FRIEDRICH-ALEXANDER
UNIVERSITÄT
ERLANGEN-NÜRNBERG

FACULTY OF SCIENCES

CHAIR IN
APPLIED ANALYSIS
FAU

Sharp Estimates in Defective Evolution Equations: From ODEs to Kinetic Equations with Uncertainties

Tobias Wöhrer Applied Analysis - AvH Professorship, FAU Erlangen-Nürnberg

Anton Arnold Analysis and Scientific Computing, TU Vienna

Shi Jin Institute Of Natural Sciences, Shanghai Jiao Tong University

Introduction

Presented is a Lyapunov functional method for linear defective ODEs. Possible applications are kinetic equations with an added uncertain parameter in the equation coefficients. Analyzing their linear sensitivity leads to defective ODE systems on a modal level. By constructing Lyapunov functionals as problem-adapted norms, one obtains sharp long time behavior of order $(1+t^M)e^{-\mu t}$, where $M \in \mathbb{N}_0$ is the defect and $\mu > 0$ is the spectral gap of the system.

Defective ODEs

$$\dot{x}(t) = -\mathbf{C}x(t), \qquad t \geq 0, \qquad x(0) = x_0 \in \mathbb{C}^d, \qquad \text{(ODE)}$$

with $\mathbf{C} \in \mathbb{C}^{d\times d}$ *positive stable*, i.e. $\mu := \min\{\operatorname{Re}\lambda \mid \lambda \in \sigma(\mathbf{C})\} > 0$.

Definition: An eigenvalue λ of $\mathbf{C}$ is *defective* if its algebraic multiplicity is strictly greater than its geometric multiplicity. In other words, λ corresponds to is at least one non-trivial Jordan block corresponding to $\mathbf{C}$.

Example 1 (Jordan transform estimate):

$$\mathbf{C} = \begin{pmatrix} 1 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} = \mathbf{V}\mathbf{J}\mathbf{V}^{-1}, \qquad \mathbf{J} = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad e^{-\mathbf{J}t} = e^{-\frac{t}{2}} \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}.$$

$$\|x(t)\|_2^2 = \|\mathbf{V}e^{-\mathbf{J}t}\mathbf{V}^{-1}x_0\|_2^2 \leq \|\mathbf{V}\|_2^2 \|\mathbf{V}^{-1}\|_2^2 \, 2(1+t^2)e^{-t}\|x_0\|_2^2, \quad t \geq 0.$$

Lyapunov functionals for defective ODEs

Lemma 1:

Case 1: If (at least) one $\lambda \in \{\lambda \in \sigma(\mathbf{C}) \mid \operatorname{Re}(\lambda) = \mu\}$ is defective, there exists a time-dependent Hermitian positive-definite matrix $\mathbf{P}(t) \in \mathbb{C}^{d\times d}$ such that a solution to (ODE) satisfies

$$x(t)^H \mathbf{P}(t) x(t) := \|x(t)\|^2_{\mathbf{P}(t)} \leq e^{-2\mu t}\|x_0\|^2_{\mathbf{P}(0)}. \qquad (1)$$

Case 2: If all $\lambda \in \{\lambda \in \sigma(\mathbf{C}) \mid \operatorname{Re}(\lambda) = \mu\}$ are non-defective, $\mathbf{P}(t) \equiv \mathbf{P}(0)$ in (1).

Remarks:

- $\mathbf{P}(t)$ can be constructed explicitly from the Jordan transformation matrices.
- Adjustable weights in $\mathbf{P}(t)$ allow for robustness in *non-defective limits* of equation parameters.

Theorem 1: Let $\mathbf{C} \in \mathbb{C}^{n\times n}$ be positive stable with maximum defect associated to $\mu > 0$ of order $M \in \mathbb{N}_0$. Then there exists an explicitly computable constant $c_{\mathbf{P}(0)} > 0$, depending on $\mathbf{P}(0)$, such that

$$\|x(t)\|_2^2 \leq c_{\mathbf{P}(0)}(1+t^{2M})e^{-2\mu t}\|x_0\|_2^2.$$

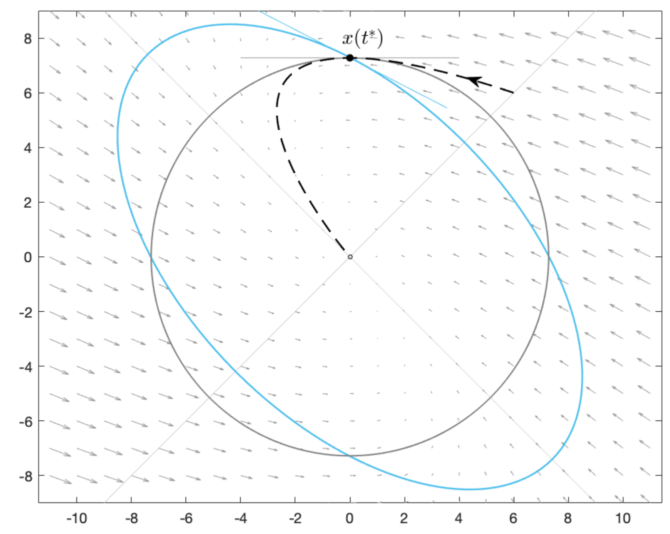
Literature

[1] Arnold, A., Jin, S., Wöhrer, T.: *Sharp Decay Estimates in Local Sensitivity Analysis for Evolution Equations with Uncertainties: from ODEs to Linear Kinetic Equations*, J. Differential Equations, vol. 268 (3), 1156–1204 (2020).

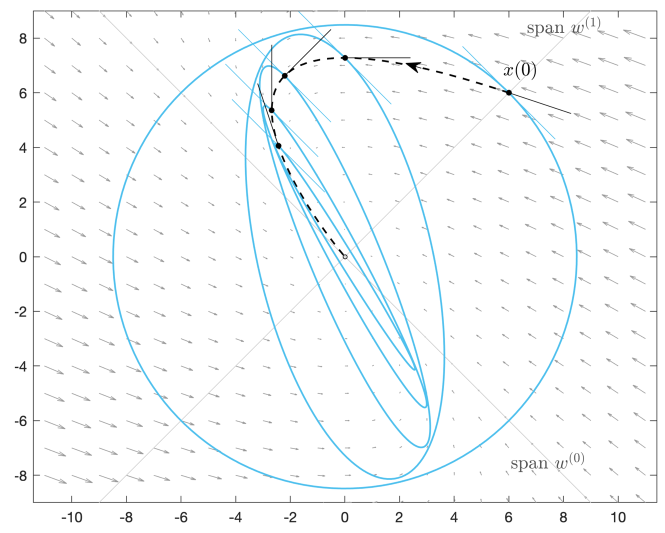
[2] Arnold, A. and Erb, J.: *Sharp Entropy Decay for Hypocoercive and Non-Symmetric Fokker–Planck Equations With Linear Drift*, arXiv:1409.5425 (2014).

Geometry of Lyapunov functionals

The dashed lines depict solution trajectories of (ODE). On the left side the system matrix $\mathbf{C}$ is non-defective (Case 2 of Lemma 1) and on the right side $\mathbf{C}$ is defective (Case 1 of Lemma 1).



Case 2: At the marked point $x(t^*)$, the solution is tangential to the Euclidean level curve. This implies non-strict decay in the Euclidean norm. The ellipse represents a level curve of the $\mathbf{P}$-norm. It modifies the geometry such that the solution is never tangential to the level curves of $|\cdot|_{\mathbf{P}}$.



Case 1: For $t = 0, 1, \ldots, 4$ the level curves of $\{x \in \mathbb{R}^2 \mid |x|^2_{P(t)} = e^{-t}|x(0)|_2^2\}$ are plotted. They intersect with the solution trajectories exactly at the marked points $x(0), x(1), \ldots, x(4)$, which corresponds to the statement of Lemma 1. The tangents of the level curves of $|\cdot|_{P(t)}$ at $x(t)$ are all parallel to each other. The intersection angle in the $P(t)$-norm is time-independent.

Sensitivity analysis for linear kinetic equations

$$\partial_t f(x,z,t) = L(z) f(x,z,t)$$

- Parameter $z \in \mathbb{R}$ (here non-stochastic) represents:
  - modelling uncertainty,
  - measurement errors etc.
- Linear evolution operator $L(z)$ that allows L^2 decomposition and has coefficients depending smoothly on z.

First order sensitivity equations (SE)

$$\partial_t \begin{pmatrix} f \\ \partial_z f \end{pmatrix} = \begin{pmatrix} L(z) & 0 \\ \partial_z L(z) & L(z) \end{pmatrix} \begin{pmatrix} f \\ \partial_z f \end{pmatrix} \qquad \text{(SE)}$$

↓ Fourier decomposition in x, $\mathcal{F}[k], k \in \mathbb{Z}$

$\partial_t x(k,z,t) = -\mathbf{C}(k,z,t)x(k,z,t)$ with $\mathbf{C}$ defective

↓ Theorem 1 yields sharp rate with uniform constant c

$\|x(k,z,t)\|_2^2 \leq c(1+t^{2M(z,k)})e^{-2\mu(k,z)t}\|x_0(k,z,t)\|_2^2.)$

↓ $\mathcal{F}^{-1}[x]$

uniform-in-z decay of (SE) with order $(1+t^{2M_{\max}})e^{-2\mu_{\min}t}$

Example: Fokker–Planck equation with linear drift

$$L_1(z)f := \partial_x[\partial_x f + C(z)xf], \quad x \in \mathbb{R}, t \geq 0.$$

Theorem 2: Solutions $\Phi(x,z,t) := (f, \partial_z f)^T \in \mathbb{R}^2$ of the system of sensitivity equations (SE) for $L_1(z)$ with drift coefficient $C \in C^1(\mathbb{R})$, $\partial_z C \in L^\infty(\mathbb{R})$, $C(z) \geq \mu > 0$ converge uniformly in z to equilibrium:

$$\sup_{z\in\mathbb{R}} \|\Phi(z,t) - \Phi^\infty(z)\|^2_{L^2(f_\infty^{-1}dx)} \leq c(1+t^2)e^{-2\mu t} \sup_{z\in\mathbb{R}} \|\Phi(z,0) - \Phi^\infty(z)\|^2_{L^2(f_\infty^{-1}dx)}.$$

caa-avh.nat.fau.eu

(03/21)

Department of DATA SCIENCE

Unterstützt von / Supported by
Alexander von Humboldt Stiftung / Foundation