UNIVERSITY OF CAMBRIDGE

CENTRE FOR MATHEMATICAL SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS
& THEORETICAL PHYSICS

On a Lie–Poisson system and its Lie algebra

Tony Bloch & Arieh Iserles

Part I:
A matrix ODE system

Let

$$X' = [N, X^2] = NX^2 - X^2N, \qquad t \geq 0,$$

where $X(0) = X_0 \in \mathrm{Sym}(n)$ and $N \in \mathfrak{so}(n)$. Here

$$\begin{aligned} \mathrm{Sym}(n): &\quad n \times n \text{ real symmetric matrices,} \\ \mathfrak{so}(n): &\quad n \times n \text{ real skew-symmetric matrices.} \end{aligned}$$

Why is this system interesting?

Reason 1: It is *isospectral:* defining a skew-symmetric matrix function

$$B(X) = NX + XN,$$

we can rewrite it at once in the form

$$X' = [B(X), X], \qquad X(0) = X_0 \in \mathrm{Sym}(n).$$

The system above is isospectral for any $B : \mathrm{Sym}(n) \to \mathfrak{so}(n)$ – its invariants are the eigenvalues of X_0.

Other isospectral systems, more well-known:

- The Toda lattice equations (Flaschka; Lax; Moser);
- The QR flow (Symes; Deift, Nanda & Tomei; Watkins);
- The double-bracket flow (Brockett; Chu & Driessel; Bloch, Brockett & Crouch; Bloch & Iserles);
- The Toeplitz annihilator flow (Chu & Driessel).

Why are isospectral ODEs isospectral?

Because they are an outcome of orthogonal group action. Thus, it is easy to verify that

$$X(t) = Q(t) X_0 Q^\top(t), \qquad t \geq 0,$$

where $Q' = B(Q X_0 Q^\top) Q, \qquad Q(0) = I$.

Since

$$A(Q) = B(QX_0Q^\top) : \mathsf{SO}(n) \to \mathfrak{so}(n)$$

and $\mathfrak{so}(n)$ is the Lie algebra of the special orthogonal group $\mathsf{SO}(n)$, it follows that Q evolves in $\mathsf{SO}(n)$ and $X(t)$ is similar to X_0.

Reason 2: The ODE is acted by congruence. Given

$$A : \mathsf{Sym}(n) \to \mathsf{M}(n),$$

where $\mathsf{M}(n)$ is the set of real $n \times n$ matrices, it is easy to verify that the solution of

$$X' = A(X)X + XA^\top(X), \quad t \geq 0,$$

where $X(0) = X_0 \in \mathsf{Sym}(n)$, is *congruent* to X_0:

$$X(t) = V(t)X_0V^\top(t), \qquad t \geq 0,$$

where

$$V' = A(VX_0V^\top)V, \qquad V(0) = I,$$

is a flow in the general linear group $\mathsf{GL}(n)$.

Congruent flows preserve the angular field of values

$$F'(X) = \{\boldsymbol{y}^* X \boldsymbol{y} \,:\, \boldsymbol{y} \in \mathbb{C}^n,\ \boldsymbol{y} \neq \boldsymbol{0}\}$$

and the signature of X_0. Also, if $X_0 = LL^\top$ is the Cholesky factorization of $X_0 \in \mathrm{Sym}_+(n)$ then a factorization of $X(t)$ is

$$X(t) = [V(t)L][V(t)L]^\top.$$

Setting $A(X) = [N, X]$, it is easy to verify that our ODE system is acted by congruence.

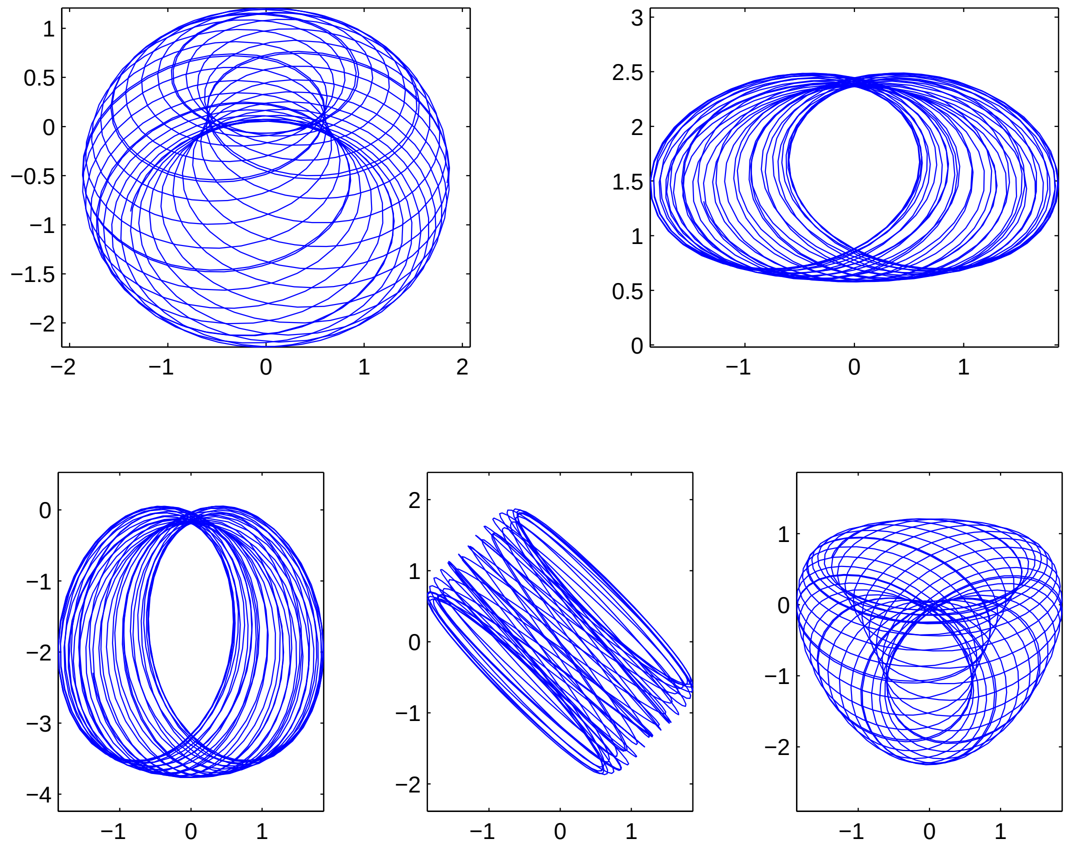
Although action by congruence is much “weaker” than action by similarity, the interesting freature of our system is that it is acted by two different groups. This is particularly interesting in the context of *Lie-group methods* since we are faced with the choice which action to retain under discretization.

Reason 3: The system is a "dual" of *generalized rigid body equations*

$$M' = [\Omega, M],$$

where $\Omega \in \mathfrak{so}(n)$ and $M = J\Omega + \Omega J$, where J lives in $\mathrm{Sym}(n)$ [Arnold].

Reason 4:



These are phase portraits (specifically, we display 2D sections $(x_{1,2}, x_{k,l})$) for $n = 3$,

$$N = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

and random initial conditions.

A persuasive observation: The solution evolves on invariant tori in $\mathbb{R}^{\frac{1}{2}n(n+1)}$.

Such behaviour is hardly ever accidental and it is reasonable to suspect that there is a deeper structure hiding within the equation $X' = [N, X^2]$. This suspicion is well founded...

Part II:
Poisson systems

Given

1 A smooth function $H : \mathbb{R}^m \to \mathbb{R}$ (a Hamiltonian); and

2 A linear, homogeneous function $S : \mathbb{R}^m \to \mathfrak{so}(m)$

the ODE system

$$\boldsymbol{x}' = S(\boldsymbol{x})\boldsymbol{\nabla} H(\boldsymbol{x}), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0 \in \mathbb{R}^m,$$

is said to be *almost Poisson.*

By "linear, homogeneous" we mean that there exist structure constants $c_{i,j}^k$ such that

$$S_{i,j}(\boldsymbol{x}) = \sum_{k=1}^{m} c_{i.j}^k x_k, \quad i, j = 1, \ldots, m.$$

Note that skew-symmetry of S implies $c_{i,j}^k + c_{j,i}^k = 0$.

We say that the structure constants obey the Jacobi condition if

$$\sum_{k=1}^{m} (c_{p,q}^{k} c_{k,r}^{l} + c_{q,r}^{k} c_{k,p}^{l} + c_{r,p}^{k} c_{k,q}^{l}) = 0$$

for all $p, q, r, l = 1, \ldots, m$. In that case the ODE is a ***Poisson system,*** a.k.a. Kostant–Kirillov–Lie–Soriau system.

Why are Poisson systems interesting?

- They represent a generalization of Hamiltonian systems. In particular, the Hamiltonian energy $H(\boldsymbol{y})$ is conserved by the flow.
- Define a Poisson bracket of two functions as

$$\{f, g\} = [\boldsymbol{\nabla} f(\boldsymbol{y})]^{\top} S(\boldsymbol{y}) \boldsymbol{\nabla} g(\boldsymbol{y}).$$

A *Casimir* is a function c which is in involution with all smooth functions:

$$\{c, g\} = 0.$$

In other words, $S\boldsymbol{\nabla} c = \mathbf{0}$. Each Casimir is a first integral of a Poisson system.

- Each Poisson system can be represented as a ***Lie–Poisson system:*** Suppose that we have square matrices $E_1, E_2, \ldots, E_m$ such that

$$[E_i, E_j] = \sum_{k=1}^{n} c_{i,j}^{k} E_k, \qquad i, j = 1, \ldots, m.$$

We generate the free Lie algebra

$$\mathcal{E} = \mathsf{FLA}(E_1, E_2, \ldots, E_m)$$

with the basis $\boldsymbol{E} = \{E_1, \ldots, E_m\}$. Thus, $\mathcal{E}$ is the closure of the basis elements (the generators) with respect to

1 Linear operations; and

2 Commutation.

Now, let $\mathcal{E}^*$ be the dual of $\mathcal{E}$: the linear space of all linear functionals acting on $\mathcal{E}$. We let

$$\langle F, E \rangle = \mathrm{tr} F^{\top} E$$

(i.e., Frobenius norm or the Killing form.)

It is possible to reformulate the Lie–Poisson flow so that it evolves in $\mathcal{E}^*$. Let $\boldsymbol{F} = \{F_1, F_2, \ldots, F_m\}$ be a dual basis of $\mathcal{E}^*$: a basis such that

$$\langle F_k, E_l \rangle = \begin{cases} 1, & k = l, \\ 0, & k \neq l. \end{cases}$$

We set

$$Y(t) = \sum_{k=1}^{m} y_k(t) F_k \in \mathcal{E}^*$$

and (abusing notation) let $H(Y) = H(\boldsymbol{y})$. Then the Lie–Poisson system can be formulated as

$$Y' = -\mathrm{ad}^*_{\mathrm{d}H(Y)} Y.$$

Here

$$\mathrm{d}H(Y) = \left(\frac{\partial H(Y)}{\partial Y_{i,j}} \right)_{i,j=1}^{m}$$

and ad^* is the dual adjoint operator which, within our context, can be taken as

$$\mathrm{ad}^*_A B = [A^\top, B].$$

Therefore a Lie–Poisson system possesses a crucial geometric feature: it evolves in a Lie algebra.

Remark: There exist several numerical methods for Lie–Poisson systems that keep the solution within $\mathcal{E}^*$ [Engø & Faltinsen].

BACK TO $X' = [N, X^2]$.
We set

$$H(X) = \frac{1}{2}\|X\|^2_{\mathrm{Frob}} = \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n} x^2_{k,l}.$$

Our equations are

$$x'_{k,l} = \sum_{i=1}^{n}\sum_{j=1}^{n}(n_{k,i}x_{i,j}x_{l,j} - n_{i,j}x_{k,i}x_{l,j} - n_{i,j}x_{k,j}x_{l,i} + n_{l,i}x_{k,j}x_{i,j})$$

and, with some algebra, we have the structure matrix

$$S_{(p,q),(r,s)} = \frac{1}{2}(n_{p,r}x_{q,s} + n_{p,s}x_{q,r} + n_{q,r}x_{p,s} + n_{q,s}x_{p,r})$$

for all $1 \le p \le q \le n$, $1 \le r \le s \le n$ (note that $\dim \mathcal{E} = m = \frac{1}{2}n(n+1)$).

The Jacobi condition:

Instead of checking directly that the Jacobi condition is satisfied – it can be done but is quite tedious and painful – we exploit the following observation due to Peter Olver: Let $N \in \mathfrak{so}(n)$ and

$$[X, Y]_N = XNY - YNX, \qquad X, Y \in \mathrm{Sym}(n).$$

It is easy to verify that $[X, Y]_N \in \mathrm{Sym}(n)$ and that it obeys all axioms of a Lie bracket: it is bilinear, skew-symmetric and obeys the Jacobi identity. Therefore, it defines a Lie algebra over $\mathrm{Sym}(n)$.

Let $G_{p,q} = \frac{1}{2}(\boldsymbol{e}_p \boldsymbol{e}_q^\top + \boldsymbol{e}_q \boldsymbol{e}_p^\top)$, $p \le q$, be a basis of this Lie algebra. Then

$$[G_{p,q}, G_{r,s}]_N = \tfrac{1}{2}(n_{p,r} G_{q,s} + n_{p,s} G_{q,r} + n_{q,r} G_{p,s} + n_{q,s} G_{p,r}),$$

THEOREM *The system $X' = [N, X^2]$ is Poisson.*

Note, however, that we cannot take the $G_{p,q}$s as a basis of our free Lie algebra, since there we require a standard matrix commutator.

Part III:
Lie–Poisson systems

Can every Poisson system be converted into a Lie–Poisson system?

This is true iff, given a set of structure constants $c_{k,l}^{i}$ that obey skew-symmetry and the Jacobi condition, we can identify matrices that form a basis of the underlying free Lie algebra. This is precisely the statement of. . .

ADO's THEOREM *Every finite-dimensional Lie algebra possesses a finite-dimensional faithful representation.*

Here, an algebra representation is a homomorphism

$$\rho : \mathfrak{g} \to \mathrm{End}\, V,$$

where $\mathfrak{g}$ is a (formal) Lie algebra, while $\mathrm{End}\, V$ is a matrix Lie algebra over the linear space V.

A Lie-algebra homomorphism a linear map s.t.

$$\rho([a,b]) = [\rho(a), \rho(b)], \qquad a, b \in \mathfrak{g}.$$

The representation is faithful if ρ is injective.

Can such a representation be derived explicitly?

The proof of Ado's theorem has been converted by Willem de Graaf into a (very complicated!) symbolic algorithm. Yet, his algorithm falls short of our needs:

- It produces matrices whose size increases exponentially as a function of the dimension of g, while we (bearing in mind eventual application to computation and geometric integration) wish to find a small (ideally, the least!) representation $\mathcal{E}$;

- We should bear it in mind that ultimately we wish to work in the dual space $\mathcal{E}^*$. In particular, we seek

$$\mathcal{E} = \mathsf{FLA}(E_1, \ldots, E_m), \qquad \mathcal{E}^* = \mathsf{FLA}(F_1, \ldots, F_m),$$

  where $m = \frac{1}{2}n(n+1)$, s.t. $\operatorname{tr} E_k^\top F_l = \delta_{k,l}$.

  Ideally, we would have liked $\operatorname{tr} E_k^\top E_l = \pi_k \delta_{k,l}$ for some $\pi_1, \ldots, \pi_m > 0$ (an orthogonal basis), since then we may identify $\mathcal{E}^*$ with $\mathcal{E}$, choosing $F_k = \pi_k^{-1} E_k$.

THE GOAL Find matrices $E_{p,q}$ s.t.

$$[E_{p,q}, E_{r,s}] = \frac{1}{2}(n_{q,r}E_{p,s} + n_{p,r}E_{q,s} + n_{q,s}E_{p,r} + n_{p,s}E_{q,r}).$$

THE ALGORITHM

Step 1: We assume without loss of generality that $\|N\|_2 = 1$: otherwise, later multiply elements of the basis by $\|N\|_2$. Consider the matrix

$$I + \mathrm{i}N.$$

This is a Hermitian matrix, since $N \in \mathfrak{so}(n)$.

Moreover, it is positive semi-definite and singular. The reason is that

$$\lambda \in \sigma(I + \mathrm{i}N) \qquad \Leftrightarrow \qquad \lambda = 1 - \mu,\ \mathrm{i}\mu \in \sigma(N).$$

However, $|\mu| \le \|N\|_2$ and $\max \mu = \|N\|_2 = 1$.

Step 2: We seek a complex upper triangular matrix R such that

$$R^*R = I + \mathrm{i}N.$$

Before you shout "Cholesky factorization!!!", we add also that R should be in the standard form. This can be done with a Cholesky-type algorithm but requires extra care.

Step 3: Note that singularity of $I + \mathrm{i}N$ implies that the bottom row of R is zero. We remove it, hence R is now an $(n-1)\times n$ complex matrix. We let

$$A = \begin{bmatrix} \mathrm{Re}\, R \\ \mathrm{Im}\, R \end{bmatrix} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n].$$

Note that $\boldsymbol{a}_k \in \mathbb{R}^{2n-2}$, $k = 1, \ldots, n$. We set

$$E_{p,q} = \tfrac{1}{2}(\boldsymbol{a}_p \boldsymbol{a}_q^\top + \boldsymbol{a}_q \boldsymbol{a}_p^\top)J, \quad 1 \le p \le q \le n,$$

where

$$J = \begin{bmatrix} O & I \\ -I & O \end{bmatrix}.$$

But why does it make sense? Letting $B = \mathrm{Re}\, R$ and $C = \mathrm{Im}\, R$, we note from

$$(B + \mathrm{i}C)^*(B + \mathrm{i}C) = R^*R = I + \mathrm{i}N$$

that

$$\begin{aligned} B^\top B + C^\top C &= I, \\ B^\top C - C^\top B &= N. \end{aligned}$$

Let

$$B = [\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n], \qquad C = [\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n].$$

Then

$$\begin{aligned} \boldsymbol{a}_p^\top J \boldsymbol{a}_q &= [\boldsymbol{b}_p^\top \, \boldsymbol{c}_p^\top] \begin{bmatrix} O & I \\ -I & O \end{bmatrix} \begin{bmatrix} \boldsymbol{b}_q \\ \boldsymbol{c}_q \end{bmatrix} = \boldsymbol{b}_p^\top \boldsymbol{c}_q - \boldsymbol{c}_p^\top \boldsymbol{b}_q \\ &= (B^\top C - C^\top B)_{p,q} = n_{p,q}. \end{aligned}$$

Therefore

$$\begin{aligned} [E_{p,q}, E_{r,s}] &= \tfrac{1}{4}[(\boldsymbol{a}_p \boldsymbol{a}_q^\top + \boldsymbol{a}_q \boldsymbol{a}_p^\top)J, (\boldsymbol{a}_r \boldsymbol{a}_s^\top + \boldsymbol{a}_s \boldsymbol{a}_r^\top)J] \\ &= \tfrac{1}{4}[(\boldsymbol{a}_q J \boldsymbol{a}_r)\boldsymbol{a}_p \boldsymbol{a}_s^\top + (\boldsymbol{a}_p^\top J \boldsymbol{a}_r)\boldsymbol{a}_q \boldsymbol{a}_s^\top + (\boldsymbol{a}_q^\top J \boldsymbol{a}_s)\boldsymbol{a}_p \boldsymbol{a}_r^\top + (\boldsymbol{a}_p^\top J \boldsymbol{a}_s)\boldsymbol{a}_q \boldsymbol{a}_r^\top \\ &\quad - (\boldsymbol{a}_s^\top J \boldsymbol{a}_p)\boldsymbol{a}_r \boldsymbol{a}_q^\top - (\boldsymbol{a}_r^\top J \boldsymbol{a}_p)\boldsymbol{a}_s \boldsymbol{a}_q^\top - (\boldsymbol{a}_s^\top J \boldsymbol{a}_q)\boldsymbol{a}_r \boldsymbol{a}_p^\top - (\boldsymbol{a}_r^\top J \boldsymbol{a}_q)\boldsymbol{a}_s \boldsymbol{a}_p^\top]J \\ &= \tfrac{1}{4}[n_{q,r}(\boldsymbol{a}_p \boldsymbol{a}_s^\top + \boldsymbol{a}_s \boldsymbol{a}_p^\top)J + n_{p,r}(\boldsymbol{a}_q \boldsymbol{a}_s^\top + \boldsymbol{a}_s \boldsymbol{a}_q^\top)J + n_{q,s}(\boldsymbol{a}_p \boldsymbol{a}_r^\top + \boldsymbol{a}_r \boldsymbol{a}_p^\top)J \\ &\quad + n_{p,s}(\boldsymbol{a}_q \boldsymbol{a}_r^\top + \boldsymbol{a}_r \boldsymbol{a}_q^\top)J] \\ &= \tfrac{1}{2}(n_{q,r}E_{p,s} + n_{p,r}E_{q,s} + n_{q,s}E_{p,r} + n_{p,s}E_{q,r}). \end{aligned}$$

We deduce that we have a representation of our Lie algebra in $\mathbb{R}^{2n-2}$.

But is it faithful? Orthogonal?

Both questions can be answered in a single calculation, since

$$\boldsymbol{a}_p^\top \boldsymbol{a}_q = \delta_{p,q}.$$

Since $JJ^\top = -J^2 = I$,

$$\begin{aligned}
\langle E_{p,q}, E_{r,s}\rangle &= \tfrac{1}{4}\mathrm{tr}(\boldsymbol{a}_p\boldsymbol{a}_q^\top + \boldsymbol{a}_q\boldsymbol{a}_p^\top)JJ^\top(\boldsymbol{a}_s\boldsymbol{a}_r^\top + \boldsymbol{a}_r\boldsymbol{a}_s^\top) \\
&= \tfrac{1}{4}\mathrm{tr}[(\boldsymbol{a}_q^\top\boldsymbol{a}_s)\boldsymbol{a}_p\boldsymbol{a}_r^\top + (\boldsymbol{a}_q^\top\boldsymbol{a}_r)\boldsymbol{a}_p\boldsymbol{a}_r^\top + (\boldsymbol{a}_p^\top\boldsymbol{a}_s)\boldsymbol{a}_q\boldsymbol{a}_r^\top \\
&\qquad + (\boldsymbol{a}_p^\top\boldsymbol{a}_r)\boldsymbol{a}_q\boldsymbol{a}_s^\top] \\
&= \tfrac{1}{2}[(\boldsymbol{a}_p^\top\boldsymbol{a}_r)(\boldsymbol{a}_q^\top\boldsymbol{a}_s) + (\boldsymbol{a}_p^\top\boldsymbol{a}_s)(\boldsymbol{a}_q^\top\boldsymbol{a}_r)] \\
&= \tfrac{1}{2}(\delta_{p,r}\delta_{q,s} + \delta_{p,s}\delta_{q,r}) \\
&= \begin{cases} 1, & p=q=r=s, \\ \tfrac{1}{2}, & p\neq q,\ p=r,\ q=s, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Therefore the representation is of full dimension $\frac{1}{2}n(n+1)$, hence faithful, and it is orthogonal.

THEOREM *The above algorithm results in a faithful orthogonal representation of the underlying Lie algebra in* $\mathbb{R}^{2n-2}$*.*

An example: Assume $a^2+b^2+c^2=1$ and set

$$N = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}, \qquad \|N\|_2 = 1.$$

We compute
$$R = \begin{bmatrix} 1 & \mathrm{i}a & \mathrm{i}b \\ 0 & \sqrt{b^2+c^2} & \frac{-ab+\mathrm{i}c}{\sqrt{b^2+c^2}} \\ 0 & 0 & 0 \end{bmatrix}$$

(verify that $R^*R = I + \mathrm{i}N$), hence (removing the bottom row)

$$A = \begin{bmatrix} \operatorname{Re} R \\ \operatorname{Im} R \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{b^2+c^2} & -\frac{ab}{\sqrt{b^2+c^2}} \\ 0 & a & b \\ 0 & 0 & \frac{c}{\sqrt{b^2+c^2}} \end{bmatrix}$$

and

$$\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{a}_2 = \begin{bmatrix} 0 \\ \sqrt{b^2+c^2} \\ a \\ 0 \end{bmatrix}, \quad \boldsymbol{a}_3 = \begin{bmatrix} 0 \\ -\frac{ab}{\sqrt{b^2+c^2}} \\ b \\ \frac{c}{\sqrt{b^2+c^2}} \end{bmatrix}.$$

Hence,

$$E_{1,1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad E_{1,2} = \begin{bmatrix} -\frac{1}{2}a & 0 & 0 & \frac{1}{2}\sqrt{b^2+c^2} \\ 0 & 0 & \frac{1}{2}\sqrt{b^2+c^2} & 0 \\ 0 & 0 & \frac{1}{2}a & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$E_{1,3} = \begin{bmatrix} -\frac{1}{2}b & -\frac{c}{2\sqrt{b^2+c^2}} & 0 & -\frac{ab}{2\sqrt{b^2+c^2}} \\ 0 & 0 & -\frac{ab}{2\sqrt{b^2+c^2}} & 0 \\ 0 & 0 & \frac{1}{2}b & 0 \\ 0 & 0 & \frac{c}{2\sqrt{b^2+c^2}} & 0 \end{bmatrix}, \qquad E_{2,2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -a\sqrt{b^2+c^2} & 0 & 0 & b^2+c^2 \\ -a^2 & 0 & 0 & a\sqrt{b^2+c^2} \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$E_{2,3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{b(a^2-b^2-c^2)}{2\sqrt{b^2+c^2}} & -\frac{1}{2}c & 0 & -ab \\ -ab & -\frac{ac}{2\sqrt{b^2+c^2}} & 0 & -\frac{b(a^2-b^2-c^2)}{2\sqrt{b^2+c^2}} \\ -\frac{ac}{2\sqrt{b^2+c^2}} & 0 & 0 & \frac{1}{2}c \end{bmatrix}, \qquad E_{3,3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{ab^2}{\sqrt{b^2+c^2}} & \frac{abc}{b^2+c^2} & 0 & \frac{a^2b^2}{b^2+c^2} \\ -b^2 & -\frac{bc}{\sqrt{b^2+c^2}} & 0 & -\frac{ab^2}{\sqrt{b^2+c^2}} \\ -\frac{bc}{\sqrt{b^2+c^2}} & -\frac{c^2}{b^2+c^2} & 0 & -\frac{abc}{b^2+c^2} \end{bmatrix}.$$

STOP PRESS: $\mathfrak{g}$ is isomorphic to $\mathfrak{sp}(n)$. [Tony Bloch, AI, Jerry Marsden & Tudor Ratiu]

Invariants

WHAT ARE THE INVARIANTS OF $X' = [N, X^2]$?

Isospectrality implies that $\mathrm{tr} X^k$ is conserved for all $k = 1, \ldots, n-1$.

However, with more work (either directly or using a technique due to Manakov) we can prove that the eigenvalues of $X + \lambda N$ are conserved for all $\lambda \in \mathbb{R}$. This results in

$$\mathrm{tr} \sum_{|\boldsymbol{i}|=k-2r} \sum_{|\boldsymbol{j}|=2r} X^{i_1} N^{j_1} X^{i_2} \cdots X^{i_s} N^{j_s}$$

for all $k = 1, \ldots, n-1$, $r = 0, \ldots \lfloor (k-1)/2 \rfloor$. We have altogether

$$\left\lfloor \frac{n+1}{2} \right\rfloor \times \left\lfloor \frac{n+2}{2} \right\rfloor$$

invariants, $\geq \frac{1}{2} m$. Were they all independent and in involution, this would have implied integrability. But are they?

STOP PRESS: Yes, they are! [Tony Bloch, AI, Jerry Marsden & Tudor Ratiu]

AND WHAT ABOUT CASIMIRS?

Computer experimentation strongly indicates that

$$\operatorname{rank} S(\boldsymbol{x}) = \tfrac{1}{2}n(n+1) - \lfloor \tfrac{1}{2}(n+1) \rfloor,$$

hence we expect $\lfloor \frac{1}{2}(n+1) \rfloor$ Casimirs. So far, just two have been identified:

- $c_1(X) = \det X$;
- $c_2(X) = \frac{1}{2}\mathbf{1}^\top(\operatorname{adj} N)X\mathbf{1}$, where $\operatorname{adj} V$ is the adjugate of the matrix V.

The proof for both cases is similar. In the first instance, up to a multiplicative factor,

$$\frac{\partial c_1(X)}{\partial x_{p,q}} = (X^{-1})_{p,q},$$

hence (with some algebra) indeed

$$S(\boldsymbol{x})\boldsymbol{\nabla} c_1(X) = \mathbf{0}.$$

For c_2 we need to replace elements of X with N on the right, taking care to avoid N^{-1} (hence the adjugate!), since N is always singular for odd n.

Conclusion

Our point of departure was the ODE matrix system $X' = [N, X^2]$, which is subject to two distinct group actions. We have proved that it is a Poisson system and constructed an algorithm for the construction of its orthogonal faithful representation in the underlying Lie algebra.

Themes for ongoing and future research:

- What are all its Casimirs? First integrals? What are features of the flow in the dual space?
- Are there bi-Hamiltonians? If so, are they nondegenerate? Can integrability follow by this route?
- Are there other isospectral flows, in addition to this one and the Toda lattice, which are Lie–Poisson? What are their Lie algebras?
- Is it possible to consider the “Ado problem” (find a faithful representation, given structure constants) in its full generality, in a linear-algebraic, numerical setting?

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