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ERLANGEN-NÜRNBERG

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Some remarks on the turnpike property

Martin Gugat `martin.gugat@fau.de`
Chair Dynamics, Control and Numerics
Friedrich-Alexander-Universität Erlangen-Nürnberg (FAU)
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OPTIMIZATION

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The Turnpike Phenomenon

Consider a **dynamic optimal control problem** with finite time horizon and objective function of integral type: $\min \int_0^T \|y\|^2 + \|u\|^2$ s.t. $y(0) = y_0$, $y' = Ay + Bu$

Wikipedia: The **New Jersey Turnpike** Creative-Commons-Lizenz

The Turnpike Phenomenon

Consider a **dynamic optimal control problem** with finite time horizon and objective function of integral type: $\min \int_0^T \|y\|^2 + \|u\|^2$ s.t. $y(0) = y_0$, $y' = Ay + Bu$

If all the time-derivatives are set to zero and initial conditions and terminal conditions are canceled, this yields a **static optimal control problem**.

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Turnpike results state **relations** between the ***static*** *optimal state/control* and the ***dynamic*** *optimal states/controls*.

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Turnpike results state **relations** between the ***static*** *optimal state/control* and the ***dynamic*** *optimal states/controls*.

Typically for large time intervals, close to its middle the ***dynamic*** *optimal states/controls* are **close** to the ***static*** *optimal state/control*.

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Turnpike results state **relations** between the ***static*** *optimal state/control* and the ***dynamic*** *optimal states/controls*.

Typically for large time intervals, close to its middle the ***dynamic*** *optimal states/controls* are **close** to the ***static*** *optimal state/control*.

In short: The influence of the *initial data* and *terminal data* becomes **small** around $\frac{T}{2}$!

Wikipedia: The **New Jersey Turnpike** Creative-Commons-Lizenz

The Turnpike Phenomenon: A historical perspective

Very early references:

JOHN VON NEUMANN (1937) *A Model of General Economic Equilibrium*

FRANK RAMSEY (1928) *A Mathematical Theory of Saving*.

Later

PAUL A. SAMUELSON (1976) *The periodic turnpike theorem*

And a quote from

LW MCKENZIE (1986) *Optimal econ. growth, turnpike thms and comparative dynamics*:

"There is a fastest route between any two points;

and if the origin and destination are *close together* and far from the turnpike, the best route may not touch the turnpike.

But if origin and destination are *far enough apart*, it will always pay to get on to the turnpike and cover distance at the best rate of travel, even if this means adding a little mileage at either end"

Inhalt

An example with L^2 objective

A 1d-example

Let $T > 0$, > 0 and $\neq 0$ be given. Consider the dynamic optimal control problem

$$\min \frac{1}{\ } \int_0^T y^2(\ ) + \ u^2(\ )\, d$$

subject to

$$y(0) = y_0;\ y'(t) = \ (y(t) + u(t))\ ;\ y(T) = y_0.$$

Without loss of generality we can put $T = 2$.

Then we can write $y(t)$ as a FOURIER series.

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With $y(t) = \frac{a_0}{2} + \sum_k a_k \cos(kt) + b_k \sin(kt)$,

$u(t) = \frac{v_0}{2} + \sum_k v_k \cos(kt) + w_k \sin(kt)$, we obtain

$$\sum_k \; k\, a_k \sin(kt) + k\, b_k \cos(kt) = \; \frac{a_0 + v_0}{2} + \sum_k \; (a_k + v_k) \cos(kt) + \; (b_k + w_k) \sin(kt).$$

This yields $v_0 = \; a_0$, $\;(a_k + v_k) = k\, b_k$ and $\;(b_k + w_k) = \; k\, a_k$.

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For the objective value we obtain

$$J = \frac{1+\;}{2} a_0^2 + \sum_k a_k^2 + b_k^2 + \;(\tfrac{k}{\;} b_k \; a_k)^2 + \;(\tfrac{k}{\;} a_k + b_k)^2 = \frac{1+\;}{2} a_0^2 + \sum_k (a_k^2 + b_k^2)(1 + \; + \;\tfrac{k^2}{\;^2}).$$

An example with L2 objective

A 1d-example: The transformed optimal control problem is

$$\min \frac{1 + }{2} a_0^2 + \sum_k (a_k^2 + b_k^2)(1 + \quad + \frac{k^2}{2})$$

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The necessary optimality conditions yield a LAGRANGE multiplier such that

$(1 + \quad) a_0 + \frac{1}{2} \quad = 0$ and for $k \in \{1, 2, 3, \dots\}$

$$2(1 + \quad + \frac{k^2}{2})\, a_k + \quad = 0;$$

$$2(1 + \quad + \frac{k^2}{2})\, b_k = 0.$$

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Thus $b_k = 0$, $\frac{a_0}{2} = \quad \frac{1}{2(1+ \quad)}\, 2$ and $a_k = \quad \frac{1}{1 + \quad + \quad \frac{k^2}{2}}\, 2$.

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This yields $y(t) = \quad \frac{}{2} \quad \frac{1}{2(1+ \quad)} + \sum_k \frac{1}{1+ \quad + \frac{}{2} k^2} \cos(k \, t) \quad .$

An example with L^2 objective (continued)

The FOURIER series of $\cosh(a(t-\pi))$ for $a \neq 0$ is

$$\cosh(a(t-\pi)) = \frac{\sinh(\pi a)}{\pi a} + \frac{2}{\pi}\sinh(\pi a)\sum_{k=1}^{\infty} \frac{a}{a^2+k^2}\cos(k\,t).$$

An example with L2 objective (continued)

The FOURIER series of $\cosh(a(t\ ))$ for $a \neq 0$ is

$$\cosh(a(t\ )) = \frac{\sinh(\ a)}{\ a} + \frac{2}{\ } \sinh(\ a) \sum_{k=1}^{\infty} \frac{a}{a^2 + k^2} \cos(k\,t):$$

Hence we have $$\frac{\ a}{2 \sinh(\ a)} \cosh(a(t\ )) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{a^2}{a^2 + k^2} \cos(k\,t):$$

An example with L2 objective (continued)

The FOURIER series of $\cosh(a(t\ ))$ for $a \neq 0$ is

$$\cosh(a(t\ )) = \frac{\sinh(\ a)}{a} + \frac{2}{\ }\sinh(\ a)\sum_{k=1}^{\infty}\frac{a}{a^2+k^2}\cos(k\,t).$$

Hence we have $\frac{a}{2\sinh(\ a)}\cosh(a(t\ )) = \frac{1}{2} + \sum_{k=1}^{\infty}\frac{a^2}{a^2+k^2}\cos(k\,t).$

For the optimal state this yields with $a = |\ |\sqrt{1+\frac{1}{\ }}$:

$$y(t) = \frac{\ }{2}\frac{1}{2(1+\ )} + \sum_{k}\frac{1}{1+\ +\frac{\ }{2}k^2}\cos(k\,t)$$

An example with L2 objective (continued)

The FOURIER series of $\cosh(a(t \quad))$ for $a \neq 0$ is

$$\cosh(a(t \quad)) = \frac{\sinh(\; a)}{\; a} + \frac{2}{\;}\sinh(\; a)\sum_{k=1}^{\infty}\frac{a}{a^2+k^2}\cos(k\,t):$$

Hence we have $$\frac{a}{2\,\sinh(\; a)}\cosh(a(t \quad)) = \frac{1}{2} + \sum_{k=1}^{\infty}\frac{a^2}{a^2+k^2}\cos(k\,t):$$

For the optimal state this yields with $a = |\;|\sqrt{1+\frac{1}{\;}}$:

$$y(t) = \text{"}\;\frac{\;}{2}\,\frac{1}{2(1+\;)} + \sum_k \frac{1}{1+\;+\frac{\;}{2}k^2}\cos(k\,t) = \frac{\;}{2}\,\frac{1}{2(1+\;)} + \frac{1}{(1+\frac{1}{\;})}\sum_k \frac{{}^2(1+\frac{1}{\;})}{{}^2\left(1+\frac{1}{\;}\right)+k^2}\cos(k\,t)$$

$$= \frac{\;}{2}\,\frac{}{2\,\sinh(\; \sqrt{1+\frac{1}{\;}})}\,\frac{}{\sqrt{\;+\;2}}\cosh\left(\sqrt{1+\frac{1}{\;}}\,(t \quad)\right)$$

An example with L2 objective (continued)

On a general time interval, for $t \in [0; T]$ we obtain an optimal state of the form

$$y(t) = \cosh\left(\sqrt{1 + \frac{1}{\ }}\; t \quad \frac{T}{2}\right) :$$

Let $y_0 = 1$, $T = 100$; $\frac{\cosh(t)}{\cosh(100)}$
Between 80 and 80, there is not much going on!

Inhalt

An example with H1 objective

A 1d-example

Let $T > 0$, $\;> 0$ and $\;> 0$ be given. Consider the dynamic optimal control problem

$$\min \frac{1}{\;} \int_0^T y^2(\;) + \;^2 |\partial_t y|^2(\;) + \; u^2(\;)\, d$$

subject to

$$y(0) = y_0;\; y'(t) = \;(y(t) + u(t));\; y(T) = y_0.$$

Again for $T = 2\;$ we write $y(t)$ as a FOURIER series.

An example with $H1$ objective

A 1d-example

Let $T > 0$, $\ > 0$ and $\ > 0$ be given. Consider the dynamic optimal control problem

$$\min \frac{1}{2}\int_0^T y^2(\ ) + {}^2|\partial_t y|^2(\ ) + \ u^2(\ )\, d$$

subject to

$$y(0) = y_0;\ y'(t) = \ (y(t) + u(t));\ y(T) = y_0.$$

Again for $T = 2$ we write $y(t)$ as a FOURIER series.

With $y(t) = \frac{a_0}{2} + \sum_k a_k \cos(kt) + b_k \sin(kt)$,
$u(t) = \frac{v_0}{2} + \sum_k v_k \cos(kt) + w_k \sin(kt)$, we obtain again

$v_0 = \ a_0$, $\ (a_k + v_k) = k\, b_k$ and $\ (b_k + w_k) = \ k\, a_k$.

An example with H1 objective

A 1d-example

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$v_0 = \ a_0$, $(a_k + v_k) = k\, b_k$ and $(b_k + w_k) = \ k\, a_k$.
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$J = \frac{1+}{2} a_0^2 + \sum_k (1 + {}^2k^2)(a_k^2 + b_k^2) + (\frac{k}{}b_k \ \ a_k)^2 + (\frac{k}{}a_k + b_k)^2 =$
$\frac{1+}{2} a_0^2 + \sum_k (a_k^2 + b_k^2)\ \ 1 + \ + \ {}^2 + \frac{}{2}\ k^2\ .$

An example with H1 objective

A 1d-example: The transformed optimal control problem is

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The necessary optimality conditions yield a LAGRANGE multiplier such that

$(1+\ \ )a_0 + \frac{1}{2}\ \ = 0$ and for $k \in \{1; 2; 3; \dots\}$

$$2\ \ 1 + \ \ + \ \ {}^2 + \frac{\ }{2}\ \ k^2\ \ a_k + \ \ = 0;$$

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Thus $b_k = 0$, $\frac{a_0}{2} = \frac{1}{2(1+\ \ )}\ \frac{\ }{2}$ and $a_k = \frac{1}{1+\ \ +\left(\ {}^2 + \frac{\ }{2}\right)k^2}\ \frac{\ }{2}$.

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Thus $b_k = 0$, $\frac{a_0}{2} = \frac{1}{2(1+\ )} \frac{}{2}$ and $a_k = \frac{1}{1+\ +(\ {}^2+\frac{}{2})k^2} \frac{}{2}$.

This yields $y(t) = \frac{}{2} \quad \frac{1}{2(1+\ )} + \sum_k \frac{1}{1+\ +(\ {}^2+\frac{}{2})k^2} \cos(k\ t) \quad :$

An H1 example (continued)

With the FOURIER series (for $a \neq 0$)

$$\frac{a}{2\,\sinh(\ a)}\cosh(a(t\qquad)) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{a^2}{a^2+k^2}\cos(k\,t)$$

An H1 example (continued)

With the FOURIER series (for $a \neq 0$)

$$\frac{a}{2\ \sinh(\ a)} \cosh(a(t \qquad)) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{a^2}{a^2 + k^2} \cos(k\ t)$$

for the optimal state we obtain with $a = |\ | \sqrt{\frac{(1+\ )}{\ ^2\ ^2 +}}$:

$$y(t) = \Big[\ \frac{}{2}\ \frac{1}{2(1+\ )} + \sum_k \frac{1}{1+\ +\frac{\ ^2\ ^2+}{\ ^2}k^2} \cos(k\ t)\Big] = \frac{}{2}\ \frac{1}{2(1+\ )} + \frac{1}{1+} \sum_k \frac{\frac{\ ^2(1+\ )}{\ ^2\ ^2+}}{\frac{\ ^2(1+\ )}{\ ^2\ ^2+} + k^2} \cos(k\ t)$$

$$= \frac{}{2}\ \frac{1}{(1+\ )} \frac{a}{2\ \sinh(\ a)} \cosh(a\,(t \qquad)))$$

An H1 example (continued)

On a general time interval, for $t\ 2\ [0; T]$ we obtain an optimal state of the form

$$y(t) = \quad \cosh \quad \sqrt{\frac{(1 + \quad)}{+ \quad 2 \quad 2}} \quad t \quad \frac{T}{2} \quad ! \quad :$$

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On a general time interval, for t 2 [0; T] we obtain an optimal state of the form

$$y(t) = \cosh\left(\sqrt{\frac{(1+\ )}{+\ ^2\ ^2}}\ \ t\ \ \frac{T}{2}\right):$$

$= 1 =\ ,\ = \frac{1}{\sqrt{2}};\ \frac{\cosh(t)}{\cosh(100)}$ and $\frac{\cosh(\sqrt{2}/\sqrt{3}\ t)}{\cosh(100\ \sqrt{2}=\sqrt{3})}$

For the L^2-case, we have $a_{L^2} = |\ |\sqrt{1+\frac{1}{\ }} = |\ |\sqrt{\frac{(1+\ )}{\ }} > |\ |\sqrt{\frac{(1+\ )}{+\ ^2\ ^2}} = a_{H^1}$.

An example with H2 objective

H^2 objective

Let $T > 0$, $\ > 0$ and $\ > 0$ be given. Consider the dynamic optimal control problem

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$J = \frac{1+}{2} a_0^2 + \sum_k (1 + {}^2 k^4)(a_k^2 + b_k^2) + (\frac{k}{} b_k \ a_k)^2 + (\frac{k}{} a_k + b_k)^2 =$
$\frac{1+}{2} a_0^2 + \sum_k (a_k^2 + b_k^2)\ 1 + + {}^2 k^4 + \frac{}{2} k^2$.

An example with H2 objective

A 1d-example: The transformed optimal control problem is

$$\min \frac{1 +}{2} a_0^2 + \sum_k (a_k^2 + b_k^2) \quad {}^2 k^4 + \frac{}{2} k^2 + 1 +$$

subject to $\frac{a_0}{2} + \sum_k a_k = y_0$.

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$$2 \quad {}^2 k^4 + \frac{}{2} k^2 + 1 + \qquad a_k \ + \ = \ 0;$$

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Thus $b_k = 0$, $\frac{a_0}{2} = \ \frac{1}{2(1+\ )} \ \frac{}{2}$ and $a_k = \ \frac{1}{{}^2 k^4 + \frac{}{2} k^2 + 1 +} \ \frac{}{2}$.

An example with H2 objective

A 1d-example: The transformed optimal control problem is

$$\min \frac{1 +}{2} a_0^2 + \sum_k (a_k^2 + b_k^2) \quad {}^2 k^4 + \frac{}{2} k^2 + 1 +$$

subject to $\frac{a_0}{2} + \sum_k a_k = y_0$.

The necessary optimality conditions yield a LAGRANGE multiplier such that

$(1 + \ )a_0 + \frac{1}{2} \ = 0$ and for $k \ 2 \ f \ 1; 2; 3; :::g$

$$2 \quad {}^2 k^4 + \frac{}{2} k^2 + 1 + \quad a_k \ + \ = \ 0;$$

$$2 \quad {}^2 k^4 + \frac{}{2} k^2 + 1 + \quad b_k \ = \ 0:$$

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This yields $y(t) = \ \frac{}{2} \ \frac{1}{2(1+\ )} + \sum_k \frac{1}{{}^2 k^4 + \frac{}{2} k^2 + 1 +} \cos(k\ t)$: Thus y is twice continuously differentiable and $y^0(0) = 0$. The |curvature| decays at 0!

An H2 example (continued)

What does the trajectory look like?

$$y(t) = \left[\frac{1}{2(1 + \ )} + \sum_{k} \frac{1}{\ ^{2} k^{4} + \frac{\ }{2} k^{2} + 1 + \ } \cos(\ k\, t) \right] :$$

If $\ = \frac{\ }{2\sqrt{1+\ ^{2}}}$ for $a = \frac{(1+\ )^{1/4}}{\sqrt{\ }}$ this implies

$$y''(t) + \sqrt{1 + \ }\, y(t) = \frac{1}{\sqrt{1+\ }} \frac{a}{2 \sinh(\ a)} \cosh(a(t\ \ ))$$ [Hidden turnpike +trigonomet.?].

An H2 example (continued)

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$$y''(t) + \sqrt{1+\ }\, y(t) = \frac{1}{\sqrt{1+\ }} \frac{a}{2 \sinh(\ a)} \cosh(a(t\ \ ))$$ [Hidden turnpike +trigonomet.?].

$$\frac{1}{4} + \frac{1}{1+(1+1^2)^2} \cos(x) + \frac{1}{1+(1+2^2)^2} \cos(2x) + \quad + \frac{1}{1+(1+16^2)^2} \cos(16x)$$

Inhalt

An example with L1-tracking term

For $T \geq 1$ and $\gamma > 0$ we consider the problem

$$(OC)_T \begin{cases} \min_{u \in L^2(0,T)} \int_0^T \frac{1}{2}|u(t)|^2 + \gamma |y(t)| \, dt \quad \text{subject to} \\ y(0) = -1; \\ y'(t) = y(t) + u(t). \end{cases}$$

An example with L1-tracking term

For T 1 and > 0 we consider the problem

$$
(OC)_T \quad \begin{cases} \min\limits_{u \in L^2(0,T)} \int_0^T \frac{1}{2}|u(t)|^2 + |y(t)|\, dt \ \text{ subject to} \\ y(0) = 1; \\ y'(t) = y(t) + u(t). \end{cases}
$$

The problem without initial conditions is

$$
(S) \quad \begin{cases} \min\limits_{u \in L^2(0,T)} \int_0^T \frac{1}{2}|u(t)|^2 + |y(t)|\, dt \ \text{ subject to} \\ y'(t) = y(t) + u(t). \end{cases}
$$

An example with L1-tracking term

For T 1 and $>$ 0 we consider the problem

$$(OC)_T \quad \begin{cases} \min_{u \in L^2(0,T)} \int_0 \frac{1}{2}|u(t)|^2 + \; |y(t)|\,dt \text{ subject to} \\ y(0) = \; 1; \\ y'(t) = y(t) + u(t). \end{cases}$$

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The solution of (S) is the turnpike ! Here the turnpike is zero, $y^{( )} = 0$ and $u^{( )} = 0$.

An example with L^1-tracking term

Solution of the problem without terminal constraint

Variation of constants yields $y(t) = e^{t}\left[1 + \int_0^t e^{s}\, u(s)\, ds\right].$

An example with L^1-tracking term

Solution of the problem without terminal constraint

Variation of constants yields $y(t) = e^t \left[1 + \int_0^t e^{-s} u(s)\, ds \right]$.

$$(OC)_T \quad \min_{u \in L^2(0,T)} \int_0^T \tfrac{1}{2}|u(t)|^2 + \left| e^t \left(1 - \int_0^t e^{-s} u(s)\, ds \right) \right| dt$$

An example with L^1-tracking term

Solution of the problem without terminal constraint

Variation of constants yields $y(t) = e^{t} \left[1 + \int_0^t e^{s}\, u(s)\, ds \right].$

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In the problem we have no terminal conditions!

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In the problem we have no terminal conditions!

Consider the time-horizon T as a parameter.

For $\ \in (0, T]$, consider the parametric optimal control problem $(OC)\ $.

As long as $y(\ )\ \ 0$, we can get rid of the absolute value and thus of the non-smoothness!

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So for suf ciently small $\ > 0$, we consider

$$(OC) \quad \min_{u \in L^2(0;\ )} \int_0^{\ } \tfrac{1}{2}u(t)^2 + \quad e^{t} \quad 1 \quad \int_0^t e^{\ s}\, u(s)\, ds \quad dt:$$

An example with L1-tracking term

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Integration by parts allows to transform the objective function:

$$(OC) \quad \min_{u \in L^2(0;\ )} \int \frac{1}{2}u(t)^2 + \ 1 \ e^{\ t} \ u(t)\, dt + \ (e \ \ 1)$$

An example with L1-tracking term

Solution of the problem without terminal constraint

Integration by parts allows to transform the objective function:

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Solution of the problem without terminal constraint

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Then the necessary optimality conditions imply

$$u(t) = \quad 1 \quad e^{\ t} \ :$$

An example with L1-tracking term

Solution of the problem without terminal constraint

Integration by parts allows to transform the objective function:

$$(OC)\quad \min_{u \in L^2(0;\ )} \int_0^{} \frac{1}{2}u(t)^2 + \quad 1 \quad e^{\ t} \quad u(t)\,dt + \quad (e \quad 1)$$

Then the necessary optimality conditions imply

$$u(t) = \quad 1 \quad e^{\ t} \ :$$

For such a control $\int_0^s e^{\ t}\,u(t)\,dt = 1$ holds if

$$= \frac{1}{\cosh(s) \quad 1}:$$

Hence for $\ = s$ we have $y(s) = 0$ and $u(s) = 0$. For $t > s$ we can continue with $u(t) = 0$ and obtain the optimal control for $(OC)_T$ for all $T \quad s$!

An example with L1-tracking term

Lemma (solution of $(OC)_T$, the problem without terminal conditions).

For > 0, de ne s > 0 as the value where $\cosh(s) = \frac{1}{} + 1$:
Assume that T s. De ne

$$\hat{u}(t) = (e^{s \quad t} \quad 1)_+ :$$

Then for the state $\hat{y}$ generated by $\hat{u}$ for t s we have $\hat{y}(t) = 0$.
Moreover, for all $t \; 2 \; (0; T)$ we have $\hat{y}(t) \quad 0$.
The control $\hat{u}$ is the unique solution of $(OC)_T$.

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Moreover, for all t 2 $(0; T)$ we have $\hat{y}(t)$ 0.

The control $\hat{u}$ is the unique solution of $(OC)_T$.

The optimal state is $y(t) = e^{t} \left[1 + \frac{e^{s}\ \ e^{s\ \ 2t}}{2} + e^{\ \ t}\ \ 1 \right] :$

The optimal state for $s = 1$

The optimal control for $s = 1$

An example with L1-tracking term

The optimal state and control for $s = 2$ and $= \frac{1}{\cosh(s)\ \ 1}$.
The weight for the non-smooth tracking term is made smaller than for $s = 1$.

An example with L1-tracking term

The exact turnpike phenomenon

For suf ciently large T, due to the L^1-norm of y that appears in the objective function, the solution has a nite–time turnpike structure :
The system is steered to zero in the nite stopping time s.

An example with L1-tracking term

The exact turnpike phenomenon

For suf ciently large T, due to the L^1-norm of y that appears in the objective function, the solution has a nite–time turnpike structure :
The system is steered to zero in the nite stopping time s.
This time s is independent of T and only depends on .

An example with L1-tracking term

The exact turnpike phenomenon

For suf ciently large T, due to the L^1-norm of y that appears in the objective function, the solution has a nite–time turnpike structure :

The system is steered to zero in the nite stopping time s.

This time s is independent of T and only depends on .

Both state and control remain at zero for all t 2 $(s; T)$.

An example with L1-tracking term

For T sufficiently large, we can add a terminal condition :

For T 1 and $>$ 0 we consider the problem

$$(FROMATOB)_T \begin{cases} \min_{u \in L^2(0,T)} \int_0^T \frac{1}{2}|u(t)|^2 + |y(t)| \, dt \text{ subject to} \\ y(0) = 1; \; y(T) = 1; \; y'(t) = y(t) + u(t). \end{cases}$$

An example with L1-tracking term

For T suf ciently large, we can add a terminal condition :

For T 1 and > 0 we consider the problem

$$(\text{FROMATOB})_T \left\{ \begin{array}{l} \min\limits_{u \in L^2(0;T)} \int_0 \frac{1}{2}|u(t)|^2 + \;\; |y(t)|\, dt \;\; \text{subject to} \\ y(0) = \;\; 1;\; y(T) = 1;\;\; y'(t) = y(t) + u(t): \end{array} \right.$$

The problem decouples into $(\text{OC})_s$ (the problem without terminal condition) on $[0; s]$ and on the remaining time interval

$[s; T]$ we consider the problem without initial condition and starting time s

$$(\text{END})_s \left\{ \begin{array}{l} \min\limits_{u \in L^2(s;T)} \int_s \frac{1}{2}|u(t)|^2 + \;\; |y(t)|\, dt \;\; \text{subject to} \\ y(T) = 1;\;\; y'(t) = y(t) + u(t);\; t \in [s; T]: \end{array} \right.$$

An example with L1-tracking term

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For the starting time s if $\cosh(T \ s) \ 1 = 1 =$, the optimal control of $(END)_s$ is $u_2(t) = (1 \ e^{s \ t})_+$.

An example with L1-tracking term

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For the starting time s if $\cosh(T \; s) \; 1 = 1 =$, the optimal control of $(END)_s$ is $u_2(t) = (1 \; e^{s \; t})_+$.
Then we have $y(s) = 0$ and $y(t) = e^{t \; T}[1 \; (\frac{e^{s \; T} \; e^{T+s \; 2t}}{2} + e^{T \; t} \; 1)]$.

An example with L1-tracking term

The solution of $(OC)_s$ reaches the turnpike after nite stopping time $s = s_{stop}$
The solution of $(END)_s$ leaves the turnpike after the nite starting time $s = s_{start}$.
Hence if s_{stop} s_{start}, the solutions can be glued together to solve $(FROMATOB)_T$

if T is suf ciently large, e.g. $\frac{1}{\ }$ 2 $(0; \cosh(T=2)$ $1)$
(then s_{stop} $T=2$ s_{start}).

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(then s_{stop} $T=2$ s_{start}).
This is the nite-time turnpike situation:
From a given initial state, the optimal state is driven to the turnpike in nite time.

An example with L1-tracking term

The solution of $(OC)_s$ reaches the turnpike after nite stopping time $s = s_{stop}$
The solution of $(END)_s$ leaves the turnpike after the nite starting time $s = s_{start}$.
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if T is suf ciently large, e.g. $\frac{1}{\ }$ 2 $(0; \cosh(T=2)$ $1)$
(then s_{stop} $T=2$ s_{start}).
This is the nite-time turnpike situation:
From a given initial state, the optimal state is driven to the turnpike in nite time.
Then it stays on the turnpike for a nite time interval.
Wir fahr'n fahr'n fahr'n auf der Autobahn
Wir fahr'n fahr'n fahr'n auf der Autobahn
Wir fahr'n fahr'n fahr'n auf der Autobahn

Text of the Song by Kraftwerk

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If the prescribed terminal state is different from the turnpike,
the state nally leaves the turnpike to reach the target state.

An example

Let $\quad = \frac{1}{\cosh(1) \quad 1}$. Then $s_{stop} = 1$ and $s_{start} = T \quad 1$:

An example

Let $\;= \frac{1}{\cosh(1)\quad 1}$. Then $s_{stop} = 1$ and $s_{start} = T \quad 1$:
Assume that T is suf ciently large that $\cosh(T=2) \quad 1 > 1=$, that is $T > 2$.

An example

Let $= \frac{1}{\cosh(1) \;\; 1}$. Then $s_{stop} = 1$ and $s_{start} = T \quad 1$:
Assume that T is suf ciently large that $\cosh(T=2) \quad 1 > 1=$, that is $T > 2$.
Then for the optimal control and the optimal state we have

$$u(t) = 0; \; y(t) = 0 \;\; \text{for all } t \; 2 \; [1; \; T \quad 1]:$$

This is the nite-time turnpike situation.
On $[0; T]$ the control u is continuous and the state is continuously differentiable.

The optimal control

Numerical examples for $\min \int \frac{1}{2}u^2 + |u| + |y|$ with the nonautomous system $y'(t) = y(t) + e^t u(t)$, $y(0) = 1$ by Michael Schuster

Numerical examples for a nonautomous system by Michael Schuster: $y'(t) = y(t) + e^t u(t)$

Numerical examples for a nonautomous system by Michael Schuster: $y'(t) = y(t) + e^{t}u(t)$

Paper on the Finite-Time Turnpike Phenomenon

The Finite-Time Turnpike Phenomenon for Optimal Control Problems: Stabilization by Non-Smooth Tracking Terms,

M. GUGAT, M. SCHUSTER, E. ZUAZUA, in
Stabilization of Distributed Parameter Systems: Design Methods and Applications, ALEXANDER ZUYEV, GRIGORY SKLYAR eds., vol. 2 of SEMA SIMAI Springer Series (2021) 17–41. arXiv:2006.07051

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Contains also results for in nite-dimensional linear systems with L^1 and L^2-norm tracking terms.

Obviously, the nite-time turnpike phenomenon can only occur for systems that are exactly controllable in the sense that the turnpike is reachable.

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Obviously, the nite-time turnpike phenomenon can only occur for systems that are exactly controllable in the sense that the turnpike is reachable.

However, this also happens for systems that are nodal pro le exactly controllable if the turnpike is prescribed through the nodal pro les in the objective function.

Inhalt

Transregio 154 Mathematical modelling, simulation and optimization using the example of gas networks

Project C03: Nodal control and the turnpike phenomenon (with RÜDIGER SCHULTZ)

Here the system dynamics on a single pipe is described by the isothermal Euler equations

$$\begin{cases} {}_t + q_x = 0 \\ q_t + \left(p + \frac{q^2}{\ } \right)_x = \frac{f_g}{2} \frac{q|q|}{\ } \end{cases}$$

or a similar (semilinear) model.

The gas pressure is increased at compressor stations.

See the results of DFG CRC 154:

Inhalt

Literature about the turnpike phenomenon

Non-smooth tracking terms: The **finite-time turnpike phenomenon** is possible.

Literature about the turnpike phenomenon

Non-smooth tracking terms: The **finite-time turnpike phenomenon** is possible.

Smooth tracking terms: The classical **turnpike phenomenon**

1. **Exponential turnpike result** (with C_0 and $\;> 0$ independent of T):

$$ku(\;)\quad u\,k^2 + ky(\;)\quad y\,k^2\quad C_0\,[\exp(\qquad) + \exp(\quad(T\qquad))]$$

For *pdes with distributed control* by PORRETTA and ZUAZUA (SICON 2013), and TRELAT, ZHANG, ZUAZUA (SICON 2018 in Hilbert space)
For *nonlinear ODEs* TRELAT, ZUAZUA in Journal of Differential Equations, 2015

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2. Another team: LARS GRÜNE and ANTON SCHIELA from Bayreuth,
   MANUEL SCHALLER and KARL WORTHMANN from Ilmenau and
   TIMM FAULWASSER from Dortmund.
   See for example *Abstract nonlinear sensitivity and turnpike analysis and an application to semilinear parabolic PDEs* ESAIM: COCV 27 (2021) 56.
   They often consider a characterization by *dissipativity inequalities* with a *storage function* and a *supply function*.

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3. ALEXANDER ZASLAVSKI (Technion). Many books!

Literature about the turnpike phenomenon

Weakest property: Measure turnpike property.

It holds if the ***measure*** *of the set where the distance between the optimal state and the turnpike is greater than a given bound* is uniformly bounded independently of the time horizon.

Literature about the turnpike phenomenon

Weakest property: Measure turnpike property.

It holds if the ***measure*** *of the set where the distance between the optimal state and the turnpike is greater than a given bound* is uniformly bounded independently of the time horizon.

Intermediate: Integral turnpike

see for example GUGAT, HANTE, (SICON 2019). Boundary control for linear 2 2 hyperbolic systems.
It holds when an **integral norm** of that measures the distance to the turnpike is uniformly bounded independently of the time horizon.

Literature about the turnpike phenomenon

Weakest property: **Measure turnpike property**.

It holds if the ***measure*** *of the set where the distance between the optimal state and the turnpike is greater than a given bound* is uniformly bounded independently of the time horizon.

Intermediate: **Integral turnpike**

see for example GUGAT, HANTE, (SICON 2019). Boundary control for linear 2×2 hyperbolic systems.
It holds when an **integral norm** of that measures the distance to the turnpike is uniformly bounded independently of the time horizon.

Turnpike Property with Interior Decay, GUGAT, (MCSS 2021)

The **turnpike property with interior decay** requires that **there exist** $C_1 > 0$ **and** $\sigma_1 \in (0, 1)$ such that for all $\eta \in (0, 1)$ and all T sufficiently large we have

$$\int_{\eta T - \sigma_1 \eta T}^{\eta T + \sigma_1 (1-\eta) T} \| \hat{y}_0(0, T, y_0, y_d)(s) - y^{(\sigma)} \|^2 \, ds \leq \frac{C_1}{\min\{\eta, (1-\eta)\}\, T}.$$

The subinterval of $[0, T]$ has the length $\sigma_1 T$.

Literature about the turnpike phenomenon

Strongest Property: **Exponential turnpike** as above and e.g. GUGAT, TRELAT, ZUAZUA, Optimal Neumann control for the 1D wave equation (2016).

Finite horizon $T \geq 2$, $\gamma > 0$. Define (P):

$$\begin{cases} \min\limits_{u \in L^2(0,T)} \int_0^T (y_x(t,0))^2 + \gamma\, u^2(t)\, dt \ \text{ subject to} \\ y(0,x) = y_0(x),\ y_t(0,x) = y_1(x),\ x \in (0,1) \\ y(t,0) = 0,\ \boxed{y_x(t,1) = u(t)},\ t \in (0,T) \\ y_{tt}(t,x) = y_{xx}(t,x),\ (t,x) \in (0,T) \times (0,1), \\ y(T,x) = 0,\ y_t(T,x) = 0,\ x \in (0,1). \end{cases}$$

Literature about the turnpike phenomenon

Strongest Property: **Exponential turnpike** as above and e.g. GUGAT, TRELAT, ZUAZUA, Optimal Neumann control for the 1D wave equation (2016).

Finite horizon T 2, > 0. Define (P):

$$\min_{u 2 L^2(0;T)} \int_0^T (y_x(t;0))^2 + u^2(t)\,dt \text{ subject to}$$

$$y(0;x) = y_0(x);\ y_t(0;x) = y_1(x);\ x\ 2\ (0;1)$$

$$y(t;0) = 0;\ \boxed{y_x(t;1) = u(t)};\ t\ 2\ (0;T)$$

$$y_{tt}(t;x) = y_{xx}(t;x);\ (t;x)\ 2\ (0;T)\quad (0;1);$$

$$y(T;x) = 0;\ y_t(T;x) = 0;\ x\ 2\ (0;1):$$

Solution of (P), Syst. & Control Lett. 2016 (with E. TRÉLAT, E. ZUAZUA)

The unique solution of (**P**) is the sum of 2 parts that grow/decay exponentially.

Literature about the turnpike phenomenon

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Solution of (P), Syst. & Control Lett. 2016 (with E. TRÉLAT, E. ZUAZUA)

The unique solution of (**P**) is the sum of 2 parts that grow/decay exponentially. Choose z_γ with

$$\boxed{z_\gamma^2 + \left(2 + \frac{4}{\gamma}\right) z_\gamma + 1 = 0.}$$

Literature about the turnpike phenomenon

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Finite horizon $T \geq 2$, $\gamma > 0$. Define (P):

$$\begin{cases} \min_{u \in L^2(0,T)} \int_0^T (y_x(t,0))^2 + \gamma\, u^2(t)\, dt \quad \text{subject to} \\ y(0,x) = y_0(x),\ y_t(0,x) = y_1(x),\ x \in (0,1) \\ y(t,0) = 0,\ \boxed{y_x(t,1) = u(t)},\ t \in (0,T) \\ y_{tt}(t,x) = y_{xx}(t,x),\ (t,x) \in (0,T) \times (0,1), \\ y(T,x) = 0,\ y_t(T,x) = 0,\ x \in (0,1). \end{cases}$$

Solution of (P), Syst. & Control Lett. 2016 (with E. TRÉLAT, E. ZUAZUA)

The unique solution of (**P**) is the sum of 2 parts that grow/decay exponentially. Choose z_γ with

$$\boxed{z_\gamma^2 + \left(2 + \frac{4}{\gamma}\right) z_\gamma + 1 = 0.}$$

For $t \in (0,2)$ let

$$H(t) = \begin{cases} \frac{y_0'(1-t) - y_1(1-t)}{2}, & t \in (0,1), \\ \frac{y_0'(t-1) + y_1(t-1)}{2}, & t \in [1,2). \end{cases}$$

Literature about the turnpike phenomenon

Strongest Property: **Exponential turnpike** as above and e.g. GUGAT, TRELAT, ZUAZUA, Optimal Neumann control for the 1D wave equation (2016).

Finite horizon $T \quad 2, \quad > 0$. Define (P):

$$\begin{cases} \min_{u \in L^2(0;T)} \int_0^T (y_x(t;0))^2 + \quad u^2(t)\,dt \text{ subject to} \\ y(0;x) = y_0(x);\ y_t(0;x) = y_1(x);\ x \in (0;1) \\ y(t;0) = 0;\ \boxed{y_x(t;1) = u(t)};\ t \in (0;T) \\ y_{tt}(t;x) = y_{xx}(t;x);\ (t;x) \in (0;T) \quad (0;1); \\ y(T;x) = 0;\ y_t(T;x) = 0;\ x \in (0;1): \end{cases}$$

Solution of (P), Syst. & Control Lett. 2016 (with E. TRÉLAT, E. ZUAZUA)

The unique solution of (**P**) is the sum of 2 parts that grow/decay exponentially. Choose z with

$$\boxed{z^2 + (2 + \tfrac{4}{\ })\, z \quad + 1 = 0.}$$

For $t \in (0;2)$ let

$$H(t) = \begin{cases} \frac{y_0'(1 \quad t) \quad y_1(1 \quad t)}{2}; & t \in (0;1); \\ \frac{y_0'(t \quad 1) + y_1(t \quad 1)}{2}; & t \in [1;2): \end{cases}$$

For $t \in (0;2)$, $k \in \mathbb{N}_0$, $t + 2k \quad T$:

$$\boxed{u(t + 2k) = \quad z^k \quad \frac{1}{z^k} \quad \frac{1+z}{1 \quad z} H(t):}$$

Literature about the turnpike phenomenon

Strongest Property: **Exponential turnpike** as above and e.g. GUGAT, TRELAT, ZUAZUA, Optimal Neumann control for the 1D wave equation (2016).

Finite horizon $T \geq 2$, $\gamma > 0$. Define (**P**):

$$\begin{cases} \min_{u \in L^2(0,T)} \int_0^T (y_x(t,0))^2 + \gamma\, u^2(t)\, dt \text{ subject to} \\ y(0,x) = y_0(x),\; y_t(0,x) = y_1(x),\; x \in (0,1) \\ y(t,0) = 0,\; \boxed{y_x(t,1) = u(t)},\; t \in (0,T) \\ y_{tt}(t,x) = y_{xx}(t,x),\; (t,x) \in (0,T) \times (0,1), \\ y(T,x) = 0,\; y_t(T,x) = 0,\; x \in (0,1). \end{cases}$$

This is an exponential turnpike structure!

Solution of (**P**), Syst. & Control Lett. 2016 (with E. TRÉLAT, E. ZUAZUA)

The unique solution of (**P**) is the sum of 2 parts that grow/decay exponentially. Choose z_γ with $|z_\gamma|<1$

$$\boxed{z^2 + \left(2 + \frac{4}{\gamma}\right) z + 1 = 0.}$$

For $t \in (0,2)$ let

$$H(t) = \begin{cases} \frac{y_0'(1-t) - y_1(1-t)}{2}, & t \in (0,1), \\ \frac{y_0'(t-1) + y_1(t-1)}{2}, & t \in [1,2). \end{cases}$$

For $t \in (0,2)$, $k \in \mathbb{N}_0$, $t + 2k \leq T$:

$$\boxed{u(t+2k) = z_\gamma^k \quad \frac{1}{z_\gamma^k} \quad \frac{1+z_\gamma}{1-z_\gamma} H(t).}$$

Inhalt