FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG
FACULTY OF SCIENCES

CHAIR IN APPLIED ANALYSIS FAU

NONLOCAL CONSERVATION LAWS (NLCL) – MODELING, SIMULATION AND OPTIMAL CONTROL

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Motivation and applications

- Main applications:
  - traffic flow: adjusting velocity of flow at a current location based on the foregoing traffic situation.
  - chemical engineering: growing of particles depending on the surface area or concentration of synthesized products.
- Temporal development of the state in the given spatial variables (also in the broad sense) depending on the information on the whole considered domain.
- Further applications: crowd dynamics, pedestrian flow, sedimentation processes, opinion formation, supply chains, etc.

Mathematical models: NLCL on $\mathbb{R}^n$...

For $(t, \boldsymbol{x}) \in (0, T) \times \mathbb{R}^n$, $n \in \mathbb{N}_{\geq 1}$ and $T \in \mathbb{R}_{>0}$

$$q_t(t,\boldsymbol{x}) + \operatorname{div}_{\boldsymbol{x}}\Big(\boldsymbol{\lambda}\big[W[q,\gamma,\mathcal{A}]\big](t,\boldsymbol{x})q(t,\boldsymbol{x})\Big) = h(t,\boldsymbol{x}) + g(t,\boldsymbol{x})q(t,\boldsymbol{x})$$

$$q(0,\boldsymbol{x}) = q_0(\boldsymbol{x})$$

$$W[q,\gamma,\mathcal{A}](t,\boldsymbol{x}) := \iint_{\mathcal{A}(t)} \gamma(t,\boldsymbol{x},\boldsymbol{y})q(t,\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$$

$$\boldsymbol{\lambda}[w](t,\boldsymbol{x}) := \boldsymbol{\lambda}(w(t,\boldsymbol{x}),t,\boldsymbol{x}).$$

Description of terms

- q: traffic density, particle size distribution (PSD), etc.
- q_0: initial datum.
- $\boldsymbol{\lambda}$: velocity function, growth function (in particle synthesis).
- W: nonlocal impact (look ahead behavior, mass of particles, etc.).
- γ: nonlocal weight.
- g: damping term (rate of vehicles on highway exits, outflow rate of particles, etc.).
- h: further source/sink term (vehicles on highway access, inflow of new nuclei, etc.).
- $\mathcal{A}$: nonlocal integration area. For $K \in \mathbb{R}_{>0}$ be $\mathcal{A} \in C([0,T];\mathscr{M}^n_K)$ with

$$\mathscr{M}^n_K := \{\mathcal{B} \in \mathscr{M}^n : \mathcal{H}^{n-1}(\partial\mathcal{B}) \leq K\}$$

  and $\mathscr{M}^n$ the set of all Lebesgue measurable subsets in $\mathbb{R}^n$ which are bounded or whose complement is bounded.

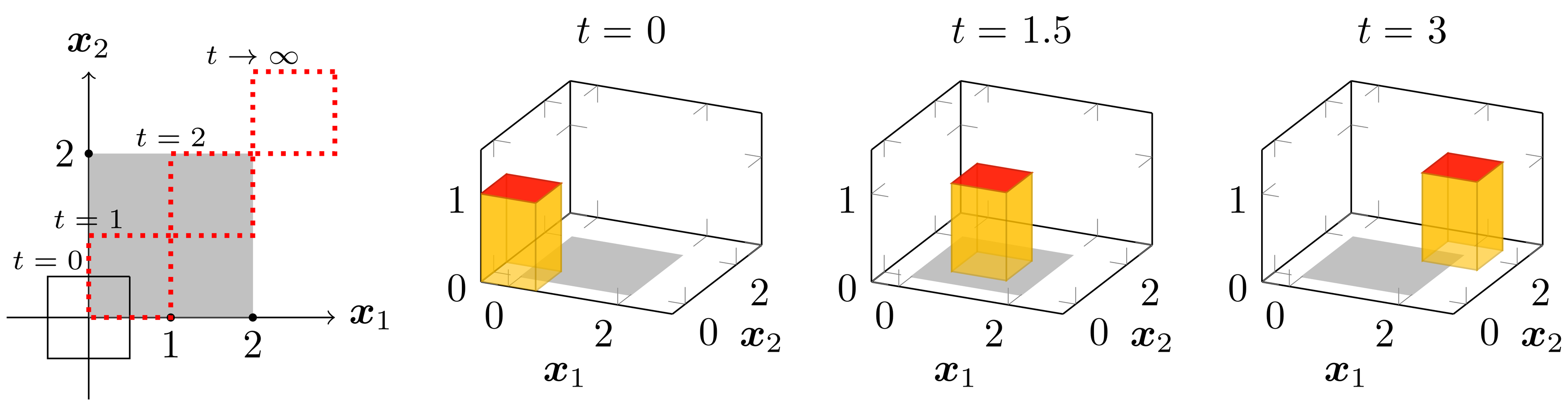

Figure 1: Graphical illustration of the nonlocal impact. Here, the speed of a generic cuboid moving in diagonal direction linearly depends on the partial volume of the cuboid located over the gray area. The motion can be described by an appropriate NLCL. **Left:** Development from the top. **Remaining:** Development from the side (cf. [3]).

...and on bounded domains in $\mathbb{R}$

For $(t, x) \in (0, T) \times (0, 1)$, and $T \in \mathbb{R}_{>0}$

$$q_t(t,x) = -\partial_x\Big(\lambda\big[W[q,v,\gamma_\eta]\big](t,x)q(t,x)\Big)$$

$$q(0,x) = q_0(x)$$

$$\lambda\big[W[q,v,\gamma_\eta]\big](t,0)q(t,0) = \lambda\big[W[q,v,\gamma_\eta]\big](t,0)u(t)$$

$$W[q,v,\gamma_\eta](t,x) := \int_x^{x+\eta} \gamma_\eta(t,x,y)\left(\begin{cases} q(t,y) & y \in (0,1) \\ v(t,y) & \text{else}\end{cases}\right)\mathrm{d}y$$

Further explanations and comments

- η: nonlocal reach (view horizon).
- u: boundary datum (inflow of new vehicles at beginning of road).
- Integration area in nonlocal term depending on x.
- Nonnegative velocity function λ. Thus, upper boundary condition more general than assuming $q(t, 0) = u(t)$ and no boundary condition in $x = 1$ to be prescribed. However, v determining the velocity of the outflow represents a kind of "boundary condition".

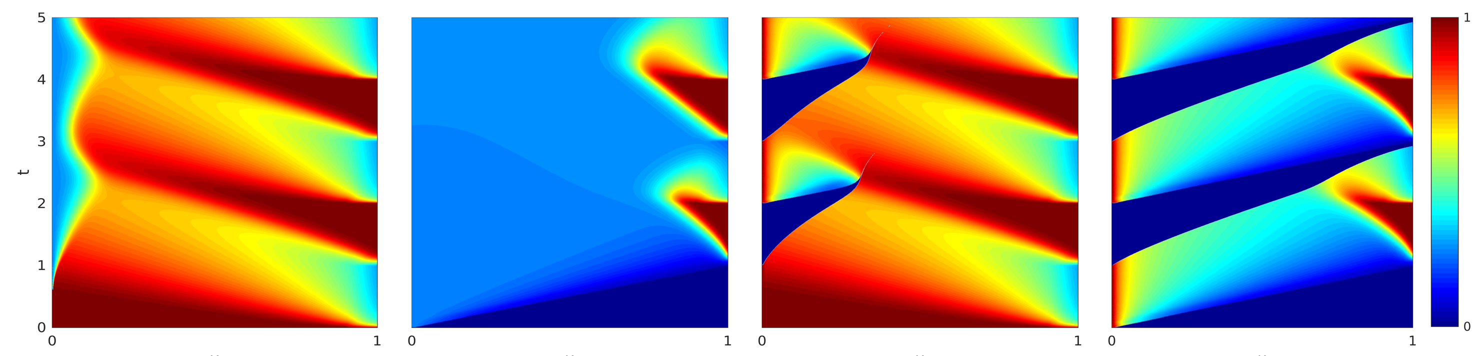

Figure 2: Simulation of different traffic scenarios with Greenshields' flux function $\lambda[w] \equiv 1 - w$, v representing traffic lights on the right end, green only for $t \in (1,2) \cup (3,4)$, and a given γ_η. **From left to right**: $(q_0, u) = (1, \frac{1}{4})$, $(q_0, u) = (0, \frac{1}{4})$, $(q_0, u) = (1, 1 - v)$ and $(q_0, u) = (0, 1 - v)$ (cf. [4]).

Analytical properties

- No fully local behavior anymore, i.e., solution has to be known on the nonlocal integration area to process in time.
- Still finite propagation of mass but infinite information flow.
- None of the usual existence results (Kružkov, etc.) applicable.
- Existence of weak solutions provable by a fixed-point argument in the nonlocal term.
- No Entropy condition required for uniqueness.
- Additional conditions on data can guarantee existence of solution on any finite time horizon (maximum principle, etc.).
- No smoothing of solutions (as with Kružkov, i.e., in contrast to the regularizing effect to solutions of local conservation laws for positive times), thus preservation of regularity.
- IBVP: very good approximation of the Lighthill–Whitham–Richards traffic flow model, significantly more reasonable due to the look ahead behavior.

Optimal control in chemical engineering

- Optimization of cost functionals depending on the solution of NLCL.
- Controls as time-dependent process conditions: temperature profiles, inflow/outflow rates,...
- Cost functionals: minimizing the difference between the resulting particle size distribution (PSD) and a target PSD, minimizing the relative standard deviation, maximizing the yield,...
- Model- and gradient-based optimization possible due to a differentiable numerical scheme.

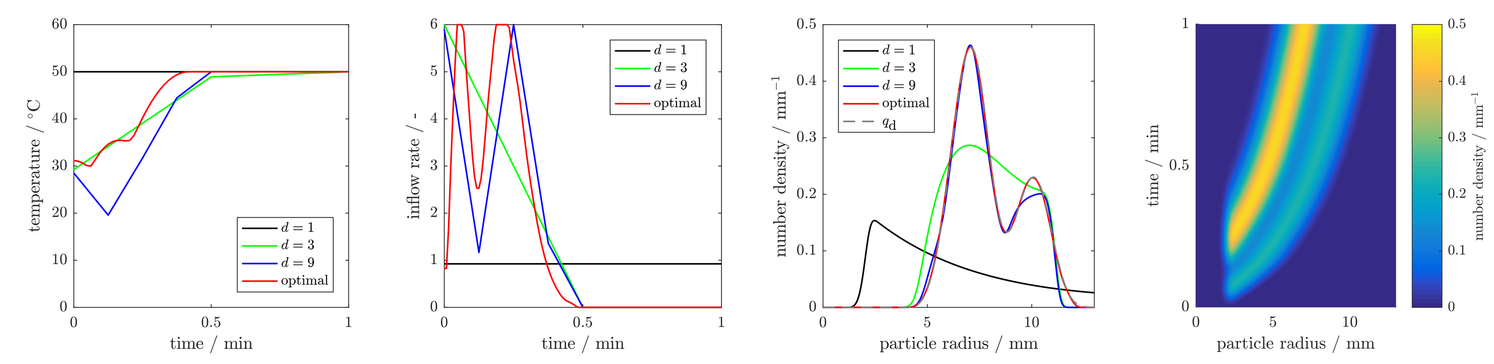

Figure 3: Final time L^2-tracking of fluidized bed spray granulation process with monomodal initial datum and bimodal target by adjusting process temperature and inflow rate. **From left to right**: optimization of controls for different degrees of freedom d, final PSD and top view of temporal development of the solution (cf. [6]).

Ongoing and further research

- Convergence of solutions of NLCL to local conservation laws (see the poster *The singular limit of nonlocal conservation laws to local conservation laws* by Coclite et al. and [1]).
- NLCL incorporating time delays for encoding data, e.g., of routing apps.
- NLCL with feedback terms for synthesis processes with residence time distribution reactors.
- Nonlocal (non mass preserving) transport equations.

Selected publications

[1] Coclite, G. M., Coron, J. M., De Nitti, N., Keimer, A., & Pflug, L. (2020). A general result on the approximation of local conservation laws by nonlocal conservation laws: The singular limit problem for exponential kernels. ArXiv preprint: 2012.13203.

[2] Keimer, A., & Pflug, L. (2017). Existence, uniqueness and regularity results on nonlocal balance laws. Journal of Differential Equations, 263(7), 4023-4069.

[3] Keimer, A., Pflug, L., & Spinola, M. (2018). Existence, uniqueness and regularity of multi-dimensional nonlocal balance laws with damping. Journal of Mathematical Analysis and Applications, 466(1), 18-55.

[4] Keimer, A., Pflug, L., & Spinola, M. (2018). Nonlocal scalar conservation laws on bounded domains and applications in traffic flow. SIAM Journal on Mathematical Analysis, 50(6), 6271-6306.

[5] Spinola, M., Keimer, A., Segets, D., Leugering, G., & Pflug, L. (2020). Model-based optimization of ripening processes with feedback modules. Chemical Engineering & Technology, 43(5), 896-903.

[6] Spinola, M., Keimer, A., Segets, D., Pflug, L., & Leugering, G. (2020). Modeling, Simulation and Optimization of Process Chains. In Dynamic Flowsheet Simulation of Solids Processes (pp. 549-578). Springer, Cham.

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