FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG

FACULTY OF SCIENCES

CHAIR IN APPLIED ANALYSIS FAU

INVERSE DESIGN FOR CONSERVATION LAWS AND HAMILTON-JACOBI EQUATIONS

Carlos Esteve-Yagüe Universidad Autónoma de Madrid
Chair of Computational Mathematics, Fundación Deusto.

Enrique Zuazua Applied Analysis - AvH Professorship
FAU Erlangen-Nürnberg

Introduction

Inverse problems consist in identifying from observations the causes that produced them. For Scalar Conservation Laws and Hamilton-Jacobi equations, we consider the inverse problem consisting in the reconstruction of the initial condition for a given observed solution at some positive time.

Given $u_T(x) = u(T,x)$, can we reconstruct $u_0(x) = u(0,x)$?

Main issues:

- Characterization of the reachable set (admissible observations).
- Lack of backward uniqueness due to the apparition of shocks.
- Reconstruction of all the initial conditions which are compatible with the given observation u_T.
- The observation might be noisy (construction of the "closest" admissible observation).

Scalar conservation laws

We consider the following scalar conservation law (Burgers equation):

$$\begin{cases} \partial_t v + \partial_x \left(\frac{v^2}{2}\right) = 0 & \text{in } (0,T)\times\mathbb{R} \\ v(0,x) = v_0(x) & \text{in } \mathbb{R}. \end{cases} \qquad \text{(SCL)}$$

For any $v_0 \in BV(\mathbb{R})$ there exists a unique **entropy solution** $v \in BV((0,T)\times\mathbb{R})$.

The reachable set

For any $T>0$ we can do the following partition of $BV(\mathbb{R})$:

$$BV(\mathbb{R}) = \{\text{Reachable targets in time } T\} \cup \{\text{Unreachable targets in time } T\}$$

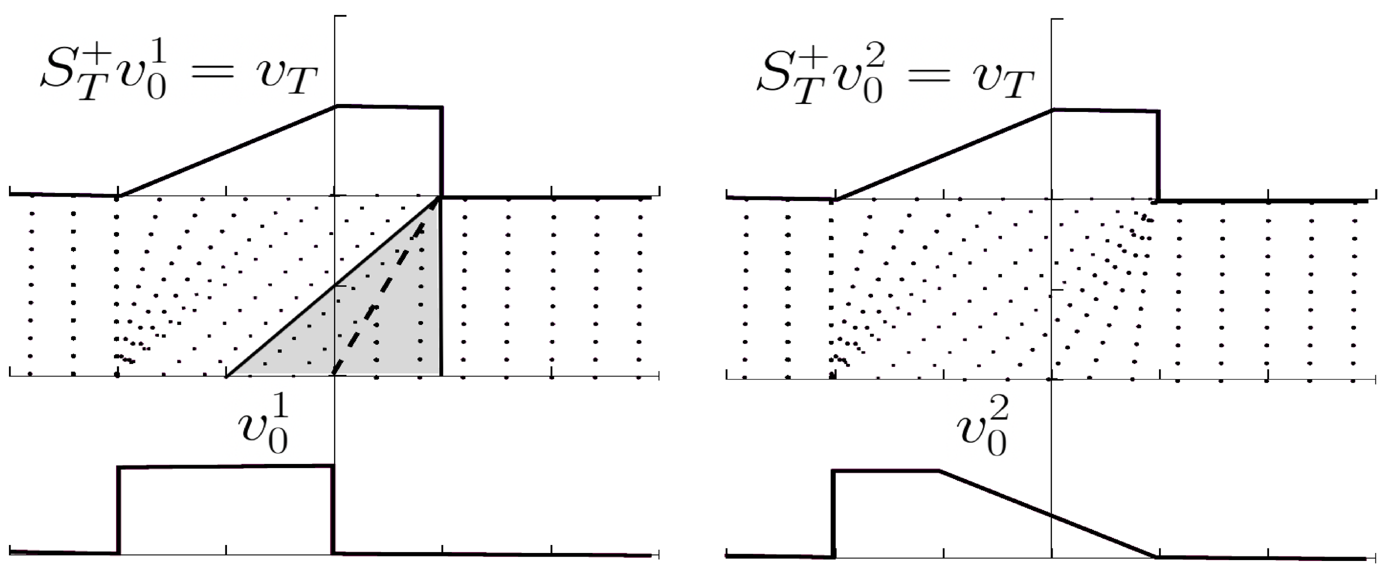
One-sided-Lipschitz condition: A function $v_T \in BV(\mathbb{R})$ is reachable if and only if

$$v_T(x) - v_T(y) \leq \frac{x-y}{T} \qquad \forall x,y \in \mathbb{R}. \qquad \text{(OSLC)}$$

It is also known as the Oleinik condition.

Lack of backward uniqueness

Backward uniqueness is lost due to the formation of shocks.
(the characteristics cross each other)



The entropy solutions associated to the initial conditions v_0^1 and v_0^2 coincide at time T, and both are indistinguishable thereafter.

The optimal inverse design

Let $T>0$ and $v_T \in BV(\mathbb{R})$. We consider the optimal control problem

$$\min_{v_0 \in BV(\mathbb{R})} \int_{\mathbb{R}} (S_T^+ v_0(x) - v_T(x))^2 dx \qquad \text{(Opt-Inv)}$$

where $S_T^+ v_0$ is the entropy solution to (SCL) at time T, with initial condition v_0.

Theorem [2, 3]: The unique solution to (Opt-Inv) is given by $v_0^* = S_T^- v_T$ where $S_T^- v_T$ is the **backward entropy solution** to (SCL) with terminal condition v_T.

Equivalence between (SCL) and (HJ)

If $N=1$, we have that $v \in BV((0,T)\times\mathbb{R})$, with compact support, is an entropy solution to (SCL) if and only if

$$u(t,x) = \int_{-\infty}^{x} v(t,y)dy$$

is a viscosity solution to (HJ).

Hamilton-Jacobi equations

We consider the Hamilton-Jacobi equation

$$\begin{cases} \partial_t u + \frac{|\nabla_x u|^2}{2} = 0 & \text{in } (0,T)\times\mathbb{R}^N \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases} \qquad \text{(HJ)}$$

For any $u_0 \in \text{Lip}(\mathbb{R}^N)$ there exists a unique **viscosity solution** $u \in \text{Lip}((0,T)\times\mathbb{R}^N)$.

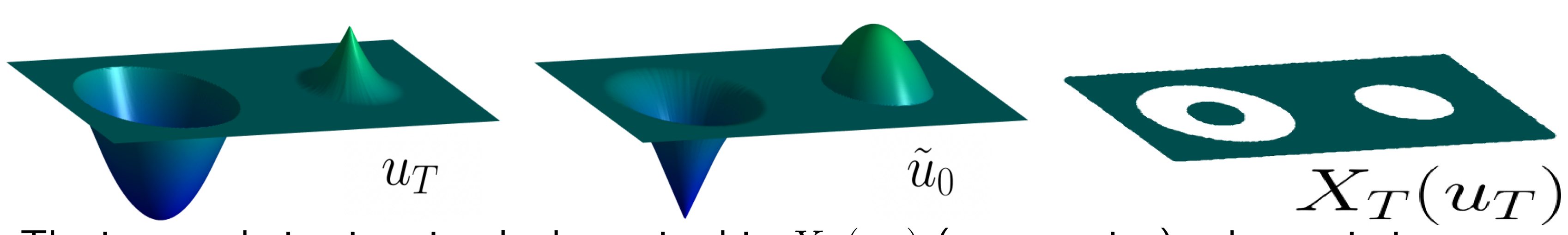
Initial data reconstruction

Theorem [1]: Let $T>0$ and $u_T \in \text{Lip}(\mathbb{R}^N)$ be a reachable target. We set the function $\tilde{u}_0(x) := S_T^- u_T(x)$, i.e. the **backward viscosity solution** to (HJ) with terminal consition u_T.

The initial condition $u_0 \in \text{Lip}(\mathbb{R}^N)$ satisfies $S_T^+ u_0 = u_T$ if and only if

- $u_0(x) \geq \tilde{u}_0(x), \quad \forall x \in \mathbb{R}^N$, and
- $u_0(x) = \tilde{u}_0(x), \quad \forall x \in X_T(u_T)$, where $X_T(u_T)$ is the set given by

$$X_T(u_T) := \{z - T\,\nabla u_T(z)\,;\ \forall z \in \mathbb{R}^N \text{ s.t. } u_T \text{ is diff. at } z\}.$$



The inverse design is uniquely determined in $X_T(u_T)$ (green region), whereas in its complementary we can only deduce a lower bound (backward uniqueness is lost).

Projection on the reachable set

If the given target u_T is unreachable, we can project it in the set of reachable targets by solving (HJ) backward and then forward in time, i.e.

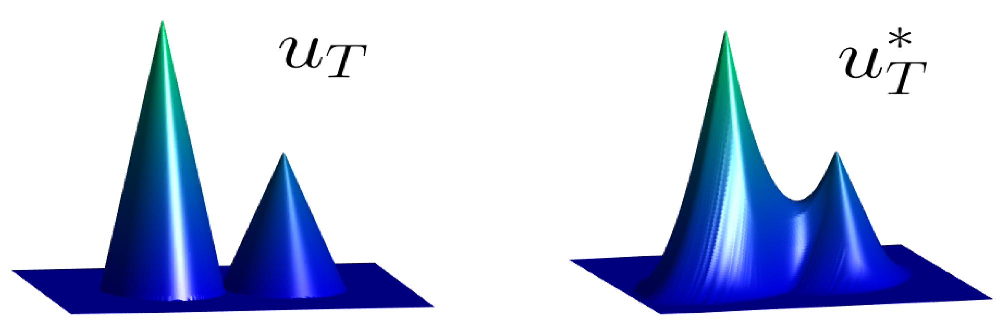
$$u_T^* = S_T^+(S_T^- u_T),$$

where S_T^+ and S_T^- are the forward and backward viscosity semigroups associated to (HJ).

Theorem [1]: Let $T>0$ and $u_T \in \text{Lip}(\mathbb{R}^N)$. The function $u_T^* = S_T^+(S_T^- u_T)$ is the unique viscosity solution to the elliptic obstacle problem

$$\min\left\{\varphi - u_T, -\lambda_N\left[D^2\varphi - \frac{1}{T}\right]\right\} = 0 \quad \text{in} \quad \mathbb{R}^N,$$

where $D^2\varphi$ is the Hessian matrix of φ, and for a symmetric matrix A, the notation $\lambda_N[A]$ stands for the greatest eigenvalue of A.



Semiconcave envelope: The function u_T^* is the smallest reachable target bounded from below by u_T.

Perspectives and open problems

Inverse design for Scalar Conservation Laws

1. **Consider convex-concave fluxes f.** More realistic choice to describe, for instance, pedestrian flows.
2. **Systems of Conservation Laws** Euler equations or shallow water equations.

Inverse design for Hamilton-Jacobi equations.

1. **L^2-projection on the reachable set.** Different from backward-forward projection.
2. **Space-depending Hamiltonians.** Implement reconstruction technique to Hamiltonians $H(x,\nabla u)$ depending on the space variable.

Selected publications

[1] C. Esteve and E. Zuazua.
The inverse problem for Hamilton–Jacobi equations and semiconcave envelopes.
SIAM Journal on Mathematical Analysis, 52(6):5627–5657, 2020.

[2] T. Liard and E. Zuazua.
Analysis and numerics solvability of backward-forward conservation laws.
Hal preprint, hal-02389808, 2020.

[3] T. Liard and E. Zuazua.
Initial data identification for the one-dimensional burgers equation.
Hal preprint, hal-03028457, 2020.

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