the flow, that is, $\text{div}(\mathbf{v}) = 0$.

Then, we may write the so called Buckley-Leverett System as

$$\begin{cases} \partial_t u - \mathrm{div}(\Theta(u)\nabla p) = 0, \\ \mathrm{div}\Big(\dfrac{\nabla p}{h(u)}\Big) = 0, \end{cases} \tag{1}$$

where $\Theta(u) = g(u)/h(u)$, and $h(u) \geq h_0$ for some $h_0 > 0$. Here, h^{-1} has the physical meaning of the total mobility, and we recall that (1) is also called the weak formulation of multidimensional Muskat problem.

One observes that, the model (1) could be posed in $(0,T) \times \Omega$, for $T > 0$ and $\Omega \subset \mathbb{R}^d$ be an open bounded domain.

Thus we prescribe some influx and outflux of the flow at the boundary of Ω, denoted by Γ, that is

$$\big(\Theta(u)\, \nabla p\big) \cdot \mathrm{n} = f \quad \text{on } \Gamma_T, \tag{2}$$

where $\mathrm{n}(r)$ denotes the unitary outer normal at $r \in \Gamma$, $\Gamma_T = (0,T) \times \Gamma$, and f is a given normal boundary flux, such that $|f(t,r)| \geq f_b$ for some $f_b > 0$.

Also, the equation (1) should be supplemented with an initial data for the density, that is, $u(0,x) = u_0(x)$ in Ω.

Difficulties and previous works

First, we observe from the Buckley-Leverett System that the velocity $\mathbf{v}(t, x)$ is expected to be a L^2 vector field. Then, we have to treat with a scalar conservation law in the class of roughly coefficients.

The mathematical problem of proving global well-posedness of weak solutions for the Buckley-Leverett system is a mathematical challenge.

In fact, the main question of showing existence of weak solutions remains open for many decades.

Moreover, due to the presence of discontinities (shock waves) the entropy formulaion is usually adopted.

In particular, if one tries to use the kinetic formulation of scalar conservation laws, then it is well known that we do not have the DiPerna–Lions renormalization property for the associated transport equation when the drift vector field is just in L^2, which is an important issue to show compactness of approaching sequences of regular solutions using, for instance, the rigidity result of Perthame-Dalibard.

We also observe that, the flux function $g(u)\mathbf{v}$ is degenerated, that is, there exists $\xi \in \mathsf{S}^{d-1}$, such that the mapping

$$\lambda \mapsto g(\lambda)\mathbf{v}(t,x) \cdot \xi$$

is constant on non-degenerated intervals, for (t,x) in an open set contained in $(0,T) \times \Omega$.

Therefore, there is no hope to apply the H-measures theory associated with the measure value solutions for conservation laws, or similarly, the kinetic theory introduced by Lions, Phertame, Tadmor.

To overcome the difficulties mentioned above, the Buckley-Leverett System has been significantly modified in some directions.

A quite general case can be described by the following partial differential equations: given non-negative (viscosity) functions μ and ν, we have

$$\begin{cases} \partial_t u + \operatorname{div}\big(g(u)\,\mathrm{v}\big) = \operatorname{div}(\mu\,\nabla u) + F(\cdot, u), \\ -\operatorname{div}(\nu\,\mathrm{D}\mathrm{v}) + h(u)\,\mathrm{v} = -\nabla p + \mathrm{G}(\cdot, u), \\ \operatorname{div}(\mathbf{v}) = 0, \end{cases} \tag{3}$$

where F and G are external forcing terms, $\mathrm{D}(\cdot)$ is the symmetric part of the gradient operator, and the above system must be supplemented with suitable initial and boundary conditions.

There exists a list of important papers retated to (3), in particular following the two main directions, that is, $\mu > 0$, $\nu = 0$, and $\mu = 0$, $\nu > 0$:

S.N. Kružkov, S.M. Sukorjanskiĭ. Boundary value problems for system of equations of two-phase filtration type: formulation of problems, questions of solvability, justification of approximate methods. Mat. Sb. (N.S.) **104** (146), (1977), 69–88,

showed that, the system (3) is well-posed in the class of semiclassical solutions, see Definition 2 in that paper, for $\mu = \mu(x, u) > 0$, and $\nu = 0$, hence they considered a parabolic–elliptic structure.

N. Chemetov, W. Neves. The Generalized Buckley–Leverett System: Solvability. Arch. Rational Mech. Anal. **208** (1), (2013), 1–24,

showed the existence of weak solutions, see Definition 6.1 in that paper, considering $\mu = F = \mathsf{G} = 0$, $\nu = const > 0$, thus considering a hyperbolic–elliptic structure.

New aproach and strategies

We consider another aproach to solve the Buckley-Leverett System (1), which is the stiff approximation, (or stiff limit solutions). Moreover, we are not going to apply the entropy formulation.

Our main motivation to use this approach comes from the paper:

B. Perthame, F. Quirós, J. L. Vázquez. The Hele-Shaw Asymptotics for Mechanical Models of Tumor Growth. Arch. Rational Mech. Anal. 212 (2014), 93–127.

In that paper, they considered both incompressible and compressible mechanical models of tumor growth.

These two classes of models are related by using a stiff pressure law, see equation (4) below.

Under some hypotheses, they succeed to pass to the limit as $\gamma \to \infty$ (called incompressibility limit, or stiff limit) in the compressible model, and obtain a weak solution to the incompressible model, which is a free boundary problem of Hely-Shaw type.

Let us present our strategy here.

We consider a companion compressible model to (1), due to a convenient pressure equation of state.

More precisely, given $\gamma > 1$ let us consider the constitutive pressure having a power law in relation with the density of the fluid, that is

$$p_\gamma = p_\gamma(u) := u^\gamma. \tag{4}$$

Under good assumptions on the initial data $u_\gamma(0)$ and also f_γ

we expect that the family of solutions $\{(u_\gamma, p_\gamma)\}$ of the companion compressible model will converge

when $\gamma \to \infty$, (in some suitable sense to be defined later),

to an element (u_∞, p_∞) solving the Buckley-Leverett System in the weak sense,

which is similarly to Perthame et al cited paper, a free boundary problem of Hele–Shaw type, (the geometric formulation of multidimensional Muskat problem).

Existence of weak solutions

We use the notations $Q_T = (0,T) \times \Omega$ for the space-time cylinder, and also $\Gamma_T = (0,T) \times \Gamma$.

We frequently use the abbreviated form $u := u(t,x)$, $p := p(t,x)$ for $(t,x) \in Q_T$.

We also define the positive and negative part of a function w as follows, $|w|^{\pm} := \max\{\pm w, 0\}$.

Moreover, we have

$$\mathrm{sgn}^{+}(w) := \begin{cases} 1, & \text{if } w > 0 \\ 0, & \text{if } w \leqslant 0, \end{cases} \qquad \mathrm{sgn}^{-}(w) := \begin{cases} -1, & \text{if } w < 0 \\ 0, & \text{if } w \geqslant 0. \end{cases}$$

Existence of weak solutions

We consider the main hypothesis.

Hypothesis A. We assume the following hypothesis on the nonlinearity coefficients:

$$g, h \in W^{1,\infty}_{\rm loc}(\mathbb{R}), \; h \geq h_0 > 0, \; g(z) > 0 \text{ for each } z > 0, \; g(0) = 0.$$

Moreover, we assume that the following function

$$\Theta(z) := \frac{g(z)}{h(z)} \text{ is non-decreasing,} \tag{5}$$

which is bounded and Lipschitz on compact sets.

Existence of weak solutions

Hypothesis B.

We assume the following hypothesis on the data:

1. For some non-negative intial condition $u_0 \in L^1(\Omega)$, the initial data $u_{0\gamma}$ satisfies

$$\begin{cases} u_{0\gamma} \geq 0, \quad p_\gamma(0) \equiv p_\gamma(u_{0\gamma}) \leq p_M, \\ \|u_{0\gamma} - u_0\|_{L^1(\Omega)} \to 0 \quad \text{as } \gamma \to \infty, \end{cases} \tag{6}$$

and further

$$\sup_\gamma ||\nabla u_{0\gamma}||_{L^1(\Omega)} < \infty. \tag{7}$$

Existence of weak solutions

2. For some $f \in L^\infty(\Gamma_T; \mathcal{H}^d)$, there exists f_γ defined in $\mathbb{R}^{d+1}_+ = (0,\infty) \times \mathbb{R}^d$ such that

$$\|f_\gamma - f\|_{L^1(\Gamma_T;\mathcal{H}^d)} \to 0 \quad \text{as } \gamma \to \infty, \tag{8}$$

and also for any compact set $K \subset \mathbb{R}^{d+1}_+$,

$$\sup_\gamma ||\nabla f_\gamma||_{L^1(K)} < \infty. \tag{9}$$

Moreover, we assume without loss of generality that, $u_{0\gamma}, f_\gamma$ are smooth functions.

One observes that **Hypothesis A** is quite general. In particular, the function $\Theta(\cdot)$ be non-decreasing is a usual assumption.

Moreover, the **Hypothesis B** is satisfied by a large set of functions. Indeed, $BV \cap L^{\infty}$ is contained in it.

In what follows we define what is meant to be a weak solution of the Buckley-Leverett System.

Definition (Weak solution).
A pair of functions $(u(t,x), p(t,x))$, with

$$(u, \nabla p) \in L^\infty(Q_T) \times L^2(Q_T)$$

is called a weak solution to the Buckley-Leverett system (1), when

$$\iint_{Q_T} \big(u\, \phi_t + \Theta(u)\, \nabla p \cdot \nabla \phi \big)\; dx\, dt = \int_{\Gamma_T} f\, \phi\, d\mathcal{H}^d, \qquad (10)$$

for any test function $\phi \in C_c^\infty((0,T) \times \mathbb{R}^d)$, and also for a. a. $t \in (0,T)$

$$\int_\Omega \frac{\nabla p(t)}{h(u(t))} \cdot \nabla \xi \, dx = 0, \qquad (11)$$

holds for any $\xi \in C_c^\infty(\Omega)$. Moreover, the initial data for the density is attained in the L^1-strong sense, that is

$$\operatorname*{ess\,lim}_{t \to 0} \int_\Omega |u(t) - u_0| \, dx = 0.$$

Theorem (Main Theorem). Under the above conditions:

1. the unique solution (u_γ, p_γ) of the compressible model converges strongly in $L^1(Q_T)$ as $\gamma \to \infty$ to (u_∞, p_∞) with $u_\infty \in C([0,\infty); L^1(\Omega)) \cap BV(Q_T)$ and $p_\infty \in BV(Q_T)$.
2. The functions u_∞ and p_∞ are such that $0 \leq u_\infty \leq 1$ and $0 \leq p_\infty \leq p_M$ a.e. in Q_T, also $p_\infty(u_\infty - 1) = 0$.
3. The ∇p_γ weakly converges in $L^2(Q_T)$ to ∇p_∞ as $\gamma \to \infty$.
4. the pair (u_∞, p_∞) solves (1) in the sense of above definition.
5. The further regularity on the time derivatives holds $\partial_t u_\infty, \partial_t p_\infty \in \mathcal{M}^1(Q_T)$, with

$$\partial_t u_\infty, \partial_t p_\infty \geq 0 \quad \text{in } \mathcal{D}'(Q_T).$$

• **Existence and uniqueness of solutions for compressible model.**

Theorem. Let $\alpha > 1$ be fixed arbitrarily, and let $\gamma > 1$. Under Hypothesis A-B, there exists a unique pair (u_γ, p_γ), solving the compressible Buckley-Leverett System with initial-condition $u_{0\gamma}$ and normal boundary flux f_γ. Moreover, one has for every $\gamma > 1$ that

$$0 \leq u_\gamma \leq \sqrt[\gamma]{p_M} \quad \text{and} \quad 0 \leq p_\gamma \leq p_M \quad \text{a.e.} - Q_T. \qquad (12)$$

We consider the compressible Buckley-Leverett System written in the following form

$$\begin{cases} \partial_t u_\gamma = \Delta\Psi(u_\gamma) + \dfrac{1}{\gamma^\alpha} u_\gamma \Phi(p_\gamma), \quad (p_\gamma = u_\gamma^\gamma), \\ \nabla\Psi(u_\gamma)\cdot \mathrm{n} = f_\gamma, \\ u_\gamma(0) = u_{0\gamma}, \end{cases} \tag{13}$$

where the function $\Psi(\cdot)$ is defined by

$$\Psi(z) := \gamma \int_0^z \Theta(s)\, s^{\gamma-1} ds. \tag{14}$$

- **L^1 bounds for u_γ and p_γ.**

Proposition. Let us assume that the set of Hypotheses A-B hold. Fix $\alpha > 1$ and for every $\gamma > 1$ let u_γ be the unique solution of the compressible Buckley-Leverett System. Then, for any $t \in [0, T]$,

$$||u_\gamma(t)||_{L^1(\Omega)} \leq e^{\frac{\Phi(0)T}{\gamma^\alpha}} \Big(||u_{0\gamma}||_{L^1(\Omega)} + 2 \int_{\Gamma_T} |f_\gamma| \; drdt \Big), \qquad (15)$$

and

$$||p_\gamma(t)||_{L^1(\Omega)} \leq (p_M)^{(\gamma-1)/\gamma} e^{\frac{\Phi(0)T}{\gamma^\alpha}} \Big(||u_{0\gamma}||_{L^1(\Omega)} + 2 \int_{\Gamma_T} |f_\gamma| \; drdt \Big). \qquad (16)$$

- **Benilan-Aronson type estimates.**

Proposition. Let us assume that the set of Hypotheses A-B hold. Fix $\alpha > 1$ and for every $\gamma > 1$ let u_γ be the unique solution of the compressible Buckley-Leverett System. Then, for every $\gamma > 1$ and a.e. in Q_T, it follows that

$$\Theta(u_\gamma)\,\Delta p_\gamma + \frac{1}{\gamma^\alpha} u_\gamma\,\Phi(p_\gamma) \geq -\frac{r_\Phi}{\gamma^\alpha} u_\gamma\,\frac{e^{-\frac{r_\Phi}{\gamma^{\alpha-1}}t}}{1 - e^{-\frac{r_\Phi}{\gamma^{\alpha-1}}t}}, \qquad (17)$$

where $r_\Phi := \min\left\{\Phi(p) - \Phi'(p)p\right\}$, which is a non-negative function by hypothesis done on Φ.

Corollary. Under the assumptions of Main Theorem, one has that $\partial_t u_\infty, \partial_t p_\infty \geq 0$ in the sense of distributions.

- **Uniform bounds for the time derivatives.**

Proposition. Under the asumptions of Main Theorem, there exists a positive constant $C = C(T)$ such that, for $\gamma > 1$,

$$||\partial_t u_\gamma||_{L^1(Q_T)} + ||\partial_t p_\gamma||_{L^1(Q_T)} \leq C.$$

- **A priori estimates for the spatial gradients.**

Proposition. Under the assumptions of Main Theorem, there exists $C = C(T) > 0$ such that, for every $\gamma > 1$ it holds

$$||\nabla u_\gamma||_{L^1(Q_T)} + ||\nabla p_\gamma||_{L^1(Q_T)} \leq C.$$

Since Γ has C^2 regularity, n is a C^1 vector value function and also could be extended to a function N, defined on the whole of $\mathbb{R}^d$ such that, $\mathrm{N} \in C^1(\mathbb{R}^d; \mathbb{R}^d)$ with $|\mathrm{N}| \leq 1$. Therefore, we may write

$$\nabla \Psi(u_\gamma) \cdot \mathrm{N} = f_\gamma \quad \text{on } \Gamma_T,$$

where $f_\gamma(t, \cdot)$ is also defined in $\mathbb{R}^d$ for each $t \in (0, T)$. Moreover, we have on Γ_T that

$$\Big|\nabla |\nabla \Psi(u_\gamma)| \cdot \mathrm{N}\Big| \leq \frac{\sqrt{d}}{|\nabla \Psi(u_\gamma) \cdot \mathrm{N}|} |D^2 \Psi(u_\gamma)\, \mathrm{N}| \leq \frac{\sqrt{d}}{f_b}\, |D^2 \Psi(u_\gamma)\, \mathrm{N}|. \tag{18}$$

• **Compactness argument and properties of the limit.**

Theorem. Under the assumptions of Main Theorem, it follows that the family of solutions $\{(u_\gamma, p_\gamma)\}$ of the compressible Buckley-Leverett System converges strongly in $L^1(Q_T)$ as $\gamma \to \infty$ to an element (u_∞, p_∞). Moreover, one has that, $u_\infty, p_\infty \in BV(Q_T)$.

The following proposition tell us in particular that u_∞, p_∞ satisfy the Hele-Shaw limit.
Proposition. Under the assumptions of Main Theorem, one has that (12) yields

$$0 \leq u_\infty \leq 1 \quad \text{and} \quad 0 \leq p_\infty \leq p_M \quad \text{a.e.} - Q_T. \tag{19}$$

Moreover the functions u_∞ and p_∞ satisfy the Hele-Shaw graph property $p_\infty(u_\infty - 1) = 0$.

- L^2 **bounds for** ∇p_γ **and weak convergence.**

Theorem. Under the assumptions of the above theorem (compactness argument), the family $\{\nabla p_\gamma\}$ is uniformly bounded in $L^2(Q_T)$. Moreover, it follows that $\nabla p_\gamma \rightharpoonup \nabla p_\infty$ weakly in $L^2(Q_T)$.

Here we use strongly the Benilan-Aronson type estimates.

- **Continuity in time and the initial strong trace.**

Proposition. Let the assumptions of Main Theorem to hold. Then, the weak limit u_∞ given by compactness theorem is such that,

$$u_\infty \in C([0,T]; L^1(\Omega)) \cap BV(Q_T)$$

and satisfies the following limit

$$\lim_{t \to 0^+} \int_\Omega |u_\infty(t) - u_0| \, dx = 0. \tag{20}$$

- **The incompressibility condition.**

Theorem. Let (u_∞, p_∞) be given by compactness theorem. Then, the pair (u_∞, p_∞) satisfies the incompressibility condition (11).

- **Proof of Main Theorem**

Finally, collecting all the previous results, we can establish the proof of the Main Theorem.

In particular, we perform a limit transition as $\gamma \to \infty$, (stiff limit), and show existence of weak solutions for the Buckley-Leverett System.

THANK YOU.