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Research Center for
Mathematics of Data | MoD

Trends in the Mathematical Sciences

Networked Hyperbolic Systems

Modeling, Control and Efficient Simulation

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June 11, 2024

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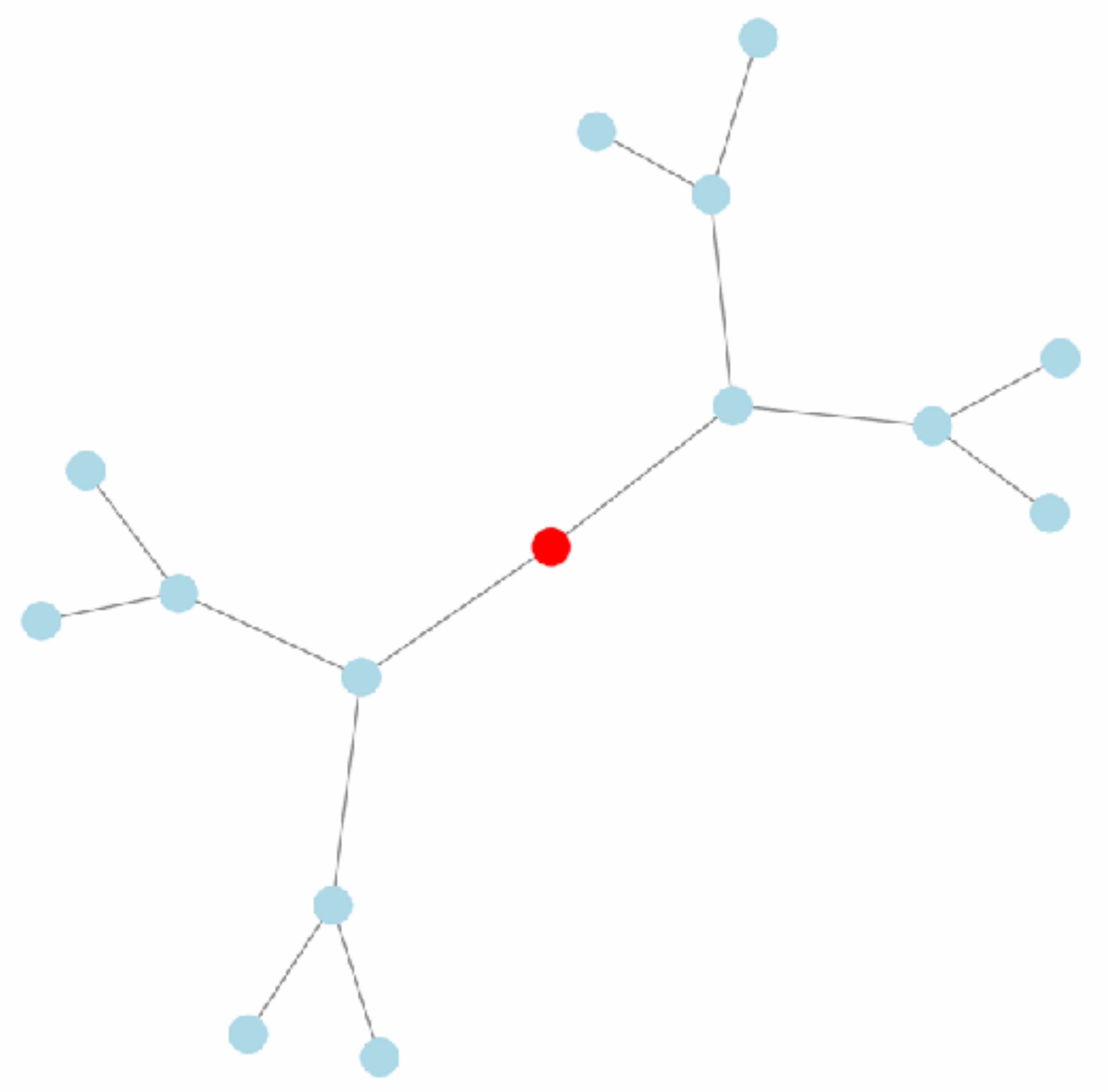
Friedrich-Alexander-Universität
DYNAMICS, CONTROL, MACHINE LEARNING AND NUMERICS

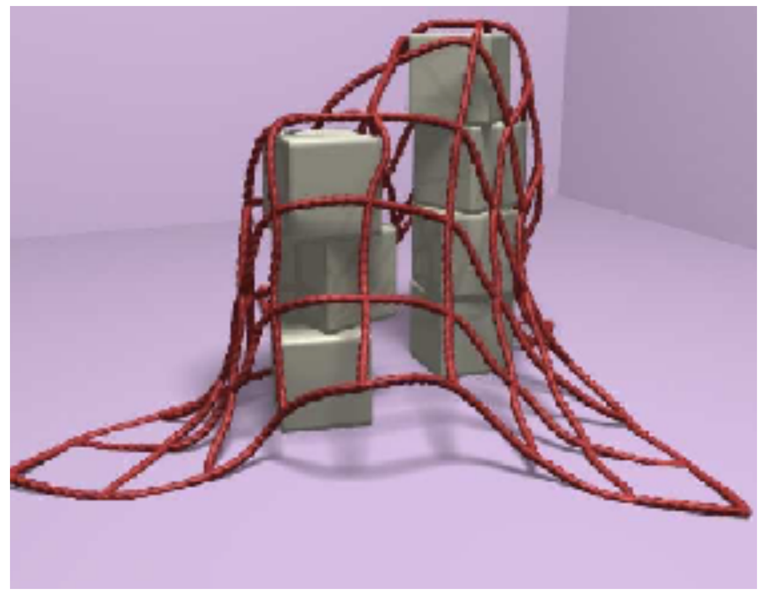
Networked Hyperbolic Systems

Hyperbolic systems are a class of partial differential equations (PDEs) that describe wave propagation and other phenomena where information travels with finite speed.

When these systems are interconnected in a network, they form networked hyperbolic systems.



Real-world Applications

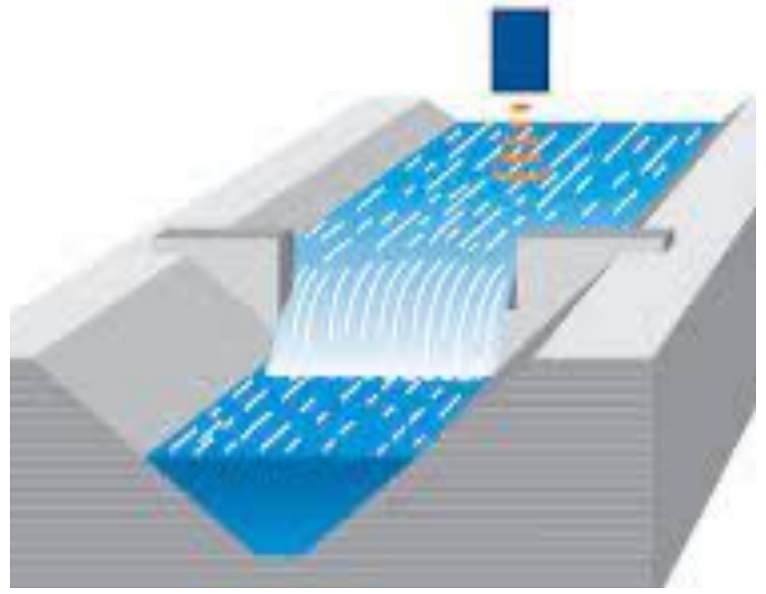


Network of Large Deflection Strings (Nonlinear coupled wave equations)

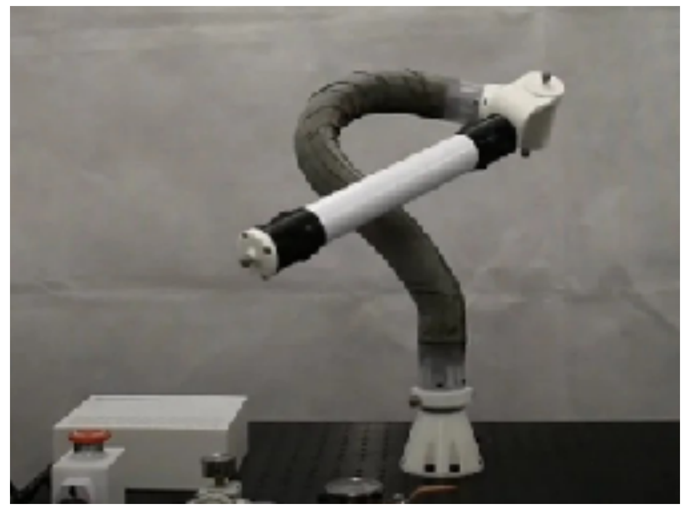
NASA Flexible Flight Device (Geometrically Exact Beams)

Wind Turbine

Gas transport networks (Isothermal Euler Equations)



Open Canal (Saint-Venant Equation)



Flexible Robotic Arm

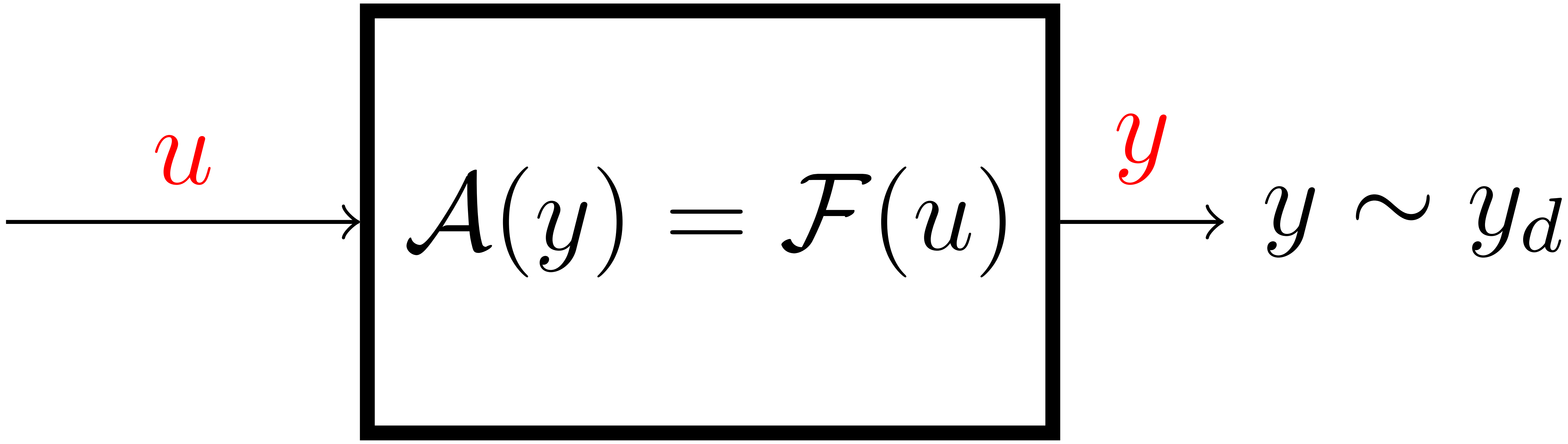
- Project ConFlex (2017–2022) and ModConFlex (2023–2027): Modeling and Control for Flexible Structures Interacting with Fluids
- Project SFB TRR154 (2018–2026): Mathematical modelling, simulation and optimization using the example of gas networks
- Project DFG WA5144/1-1 (2022–2024): Analysis and Control of Nonlinear Hyperbolic Systems with Degeneration on Networks
- Sino-German Mobility Project CIN-PDE (2022-2025): Control, Inversion and Numerics for Partial Differential Equations (Coming workshop in October 8th. -10th. at Shanghai)

Significant Interests

$$u \longrightarrow \boxed{\mathcal{A}(y) = \mathcal{F}(u)} \xrightarrow{\;y\;} y \sim y_d$$

- Modeling and Analysis:
  1. Physics-driven & Data-driven (e.g. Physic Laws & Machine Learning).
  2. For analysis, difficulties may arise on networks with
     - **nonlinear** elements,
     - non-trivial **boundary conditions** and **coupling**,
     - complex **topological structure**.

Significant Interests



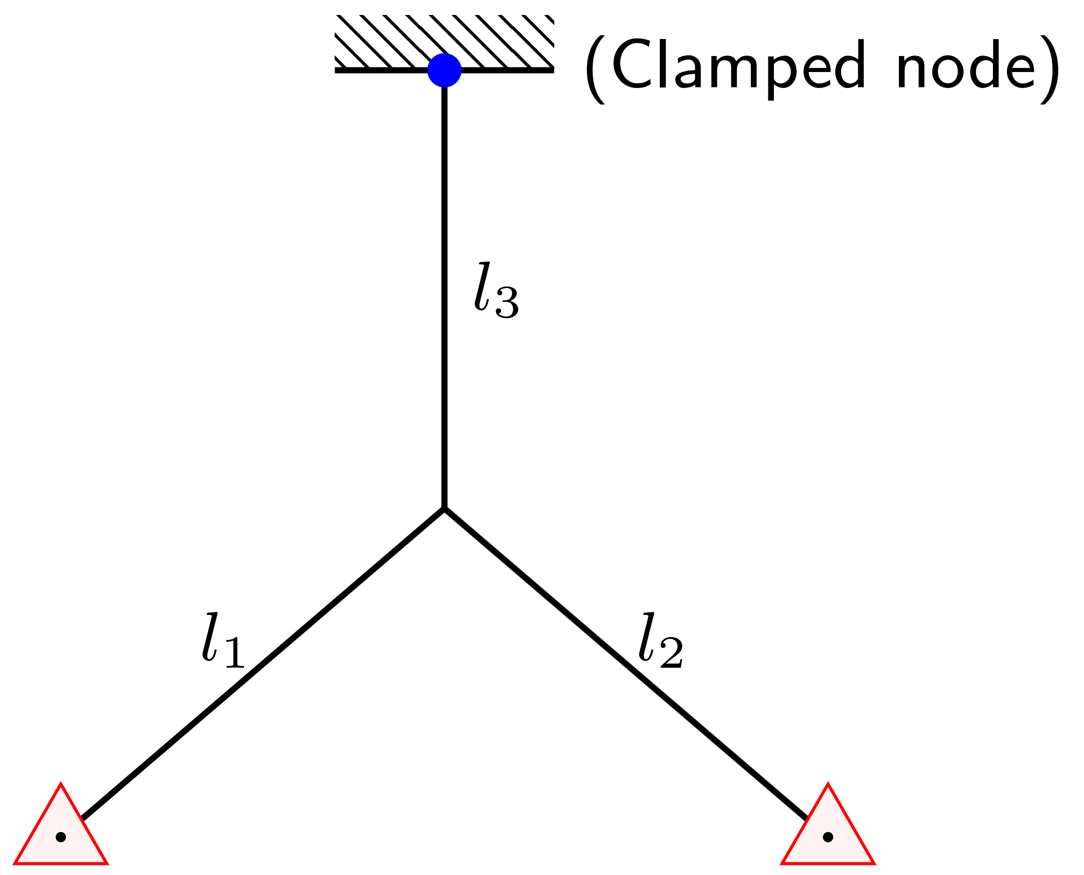
- Modeling and Analysis
- Control Theory and Optimal Design

1. Feasibility $\longrightarrow$ Controllability (To find at least one way to reach the target, e.g $y(x,T) = y_d(x)$ (Exact Controllability), or $y(N_i,t) = y_d(t)$(Nodal Profile Control));
2. Optimality $\longrightarrow$ Optimal control (To find the best way, in some sense, to reach the target. e.g. controllability time, minimum number of controls, control design on networks, constrained optimization)

Example: Networks of vibrating strings

Two boundary control problems (linear case)

	Null Controllability	Controllability of Nodal Profile
Control Target	$y^i(T,x) = y^i_t(T,x) = 0$	$y^3_x(t,l_3) = y_d, \quad t > T$
(sharp) controllability time $T \geq T^*$	$T^* = 2(l_3 + \max\{l_1, l_2\})$	$T^* = l_3 + \min\{l_1, l_2\}$
minimum number of required controls	2	1



$$
\begin{cases}
y^i_{tt} - y^i_{xx} = 0, & (t,x) \in (0,T) \times (0,l_i), \quad i = 1,2,3, \\
x = 0 : y^1(t,0) = y^2(t,0) = y^3(t,0), & t \in (0,T), \\
y^0_x(t,0) + y^1_x(t,0) + y^2_x(t,0) = 0, & t \in (0,T), \\
x = l_3 : y^3(t,l_3) = 0, & t \in (0,T), \\
x = l_i : y^i_x(t,l_i) = u^i(t), & t \in (0,T), \quad i = 1,2,
\end{cases}
$$

- *René Dáger, Enrique Zuazua. Wave Propagation, Observation and Control in 1-d Flexible Multi-Structures (2006)*
- *Yue Wang, Günter Leugering. Boundary controllability and observability of nodal profile for wave equation (2022)*

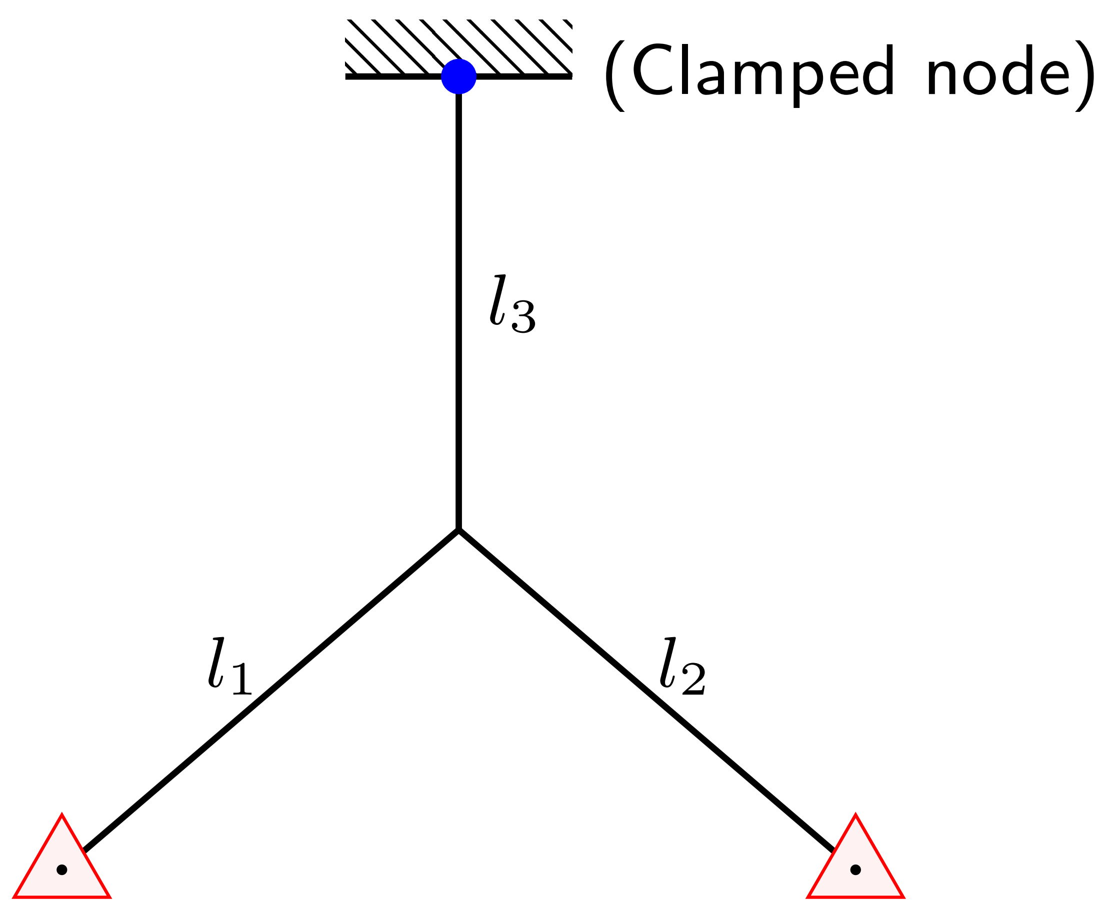
Key Techniques: Set suitable Hilbert space, duality between controllability and observability, observability Inequality.

Example: Networks of vibrating strings

Nonlinear Case

Consider the following coupled system of 1-D quasilinear wave equations $(i = 1, ..., n)$:

$$(\mathbf{E}) \begin{cases} y^i_{tt} - (K^i(y^i, y^i_x))_x = F(\mathbf{y}, \mathbf{y}_x, \mathbf{y}_t), \quad x \in [0, L_i], t \in [0, T] \\ \sum K^i(y^i(t,0), y^i_x(t,0)) = 0, \ t \in [0, T] \\ y^j(t, 0) = y^i(t, 0), i \neq j, \\ y^i(t, L_i) = u^i(t), \ t \in [0, T] \\ (y^i, y^i_t)(0, x) = (\phi^i(x), \psi^i(x)), \quad x \in [0, L_i]. \end{cases}$$



where

- $\mathbf{y} = (y^1, ..., y^n)^{\mathrm{T}}$ is an unknown vector function of (t, x),
- $K^i = K^i(y^i, y^i_x)$ are given C^2 functions of y^i and y^i_x,
- $\frac{\partial}{\partial y^i_x} K^i(y^i, y^i_x) > 0$,
- u^i can be considered as 0 (no control) or control function.

! ! HUM method (J.Lions, 1980s) and duality method (E.Zuazua, 1990s) can not be applied on this case.

Difficulties may arise in...

- **Nonlinearity.**
  - **Weak solutions**. [of quasilinear hyperbolic systems $\to$ shock waves $\to$ an irreversible process $\to$ Impossible to get exact boundary controllability for any arbitrarily given initial and final states [A. Bressan, G. M. Coclite, '02]

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  - **Classical solution** exists only locally in time (P. D. Lax, '64; F. John, '90; T. Li, '94) $\rightarrow$ semi-global classical solution ($T > 0$ might be suitably large) [M. Cirinà, '70, T.Li, Y.Jin, B.Rao, '00, '01] $\rightarrow$ Local exact controllability in the quasilinear case.

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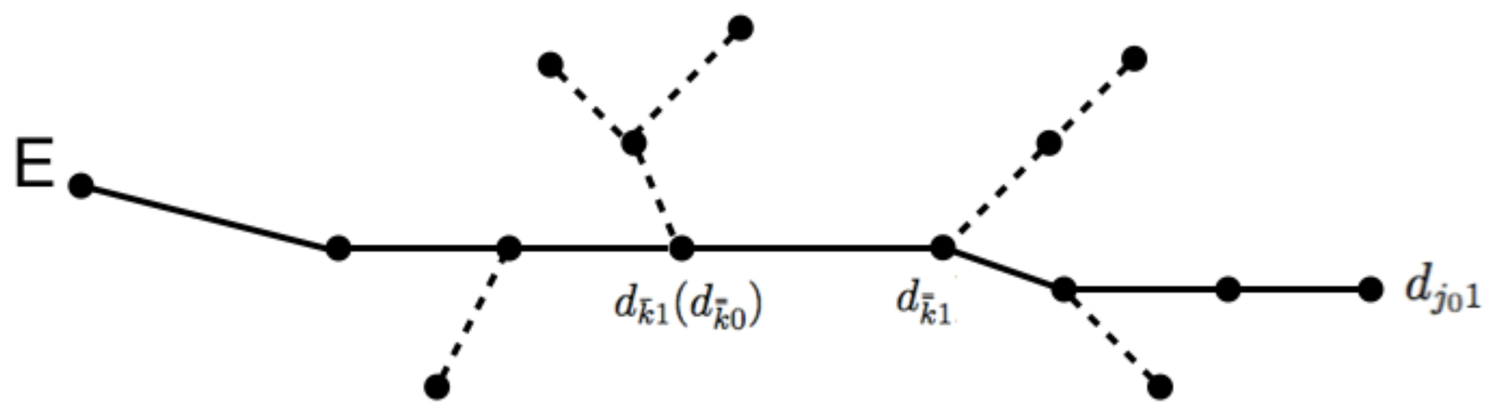
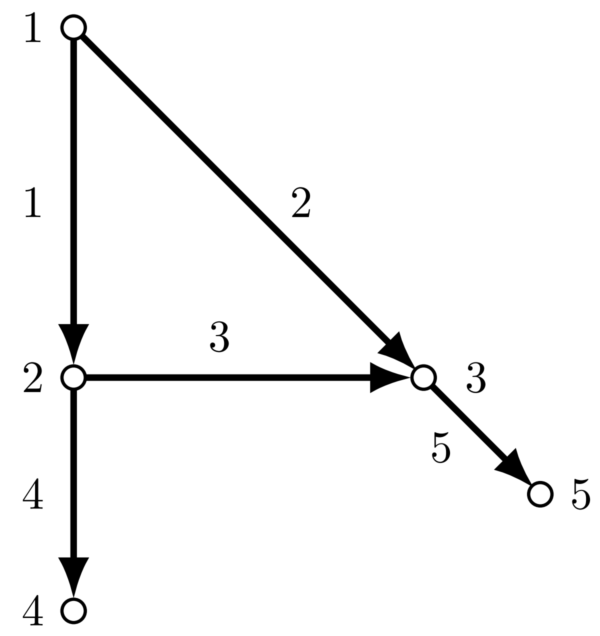
- **Networked Structure.**
  - **Coupling at the junction**. Complexity and Nonlinearity in **interface conditions**.
  - Complex topological structure of networks $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ may change the controllability results [Lagnese-Leugeing-Schmidt, '94]





Example: Networks of vibrating strings

New boundary conditions + coupling

Consider the following **coupled** system of 1-D **quasilinear** wave equations $(i = 1, ..., n)$:

$$
(\mathbf{E}) \begin{cases} y^i_{tt} - (K^i(y^i, y^i_x))_x = F(\mathbf{y}, \mathbf{y}_x, \mathbf{y}_t), \quad x \in [0, L_i], t \in [0, T] \\ y^i_{tt}(t, 0) = G^i(t, \mathbf{y}(t, 0), \mathbf{y}_x(t, 0), \mathbf{y}_t(t, 0)) \\ \qquad\qquad + \displaystyle\int_0^t H^i(t, s, \mathbf{y}(s, 0)) \mathrm{d}s, \ t \in [0, T] \\ y^i(t, L_i) = u^i(t), \ t \in [0, T] \\ (y^i, y^i_t)(0, x) = (\phi^i(x), \psi^i(x)), \quad x \in [0, L_i]. \end{cases}
$$

Second-order differential operators

(temporal) non-locality

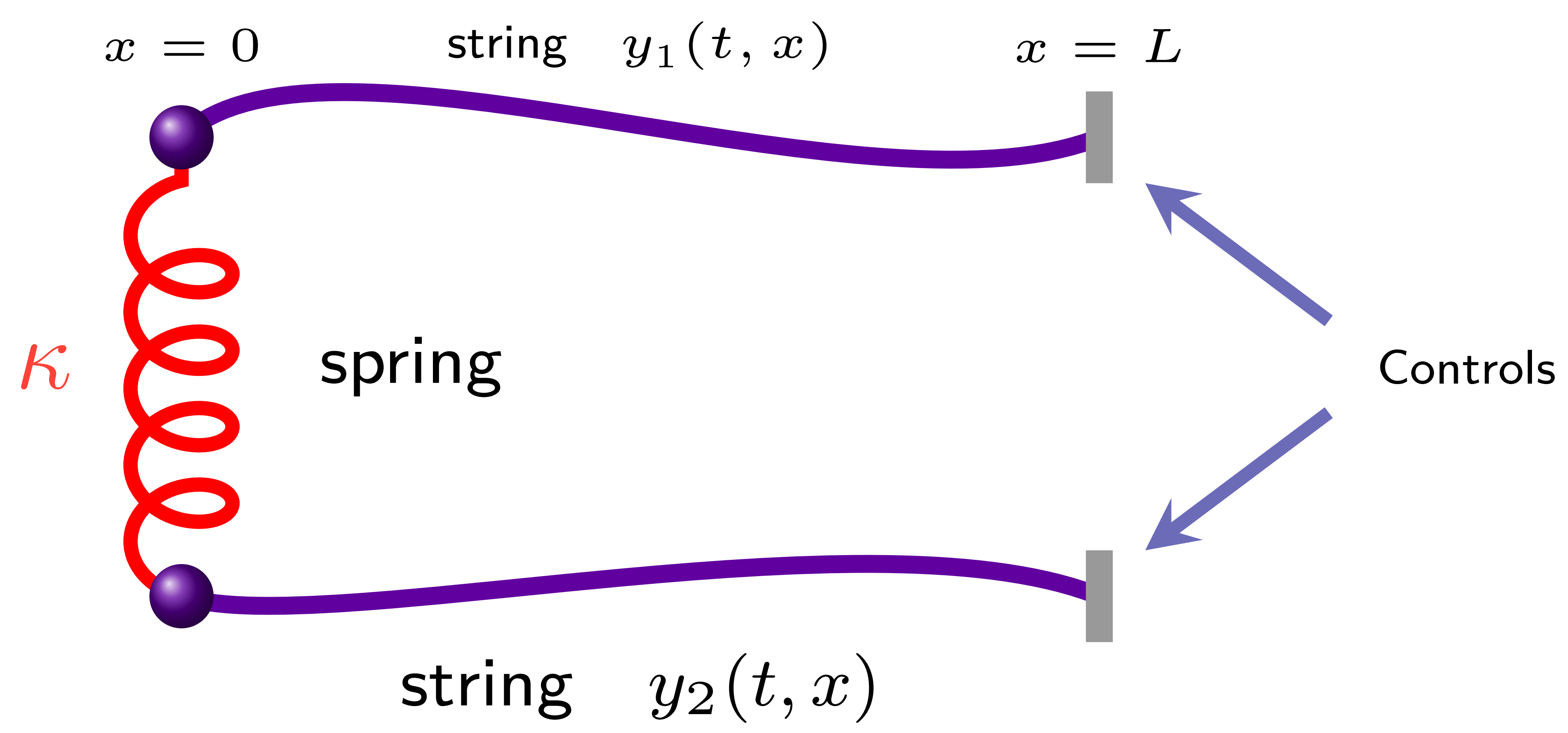
where

- $\mathbf{y} = (y^1, ..., y^n)^{\mathrm{T}}$ is an unknown vector function of (t, x),
- $K^i = K^i(y^i, y^i_x)$ are given C^2 functions of y^i and y^i_x,
- $\frac{\partial}{\partial y^i_x} K^i(y^i, y^i_x) > 0$,
- F^i, G^i, H^i are given C^1 functions of their arguments and 0 value at null state (i.e. 0 is an equiblium).

Example: String-mass-spring system

$$\begin{cases} y^i_{tt} - K_i(y^i_x)_x = 0, \quad 0 \leq x \leq L, t > 0, \qquad i = 1, 2, \\ x = 0 : y^1_{tt}(0,t) = K_1(y^1_x(0,t)) - \kappa(y^1(0,t) - y^2(0,t)), \\ \qquad y^2_{tt}(0,t) = K_2(y^2_x(0,t)) + \kappa(y^1(0,t) - y^2(0,t)), \\ x = L : y^i = u^i(t), \qquad i = 1, 2. \end{cases}$$

Dynamical transmission conditions



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- If the spring stiffness tends to infinity, formally the system tends to the classical string-mass problem. [1]
- For string-mass system it is known that the mass smoothens the waves while crossing the mass-point.[2]
- If the spring stiffness tends to zero, the strings become uncoupled.
- This system is controllable by only 1 control, and in this case, we discovered **asymmetric solution space** (smoothing effect of mass on waves) and **controllable space**.[3]

---

[1] G. Leugering, 1998; F. Almusallams, 2015; Y.Wang, T.Li, 2018

[2] S. Hansen, E.Zuazua 1995

[3] G.Leugering,S.Micu, I.Roventa, Y.Wang, 2022

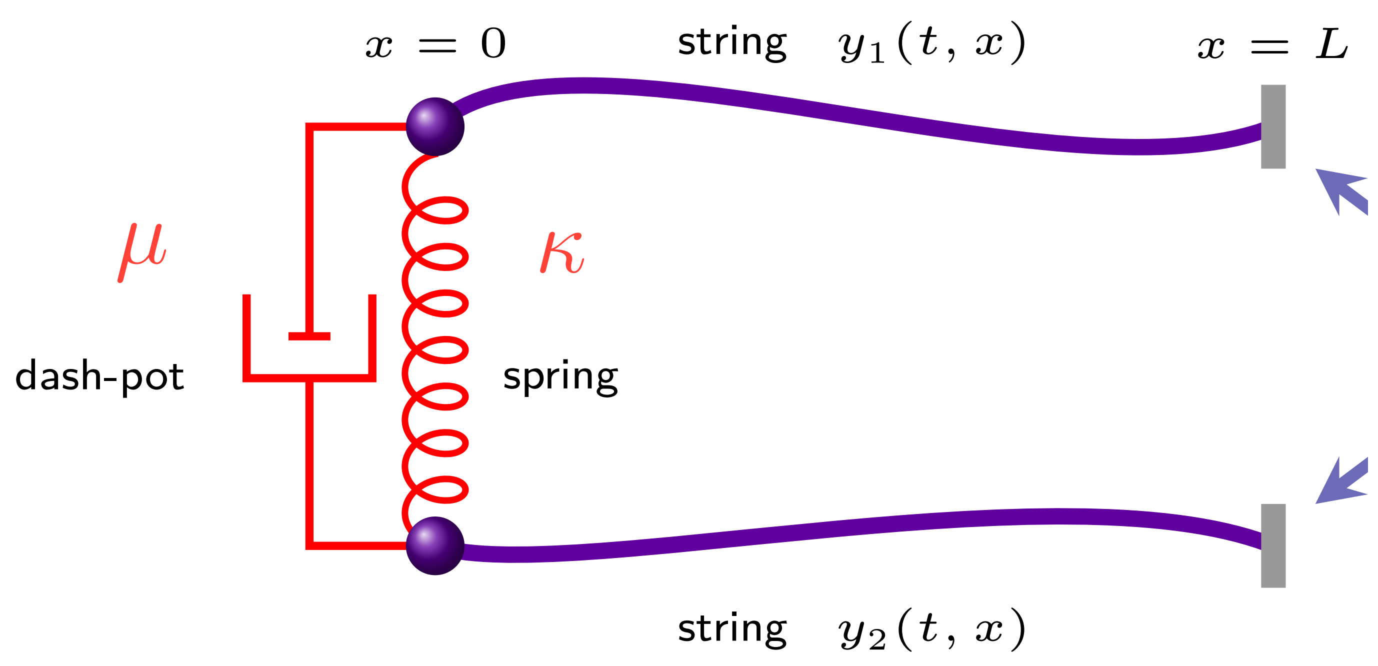
Other examples for dynamical boundary conditions

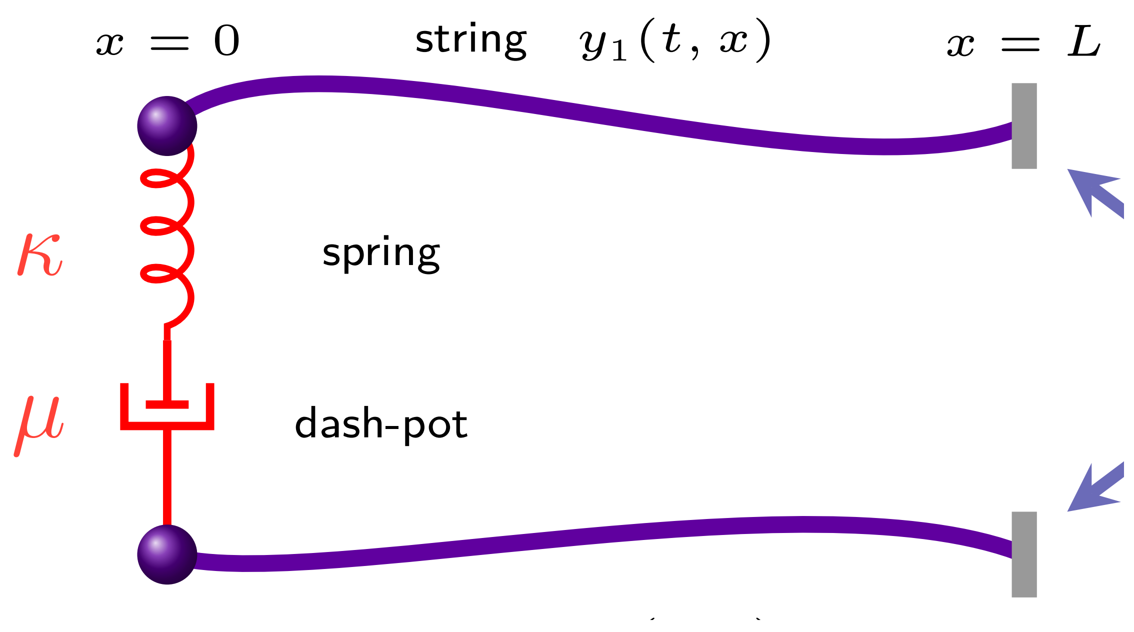
- **Kelvin Model**: a classical class of viscoelastic **solid** models.

$$\begin{cases} y^i_{tt} - K_i(y^i_x)_x = 0, \quad 0 \le x \le L, t > 0, \qquad i = 1,2,\\ x = 0: y^1_{tt}(t,0) = K_1(y^1_x(t,0)) - \kappa(y^1(t,0) - y^2(t,0)) + \mu(y^1_t(t,0) - y^2_t(t,0)),\\ \quad\quad\; y^2_{tt}(t,0) = K_2(y^2_x(t,0)) + \kappa(y^1(t,0) - y^2(t,0)) + \mu(y^1_t(t,0) - y^2_t(t,0)),\\ x = L: y^i = u^i(t), \qquad i = 1,2. \end{cases}$$

- **Maxwell Model**: a classical class of viscoelastic **fluid** models.

$$\begin{cases} y^i_{tt} - K_i(y^i_x)_x = 0, \quad 0 \le x \le L, \qquad i = 1,2,\\ x = 0: y^1_{tt}(0,t) = K_1(y^1_x(t,0)) - \kappa(y^1(t,0) - y^2(t,0))\\ \qquad\qquad + \dfrac{\kappa^2}{\mu}\displaystyle\int_0^t e^{-\frac{\kappa}{\mu}(t-\tau)}(y^1(\tau,0) - y^2(\tau,0))\,\mathrm{d}\tau\\ \quad\quad\; y^2_{tt}(0,t) = \cdots\\ x = L: y^i = u^i(t), \qquad i = 1,2. \end{cases}$$





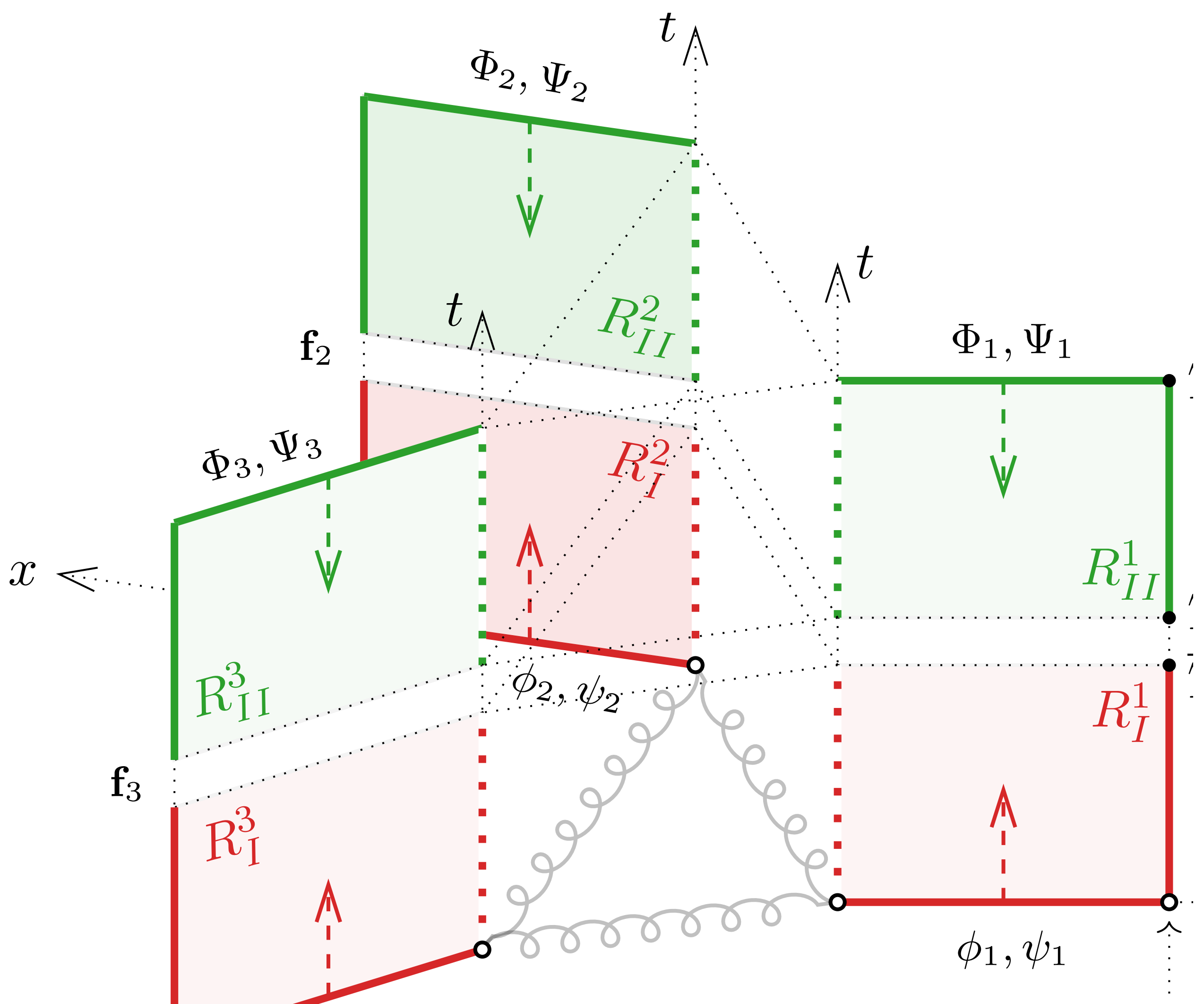
Exact boundary controllability

$$
(\mathbf{E}) \begin{cases} y^i_{tt} - (K^i(y^i, y^i_x))_x = F(\mathbf{y}, \mathbf{y}_x, \mathbf{y}_t), \quad x \in [0, L_i], t \in [0, T] \\ y^i_{tt}(t,0) = G^i(t, \mathbf{y}(t,0), \mathbf{y}_x(t,0), \mathbf{y}_t(t,0)) \\ \qquad\qquad + \int_0^t H^i(t, s, \mathbf{y}(s,0))\mathrm{d}s, \ t \in [0,T] \\ y^i(t, L_i) = u^i(t), \ t \in [0,T] \\ (y^i, y^i_t)(0,x) = (\phi^i(x), \psi^i(x)), \quad x \in [0, L_i]. \end{cases}
$$

The system (E) is locally exact controllable

- with n controls [G.Leugering, T.Li, Y.Wang, '18,'19].

Controllability Time (sharp): $T^* = \max_{i=1,\dots n} \frac{2L_i}{\sqrt{K^i_{y_x}(0,0)}}$



Wellposedness

We introduce $\mathbf{w}^i = (w_1^i, w_2^i, w_3^i)^T := (y^i, y_x^i, y_t^i)^T$. Then we get

$$\frac{\partial}{\partial t}\begin{pmatrix} w_1^i \\ w_2^i \\ w_3^i \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -K^i_{w_2^i} & 0 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} w_1^i \\ w_2^i \\ w_3^i \end{pmatrix} = \begin{pmatrix} w_2^i \\ 0 \\ F^i(\mathbf{w}^i) + K^i_{w_1^i} w_2^i \end{pmatrix}$$

with $(t, x) \in [0, T] \times [0, L_i]$. This, in turn, can be rewritten in the form of a quasilinear hyperbolic system

$$\mathbf{w}_t^i + A^i(x, \mathbf{w}^i)\mathbf{w}_x^i = \tilde{F}(\mathbf{w}^i),$$

where A^i has 3 distinct real eigenvalues:

$$\lambda_i^- = -\sqrt{K^i_{w_2^i}(w_1^i, w_2^i)}, \quad \lambda_i^0 = 0, \quad \lambda_i^+ = \sqrt{K^i_{w_2^i}(w_1^i, w_2^i)}.$$

Wellposedness ctd.

We may integrate the boundary conditions w.r.t. time and obtain a kind of **non-local (of time)** boundary condition in the first order system (FOS):

$$
(\mathbf{FOS})\begin{cases}
\mathbf{w}^i_t + A^i(x,\mathbf{w}^i)\mathbf{w}^i_x = \tilde{F}(\mathbf{w}^i), \quad x \in [0,L_i], t \in [0,T] \\
w^i_2(t,0) = \psi^i(0) + \int_0^t G^i(\tau,\mathbf{w}^i(\tau,0))\,\mathrm{d}\tau \\
\qquad\qquad + \int_0^t \int_0^\tau H^i(\tau,s,w^i_1(s,0))\,\mathrm{d}s\,\mathrm{d}\tau,\ t \in [0,T] \\
w^i_1(t,L_i) = u^i(t),\ t \in [0,T] \\
\mathbf{w}^i(0,x) = \mathbf{w}^{0,i}(x) = (\phi^i(x),\psi^i(x),\phi^{i\prime}(x)), \quad x \in [0,L_i].
\end{cases}
$$

- Local existence of C^1 solution to (FOS) (T.Li, '85): $\exists!$ C^1 solution on $\mathcal{R}(\delta) = \{(t,x) | 0 \le t \le \delta, 0 \le x \le L\}$, where δ depends on the initial and boundary data.
- For given $T > 0$, NO results on existence of semi-global classical solutions before.

Lemma: A uniform priori estimate of solution to (FOS) [Y.W.'19]

$$
\|w(t,\cdot)\|_1 \triangleq \|w(t,\cdot)\| + \left\|\frac{\partial w}{\partial x}(t,\cdot)\right\| \le C(T), \qquad 0 \le t \le T,
$$

where $\|\cdot\|$ denotes C^0-norm.

Exact boundary controllability

$$
(\mathbf{E}) \begin{cases}
y^i_{tt} - (K^i(y^i, y^i_x))_x = F(\mathbf{y}, \mathbf{y}_x, \mathbf{y}_t), \quad x \in [0, L_i], t \in [0, T] \\
y^i_{tt}(t, 0) = G^i(t, \mathbf{y}(t, 0), \mathbf{y}_x(t, 0), \mathbf{y}_t(t, 0)) \\
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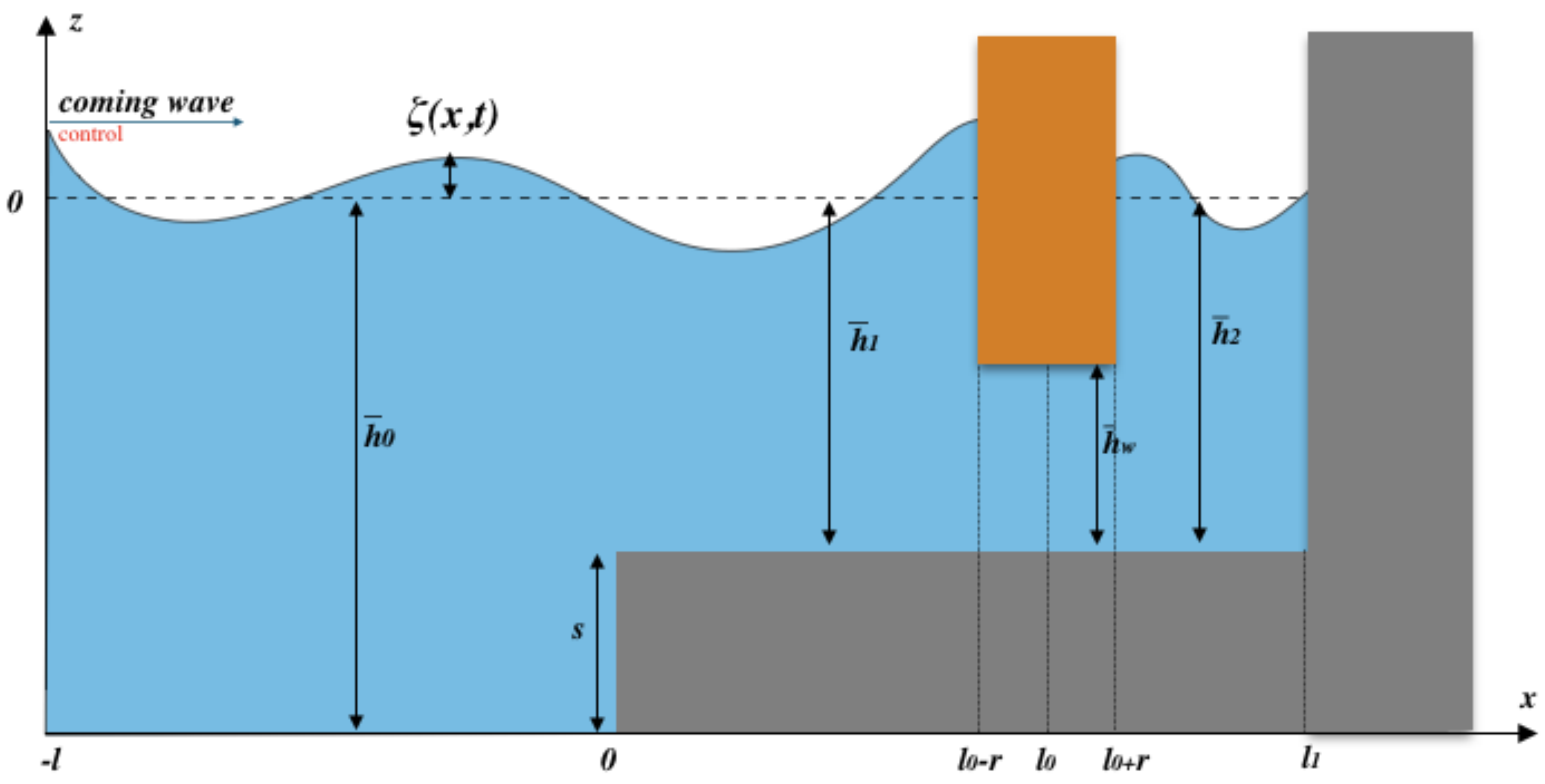
y^i(t, L_i) = u^i(t), \ t \in [0, T] \\
(y^i, y^i_t)(0, x) = (\phi^i(x), \psi^i(x)), \quad x \in [0, L_i].
\end{cases}
$$

The system (E) is locally exact controllable

- with n controls [G.Leugering, T.Li, Y.Wang, '18,'19].
- This result can be improved by **reducing the number of controls to** $n - 1$, but the space of controlled initial data is **asymmetric** [G.Leugering, S.Micu, I.Robenta, Y.Wang, '22] [G.Leugering, C.Rodriguez, Y.Wang, '22].

Extension: dynamical boundary conditions in 1st-order hyperbolic systems

- Project Conflex. [G.Vergara-Hermosilla, G.Leugering, Y.Wang, COCV '21].
- One dimensional nonlinear shallow water system, describing the free surface flow of water as well as the flow under a fixed gate structure.



$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x \left(q^2/h \right) + gh\partial_x \zeta = 0, \end{cases}$$

where $\zeta(t, x)$ is the free surface elevation, $h(t, x)$ is the fluid height, $q(t, x)$ is the horizontal discharge.

$$\zeta_0(t,0) = \zeta_1(t,0), \quad q_0(t,0) = q_1(t,0),$$

$$\begin{cases} q_2(t, l_0 + r) = q_1(t, l_0 - r) = q_w(t), \\ \left[\dfrac{q_i^2}{2h_i^2} + g\zeta_i \right]_{i=1, x=l_0-r}^{i=2, x=l_0+r} = -\alpha \dfrac{\mathrm{d}}{\mathrm{d}t} q_w(t), \end{cases}$$

Remark: Controllability of Nodal Profile

Nodal Control Problem: Let $T > T^* > 0$. For given **desired profile function** $y_d(t)$ to find boundary controls $u^1, ..., u^m$ so that

$$\mathscr{F}(u^1, \ldots, u^m) = y^n(t, L_n) = y_d(t), t \in [T^*, T]$$

$$or \quad \mathscr{F}(u^1, \ldots, u^m) = y^n_x(t, L_n) = y_d(t), t \in [T^*, T].$$

Theorem

In a neighbourhood of an equilibrium (around 0), the system (E) is locally exact boundary controllable of nodal profile by **only 1 control** when (controllability time, sharp)

$$T > \bar{T}.$$

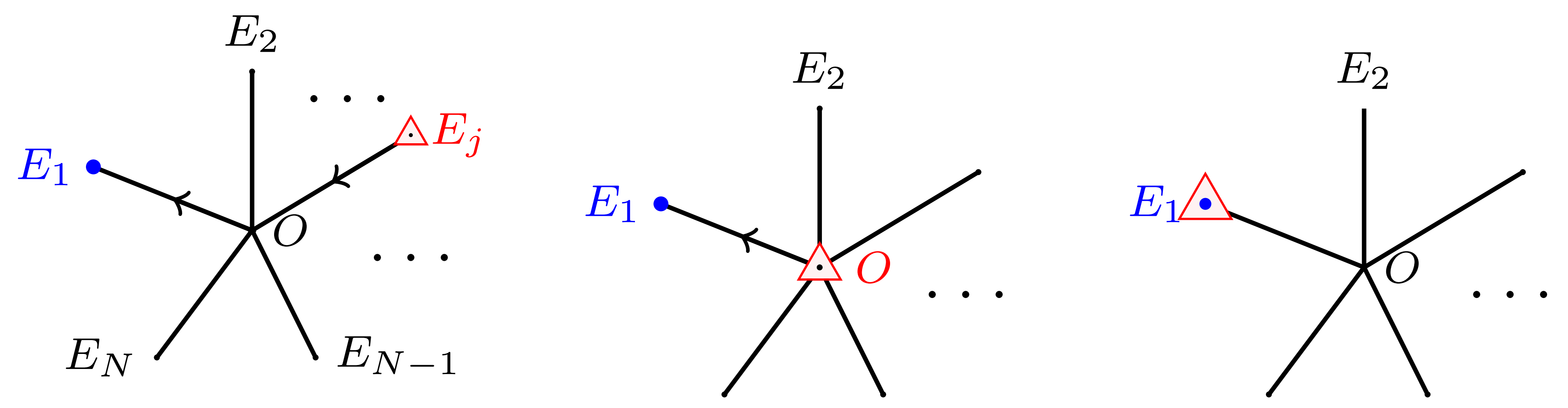
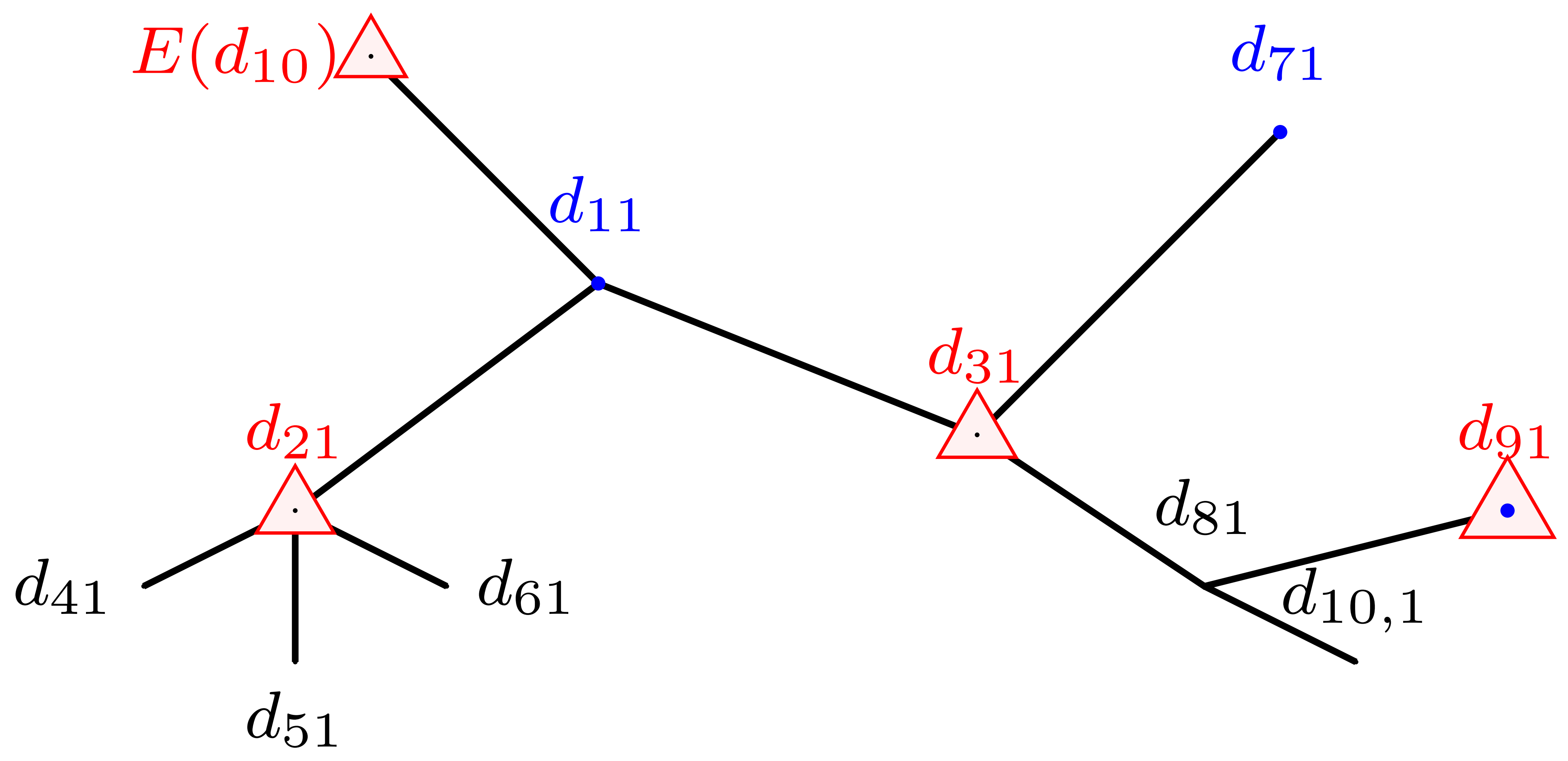


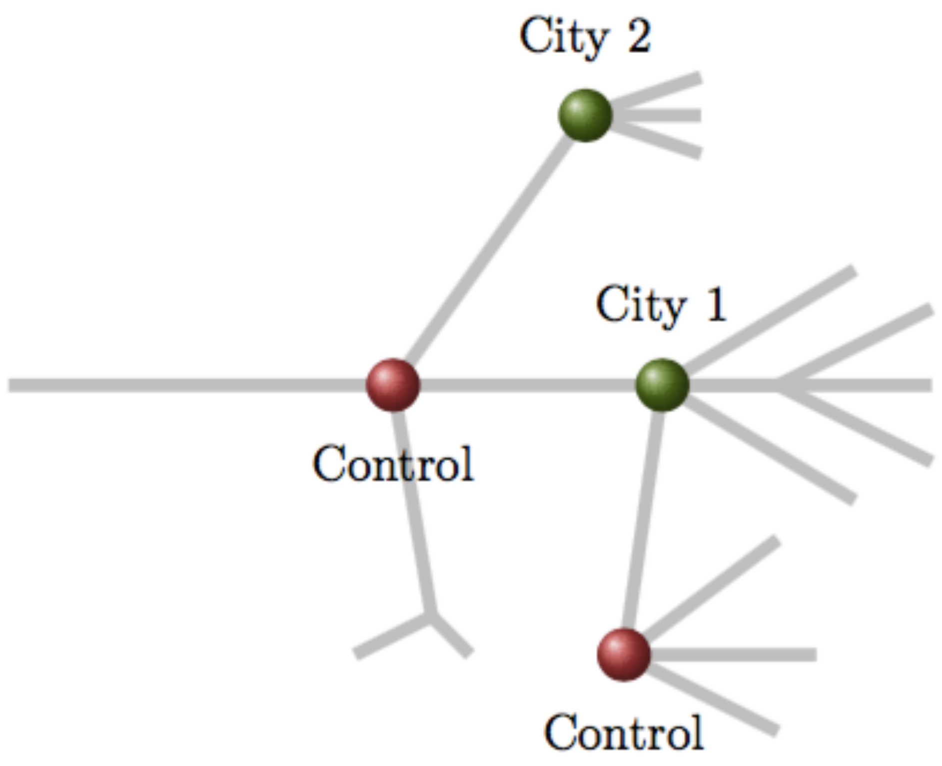
Fig.	Charged node	Controlled node	Controllability Time T
(a)	E_1	$E_j (j \neq 1)$	$\overline{T} > \frac{L_1}{\sqrt{K^1_{y^1_x}(0,0)}} + \frac{L_j}{\sqrt{K^j_{y^j_x}(0,0)}}$
(b)	E_1	O	$\overline{T} > \frac{L_1}{\sqrt{K^1_{y^1_x}(0,0)}}$
(c)	E_1	E_1(in-situ)	$\overline{T} > 0$



* Optimal controllability time T^*.
* Minimum number of controls.
* Placement of controls.
* Calculation of controls.

- Nodal Profile Control: Our aim is to fit (a part of) the boundary traces to a given profile after a suitably long time $t = T$ by means of boundary controls. **[Project: Control theory on planar or spacial string networks: controllability and partial nodal control for quasilinear hyperbolic systems. (Individual funding & NSFC-1121101.Joint work with T.Li.]**

Extension: Flow Control on gas network and Control Design

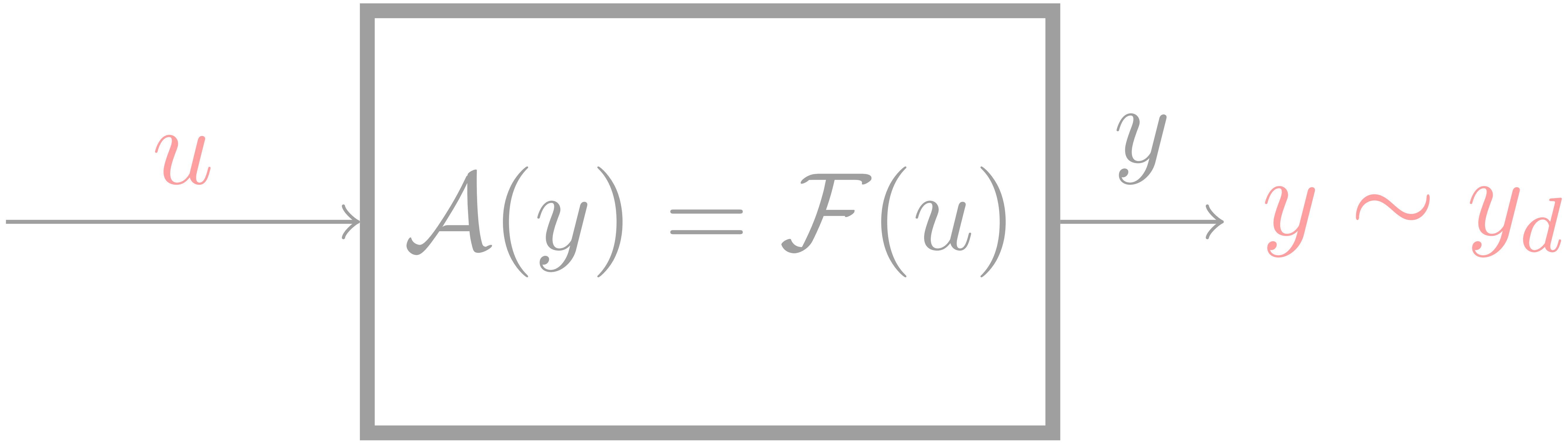


- The coupling of gas pipes.
- State function [isothermal Euler equations]
  - $\rho(t,x)$: the density of the gas,
  - $q(t,x)$: the flux in the pipe.
- Nodal Controls $u(t)$: Pressure increases at the compressor stations.

- Q: Can we find controls to satisfy the given demand of the cities?
  Aim: T*he boundary traces of state* to *exactly* fit any given profile as function of time on a node *after a suitable time* $t = T$ by means of *boundary controls*. [= Exact boundary controllability of nodal profile]
- Answer: Yes! (in local sense, and at least after a waiting time T^*). [M.Gugat 2010, 2014, T.Li 2010]

* Minimum number of controls.
* Placement of controls.
* Calculation of controls.

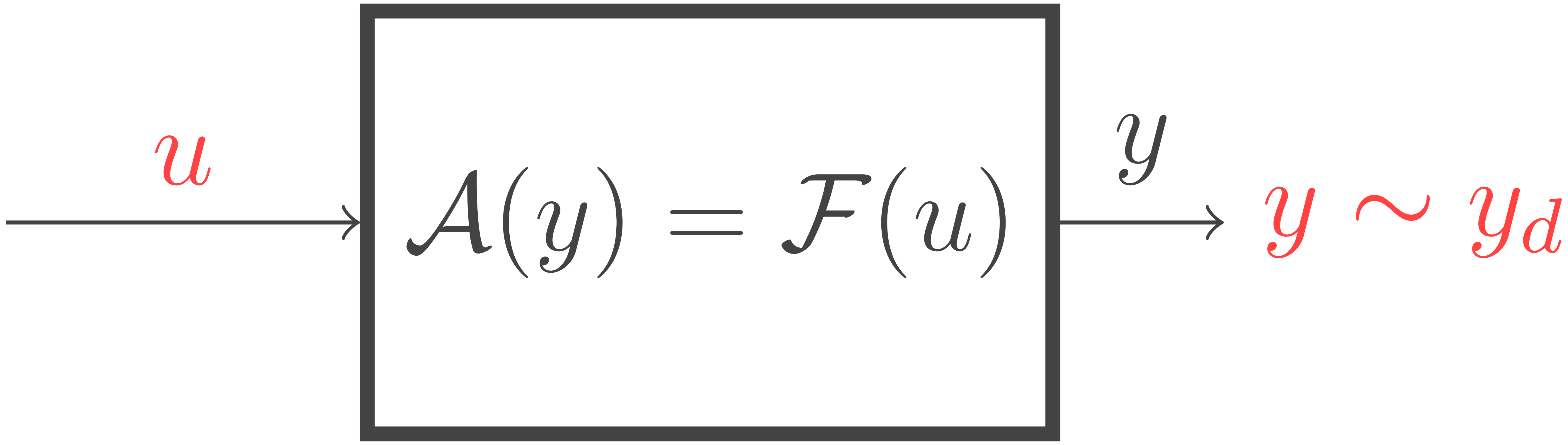
Significant Interests



- Modeling and Analysis
- Control Theory and Optimal Design

Key issue: developing and applying mathematical methods, including nonlinear functional analysis, new control theory and strategy to **model**, **understand** and **control** the dynamics of PDEs.

Significant Interests

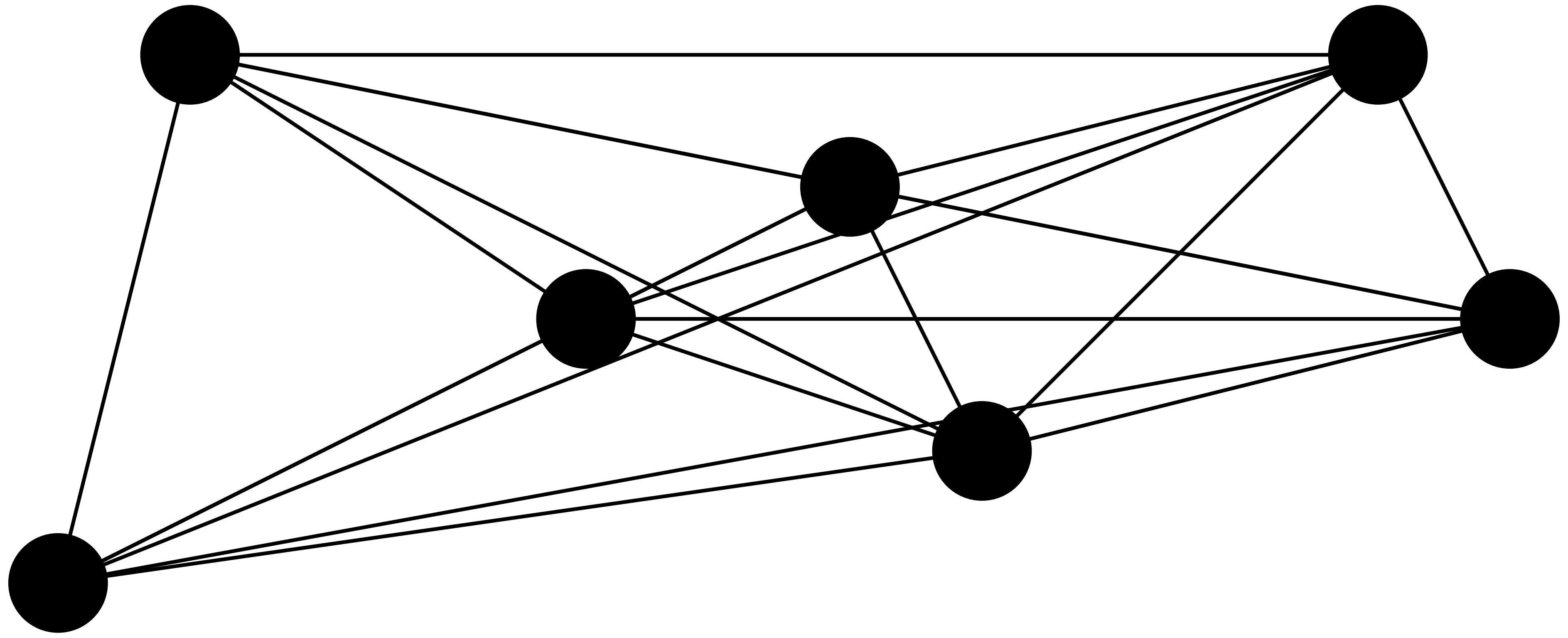


- Modeling and Analysis
- Control Theory and Optimal Design
- Accurate and Fast Prediction of Numerical Solutions/ Optimal Control for Network PDEs
  1. A stochastic method inspired by Random Batch Method
  2. PINN approach

A Stochastic Algorithm for the Efficient Simulation for Networked Linear hyperbolic systems

Origins of the Random Batch Method

Initial Motivation: Simulation and control of large *interacting particle systems* can be computationally demanding.

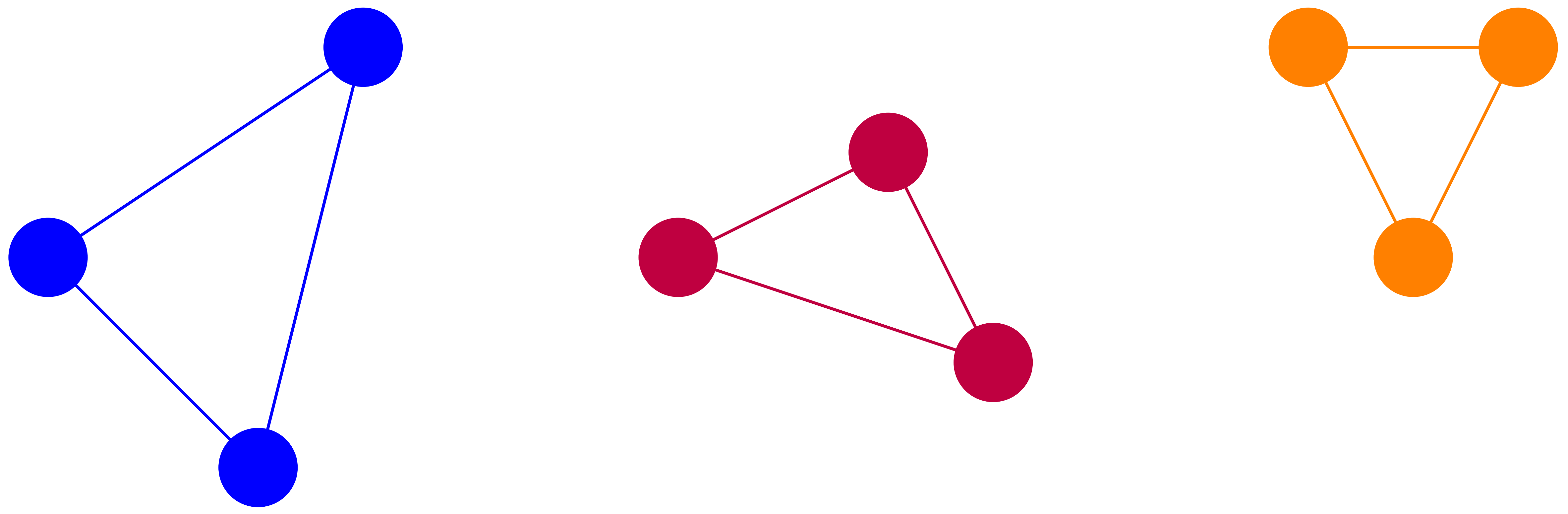


There are $N(N-1)/2$ interaction forces between N particles.
$\Rightarrow$ Computational cost grows rapidly when N is large.

Origins of the Random Batch Method

Proposed simulation method: The Random Batch Method

[Shi Jin, Lei Li, Jian-Guo Liu, J. of Computational Physics, 2020]

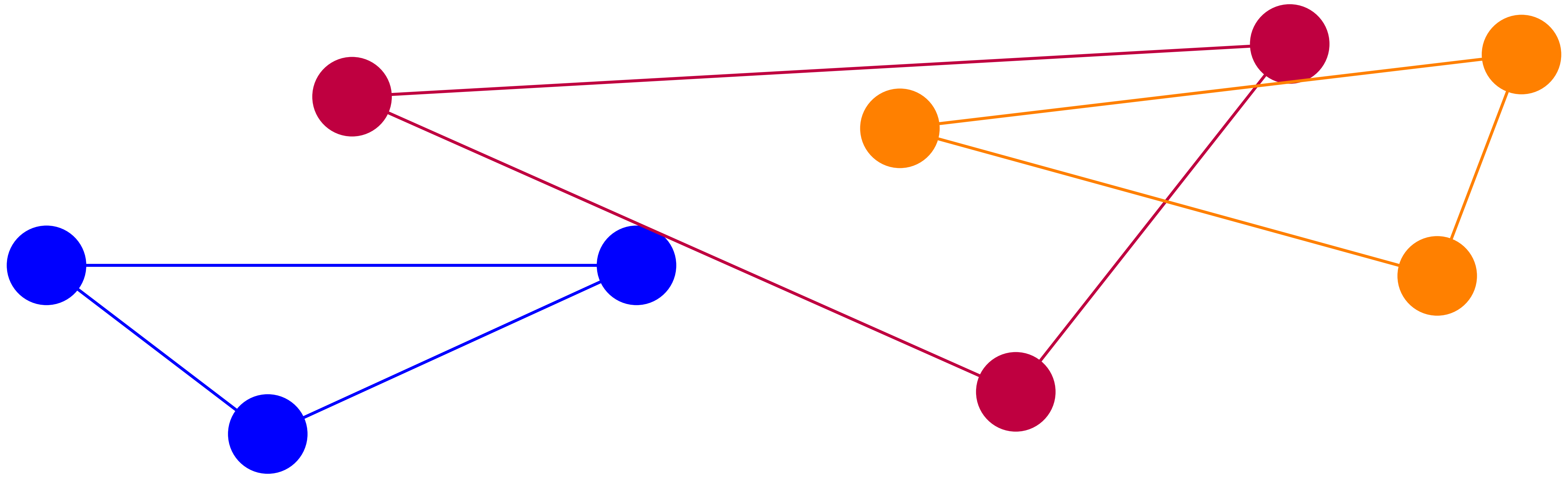


- Divide the N particles randomly into batches of size $P \geq 2$.
- Consider only interactions between particles in the same batch.
- Do a simulation over a short time interval of length h.

Origins of the Random Batch Method

Proposed simulation method: The Random Batch Method

[Shi Jin, Lei Li, Jian-Guo Liu, J. of Computational Physics, 2020]



- Divide the N particles randomly into batches of size $P \geq 2$.
- Consider only interactions between particles in the same batch.
- Do a simulation over a short time interval of length h.
- Repeat.

Origins of the Random Batch Method

Proposed simulation method: The Random Batch Method

[Shi Jin, Lei Li, Jian-Guo Liu, J. of Computational Physics, 2020]

- the RBM-solution converges to the solution of the original problem as $h \to 0$.
- the RBM reduces the computational cost from $O(N^2)$ to $O(PN)$.

RBM for Optimal Control

- The RBM can speed up the solution of optimal control problems governed by interacting particles systems [D. Ko, E. Zuazua, Math. Models Methods Appl. Sci., Vol. 31, No. 8, 2021] (only numerical experiments).
- The first convergence proof is given in [D.Veldman, E.Zuazua, Numerische Mathematik, 2022] for finite dimensional linear-quadratic optimal control in the operator-splitting setting.

$$\min_u \int_0^T (|x(t) - x_d(t)|^2) + |u(t)|^2 \mathrm{d}t,$$
$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

.

- Whether this algorithm can accelerate the simulation and optimization of nonlinear dynamics and **networked** infinite-dimensional systems, its convergence theory and applications are still open issues! (A **is unbounded operator**).

RBM for Hyperbolic equations: Toy Example

Consider the transport equation

$$
\begin{aligned}
&y_t(t,x) + v(x) y_x(t,x) = 0, && t \in (0,T), x \in \mathbb{R},\\
&y(0,x) = y_0(x), && x \in \mathbb{R},
\end{aligned}
$$

where $v(x)$ is bounded and Lipschitz, y_0 is globally Lipschitz.

- We split the generator of the semi-group as

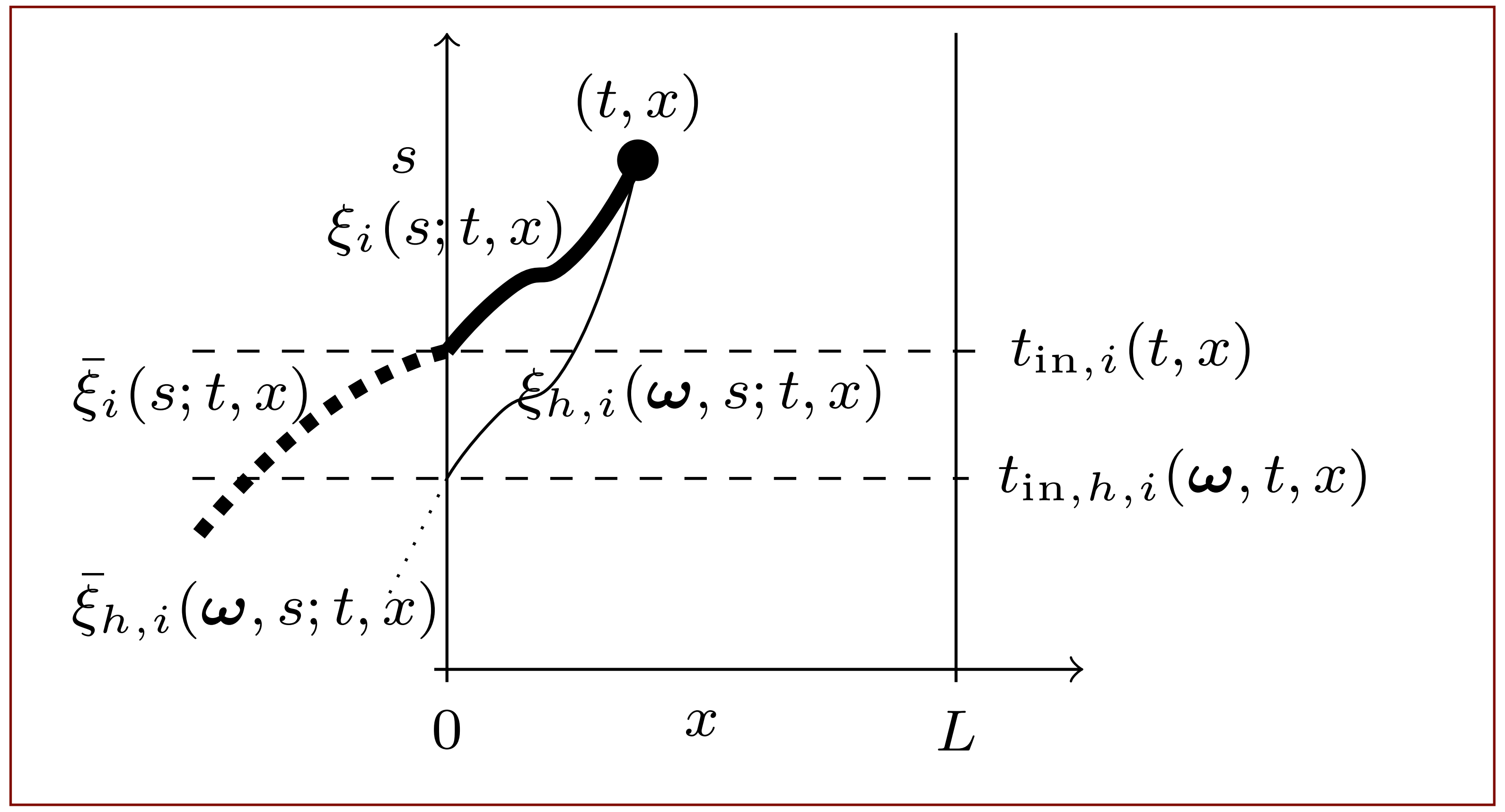
$$
\underbrace{-v(x)\frac{\partial}{\partial x}}_{A} = \sum_{m=1}^{M} \underbrace{-v_m(x)\frac{\partial}{\partial x}}_{A_m},
$$

  where the $v_m(x)$ are Lipschitz and bounded.
- In each time step, we randomly choose batch B_k, subset of $\{1, .., M\}$, of size P and consider the velocity field as

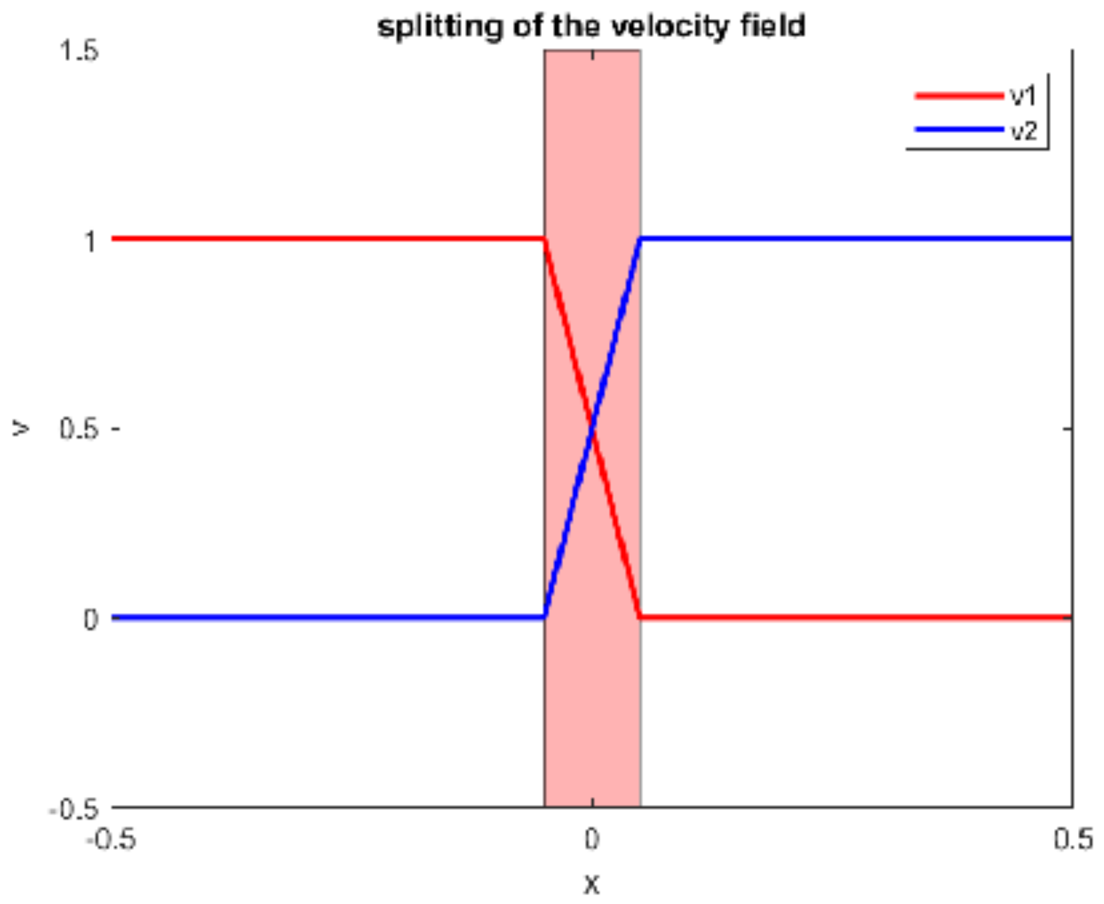
$$
v_h(\boldsymbol{\omega}, x) = \frac{M}{P} \sum_{m \in B_k} v_m(x), \qquad t \in [t_{k-1}, t_k).
$$

Let $y_{h,t}(\omega, t, x)$ be the solution resulting from $v_h(\boldsymbol{\omega}, t, x)$, then

$$
\mathbb{E}[\,|y_h(t) - y(t)|^2_{L^\infty}] \leq Ch
$$



Toy example: visualization



$$v(x) \equiv 1 = v_1(x) + v_2(x)$$

h=0.01

h=0.001

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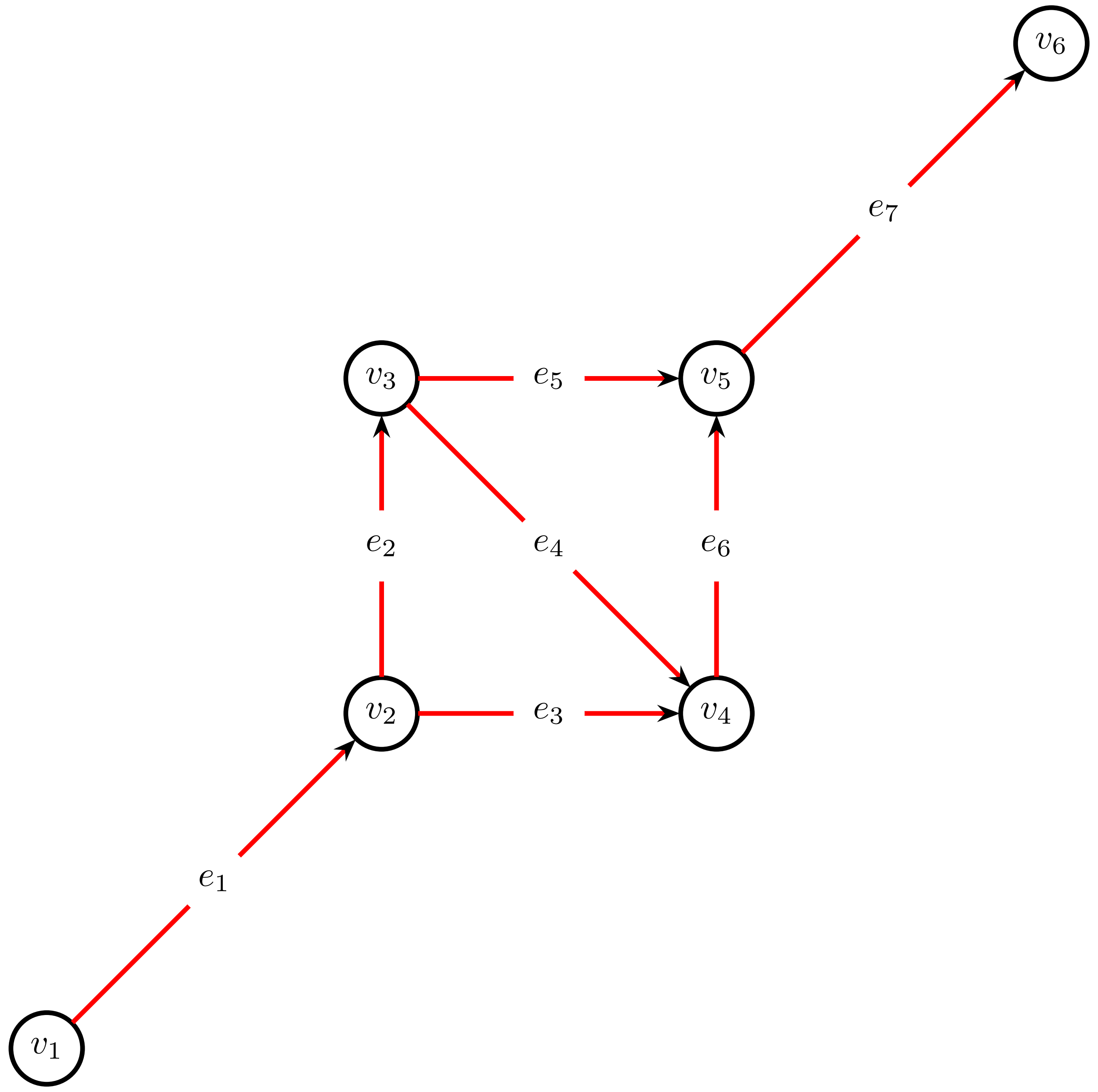
RBM for Networks

Coupled wave equations on diamond networks [D. W. M. Veldman, Y. Wang, 2024]



Diamond Directed Graph

$$\begin{cases} y^{e_i}_{tt}(t,x) - c^2_{e_i} y^{e_i}_{xx}(t,x) = 0 & e_i \in E, \\ \displaystyle\sum_{e_i \in E(v_j)} D_{ji} c_{e_i} y^{e_i}_x(t, v_j) = \bar{u}^{v_j}(t) & v_j \in V, \\ y^{e_i}(t, v_j) = y^{e_k}(t, v_j), & \forall e_i, e_k \in E(v_j), v_j \in V, \\ y^{e_i}(0,x) = y^{e_i}_0(x), \quad y^{e_i}_t(0,x) = y^{e_i}_1(x), & e_i \in E, \end{cases}$$

$$\mathbf{w}^{e_i}(t,x) = \begin{pmatrix} w^{e_i}_-(t,x) \\ w^{e_i}_+(t,x) \end{pmatrix} = \begin{pmatrix} y^{e_i}_t(t,x) + c_{e_i} y^{e_i}_x(t,x) \\ y^{e_i}_t(t,x) - c_{e_i} y^{e_i}_x(t,x) \end{pmatrix}$$

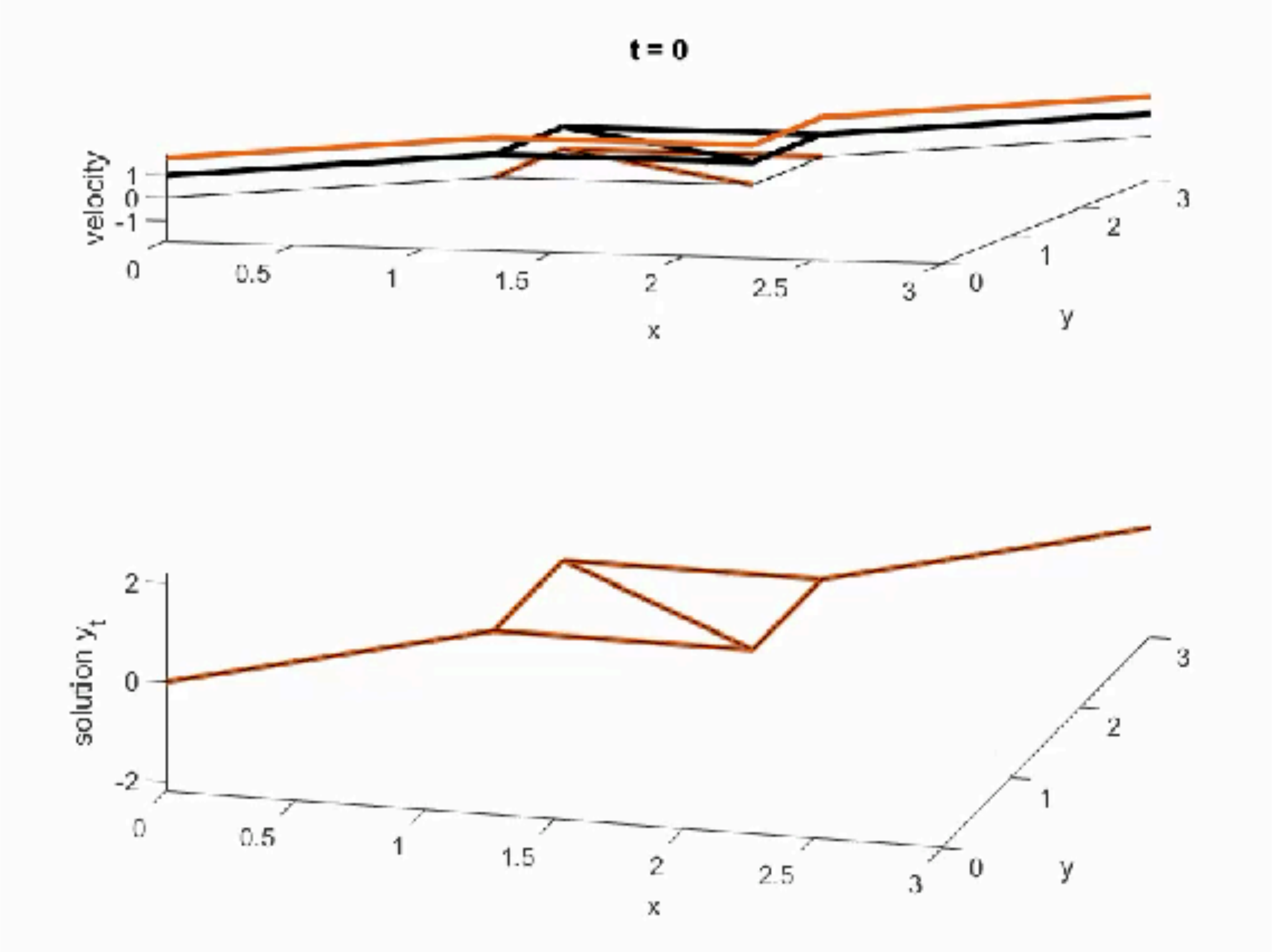
$$\begin{cases} w^{e_i}_{-,t}(t,x) - c_{e_i} w^{e_i}_{-,x}(t,x) = 0, \\ w^{e_i}_{+,t}(t,x) + c_{e_i} w^{e_i}_{+,x}(t,x) = 0. \end{cases}$$

Riemann Variables for the Wave Equation

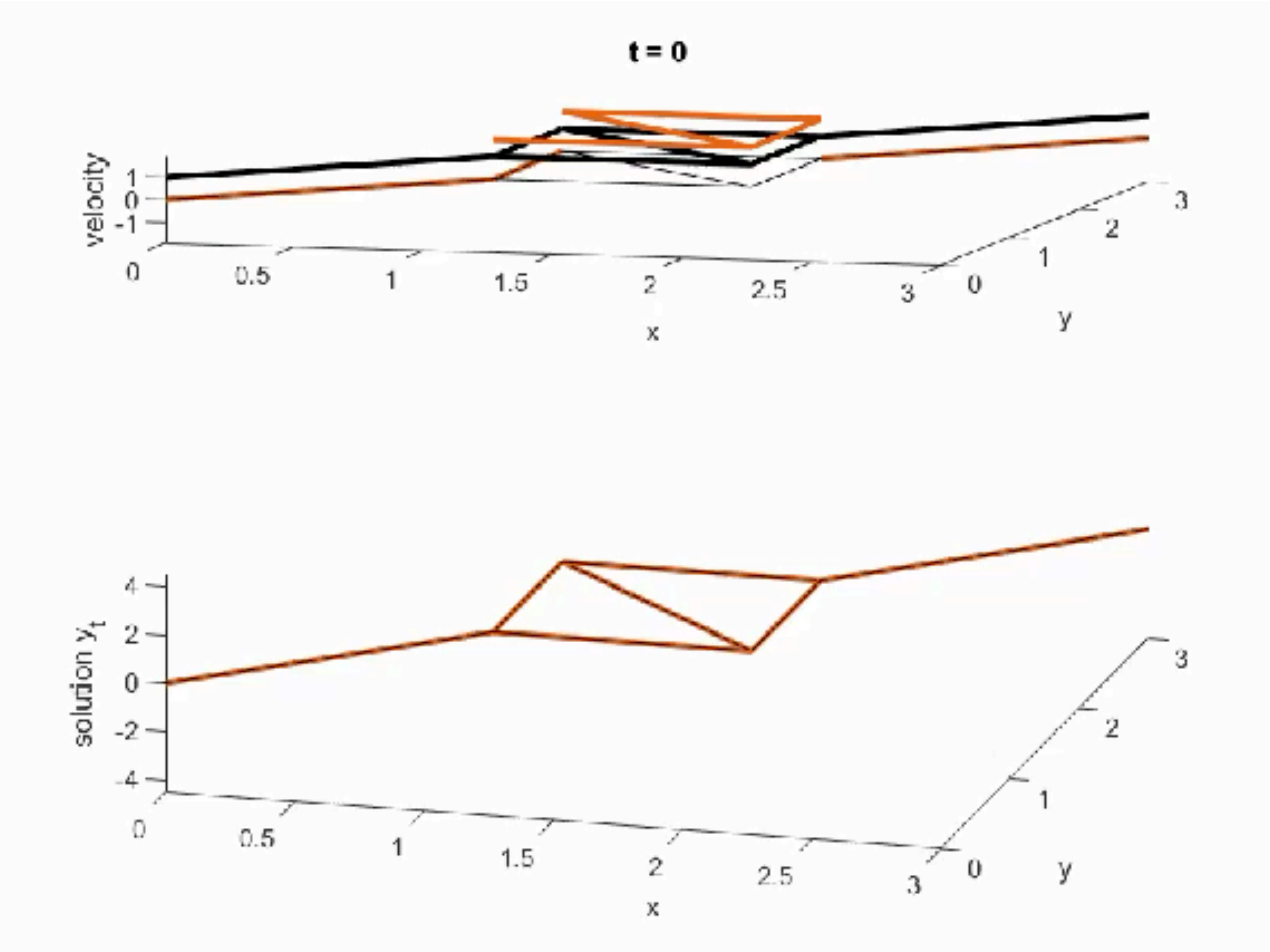
Numerical Illustration

Coupled wave equations on diamond [D. W. M. Veldman, Y. Wang, 2024]

- Split the velocity field per edge.
- P = 4 of M= 7 edges are active simultaneously.



- $h = 0.01, dx = 0.1$
- full Model (Black): 0.36 s
- RBM-approximation: 0.27 s
- reduction: 24%
- error: 22%



- $h = 0.001, dx = 0.01$
- full Model (Black): 551 s
- RBM-approximation: 397 s
- reduction: 28%
- error: 18%

Convergence Results

$$\min_{\mathbf{u}} J(\mathbf{u}) = \frac{1}{2}\|\mathbf{y} - \mathbf{y}_d\|^2_{L^2(Q)} + \frac{s_0}{2}|\mathbf{u}|^2_{L^2(0,T)} + \frac{s_1}{2}|\mathbf{u}_t|^2_{L^2(0,T)}$$

- If the original system admits an H^1 solution $y(t,x)$, then
$$\lim_{h\to 0} \mathbb{P}[|y_h^{e_i}(t) - y^{e_i}(t)|_{L^2(0,\ell_{e_i})} > \varepsilon] = 0$$
- If the original system admits an H^2 solution, then
$$\mathbb{E}[|y_h^{e_i}(t) - y^{e_i}(t)|^2] \leq Ch$$
- Remark: Markovs inequality
$$\mathbb{P}[X \geq a] \leq \frac{\mathbb{E}[X]}{a}$$

- If $s_0 > 0$ and $s_1 = 0$, then
$$\lim_{h\to 0} \mathbb{P}[|\mathbf{u}_h^* - \mathbf{u}^*|_{L^2(0,T)} > \varepsilon] = 0.$$
- If $s_1 > 0$, then
$$\mathbb{E}[|\mathbf{u}_h^* - \mathbf{u}^*|_{L^2(0,T)}] \leq Ch$$

y: original solution
y_h: solution to randomized system
$\mathbf{u}^*$: optimal control to original system
$\mathbf{u}_h^*$: optimal control to randomized system

Summary and Perspectives

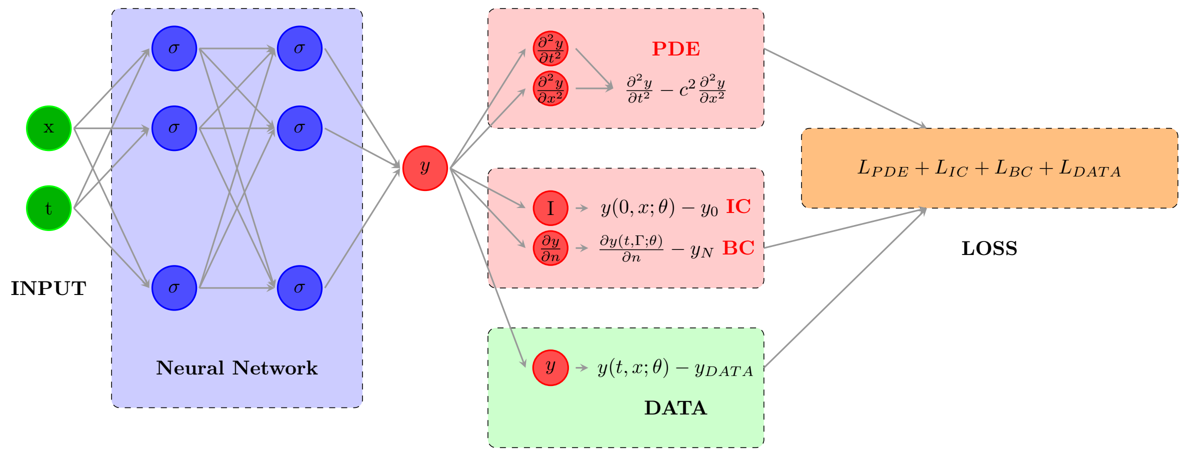
- The application of the RBM to (networked) hyperbolic PDEs combines
  - **operator splitting for PDEs**
  - **stochastic methods for large-scale optimization**
  - **characteristic method for 1d Hyperbolic type PDEs.**
- We **efficiently approximate** the solution to networked linear hyperbolic equations and associated optimal control problems, and obtain the **convergence results**
  - $\mathbf{y}_h(\omega, t)$ converges to $\mathbf{y}(t)$ for $h \to 0$
  - Convergence in the optimal controls can be proven along the lines of [E.Zuazua, D.Veldman 2022], but some regularity properties need to be verified.
- Extensions to nonlinear setting:
  - semi-linear case is straight forward, e.g. $y_t + \Lambda y_x = f(t, x, y)$ with f Lipschitz in y.
  - quasi-linear case is more challenging but appears in many real-world applications (nonlinear transport equations / conservation laws, and networks of incompressible Euler equations)
- Extension to non-overlapping domain decomposition on complex spatial structures and XPINNs

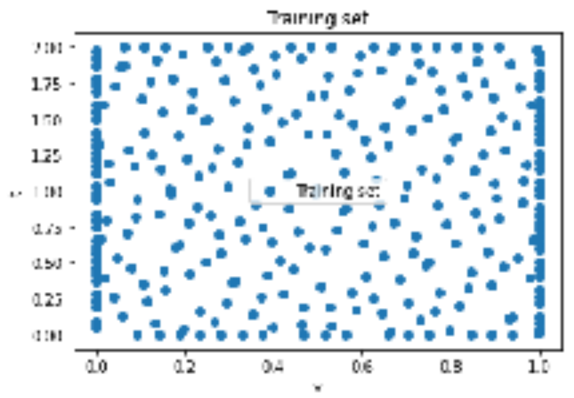
Projects on Real-time capable methods and algorithms

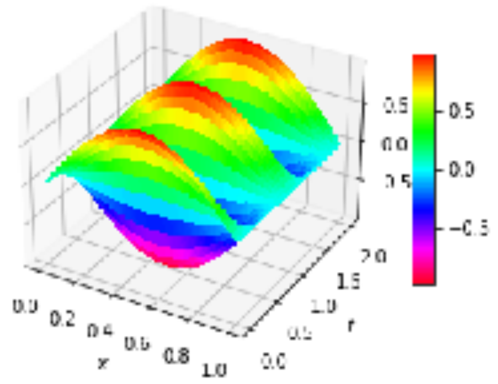
- **Simulation, inverse problems, and control for (degenerate) 1-D wave equations using PINNs.**
  https://github.com/DCN-FAU-AvH/PINNs_wave_equation

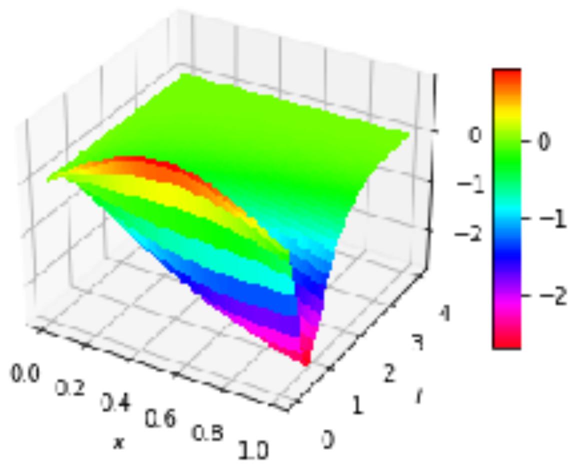
- Dania Sana (Jun. - Sep. 2022)
  supervised by Y. Wang and E. Zuazua

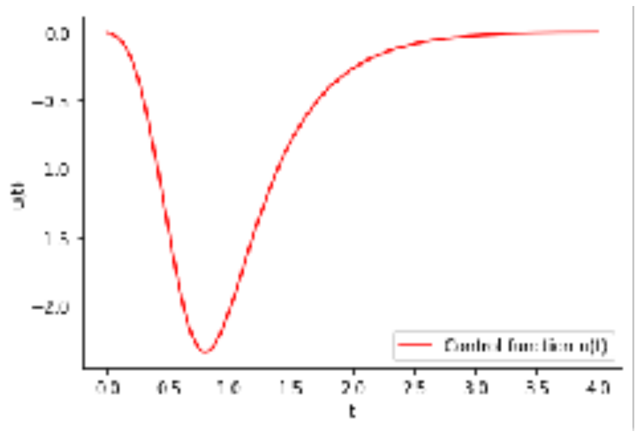
- Internship for young female researchers at FAU-MoD (Center for Mathematics of Data)











- **PGML: Physics-Guided Machine Learning (2024-2027)**
  focusing on
  Simulation and modeling of electrochemical cells and of mechanical systems

- **SHARE at FAU (Schäffler Hub for Advanced Research at Friedrich-Alexander University)**

Thank you!

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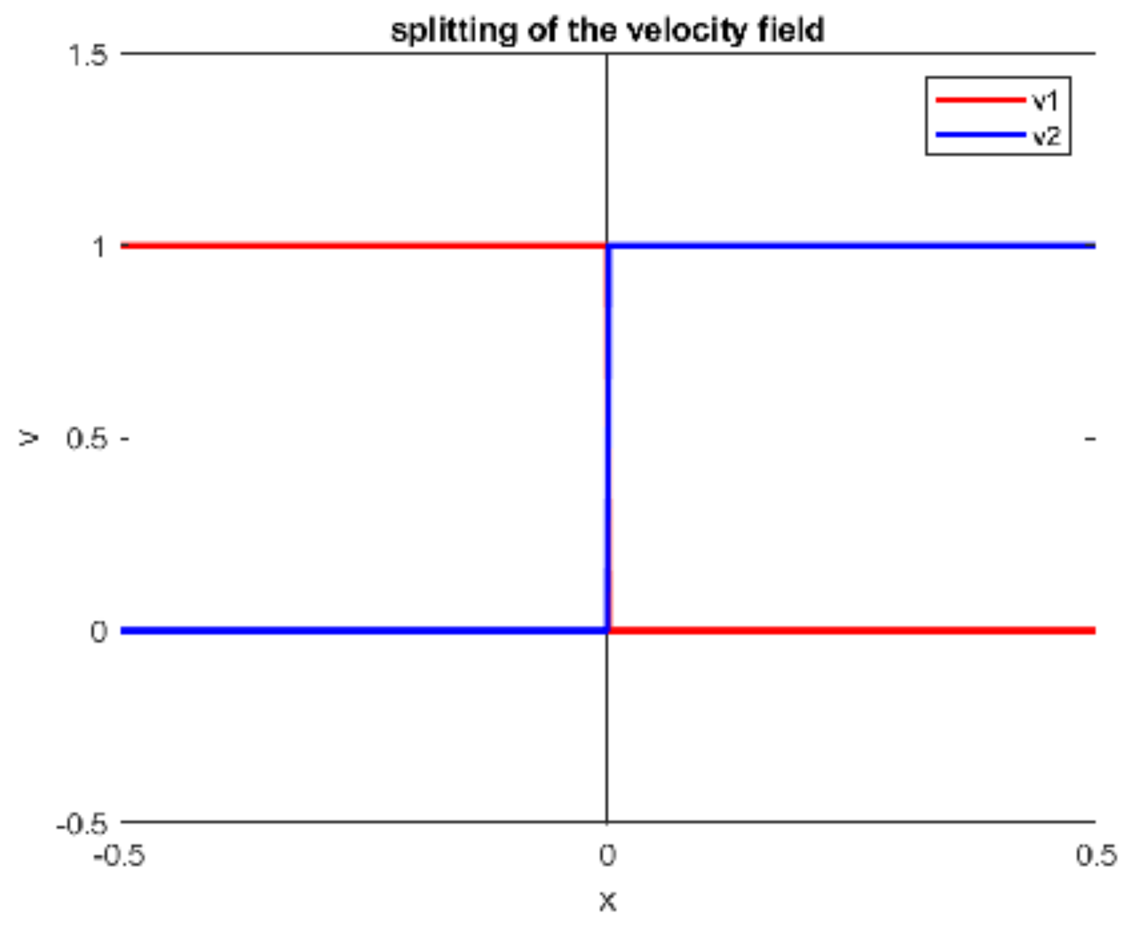
Trends in the Mathematical Sciences

ERLANGEN. JUNE 10 - 14, 2024

Toy example: visualization



$$v(x) \equiv 1 = {\color{red} v_1(x)} + {\color{blue} v_2(x)}$$

h=0.001

Lemma: A uniform priori estimate of solution to (FOS) [Y.W.'19]

$$\|w(t,\cdot)\|_1 \triangleq \|w(t,\cdot)\| + \left\|\frac{\partial w}{\partial x}(t,\cdot)\right\| \leq C(T), \qquad 0 \leq t \leq T,$$

where $\|\cdot\|$ denotes C^0-norm.
Main Idea in the Proof: We apply

$$\mathbf{l}_i^- = (\mathbf{0}, \sqrt{K^i_{w_2^i}}, 1), \quad \mathbf{l}_i^0 = (1, \mathbf{0}, \mathbf{0}), \quad \mathbf{l}_i^+ = (\mathbf{0}, -\sqrt{K^i_{w_2^i}}, 1)$$

to (FOS) and define **Riemann variables** as

$$v_i = \mathbf{l_i}(w)w, \quad \bar{v}_i = \mathbf{l_i}(w)w_x.$$

They follow

$$\frac{Dv_i}{D_i t} = \sum_{j,k=1}^{n} \beta_{ijk}(w)\bar{v}_j\bar{v}_k + \sum_{j=1}^{n} \tilde{\beta}_{ij}(w)\tilde{F}_j(w) \qquad (i = 1, ..., n),$$

$$\frac{D\bar{v}_i}{D_i t} = \sum_{j,k=1}^{n} \gamma_{ijk}(w)\bar{v}_j\bar{v}_k + \sum_{j=1}^{n} \tilde{\gamma}_{ij}(w)\bar{v}_j \qquad (i = 1, ..., n),$$

along the characteristic curves, where

$$\frac{D}{D_i t} = \frac{\partial}{\partial t} + \lambda_i(u)\frac{\partial}{\partial x}.$$

Uniform Priori Estimate (ctd.)

Let

$$T_1 = \min_{\substack{i=1,\ldots,n;\\ \|w\|\le\eta_0}} \frac{L}{|\lambda_i(w)|} > 0.$$

For $(t,x)\in\mathcal{R}(T_1)$, we estimate $|v_i(t,x)|$ by integrating (backward) along the characteristic curve (three cases, $\lambda_i <,=,> 0$). It will arrive at $(0,\alpha)$, or (t_*,L), or $(t_*,0)$. In different cases, we could obtain

$$|v_i(t,x)| \le \|v(0,\cdot)\| + C_1\int_0^t v_i(\tau)\,d\tau,$$

or

$$|v_i(t,x)| \le A\|v_i(0,\cdot)\| + \|u'\| + C_2\int_0^t v_i(\tau)\,\mathrm{d}\tau, \qquad \forall t\in[0,T_1],$$

where $v(\tau) = \sup_{0\le t\le\tau}\|v(t,\cdot)\|$.
Using **Gronwall inequality** it follows that

$$|v(t,x)| \le C\max\{\|u'\|,\|v(0,\cdot)\|\} \triangleq C\alpha_0, \qquad \forall t\in[0,T_1],$$

with $C>1$.
Then **repeating** $N = \left[\frac{T}{T_1}\right]+1$ times, we have

$$|v(t)| \le C^N\alpha_0, \qquad \forall t\in[(N-1)T_1, T].$$