FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG

FACULTY OF SCIENCES

CHAIR IN APPLIED ANALYSIS FAU

NON-LOCAL PDE AND CONTROL

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Why non-local?

Some relevant models in sciences and engineering are of non-local nature. They describe complex phenomena for which a local approach is inappropriate or limiting.

- Peierls-Nabarro equation in elasticity.
- Population dynamics.
- Laser implementation.
- Porous media flow.
- Finance: pricing of American options.
- ...

In this setting, classical PDE theory fails because of non-locality. Yet many of the existing techniques can be tuned and adapted, although this is often a delicate matter.

1. Fractional Schrödinger equation

In [Bic21], we analyze interior control properties for the fractional Schrödinger equation

$$iz_t + (-\Delta)^s z = u\chi_\omega, \quad (x,t) \in \Omega \times (0,T) \tag{1}$$

$\Omega \subset \mathbb{R}^N$ being a bounded and regular domain, with $\omega \subset \Omega$ a neighborhood of $\partial\Omega$ satisfying GCC, and where for any $s \in (0,1)$ we denote with $(-\Delta)^s$ the fractional Laplace operator

$$(-\Delta)^s z(x,t) := \mathcal{C}(N,s) P.V. \int_{\mathbb{R}^N} \frac{z(x,t) - z(y,t)}{|x-y|^{N+2s}}\, dy.$$

Through the Hilbert Uniqueness Method, we obtain the following controllability results:

- When $s = 1/2$, the equation (1) is null-controllable at time $T_0 > 0$ large enough, i.e. for any $T \geq T_0$ there exists a control function $u \in L^2(\omega \times (0,T))$ such that $y(\cdot,T) = 0$ a.e. in Ω.
- When $s \in (1/2, 1)$, the equation (1) is null-controllable at any time $T > 0$.
- When $s \in (0, 1/2)$, the equation (1) is not null-controllable.

The attainment or failure of null-controllability in dependence of s is related with the propagation properties of the rays of geometric optics, solutions to the Hamiltonian system associated with (1) (see Figure 1). Indeed, through a stationary phase approach, we can see that the group velocity of the solutions to (1) is $v \sim |\xi|^{2s-1}$, ξ being the solution frequency, and therefore

- For $s > 1/2$, the group velocity increases when ξ increases.
- For $s = 1/2$, the group velocity remains constant with respect to ξ.
- For $s < 1/2$, the group velocity decreases when ξ increases.

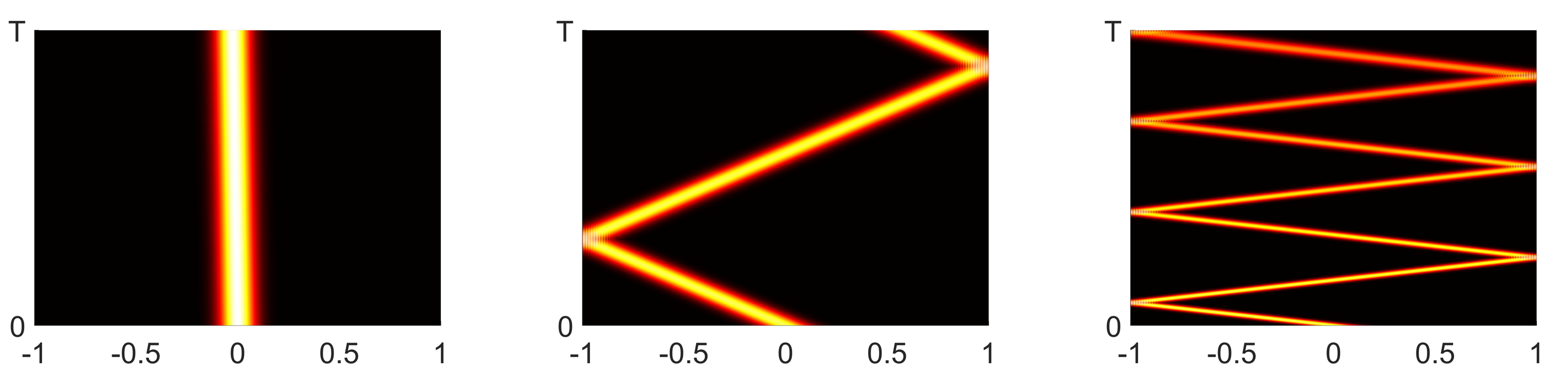

FIGURE 1. Rays of geometric optics associated with (1) in space dimension $N = 1$ on the interval $\Omega = (-1,1)$ for $s < 1/2$ (left), $s = 1/2$ (middle) and $s > 1/2$ (right).

In particular, when $s < 1/2$, the high-frequency solutions may not reach the control region ω, thus making impossible to control them.

2. PDE with nonlocal potentials

In [BH19a], we study the null controllability of the following heat equation with non-local potential

$$z_t - \Delta z + \int_\Omega K(x,\theta,t) z(\theta,t)\, d\theta = u\chi_\omega, \quad (x,t) \in \Omega \times (0,T), \tag{2}$$

ω being an open subset of the bounded and regular domain $\Omega \subset \mathbb{R}^N$.

Assuming that $K \in L^\infty(\Omega \times \Omega \times (0,T))$ satisfies suitable time-decay conditions, we prove that (2) is null-controllable at any time $T > 0$.

Our approach, based on a Carleman inequality for the solutions of (2), also provides explicit estimates on the cost of null-controllability, thus allowing us to consider non-linear models as well:

$$z_t - \Delta z + \int_\Omega K(x,\theta,t) z(\theta,t)\, d\theta = f(z) + u\chi_\omega, \quad (x,t) \in \Omega \times (0,T),$$

$$z_t - \Delta z + \int_\Omega K(x,\theta,t) f(z(\theta,t))\, d\theta = u\chi_\omega, \quad (x,t) \in \Omega \times (0,T),$$

with $f \in C^1(\mathbb{R})$ globally Lipschitz.

3. Fractional heat equations

In [BH19], we analyze controllability properties of the fractional heat equation

$$z_t + (-\Delta)^s z = u\chi_\omega, \quad (x,t) \in (-1,1) \times (0,T) \tag{3}$$

We employ spectral techniques and parabolic Ingham inequalities to prove that, when $s \in (1/2,1)$, the equation (3) is null-controllable for any positive time $T > 0$ by means of a control function $u \in L^2(\omega \times (0,T))$. On the other hand, for $s \in (0,1/2]$ only approximate controllability holds, as a consequence of unique continuation properties for the fractional Laplacian.

Furthermore, in [BWZ20], we study the controllability of (3) under positivity constraints on the control and we show that:

- For $s \in (1/2,1)$ and $T \geq T_0 > 0$ large enough, (3) is null-controllable to trajectories by means of a non-negative control function $u \in L^\infty(\omega \times (0,T))$.
- For $s \in (1/2,1)$ and $T = T_0$, the non-negative control u belongs to the space of Radon measures $\mathcal{M}(\omega \times (0,T))$.

The same results as in [BWZ20] are obtained in [ABPWZ20] for the case of exterior controls, that is, when the control is placed outside $(-1,1)$ (see Figure 2). This is an generalization of the notion of boundary control for local PDE. It copes with the fact that in the presence of a fractional Laplacian, due to the non-locality of the operator, it does not make sense to localize the control in a subset of the boundary, as the resulting model would be ill-posed.

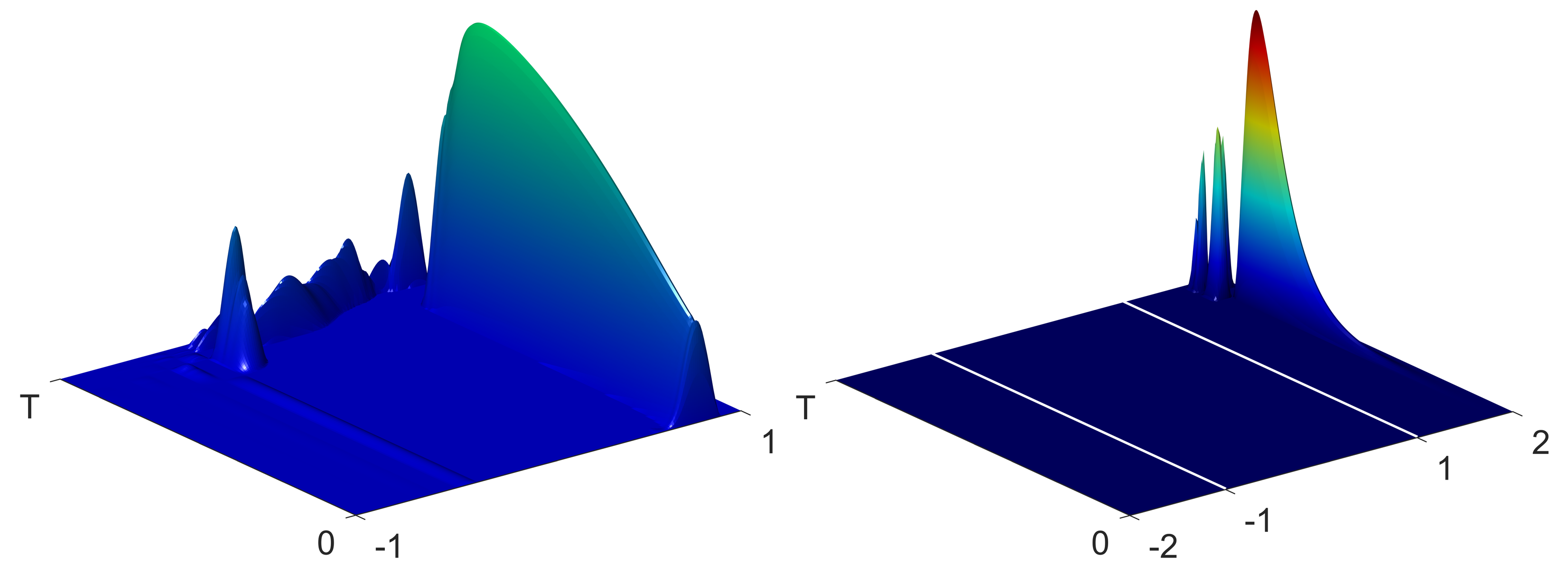

FIGURE 2. Non-negative interior (left) and exterior (right) controls for the fractional heat equation.

4. Memory and hybrid PDE-ODE models

Evolution equations involving memory terms model a large spectrum of phenomena which, apart from their current state, are influenced also by their history.

Given a positive self-adjoint operator $\mathcal{A}$, we consider the following wave-type equation with memory

$$y_{tt} + \mathcal{A}y - \int_0^t \mathcal{A}y(\cdot,\tau)\, d\tau = u\chi_{\omega(t)}, \quad (x,t) \in \Omega \times (0,T). \tag{4}$$

The memory term in (4) produces accumulation phenomena that affect the stability of the system, thus rendering the classical notion of controllability not suitable in this setting. Indeed, driving the solution to zero is no longer sufficient to guarantee that the dynamics of the system reaches an equilibrium, as one needs to ensure that the control shuts down also the memory effects. This motivates the introduction of the so-called **memory-type null controllability** property:

$$y(\cdot,T) = y_t(\cdot,T) = \int_0^T \mathcal{A}y(\cdot,\tau)d\tau = 0. \tag{5}$$

To prove (5), a standard approach is to rewrite (4) as a hybrid PDE-ODE system which couples a wave equation with an infinite-dimensional ordinary differential equation

$$\begin{cases} y_{tt} + \mathcal{A}y - z = u\chi_{\omega(t)}, & (x,t) \in (-1,1) \times (0,T) \\ z_t = \mathcal{A}y, & (x,t) \in (-1,1) \times (0,T). \end{cases} \tag{6}$$

This approach enhances other peculiar behaviors associated with the memory term, as the ODE component in (6) accumulates along vertical characteristics and does not propagate in space. From here, the necessity of employing a **moving control strategy**.

All these features of memory type equations are taken into account in [BM19,BW20], where we discuss the theoretical controllability properties of (4) in space dimension $N = 1$ and when

$$\mathcal{A} = -\Delta \quad \text{or} \quad \mathcal{A} = (-\Delta)^s.$$

Assuming that the support $\omega(t)$ of the control moves in space with a constant velocity c, we prove that:

- When $\mathcal{A} = -\Delta$, (4) is memory-type null controllable at time T provided that the time horizon is large enough, namely $T \geq T_0(c) > 0$.
- When $\mathcal{A} = (-\Delta)^s$, (4) is memory-type null controllable at time T provided that $s \in (1/2,1)$ and the time horizon is large enough, namely $T \geq T_1(c) > 0$.

Open problems

- Geometric optics for fractional wave-type PDE: rigorous micro-local analysis of fractional models to study propagation properties of the solutions and interactions with boundaries or interfaces in the physical domain.
- Heat equations with high-order non-local potentials: $z_t - \Delta z + \int_\Omega K(x,t)\Delta z(\theta,t)\, d\theta = u\chi_\omega$. Study of controllability properties in the linear and non-linear setting.
- Constrained and unconstrained controllability for multi-dimensional (possibly non-linear) fractional heat equations: can Carleman inequalities handle non-local terms?
- Feedback stabilization of PDE with memory: can the moving control strategy be integrated in the design of efficient feedback laws?

Selected publications

[ABPWZ20] Antil, H, Biccari, U., Ponce, R., Warma, M. & Zamorano, S. (2020). Controllability properties from the exterior under positivity constraints for a 1-D fractional heat equation. *Preprint.* arXiv:1910.14529.

[Bic21] Biccari, U. (2021). Internal control for a non-local Schrödinger equation involving the fractional Laplace operator. *Evol. Eq. Control. Theo.* to appear.

[BH19] Biccari, U. & Hernández-Santamaría, V. (2019). Controllability of a one-dimensional fractional heat equation: theoretical and numerical aspects. *IMA J. Math. Control Inf.*, 36(4): 1199–1235.

[BH19a] Biccari, U. & Hernández-Santamaría, V. (2019). Null Controllability of linear and semilinear nonlocal heat equations with an additive integral kernel. *SIAM J. Control Optim.*, 57(4):2924–2938.

[BM19] Biccari, U & Micu, S. (2019). Null-controllability properties of the wave equation with a second order memory term. *J. Differential Equations*, 267(2):1376–1422.

[BW20] Biccari, U. & Warma, M. (2020). Null-controllability properties of a fractional wave equation with a memory term. *Evol. Eq. Control. Theo.* 9(2): 399–430.

[BWZ20] Biccari, U, Warma, M. Zuazua, E. (2020). Controllability of the one-dimensional fractional heat equation under positivity constraints. *Commun. Pure Appl. Anal.*, 19(4):1949–1978.

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