SPARSE INITIAL SOURCE IDENTIFICATION FOR A DIFFUSION-ADVECTION EQUATION

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Motivation and Introduction

- **Background** An important inverse problem arising in scientific computing is the identification of moving pollution sources in fluids that can be described by diffusion-advection systems. Such kind problems can be mathematically modeled by initial source identification problems of diffusion-advection systems, where the initial source is assumed to be sparse, i.e., its support is zero in Lebesgue measure.
- **Diffusion-advection equation.** Let $\Omega \subset \mathbb{R}^N$ with $N \geq 1$ be a bounded domain and $\partial\Omega$ its boundary. Consider the linear diffusion-advection equation:

$$\begin{cases} \partial_t u - d\Delta u + v\cdot\nabla u = 0 & \text{in } \Omega\times(0,T), \\ u = 0 & \text{on } \partial\Omega\times(0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where $0 < T < +\infty$, $d > 0$, and $v \in \mathbb{R}^N$. We further assume that

$$u_0(x) = \sum_{i=1}^{l} \alpha_i \delta_x(x_i),$$

where $\{\alpha_i\}_{i=1}^l \in \mathbb{R}^l$ and $x_i \in \Omega, 1 \leq i \leq l$, are the intensities and locations, respectively, with $1 \leq l < +\infty$ the number of locations. The Dirac measure $\delta_x(x_i)$ is defined by $\delta_x(x_i) = 1$ if $x = x_i$, and $\delta_x(x_i) = 0$ otherwise.
- **Sparse Initial Source Identification Problem.** Consider the diffusion-advection equation (1). Let u_T be a given or observed function. We aim at identifying an initial condition $\widehat{u}_0^*$ satisfying

$$\widehat{u}_0^*(x) = \sum_{i=1}^{l} \widehat{\alpha}_i^* \delta_x(\widehat{x}_i^*) \quad \text{with} \quad \widehat{\alpha}_i^* \in \mathbb{R},\ \widehat{x}_i^* \in \Omega$$

such that the corresponding final state $\widehat{u}^*(\cdot;T)$ of (1) is close to u_T, in the sense that for $\varepsilon > 0$ arbitrary small we have

$$\|\widehat{u}^*(\cdot;T) - u_T\|_{L^2(\Omega)} \leq \varepsilon, \text{ a.e in } \Omega.$$

- **Our numerical approach** It is well-known that the sparse initial source identification problem is exponentially ill-posed and thus challenging to design some efficient numerical algorithms for solving it. We propose a new two-stage numerical approach consisting of a sparsity promotion stage and a structure enhancement stage.

Sparsity Promotion Stage

- **Optimal control problem.** We formulate the sparse initial source identification problem as the following optimal control problem:

$$u_0^* = \arg\min_{u_0\in L^2(\Omega)} = \frac{1}{2}\int_\Omega |u(\cdot,T) - u_T|^2\,dx + \frac{\tau}{2}\int_\Omega |u_0|^2\,dx + \beta\int_\Omega |u_0|\,dx, \tag{2}$$

  - $u(\cdot,T)$ is the final state of equation (1) corresponding to u_0, the constants $\tau > 0$ and $\beta > 0$ are regularization parameters;
  - the first term $\frac{1}{2}\int_\Omega |u(\cdot,T) - u_T|^2\,dx$ seeks for an initial condition u_0 such that the corresponding $u(\cdot,T)$ is as close as possible to u_T;
  - the L^1-regularization term promotes the sparsity of u_0;
  - the L^2-regularization term is introduced to guarantee the well-posedness while improving the conditioning of the optimal control problem.
- For solving (2), we introduce a generalized primal-dual algorithm. The main computation at each iteration is solving only two PDEs. Hence, its implementation is easy and computationally cheap; its strong global convergence and worst-case convergence rate are also analyzed.
- Due to the presence of the L^2-regularization term and its smoothing property, the recovered initial condition u_0^* by solving (2) is not sparse as desired (see the results reported below). Hence, a structure enhancement stage is necessary to identify the optimal locations $\{\widehat{x}_i^*\}_{i=1}^l$ and the intensities $\{\widehat{\alpha}_i^*\}_{i=1}^l$.

Figure: Numerical results by solving (2) with $T = 0.01$, $d = 1$, $v = (0,0)^\top$, $\tau = 10^{-2}$ and $\beta = 3\times 10^{-1}$

(a) Reference initial datum $\widehat{u}_0$ (b) Recovered initial datum u_0^*

Structure Enhancement Stage

- **Optimal locations identification.** It was shown in [4] that the local maxima of $|u_0^*(x)|$ fall into the optimal locations. Consequently, one can identify the optimal locations $\widehat{x}^*$ by solving

$$\widehat{x}^* = \arg\max_{x\in \text{supp}(u_0^*)} |u_0^*(x)|, \tag{3}$$

where we denote by $\text{supp}(u_0^*)$ the support of u_0^*.
- **Optimal intensities identification.** Since equation (1) is linear, the corresponding solution operator $\mathcal{L}$ verifies $\mathcal{L}u_0 = \sum_{i=1}^l \alpha_i \mathcal{L}\delta_x(x_i)$, for any $u_0(x) = \sum_{i=1}^l \alpha_i\delta_x(x_i)$ with $\alpha_i \in \mathbb{R}$ and $x_i \in \Omega$. To find the optimal intensities , it is sufficient to consider the following least squares problem:

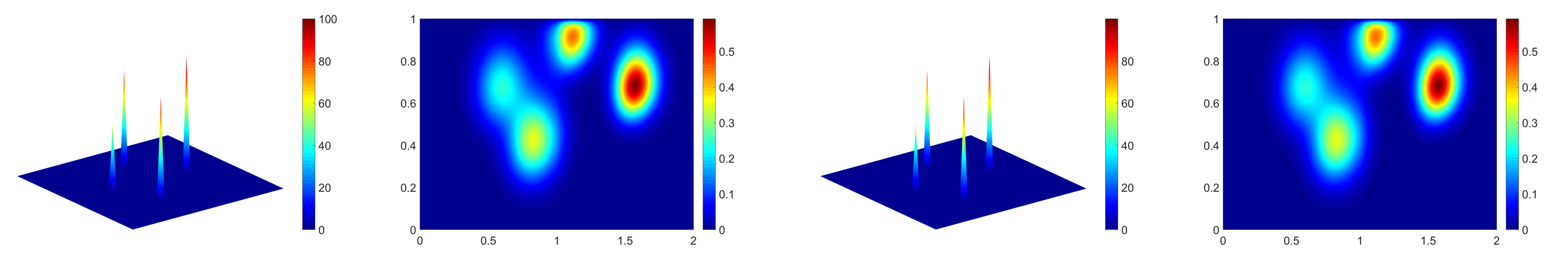
$$\{\widehat{\alpha}_i^*\}_{i=1}^l = \arg\min_{\{\alpha_i\}_{i=1}^l\in\mathbb{R}^l} \frac{1}{2}\left\|\sum_{i=1}^l \alpha_i\mathcal{L}\delta_x(\widehat{x}_i^*) - u_T\right\|_{L^2(\Omega)}^2. \tag{4}$$

With the computed locations $\{\widehat{x}_i^*\}_{i=1}^l$ and intensities $\{\widehat{\alpha}_i^*\}_{i=1}^l$, the recovered initial source is thus given by

$$\widehat{u}_0^* = \sum_{i=1}^l \widehat{\alpha}_i^*\delta_x(\widehat{x}_i^*).$$

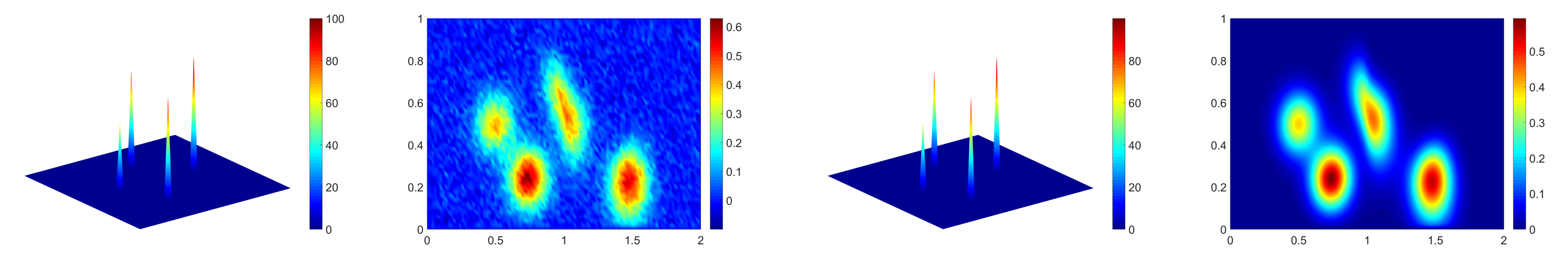
Numerical Results

Figure: Numerical results for $d = 0.08$ on $\Omega_1 = (0,1)\times(0,1)$, $d = 0.05$ on $\Omega_2 = (1,2)\times(0,1)$, $v = (1,2)^\top$ on Ω, and a reachable target u_T at $T = 0.1$.



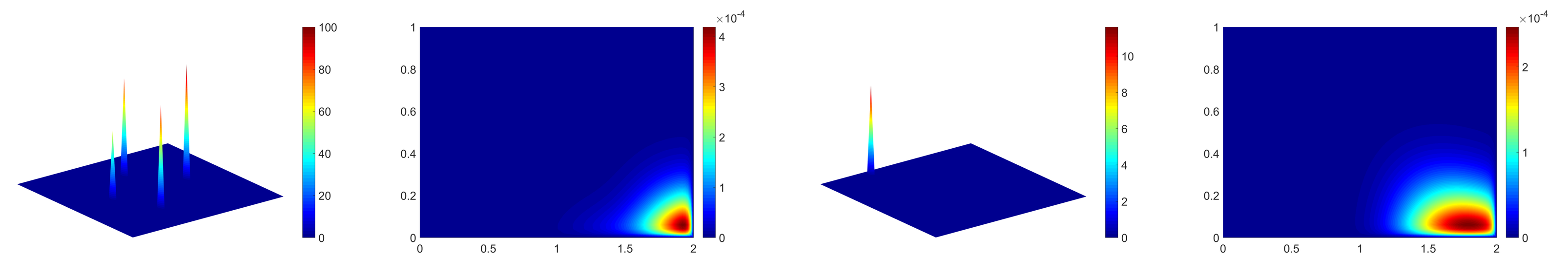
(a) Reference initial state (b) Reachable target u_T (c) Recovered initial state (d) Recovered final state

Figure: Numerical results for $d = 0.05$ on Ω, $v = (0,0)^\top$ on $\Omega_1 = (0,1)\times(0,1)$, $v = (0,-3)^\top$ on $\Omega_2 = (1,2)\times(0,1)$), and a noisy observation u_T at $T = 0.1$.



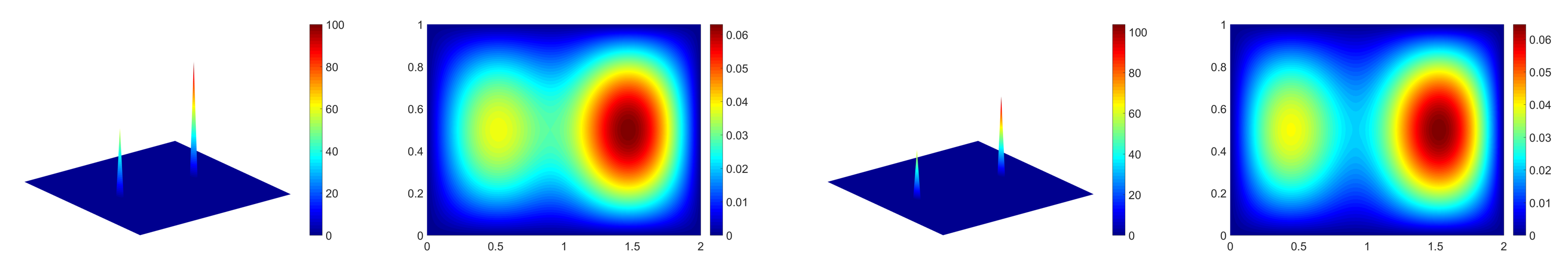
(a) Reference initial state (b) Noisy observation u_T (c) Recovered initial state (d) Recovered final state

Figure: Numerical results for $d = 0.05$, $v = (2,-2)^\top$ on Ω, and a reachable target u_T at $T = 1$.



(a) Reference initial datum $\widehat{u}_0$ (b) Reference final state u_T (c) Recovered initial datum $\widehat{u}_0^*$ (d) Recovered final state $u(T)$

Figure: Numerical results for $d = 0.05$ and $v = (0,0)^\top$ on Ω, and a reachable target u_T at $T = 1$.



(a) Reference initial state (b) Reachable target u_T (c) Recovered initial state (d) Recovered final state

Conclusions and Perspectives

- Conclusions
  - When the final time T is small (e.g., $T = 0.1$), the initial sources from reachable targets or noisy observations were efficiently and accurately identified by our proposed two-stage numerical approach, even for some heterogeneous materials or coupled models.
  - When the final time T gets larger (e.g., $T = 1$), the problem becomes increasingly ill-posed, and the sparse initial source may not be identified correctly. We observe that the admissible final time, for which the sparse initial source can be identified accurately, varies from case to case.
- Perspectives
  - Design novel and efficient algorithms allowing to address the sparse initial source identification of advection-diffusion systems in some relatively longer time horizons.
  - Address a complete analysis of the maximum admissible final time, for which the sparse initial source can still be accurately identified.
  - Discuss the optimal combination of the L^2- and L^1-regularization parameters.
  - Design algorithms for the sparse initial source identification of equations that are nonlinear or modeled on more complicated geometries.

Literature

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