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Titre

Homogenization for the Poisson Equation in Perforated Domains

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Contents

Introduction

The aim of homogenization theory is to study the macroscopic behaviour of a system form its microscopic one. In other words, if we consider an heterogeneous problem P_ε where ε is a very small parameter and if u_ε is a solution of P_ε, the homogenization theory is an asymptotic tool giving us some answers to the following : Do the solution u_ε converges in some specified topology to a limit u? What is the limiting problem that u is a solution?

The objectif of this master thesis is to study the homogenization of the following Dirichlet problem:

$$\begin{cases} -\Delta u_\varepsilon = 0 & \text{in } \mathcal{D}'\left(D^\varepsilon\right), \\ u^\varepsilon \in H_0^1(D^\varepsilon), \end{cases}$$

where D^ε is a perforated domain obtained by removing a region $D \subseteq \mathbb{R}^d, d \geq 2$ the closures of spherical holes T_i^ε.

In the first chapter, the simplest case of perforated domains is considered, that is where the holes are *periodically* distributed. It is well-known that in this case there are three typical situations depending on the size of the holes: (1) Either the holes are too small and u_ε converges to a solution of a Dirichlet problem with Laplace operator as the first problem, (2) or the holes are too big and the solution u_ε converges to zero, (3) between these two situations there is a critical size in u_ε converges to a solution of a Dirichlet problem with an extra-term of ordrer zero, see D. Cioranescu and Murat [5]. So we only focus our attention on the third case which is, at our opinion, the most interesting one. In the second chapter, another type of a perforated domain is studied. Here, the holes, considered as balls, are randomly distributed in such a way that the centers and the radii of these balls denoted T_i^ε are generated by a marked point process $(\Phi, \mathcal{R})$ (see, Appendix B). We present in this chapter the more recently studies of A. Giunti, R. Höfer, and J.J.L. Velazquez [10] which generalizes those studies of D. Cioranscu and F. Murat introduced in the first chapter into the case of *random* holes.

Chapter 1

Homogenization of a Dirichlet problem in a perforated domain with periodic structure

We study in this chapter is the homogenization of a Dirichlet problem in a perforated domain with spherical holes distributed periodically in the volume. This work was done by D. Cioranscu and F. Murat [5].

1.1 Setting of the problem

Let D be an open bounded set of $\mathbb{R}^d$ where $d \geq 2$. For every $\varepsilon > 0$, we cover $\mathbb{R}^d$ by cubes P_k^ε of size 2ε. For example we can write:

$$\mathbb{R}^d = \bigcup_{k \in \mathbb{Z}^d} \left\{ \prod_{i=1}^{d} [2\varepsilon k_i - \varepsilon, 2\varepsilon k_i + \varepsilon[\right\},$$

where $\prod_{i=1}^d$ is the cartesian product. one has

$$P_k^\varepsilon = \prod_{i=1}^{d} [2\varepsilon k_i - \varepsilon, 2\varepsilon k_i + \varepsilon[\,,\ k = (k_1, ..., k_d) \in \mathbb{Z}^d.$$

Indeed, for every $x = (x_1, ..., x_d) \in \mathbb{R}^d$ and for every $\varepsilon > 0$ there exists $k = (k_1, ..., k_d) \in \mathbb{Z}^d$ such that $x \in P_k^\varepsilon$. It suffices to take for every $i = 1, ..., d$

$$k_i = \left(\left\lfloor \frac{1}{2} \left(\frac{x_i}{\varepsilon} + 1 \right) \right\rfloor \right),$$

where $\lfloor . \rfloor$ is the integer part. For every $k \in \mathbb{Z}^d$ and each cube P_k^ε we consider the closed balls $T_k^\varepsilon \subset P_k^\varepsilon$ with radii a^ε where $0 < a^\varepsilon < \varepsilon$ and the center is the point $(2\varepsilon k_1, 2\varepsilon k_2, ..., 2\varepsilon k_d)$ which is also the center of the cube P_k^ε. We set

$$Q^\varepsilon = \mathbb{R}^d \backslash \bigcup_{k\in\mathbb{Z}^d} T_k^\varepsilon, \qquad D^\varepsilon = D\cap Q^\varepsilon = D\backslash \bigcup_{k\in\mathbb{Z}^d\cap\frac{1}{2\varepsilon}D} T_k^\varepsilon \tag{1.1}$$

where

$$\frac{1}{2\varepsilon}D := \left\{ x\in\mathbb{R}^d, 2\varepsilon k\in D \right\}.$$

Let $f\in L^2(D)$. We consider the Dirichlet problem in D^ε: Find u^ε such that

$$\begin{cases} -\Delta u^\varepsilon = f \ \text{ in } \mathcal{D}'(D^\varepsilon), \\ u^\varepsilon \in H_0^1(D^\varepsilon). \end{cases} \tag{1.2}$$

The equivalent variational formulation of (1.2) is

$$\begin{cases} \text{Find } u^\varepsilon \in H_0^1(D^\varepsilon), \\ \int_{D^\varepsilon} \nabla u^\varepsilon \nabla v \ dx = \int_{D^\varepsilon} f v^\varepsilon \ dx, \ \forall\ v^\varepsilon \in H_0^1(D^\varepsilon). \end{cases} \tag{1.3}$$

Applying Lax-Milgram Lemma, we can easily show that the problem (1.3) has a unique weak solution $u^\varepsilon\in H_0^1(D^\varepsilon)$. Now, denote by $\tilde{u}^\varepsilon$ the extension of u^ε by 0 inside the holes, i.e

$$\tilde{u}^\varepsilon(x) = \begin{cases} u^\varepsilon(x) \quad \text{a.e } x\in D^\varepsilon, \\ 0 \quad \text{a.e } x\in\ T_k^\varepsilon,\ k\in\mathbb{Z}^d\cap\frac{1}{2\varepsilon}D. \end{cases}$$

It is clear that $\tilde{u}^\varepsilon\in H_0^1(D)$. Since D is bounded we can use Poincaré's inequality : there exists a constant $\alpha>0$ independant of ε, such that

$$\alpha\left\|\tilde{u}^\varepsilon\right\|_{H_0^1(D)} \leq \left\|\nabla\tilde{u}^\varepsilon\right\|_{(L^2(D))^d}. \tag{1.4}$$

Let us return to (1.3) and take $v^\varepsilon = u^\varepsilon$, we obtain

$$\int_D |\nabla\tilde{u}^\varepsilon|^2\,dx = \int_{D^\varepsilon}|\nabla u^\varepsilon|^2\,dx = \int_{D^\varepsilon} f u^\varepsilon dx = \int_D f\tilde{u}^\varepsilon dx \leq \|f\|_{L^2(D)}\left\|\tilde{u}^\varepsilon\right\|_{H_0^1(D)}$$

and using (1.4) we get immediatly

$$\left\|\tilde{u}^\varepsilon\right\|_{H_0^1(D)} \leq \frac{1}{\alpha}\|f\|_{L^2(D)}.$$

Hence by Rellich-Kondrachov Theorem, we can extract a subsequence still denoted by $\tilde{u}^\varepsilon$ such that

$$\tilde{u}^\varepsilon \to u \text{ strongly in } L^2(D), \tag{1.5}$$

then

$$\nabla\tilde{u}^\varepsilon \rightharpoonup \nabla u \text{ weakly in } \left(L^2(D)\right)^d.$$

The main objectif of homogenization theory is to construct the limit problem that u is a solution.

Remark 1.1 *We cannot pass to the limit in (1.3), because we only have weak convergence in the gradient. To overcome this difficulty, we take some special test functions of the form :* φw^ε *where* $\varphi \in \mathcal{D}(D)$ *and* w^ε *is some functions called correctors, which are specifically constructed from the microscopic description of the initial problem. This technique is called the energy method of Tartar or oscillating test functions introduced by L. Tartar in [13] in the context of the homogenization of linear elliptic equations.*

1.2 Construction of a test function

In this section, we shall give an explicit expression of an oscillating test function which shall be used in the homogenization process. It is given by the following technical Lemma.

Lemma 1.2 *For* $\varepsilon > 0$, *there exists a sequence of functions* w^ε *and a distribution* μ *such that*

$$(P1)\ w^\varepsilon \in H^1(D),$$
$$(P2)\ w^\varepsilon = 0 \text{ in the holes } T_k^\varepsilon,\ k \in \mathbb{Z}^d \cap \tfrac{1}{2\varepsilon}D,$$
$$(P3)\ w^\varepsilon \rightharpoonup 1 \text{ weakly in } H^1(D),$$
$$(P4)\ \mu \in W^{-1,\infty}(D),$$
$$(P5) \begin{cases} \text{For a sequence } v^\varepsilon \text{ with } v^\varepsilon = 0 \text{ in } T_k^\varepsilon, k \in \mathbb{Z}^d \cap \frac{1}{2\varepsilon}D, \\ \text{satisfies } v^\varepsilon \rightharpoonup v \text{ weakly in } H^1(D) \text{ with } v \in H^1(D), \text{ we obtain} \\ \langle -\Delta w^\varepsilon, \varphi v^\varepsilon \rangle_{H^{-1}(D),\, H_0^1(D)} \to \langle \mu, \varphi v \rangle_{H^{-1}(D),\, H_0^1(D)} \\ \text{for every } \varphi \in \mathcal{D}(D). \end{cases}.$$

Proof. As a first step of the proof, we define the function w_k^ε on each cube P_k^ε and we put

$$\begin{cases} w_k^\varepsilon = 0 \text{ in } T_k^\varepsilon, \\ \Delta w_k^\varepsilon = 0 \text{ in } B_k^\varepsilon - T_k^\varepsilon, \\ w_k^\varepsilon = 1 \text{ in } P_k^\varepsilon - B_k^\varepsilon, \\ w_k^\varepsilon \text{ is continuous in the interfaces } \partial B_k^\varepsilon, \partial T_k^\varepsilon, \end{cases} \tag{1.6}$$

where $B_k^\varepsilon \subset P_k^\varepsilon$ is the closed ball of radius ε with same center of T_k^ε, $k \in \mathbb{Z}^d$:

$$B_k^\varepsilon = \left\{ x \in \mathbb{R}^d, |x - 2\varepsilon k| \leq \varepsilon \right\}.$$

See figure 1.2.

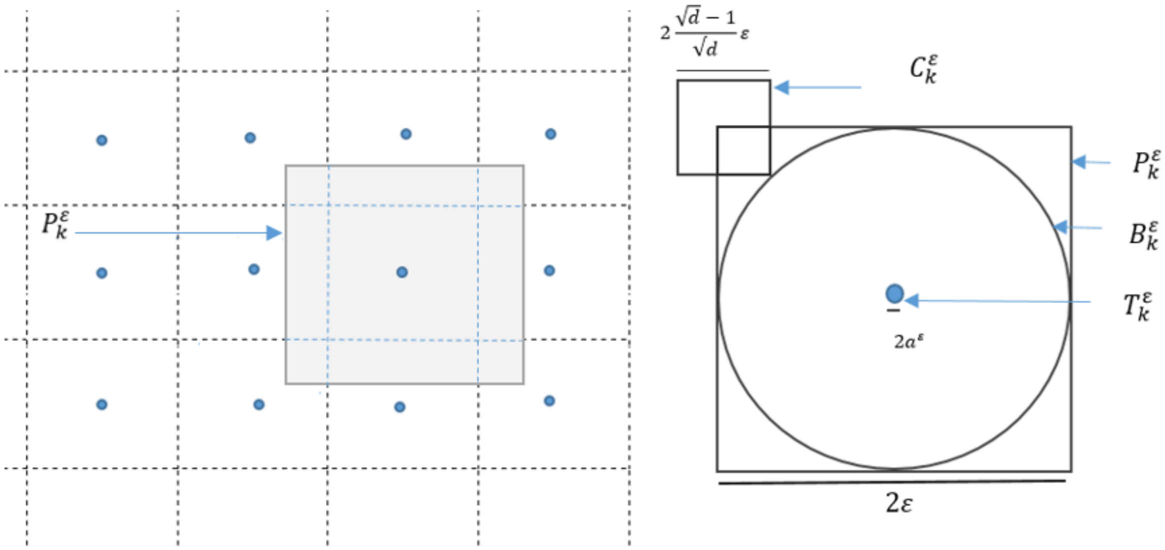

Figure 1.2.2 : This figure represente a zoom in the cell Q_k^ε perforated by a spherical hole $T_k^\varepsilon \subseteq B_k^\varepsilon \subseteq P_k^\varepsilon$.

Then, we define w^ε in the whole set $\mathbb{R}^d$ by

$$w^\varepsilon(x) = w_k^\varepsilon(x), \ x \in P_k^\varepsilon.$$

It follows then

$$\begin{cases} \Delta w^\varepsilon = 0 & \text{in } \bigcup_{k\in\mathbb{Z}^d} B_k^\varepsilon - T_k^\varepsilon, \\ 1 & \text{in } \mathbb{R}^d \setminus \bigcup_{k\in\mathbb{Z}^d} B_k^\varepsilon, \\ 0 & \text{in } \bigcup_{k\in\mathbb{Z}^d} T_k^\varepsilon. \end{cases}$$

Let us now give an explicit formulae for w^ε. Let $r = |x - x_k|$ where x_k is the center of the ball T_k^ε and search for w^ε as a radial solution

$$w^\varepsilon(x) = v(|x - x_k|),$$

where v is an unknown scalar function to be determined. Note that we dropped the $\varepsilon-$index just to simplify the presentation. We get the following initial-value problem

$$\begin{cases} -\Delta w^{\varepsilon}(x) = -v''(r) + \frac{1-d}{r}v'(r) = 0 \quad \text{in }]a^{\varepsilon}, \varepsilon[\,, \\ v(a^{\varepsilon}) = 0, \\ v(\varepsilon) = 1. \end{cases}$$

Solving the latter gives us

$$\begin{cases} w^{\varepsilon} = \dfrac{\ln a^{\varepsilon} - \ln |x - x_k|}{\ln a^{\varepsilon} - \ln \varepsilon} \text{ if } d = 2, \\ \\ w^{\varepsilon} = \dfrac{(a^{\varepsilon})^{-(d-2)} - |x - x_k|^{-(d-2)}}{(a^{\varepsilon})^{-(d-2)} - \varepsilon^{-(d-2)}} \text{ if } d \geq 3. \end{cases} \tag{1.7}$$

Now let us choose

$$\begin{cases} a^{\varepsilon} = \exp(-\frac{C_0}{\varepsilon^2}) \text{ if } d = 2, \\ a^{\varepsilon} = C_0 \varepsilon^{\frac{d}{d-2}} \text{ if } d \geq 3 \end{cases}$$

where C_0 is a positive constant independent of ε. Thus w^{ε} satisfies the properties $(P1) - (P5)$with

$$\begin{cases} \mu = \frac{\pi}{2}\frac{1}{C_0} \text{ if } d = 2, \\ \mu = \frac{\sigma_d (d-2)}{2^d} C_0^{d-2} \text{ if } d \geq 3. \end{cases} \tag{1.8}$$

For more details, we refer the reader to D. Cioranescu and F. Murat [5] ■

1.3 Passage to the limit

In what follows w^{ε} and μ are as in the previous section, namely they satisfy the properties $(P1)$-$(P5)$ of Lemma 1.2.

Proposition 1.3 *We have*

$$\langle \mu, \varphi \rangle_{\mathcal{D}'(D), \mathcal{D}(D)} = \lim_{\varepsilon \to 0} \int_D |\nabla \tilde{u}^{\varepsilon}|^2 \varphi dx, \ \forall \varphi \in \mathcal{D}(D). \tag{1.9}$$

Remark 1.4 *Before proving this result, we mention that the limit μ of $|\nabla \tilde{u}^{\varepsilon}|^2$ in the sense of distribution is a Radon measure.*

Proof. From $(P5)$ it is easily seen that (for $v^\varepsilon = w^\varepsilon$, $v = 1$, $\varphi \in \mathcal{D}(D)$)

$$\begin{aligned}\int_D |\nabla w^\varepsilon|^2 \varphi dx + \int_D w^\varepsilon \nabla w^\varepsilon \nabla(\varphi) dx &= \int_D \nabla w^\varepsilon \nabla (w^\varepsilon \varphi) dx \\ &= \langle -\Delta w^\varepsilon, \varphi w^\varepsilon \rangle_{H^{-1}(D), H_0^1(D)} \\ &\to \langle \mu, \varphi \rangle_{H^{-1}(D), H_0^1(D)} .\end{aligned}$$

taking into account that we have

$$\nabla w^\varepsilon \rightharpoonup 0 \text{ weakly in } (L^2(D))^d$$

and by Rellich-Kondrachov theorem we have

$$w^\varepsilon \to 1 \textit{ strongly in } L^2(D).$$

We pass to the limit, we obtain

$$\int_D w^\varepsilon \nabla w^\varepsilon \nabla(\varphi) dx \to 0.$$

Therefore the result 1.9 holds true. ■

Theorem 1.5 *Under the hypothesis* $(P1)$ *to* $(P5)$*, the solution* $\tilde{u}^\varepsilon$ *of (1.2) converges weakly in* $H_0^1(D)$ *to* u *the unique solution of*

$$\begin{cases} -\Delta u + \mu u = f \quad \textit{in } \mathcal{D}'(D), \\ \\ u \in H_0^1(D). \end{cases} \tag{1.10}$$

Proof. We have proved before in section 1.1 that $\|\tilde{u}^\varepsilon\|_{H_0^1(D)}$ is bounded. Then by Eberlein-S̃muljan theorem there exists a subsequence denoted also $\tilde{u}^\varepsilon$ and $u \in H_0^1(D)$ such that $\tilde{u}^\varepsilon$ converges weakly to u in $H_0^1(D)$ and by Rellich-Kondrachov theorem $\tilde{u}^\varepsilon$ converge strongly to u in $L^2(D)$. Now we identify the equation statisfied by the limit u. If $\varphi \in \mathcal{D}(D)$ and $w^\varepsilon \in H^1(D)$ then we have $w^\varepsilon \varphi \in H_0^1(D)$, furthermore w^ε satisfies hypothesis $(H2)$ it follows $w^\varepsilon \varphi \in H_0^1(D^\varepsilon)$. Then, we can substitute $w^\varepsilon \varphi$ in variationnal formulation (1.3), one has

$$\begin{aligned}\int_{D^\varepsilon} f w^\varepsilon \varphi dx &= \int_{D^\varepsilon} \nabla u^\varepsilon \nabla (w^\varepsilon \varphi) dx \\ &= \int_D \varphi \nabla \tilde{u}^\varepsilon \nabla w^\varepsilon dx + \int_D w^\varepsilon \nabla \tilde{u}^\varepsilon \nabla \varphi dx.\end{aligned} \tag{1.11}$$

Using the following result

$$w^\varepsilon \to 1 \text{ strongly in } L^2(D)$$

and (1.5) we can pass to the limit in the first and the last integral of (1.11) then, one has

$$\begin{aligned} \int_D w^\varepsilon \nabla \tilde{u}^\varepsilon \nabla \varphi \, dx &\to \int_D \nabla u \nabla \varphi \, dx, \\ \int_{D^\varepsilon} f w^\varepsilon \varphi dx &\to \int_D f \varphi dx. \end{aligned} \tag{1.12}$$

Applying Green's formula we get

$$\int_D \varphi \nabla \tilde{u}^\varepsilon \nabla w^\varepsilon dx = \langle -\Delta w^\varepsilon, \varphi \tilde{u}^\varepsilon \rangle_{H^{-1}(D), H_0^1(D)} - \int_D \tilde{u}^\varepsilon \nabla \varphi \nabla w^\varepsilon dx. \tag{1.13}$$

We can pass easily to the limit in the right hand side of (1.13), using $(P5)$ for the first integral it follows that

$$\langle -\Delta w^\varepsilon, \varphi \tilde{u}^\varepsilon \rangle_{H^{-1}(D), H_0^1(D)} \to \langle \mu, u\varphi \rangle_{H^{-1}(D), H_0^1(D)} \,. \tag{1.14}$$

For the second integral of (1.13), we use $(P3)$, i.e ∇w^ε converges weakly to 0 in $\left(L^2(D)\right)^d$ and the strong convergence of $\tilde{u}^\varepsilon$ in $L^2(D)$, we obtain then

$$\int_D \tilde{u}^\varepsilon \nabla \varphi \nabla w^\varepsilon dx \to 0. \tag{1.15}$$

Summing these convergences (1.12), (1.14) and (1.15), we get

$$\int_D \nabla u \nabla \varphi dx + \langle \mu u, \varphi \rangle_{H^{-1}(D), H_0^1(D)} = \int_D f \varphi dx, \quad \forall \varphi \in \mathcal{D}(D),$$

and it follows that

$$\langle -\Delta u, \varphi \rangle_{\mathcal{D}'(D), \mathcal{D}(D)} + \langle \mu u, \varphi \rangle_{H^{-1}(D), H_0^1(D)} = \langle f, \varphi \rangle_{\mathcal{D}'(D), \mathcal{D}(D)} \,, \quad \forall \varphi \in \mathcal{D}(D).$$

We can remark that the product μu of $\mu \in W^{-1,\infty}(D)$ and $u \in H_0^1(D)$ belongs to $H^{-1}(D)$, then the duality pairing $\langle \mu u, \varphi \rangle_{H^{-1}(D), H_0^1(D)}$ is well-defined which allows to write

$$-\Delta u + \mu u = f \quad \text{in } \mathcal{D}'(D).$$

Let us prove now the uniqueness of the solution u. Indeed, Let u_1, $u_2 \in H_0^1(D)$ two solutions of (1.10). One has

$$\begin{aligned} \int_D \nabla u_1 . \nabla \varphi \, dx + \langle \mu, u_1 \varphi \rangle_{\mathcal{D}'(D), \mathcal{D}(D)} &= \int_D f \varphi dx, \quad \forall \varphi \in \mathcal{D}(D), \\ \int_D \nabla u_2 . \nabla \varphi \, dx + \langle \mu, u_2 \varphi \rangle_{\mathcal{D}'(D), \mathcal{D}(D)} &= \int_D f \varphi dx, \quad \forall \varphi \in \mathcal{D}(D). \end{aligned}$$

By substracting, we get

$$\int_D \nabla(u_1 - u_2).\nabla\varphi \, dx + \langle \mu, (u_1 - u_2)\varphi \rangle_{\mathcal{D}'(D),\mathcal{D}(D)} = 0, \quad \forall \varphi \in \mathcal{D}(D).$$

For $\varphi = u_1 - u_2 \in H_0^1(D)$, it follows

$$\int_D |\nabla(u_1 - u_2)|^2 \; dx + \left\langle \mu, (u_1 - u_2)^2 \right\rangle_{W^{-1,\infty}(D),W_0^{1,1}(D)} = 0.$$

by (1.9), μ is a positive measure. Therefore

$$u_1 = u_2.$$

Thus, we get the uniqueness of solution. ■

1.4 Weak lower semi-continuity of the energy: correctors

In this section, we assume that the construction of w^ε and μ^ε satisfying hypotheses $(H1)$ to $(H5)$ introduced in section 1.2 holds true.

Proposition 1.6 *For every sequence z^ε and z such that:*

$$\begin{aligned} z^\varepsilon &\rightharpoonup z \;\; \text{weakly in } H_0^1(D), \\ z^\varepsilon &= 0 \;\text{ on the holes } T_k^\varepsilon, \; \forall k \in \mathbb{Z}^d \cap \frac{1}{2\varepsilon} D, \end{aligned} \tag{1.16}$$

One has

$$\liminf_{\varepsilon \to 0} \int_D |\nabla z^\varepsilon|^2 \, dx \geq \int_D |\nabla z|^2 \, dx + \left\langle \mu, z^2 \right\rangle_{W^{-1,\infty}(D),W_0^{1,1}(D)} . \tag{1.17}$$

Remark 1.7 *The classical weak lower semicontinuity of the energy defined as follows: For every sequence z^ε and z satisfies*

$$z^\varepsilon \to z \text{ weakly in } H_0^1(D),$$

then

$$\liminf_{\varepsilon \to 0} \int_D |\nabla z^\varepsilon|^2 \, dx \geq \int_D |\nabla z|^2 \, dx,$$

We can remark the fact that z^ε vanishes in the holes introduce a new energy. Since $\mu \in W^{-1,\infty}(D)$ and $z^2 \in W_0^{1,1}(D)$ then $\left\langle \mu, z^2 \right\rangle$ is well-defined.

Proof. (of Proposition 1.6). Let $\varphi \in \mathcal{D}(D)$. We consider the following integral

$$\begin{aligned}\int_D |\nabla(z^\varepsilon - w^\varepsilon\varphi)|^2\,dx &= \int_D |\nabla z^\varepsilon - \varphi\nabla w^\varepsilon - w^\varepsilon\nabla\varphi|^2\,dx\\ &= \int_D |\nabla z^\varepsilon|^2\,dx + \int_D |\nabla\varphi|^2\,|w^\varepsilon|^2\,dx + \int_D |\varphi|^2\,|\nabla w^\varepsilon|^2\,dx\\ &\quad -2\int_D w^\varepsilon\nabla z^\varepsilon w^\varepsilon\nabla\varphi dx + 2\int_D w^\varepsilon\varphi\nabla w^\varepsilon\nabla\varphi dx\\ &\quad -2\int_D \nabla z^\varepsilon\nabla w^\varepsilon\varphi dx,\end{aligned}$$

taking into account

$$\langle -\Delta w^\varepsilon, \varphi z^\varepsilon\rangle_{H^{-1}(D),H_0^1(D)} = \int_D \varphi\nabla z^\varepsilon\ \nabla w^\varepsilon dx + \int_D z^\varepsilon\nabla\varphi\ \nabla w^\varepsilon dx,$$

we obtain

$$\begin{aligned}\int_D |\nabla(z^\varepsilon - w^\varepsilon\varphi)|^2\,dx &= \int_D |\nabla z^\varepsilon - \varphi\nabla w^\varepsilon - w^\varepsilon\nabla\varphi|^2\,dx\\ &= \int_D |\nabla z^\varepsilon|^2\,dx + \int_D |\nabla\varphi|^2\,|w^\varepsilon|^2\,dx + \int_D |\varphi|^2\,|\nabla w^\varepsilon|^2\,dx\\ &\quad -2\int_D w^\varepsilon\nabla z^\varepsilon\ w^\varepsilon\nabla\varphi dx + 2\int_D w^\varepsilon\varphi\nabla w^\varepsilon\nabla\varphi dx\\ &\quad +2\int_D z^\varepsilon\ \nabla w^\varepsilon\nabla\varphi dx - 2\,\langle -\Delta w^\varepsilon, \varphi z^\varepsilon\rangle_{H^{-1}(D),H_0^1(D)}.\end{aligned} \tag{1.18}$$

We choose ε such that $\int_D |\nabla z^\varepsilon|^2\,dx$ converges, then using Rellich-Kondrachov theorem and $(P5)$ to pass to the limit in each term, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_D |\nabla(z^\varepsilon - w^\varepsilon\varphi)|^2\,dx &= \lim_{\varepsilon\to 0}\int_D |\nabla z^\varepsilon|^2\,dx + \int_D |\nabla\varphi|^2\,dx\\ &\quad + \lim_{\varepsilon\to 0}\Big(\int_D \nabla(w^\varepsilon\varphi^2)\nabla w^\varepsilon dx - \int_D w^\varepsilon\,|\nabla\varphi|^2\,\nabla w^\varepsilon dx\Big)\\ &\quad -2\int_D \nabla z\ \nabla\varphi dx - 2\,\langle \mu, \varphi z\rangle_{H^{-1}(D),H_0^1(D)}.\end{aligned}$$

Then

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_D |\nabla(z^\varepsilon - w^\varepsilon\varphi)|^2\,dx &= \lim_{\varepsilon\to 0}\int_D |\nabla z^\varepsilon|^2\,dx + \int_D |\nabla\varphi|^2\,dx + \langle \mu, \varphi^2\rangle_{H^{-1}(D),H_0^1(D)}\\ &\quad -2\int_D \nabla z\nabla\varphi dx - 2\,\langle \mu, \varphi z\rangle_{H^{-1}(D),H_0^1(D)}.\end{aligned} \tag{1.19}$$

Now we choose a subsequence denoted also $\varepsilon > 0$ such that:

$$\lim_{\varepsilon\to 0}\int_D |\nabla z^\varepsilon|^2\,dx = \liminf_{\varepsilon\to 0}\int_D |\nabla z^\varepsilon|^2\,dx.$$

Since the left hand side of (1.19) is positive, we get

$$\begin{aligned}\liminf_{\varepsilon\to 0}\int_D |\nabla z^\varepsilon|^2\,dx \quad &\geq \quad 2\int_D \nabla z\ \nabla\varphi dx - \int_D |\nabla\varphi|^2\,dx \\ &+2\left\langle \mu,\varphi z\right\rangle_{H^{-1}(D),H_0^1(D)} - 2\left\langle \mu,\varphi^2\right\rangle_{H^{-1}(D),H_0^1(D)}. \end{aligned} \tag{1.20}$$

This result holds true for every $\varphi \in \mathcal{D}(D)$. If we choose φ such that φ converges strongly to z in $H_0^1(D)$, one has

$$\liminf_{\varepsilon\to 0}\int_D |\nabla z^\varepsilon|^2\,dx \geq \int_D |\nabla z|^2\,dx + \left\langle \mu, z^2\right\rangle_{H^{-1}(D),H_0^1(D)}.$$

If z belongs only to $H_0^1(D)$, one has $w^\varepsilon z \notin H_0^1(D)$ under $(P1)$, then, (1.18) does not make any sense with $\varphi = z$. This is the reason why we had approximate z by smooth functions φ. If $z = \varphi$ from the beginning and this is impossible if $z \in \mathcal{D}(D)$, we obtain (1.17) from (1.20) directly without passing to the limit. ■

Proposition 1.8 *If moreover z^ε satisfies:*

$$\lim_{\varepsilon\to 0}\int_D |\nabla z^\varepsilon|^2\,dx \to \int_D |\nabla z|^2\,dx + \langle \mu, z^2\rangle_{W^{-1,\infty}(D),W_0^{1,1}(D)} \tag{1.21}$$

Then

$$z^\varepsilon - w^\varepsilon z \to 0 \text{ strongly in } W_0^{1,1}(D) \tag{1.22}$$

Remark 1.9 *By (1.22) we have a stong convergence only in $W_0^{1,1}(D)$, but we would like to have this convergence in $H_0^1(D)$, which is the natural space for the problem. We shall see at the end of the proof of proposition a strong convergence result in $W_0^{1,q}(D)$ with $q = \frac{d-1}{d}$ thanks to Gagliardo-Nirenberg-Sobolev theorem (See Appendix A).*

Proof. Let us return to (1.18) and taking into account the hypothesis (1.21). We can establishes for $\varphi \in D\,(D)$

$$\lim_{\varepsilon\to 0}\int_D |\nabla(z^\varepsilon - w^\varepsilon\varphi)|^2\,dx = \int_D |\nabla(z-\varphi)|^2\,dx + \langle \mu,(z-\varphi)^2\rangle_{W^{-1,\infty}(D),W_0^{1,1}(D)}.$$

If $z \in \mathcal{D}(D)$, we can take $\varphi = z$ and we have proved

$$z^\varepsilon - w^\varepsilon z \to 0 \text{ strongly in } H_0^1(D).$$

If z is not regular, we fix φ such that there exists a constant $\delta > 0$ such that

$$\|z-\varphi\|_{H_0^1(D)} \leq \delta.$$

Using the embedding of $H_0^1(D)$ in $W_0^{1,1}(D)$. It follows

$$\lim_{\varepsilon\to 0}\int_D |\nabla(z^\varepsilon - w^\varepsilon\varphi)|^2\,dx \leq (1+2\,\|\mu\|_{W^{-1,\infty}(D)})\delta^2.$$

thanks to Poincaré inequality, one has

$$\lim_{\varepsilon\to 0}\int_D |(z^\varepsilon - w^\varepsilon\varphi)|^2\,dx \leq (1+2\,\|\mu\|_{W^{-1,\infty}(D)})\delta^2.$$

Using definition of the limit concept: For $C_1 = (1+2\,\|\mu\|_{W^{-1,\infty}(D)})\delta^2 > 0$, there exist ε_0 such that for every $\varepsilon \leq \varepsilon_0$, one has

$$\|z^\varepsilon - w^\varepsilon\varphi\|^2_{H_0^1(D)} \leq C_1.$$

In the other hand

$$\begin{aligned} \|z^\varepsilon - w^\varepsilon z\|_{W_0^{1,1}(D)} &\leq \|z^\varepsilon - w^\varepsilon\varphi\|_{W_0^{1,1}(D)} + \|w^\varepsilon(z-\varphi)\|_{W_0^{1,1}(D)} \\ &\leq \|z^\varepsilon - w^\varepsilon\varphi\|_{H_0^1(D)} + \|w^\varepsilon\|_{H_0^1(D)}\,\|z-\varphi\|_{H_0^1(D)} \\ &\leq C_1'\delta + C_2\delta, \end{aligned} \tag{1.23}$$

where $C_1', C_2 > 0$, for every $\varepsilon \leq \varepsilon_0$, which prove (1.22). We have used in (1.23) an estimation of $w^\varepsilon(z-\varphi)$ in $W_0^{1,1}(D)$. Thanks to Gagliardo-Nirenberg-Sobolev theorem, we have $H_0^1(D) \subset L^{2\star}(D)$ where $2\star = \frac{2d}{(d-2)}$ puting $\frac{1}{q} = \frac{1}{2} + \frac{1}{2\star}$, we can write

$$\begin{aligned} \|w^\varepsilon(z-\varphi)\|_{W_0^{1,q}(D)} &= \|\nabla(w^\varepsilon(z-\varphi))\|_{L^q(D)} \\ &\leq \|\nabla w^\varepsilon\|_{(L^2(D))^d}\,\|z-\varphi\|_{L^{2\star}(D)} + \|z-\varphi\|_{L^2(D)}\,\|w^\varepsilon\|_{L^{2\star}(D)}, \end{aligned}$$

which allows to

$$z^\varepsilon - w^\varepsilon z \to 0 \text{ strongly in } W_0^{1,q}(D),$$

where $q = \frac{d-1}{d}$. ■

Assume Propositions 1.6 and 1.8 are satisfied, then we obtain the following corrector result

Corollary 1.10 *Let u^ε be the solution of the Dirichlet problem (1.2). Then there exists r^ε such that*

$$\begin{cases} \tilde{u}^\varepsilon = w^\varepsilon u + r^\varepsilon, \\ \\ r^\varepsilon \to 0 \text{ strongly in } W_0^{1,1}(D), \end{cases}$$

where u is a solution of (1.10).

Proof. Using theorem 1.5, we have $\tilde{u}^\varepsilon$ converges weakly to u in $H_0^1(D)$, where u is a solution of (1.10). Multiplying the equation of (1.2) by u^ε and (1.10) by u using Green formula one has

$$\int_D |\nabla \tilde{u}^\varepsilon|^2 \, dx = \int_D f \tilde{u}^\varepsilon dx \to \int_D f u dx = \int_D |\nabla u|^2 \, dx + \left\langle \mu, u^2 \right\rangle_{W^{-1,\infty}(D), W_0^{1,1}(D)}.$$

Applaying proposition 1.8, taking $z^\varepsilon = \tilde{u}^\varepsilon$and $z = u$, we get

$$\tilde{u}^\varepsilon - w^\varepsilon u \to 0 \text{ strongly in } W_0^{1,1}(D).$$

Taking

$$r^\varepsilon = \tilde{u}^\varepsilon - w^\varepsilon u \to 0 \text{ strongly in } W_0^{1,1}(D).$$

Then we get our result. ■

Chapter 2

Homogenization for Dirichlet problem in randomly perforated domain

This chapter deals with the homogenization of the Poisson equation in a bounded domain of $\mathbb{R}^d$, $d \geq 3$, which is perforated by a random number of small spherical holes with random radii and positions studied by A. Giunti et al in [10] using the oscillating test functions method. We recover in the homogenized limit an averaged analogue of the "strange term" obtained by D. Cioranescu and F. Murat in the periodic case [5]. In addition, we put a minimal assumption on the size of the holes in order to ensure that the homogenized equation has a sens and thus the homogenization occurs.

2.1 Setting of the problem

Let $D \subseteq \mathbb{R}^d$, $d \geq 3$, be an open and bounded set that it is star-shaped with respect to the origin. For $\varepsilon > 0$, let us define the set of closed small spherical holes $H^\varepsilon \subseteq \mathbb{R}^d$ of the form

$$H^\varepsilon = \bigcup_{z_j \in \Phi \cap \frac{1}{\varepsilon} D} B_{\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j), \tag{2.1}$$

where $\frac{1}{\varepsilon} D = \left\{ x \in \mathbb{R}^d, \varepsilon x \in D \right\}$, the set $\Phi \subseteq \mathbb{R}^d$ is a random collection of points and the radii $\left\{ \rho_j \right\}_{z_j \in \Phi} \subseteq \mathbb{R}^+$ are random variables. We may thus be thought that the set H^ε being generated by a marked point process $(\Phi, \mathcal{R})$ on $\mathbb{R}^d \times \mathbb{R}^+$ where Φ is a point process on $\mathbb{R}^d$ for the center of balls and the marks $\mathcal{R} = \left\{ \rho_j \right\}_{z_j \in \Phi} \subseteq \mathbb{R}^+$ are the radii associated to each center. For a precise definition we refer the reader to Appendix B. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where Ω denotes the set of events, $\mathcal{F}$ is σ-algebra and $\mathbb{P}$ is a probability measure, associated to the process $(\Phi, \mathcal{R})$ satisfying the following properties:

a. The process Φ is stationary: For every $x \in \mathbb{R}^d$ and each $\{z_j\}_{j \in \mathbb{N}} \subseteq \mathbb{R}^d$, the translation operators τ_x are defined as follow

$$\tau_x\left(\{z_j\}_{j\in\mathbb{N}}\right) = \{z_j + x\}_{j\in\mathbb{N}}.$$

So

$$\tau_x(\Phi) = \Phi. \tag{2.2}$$

b. There exists $\lambda < +\infty$ such that for any unitary cube $Q \subseteq \mathbb{R}^d$

$$\left\langle \#(\Phi \cap Q)^2 \right\rangle^{\frac{1}{2}} \leq \lambda, \tag{2.3}$$

where $\#S \in \mathbb{N} \cup \infty$ denotes the cardinality of a set S and $\langle . \rangle$ is the integration over Ω with respect to the probability measure $\mathbb{P}$.

c. The point process Φ satisfies a strong mixing condition: For any bounded Borel set $A \subseteq \mathbb{R}^d$, $\mathcal{F}(A)$ be the smallest σ-algebra with respect to which the random variables

$$N(B)(\omega) = \#(\Phi(\omega) \cap B),$$

are measurable for every Borel set $B \subseteq A$. Then, there exists $C_1 < +\infty$ and $\gamma > d$ such that for every $A \subseteq \mathbb{R}^d$ as above, every $x \in \mathbb{R}^d$, with $|x| > diam(A)$ and every ξ_1, ξ_2 are measurable function with respect to $\mathcal{F}(A)$ and $\mathcal{F}(\tau_x A)$, respectively, we have

$$|\langle \xi_1 \xi_2 \rangle - \langle \xi_1 \rangle \langle \xi_2 \rangle| \leq \frac{C_1}{1 + (|x| - diam(A))^\gamma} \left\langle \xi_1^2 \right\rangle^{\frac{1}{2}} \left\langle \xi_2^2 \right\rangle^{\frac{1}{2}}. \tag{2.4}$$

d. The marginal $\mathbb{P}_{\mathcal{R}}$ of the marks has two correlation functions, the first is the density function of a random variable $\rho \in \mathcal{R}$ denoted by h_ρ satisfies

$$\int_0^{+\infty} x^{d-2} h_\rho(x) dx < +\infty. \tag{2.5}$$

The second is the joint density function of two variables ρ_i, ρ_j depend on the centers z_i and z_j denoted by $h_{\rho_i \rho_j}$ and satisfies for $x, y \in \mathbb{R}^+$

$$h_{\rho_i \rho_j}(x, y) = h_{\rho_i} h_{\rho_j}(x, y) + g(z_i, z_j, x, y), \tag{2.6}$$

with

$$|g(z_i, z_j, x, y)| \leq \frac{c}{(1 + |z_i - z_j|^\gamma)(1 + x^p)(1 + y^p)},$$

for $p > d-1$, $\gamma > d$, $c \in \mathbb{R}^+$ and g is an integrable function with respect to the variable $r = |z_i - z_j|$ and vanishes when the distance $|z_i - z_j| \to +\infty$. For $f \in H^{-1}(D)$, we introduce our main problem as follow: Find u_ε such that

$$\begin{cases} -\Delta u_\varepsilon(\omega,.) = f(.) & \text{in } D^\varepsilon(\omega), \\ u_\varepsilon(\omega,.) = 0 & \text{in } \partial D^\varepsilon(\omega), \end{cases} \tag{2.7}$$

where $D^\varepsilon(\omega)$ is a punctured domain obtained by removing from D the set $H^\varepsilon(\omega)$. We write

$$D^\varepsilon(\omega) = D \backslash \bigcup_{z_j \in \Phi(\omega) \cap \frac{1}{\varepsilon} D} B_{\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j). \tag{2.8}$$

The equivalent variational formulation is

$$\begin{cases} \text{find } u_\varepsilon \in H_0^1(D^\varepsilon(\omega)) \text{ such that} \\ \int_{D^\varepsilon(\omega)} \nabla u_\varepsilon \nabla v dx = \langle f, v \rangle_{H^{-1}(D^\varepsilon(\omega)), H_0^1(D^\varepsilon(\omega))}, \quad \forall v \in H_0^1(D^\varepsilon(\omega)). \end{cases} \tag{2.9}$$

Denote by $\tilde{u}_\varepsilon$ the extension by zero of u_ε to the whole set D

$$\tilde{u}_\varepsilon = \begin{cases} u_\varepsilon & \text{in } D^\varepsilon(\omega), \\ 0 & \text{in } H^\varepsilon(\omega), \end{cases}$$

then $\tilde{u}_\varepsilon \in H_0^1(D)$. In order to simplify the presentation, we denote for $\mathbb{P}$-almost every $\omega \in \Omega$ D^ε and H^ε instead of $D^\varepsilon(\omega)$ and $H^\varepsilon(\omega)$.

2.2 Some preliminaries results

In this section, we give our main result of homogenization and some lemmas in order to use it in the proof of the following theorem.

Theorem 2.1 *Let the holes in (2.1) be generated by a marked point process $(\Phi, \mathcal{R})$. Let Φ satisfy (2.2), (2.3) and (2.4), and let the marginal $\mathbb{P}_{\mathcal{R}}$ satisfy (2.5) and (2.6). Assume that the expectation of each radius ρ_i satisfies*

$$\left\langle \rho_i^{d-2} \right\rangle = \int_0^{+\infty} x^{d-2} h_{\rho_i}(x) dx < +\infty. \tag{2.10}$$

For $f \in H^{-1}(D)$ and $\varepsilon > 0$, let $u_\varepsilon = u_\varepsilon(\omega,.) \in H_0^1(D^\varepsilon(\omega))$ solves (2.7). Then, there exists a constant $C_0 > 0$ and $u_h \in H_0^1(D)$ solving

$$\begin{cases} -\Delta u_h + C_0 u_h = f & \textit{in } D, \\ u_h = 0 & \textit{in } \partial D, \end{cases} \tag{2.11}$$

such that for $\mathbb{P}-$*almost every* $\omega \in \Omega$

$$\tilde{u}_\varepsilon(\omega,.) \rightharpoonup u_h \text{ weakly in } H_0^1(D), \text{ for } \varepsilon \downarrow 0^+.$$

Moreover, we have that the constant C_0 *in (2.11) is defined as*

$$C_0 = (d-2)\sigma_d \langle N(Q) \rangle \left\langle \rho^{d-2} \right\rangle,$$

where σ_d *is the* $(d-1)$*-dimensional area of the unit sphere of* $\mathbb{R}^d$, $N(Q)$ *is the number of centers falling into any fixed unitary cube* Q *and* $\rho \in \mathcal{R}$.

To prove this theorem we give the following Lemma.

Lemma 2.2 *Let* $H^\varepsilon = H^\varepsilon(\omega)$ *be as in . Then, for* $\mathbb{P}-$*almost every* $\omega \in \Omega$, *there exists a sequence* $\{w^\varepsilon(\omega,.)\}_{\varepsilon>0} \subseteq H^1(D)$ *which satisfies*

(H1) For every $\varepsilon > 0$, $w^\varepsilon(\omega,.) = 0$ *in* H^ε;

(H2) $w^\varepsilon(\omega,.) \rightharpoonup 0$ *in* $H^1(D)$ *for* $\varepsilon \downarrow 0^+$;

(H3) For every sequence $v_\varepsilon \rightharpoonup v$ *in* $H_0^1(D)$ *such that* $v_\varepsilon = 0$ *in* H^ε, *it holds that*

$$(-\Delta w^\varepsilon(\omega,.), v_\varepsilon)_{H^{-1}(D),H_0^1(D)} \longrightarrow C_0 \int_D v,$$

for $\varepsilon \downarrow 0^+$ *and where* C_0 *defined as in theorem .*

The construction of w^ε is given in two steps. The first step is to give an argument in the simplest case of the random holes where the centers of balls are distributed periodically and the radii are associated as an i.i.d random variables. We then generalize this argument to an arbitrary marked point process $(\Phi, \mathcal{R})$ that satisfies the assumption of theorem 2.2. We first fix the following notation: For any two open sets $A \subseteq B \subseteq \mathbb{R}^d$, we define the capacity of the condenser (A, B)

$$cap(A,B) = \inf\left\{\int_B |\nabla v| : v \in \mathcal{C}_0^\infty(B), \quad v \geq \mathbf{1}_A\right\} \tag{2.12}$$

where $\mathcal{C}_0^\infty(B)$ is the space of infinitely differentiable functions with compact support. The minimizer of (2.12) is given as a solution of the following problem

$$\begin{cases} -\Delta u = 0 & \text{in } B\backslash A, \\ u = 1 & \text{in } \partial A, \\ u = 0 & \text{in } \partial B. \end{cases}$$

The solution u, called harmonic function; satisfies $0 \leq u \leq 1$ (Maximum principle, see for instance [3] p.172-173). For a point process Φ on $\mathbb{R}^d$ and any bounded set $E \subseteq \mathbb{R}^d$, we define

$$\begin{aligned} \Phi(E) &= \Phi \cap E, & \Phi^\varepsilon(E) &= \Phi^\varepsilon \cap (\tfrac{1}{\varepsilon}E), \\ N(E) &= \#(\Phi(E)), & N^\varepsilon(E) &= \#(\Phi^\varepsilon(E)). \end{aligned} \tag{2.13}$$

For $\delta > 0$, we denote by Φ_δ a thinning for the process Φ obtained as

$$\Phi_\delta = \left\{ x \in \Phi : \min_{\substack{y \in \Phi(\omega) \\ y \neq x}} |x - y| \geq \delta \right\}, \tag{2.14}$$

i.e. The points of $\Phi(\omega)$ whose minimal distance from the other points is at least δ. For a fixed $M > 0$, we define the truncated marks

$$\mathcal{R}^M = \left\{\rho_{j,M}\right\}_{z_j \in \Phi}, \qquad \rho_{j,M} = \rho_j \wedge M = \min\left\{\rho_j, M\right\}. \tag{2.15}$$

2.2.1 Case(a): Periodic centers

In this setting the holes H^ε are generated by $\Phi = \mathbb{Z}^d$ and a collection of i.i.d. random variables $\{\rho_i\}_{z_i \in \mathbb{Z}^d}$ satisfying the assumption (2.10). Since the centers of the holes are periodically distributed, the only chalenge in the construction of the functions w^ε is due to the random variables $\{\rho_i\}_{z_i \in \mathbb{Z}^d}$ which might generate very large holes under the mere condition (2.10). We introduce the following lemma which might simplify the construction of w^ε.

Lemma 2.3 *Let $\delta \in \left(0, \frac{2}{d-2}\right)$ be fixed. Then, there exists $\varepsilon_0 = \varepsilon_0(\delta) > 0$ such that $\mathbb{P}$-almost every $\omega \in \Omega$ and for all $\varepsilon \leq \varepsilon_0$ there exist $H_g^\varepsilon(\omega), H_b^\varepsilon(\omega), D_b^\varepsilon(\omega) \subseteq \mathbb{R}^d$ such that*

$$\begin{aligned} &H^\varepsilon(\omega) = H_g^\varepsilon(\omega) \cup H_b^\varepsilon(\omega), H_b^\varepsilon(\omega) \subset D_b^\varepsilon(\omega), \\ &dist(H_g^\varepsilon(\omega), D_b^\varepsilon(\omega)) \geq \tfrac{\varepsilon}{2}, \end{aligned} \tag{2.16}$$

where

$$\lim_{\varepsilon \to 0} cap(H_b^\varepsilon(\omega), D_b^\varepsilon(\omega)) = 0. \tag{2.17}$$

Moreover, $H_g^\varepsilon(\omega)$ may be written as the following union of disjoint balls centered in $n^\varepsilon(\omega) \subseteq \mathbb{Z}^d \cap \frac{1}{\varepsilon}D$

$$\begin{aligned} &H_g^\varepsilon(\omega) = \bigcup_{z_j \in n^\varepsilon} B_{\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j), \ \varepsilon^{\frac{d}{d-2}}\rho_j \leq \varepsilon^{\delta+1} < \tfrac{\varepsilon}{2}, \\ &\lim_{\varepsilon \to 0} \varepsilon^d \#(n^\varepsilon) = |D|, \end{aligned} \tag{2.18}$$

where $|D|$ denoted for the measure of the set D.

Remark 2.4 *This lemma ensures that $H^{\varepsilon}(\omega)$ may be almost surely partitioned into two subsets, a good and bad sets of holes which we denote by $H_g^{\varepsilon}(\omega)$ and $H_b^{\varepsilon}(\omega)$, respectively. The set $H_g^{\varepsilon}(\omega)$ is made of small balls where the construction of w^{ε} may be carried out similarly as the first chapter. The remaining holes are included in $H_b^{\varepsilon}(\omega)$ in addition, this set is well separated from $H_g^{\varepsilon}(\omega)$ and small with respect to the macroscopic size of the domain D.*

Proof. In what follows for each $z_i = \left(z_i^1, ..., z_i^d\right) \in \mathbb{Z}^d$, we denote by Q_i^{ε} the cube of length ε centered at εz_i, namely

$$Q_i^{\varepsilon} = \prod_{k=1}^{d} \left[\varepsilon z_i^k - \frac{\varepsilon}{2}, \varepsilon z_i^k + \frac{\varepsilon}{2}\right],$$

with $\prod_{k=1}^{d}$ is a cartesian product. In all what follows we use for $\mathbb{P}-$almost every event $\omega \in \Omega$ the notation $H_b^{\varepsilon}, H_g^{\varepsilon}$ and D_b^{ε} instead of $H_b^{\varepsilon}(\omega), H_g^{\varepsilon}(\omega)$ and $D_b^{\varepsilon}(\omega)$. We give the proof of this lemma in three steps.

Step 1: Construction of the sets H_b^{ε} and its "safety layer" D_b^{ε}. We denote by I_b^{ε} the set of points of $\mathbb{Z}^d \cap \frac{1}{\varepsilon}D$ which generate the set H_b^{ε} and its safety layer D_b^{ε}. We start by requiring that I_b^{ε} contains the set J_b^{ε} of points z_j where the corresponding balls $B_{\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j)$ are too large campared to the size of the cubes Q_j^{ε}. So for $\delta \in \left(0, \frac{2}{d-2}\right)$, we write

$$J_b^{\varepsilon} = \left\{z_j \in \mathbb{Z}^d \cap \frac{1}{\varepsilon}D : \varepsilon^{\frac{d}{d-2}}\rho_j \geq \varepsilon^{\delta+1}\right\} \subseteq I_b^{\varepsilon}. \tag{2.19}$$

Bad holes are not only balls with large radii, we can find a ball with small radii that has a non empty intersection with other balls with small or large radii. Namely, there exists $z_i \in (\mathbb{Z}^d \cap \frac{1}{\varepsilon}D) \backslash J_b^{\varepsilon}$ and $z_j \in J_b^{\varepsilon}$

$$B_{\varepsilon^{\frac{d}{d-2}}\rho_i}(\varepsilon z_i) \cap B_{\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j) \neq \emptyset. \tag{2.20}$$

For that reason, we can extend J_b^{ε} into the centers which might are close to $\tilde{H}_b^{\varepsilon}$, with

$$\tilde{H}_b^{\varepsilon} = \bigcup_{z_j \in J_b^{\varepsilon}} B_{2\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j)$$

and put

$$\tilde{I}_b^{\varepsilon} = \left\{z_j \in \mathbb{Z}^d : Q_j^{\varepsilon} \cap \tilde{H}_b^{\varepsilon} \neq \emptyset\right\} \supseteq J_b^{\varepsilon}, \quad I_b^{\varepsilon} = \tilde{I}_b^{\varepsilon} \cap \frac{1}{\varepsilon}D. \tag{2.21}$$

We finally set

$$H_b^{\varepsilon} = \bigcup_{z_j \in I_b^{\varepsilon}} B_{\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j), \quad D_b^{\varepsilon} = \bigcup_{z_j \in \tilde{I}_b^{\varepsilon}} Q_j^{\varepsilon}. \tag{2.22}$$

Step 2: We show (2.17). We first show that for any $\varepsilon \le \varepsilon_0(\delta)$ such that $2\varepsilon_0^\delta \le 1$

$$B_{2\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j) \subseteq D_b^\varepsilon,\ \forall z_j \in I_b^\varepsilon. \tag{2.23}$$

Indeed, if $z_j \in J_b^\varepsilon$ it follows by definition of $\tilde{H}_b^\varepsilon$ and D_b^ε that

$$B_{2\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j) \subseteq D_b^\varepsilon.$$

If $z_j \in I_b^\varepsilon \backslash J_b^\varepsilon$, we claim that

$$B_{2\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j) \subseteq Q_j^\varepsilon \subseteq D_b^\varepsilon. \tag{2.24}$$

By definition of δ, the corresponding radii satisfies

$$\varepsilon^{\frac{d}{d-2}}\rho_j \le \varepsilon^{1+\delta} < \varepsilon^{1+\frac{2}{d-2}} = \varepsilon^{\frac{d}{d-2}},$$

so it is sufficient to prove

$$B_{2\varepsilon^{\frac{d}{d-2}}}(\varepsilon z_j) \subseteq Q_j^\varepsilon.$$

We also have $2\varepsilon^{\frac{d}{d-2}} \le \varepsilon_0$, so we fix ε_0 such that $2\varepsilon^{\frac{d}{d-2}} \le \frac{\varepsilon}{2}$. Then, (2.24) is established and hence yields (2.23). Let us return to show our main result (2.17), using (2.22), (2.24) and the subadditivity of capacity we can write

$$\begin{aligned} cap(H_b^\varepsilon, D_b^\varepsilon) &= \inf\left\{\int_{D_b^\varepsilon} \nabla v\ ,\ v \in \mathcal{C}_0^1(D_b^\varepsilon)\ v1_{H_b^\varepsilon} \ge 1\right\} \\ &= \sum_{z_j \in I_b^\varepsilon} cap(B_{\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j), D_b^\varepsilon) \\ &\le \sum_{z_j \in I_b^\varepsilon} cap(B_{\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j), B_{2\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j)). \end{aligned}$$

We have

$$cap(B_{\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j), B_{2\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j)) = \int_{B_{2\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j)\backslash B_{\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j)} |\nabla u|^2,$$

with u is the solution to

$$\begin{cases} -\Delta u = 0 & \text{in } B_{2\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j) \backslash B_{\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j), \\ u = 1 & \text{in } B_{\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j), \\ u = 0 & \text{in } \mathbb{R}^d \backslash B_{\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j), \end{cases} \tag{2.25}$$

Also, u has the explicit expression:

$$u(x)=\frac{|x-\varepsilon z_j|^{-(d-2)}-\left(2\varepsilon^{\frac{d}{d-2}}\rho_j\right)^{-(d-2)}}{\left(\varepsilon^{\frac{d}{d-2}}\rho_j\right)^{-(d-2)}-\left(2\varepsilon^{\frac{d}{d-2}}\rho_j\right)^{-(d-2)}},\ \varepsilon^{\frac{d}{d-2}}\rho_j<|x-\varepsilon z_j|<2\varepsilon^{\frac{d}{d-2}}\rho_j;$$

from which we get

$$\begin{aligned}
cap(B_{\varepsilon^{\frac{d}{d-2}}\rho_j},B_{2\varepsilon^{\frac{d}{d-2}}\rho_j}) &= \int_{B_{2\varepsilon^{\frac{d}{d-2}}\rho_j}\setminus B_{\varepsilon^{\frac{d}{d-2}}\rho_j}}\sum_{j=1}^{d}\left|\partial_{x_j}w_2^{\varepsilon,i}(x)\right|^2dx\\
&= \frac{(d-2)^2}{\left(\left(\varepsilon^{\frac{d}{d-2}}\rho_j\right)^{-(d-2)}-\left(2\varepsilon^{\frac{d}{d-2}}\rho_j\right)^{-(d-2)}\right)^2}\int_{B_{2\varepsilon^{\frac{d}{d-2}}\rho_j}\setminus B_{\varepsilon^{\frac{d}{d-2}}\rho_j}}\frac{1}{|x-\varepsilon z_j|^{2(d-1)}}dx\\
&= \frac{(d-2)^2\sigma_d}{\left(\left(\varepsilon^{\frac{d}{d-2}}\rho_j\right)^{-(d-2)}-\left(2\varepsilon^{\frac{d}{d-2}}\rho_j\right)^{-(d-2)}\right)^2}\int_{\varepsilon^{\frac{d}{d-2}}\rho_j}^{2\varepsilon^{\frac{d}{d-2}}\rho_j}\frac{1}{r^{(d-1)}}dr.\\
&= \frac{(d-2)\sigma_d}{\left(\varepsilon^{\frac{d}{d-2}}\rho_j\right)^{-(d-2)}-\left(2\varepsilon^{\frac{d}{d-2}}\rho_j\right)^{-(d-2)}}=\frac{(d-2)\sigma_d}{\left(1-2^{-(d-2)}\right)}\varepsilon^d\rho_j^{d-2}
\end{aligned}$$

where σ_d is the $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$. Then

$$\begin{aligned}
cap(H_b^\varepsilon,D_b^\varepsilon) &= \sum_{z_j\in I_b^\varepsilon}cap(B_{\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j),B_{2\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j))\\
&\leq \frac{(d-2)\sigma_d}{\left(1-2^{-(d-2)}\right)}\sum_{z_j\in I_b^\varepsilon}\varepsilon^d\rho_j^{d-2}.
\end{aligned}$$

To apply lemma 2.10, we need to argue

$$\lim_{\varepsilon\to 0}\varepsilon^d\#I_b^\varepsilon=0. \tag{2.26}$$

Indeed, by (2.19) and (2.21), we have

$$\begin{aligned}
\varepsilon^d\#I_b^\varepsilon &= \varepsilon^d\#J_b^\varepsilon+\varepsilon^d\#(I_b^\varepsilon\setminus J_b^\varepsilon)=\varepsilon^d\#J_b^\varepsilon+\varepsilon^d\sum_{z_j\in I_b^\varepsilon\setminus J_b^\varepsilon}\\
&= \varepsilon^d\#J_b^\varepsilon+\sum_{z_j\in(I_b^\varepsilon\setminus J_b^\varepsilon)}\left|Q_j^\varepsilon\right|
\end{aligned}$$

since $\left|Q_j^\varepsilon\right|=\varepsilon^d$. But, for $z_j\in I_b^\varepsilon$, there exists a constant $c=c(d)>0$ and $y_j\in J_b^\varepsilon$ such that

$$Q_j^\varepsilon\subseteq B_{2c\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon y_j),$$

and it follows that

$$\begin{aligned} \varepsilon^d \# I_b^\varepsilon &\leq \varepsilon^d \# J_b^\varepsilon + \sum_{z_j \in J_b^\varepsilon} \left| B_{2c\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon y_j) \right| \\ &\leq \varepsilon^d \# J_b^\varepsilon + (2c)^d \sum_{z_j \in J_b^\varepsilon} (\varepsilon^{\frac{d}{d-2}} \rho_j)^d. \end{aligned} \tag{2.27}$$

We have

$$\begin{aligned} (\varepsilon^{\frac{d}{d-2}} \rho_j)^d &= \varepsilon^{\frac{d^2-2d+2d}{d-2}} \rho_j{}^{d-2} \rho_j{}^2 \\ &\leq (\varepsilon^{\frac{d}{d-2}} \max_{z_j \in \mathbb{Z}^d \cap \frac{1}{\varepsilon} D} \rho_j)^2 \varepsilon^d \rho_j{}^{d-2} \\ &\leq (\varepsilon^d \sum_{z_j \in \mathbb{Z}^d \cap \frac{1}{\varepsilon} D} \rho_j^{d-2})^{\frac{2}{d-2}} \varepsilon^d \rho_j{}^{d-2}. \end{aligned} \tag{2.28}$$

Since by lemma 2.9, we have

$$\lim_{\varepsilon \to 0} \varepsilon^d \sum_{z_j \in \mathbb{Z}^d \cap \frac{1}{\varepsilon} D} \rho_j^{d-2} = \left\langle \rho^{d-2} \right\rangle |D| \ \text{ almost surely.} \tag{2.29}$$

Then, it follows for ε small enough that

$$(\varepsilon^{\frac{d}{d-2}} \rho_j)^d \leq \left(\left\langle \rho^{d-2} \right\rangle |D| \right)^{\frac{2}{d-2}} \varepsilon^d \rho_j{}^{d-2}. \tag{2.30}$$

Substituting (2.30) in (2.27), it holds

$$\varepsilon^d \# I_b^\varepsilon \leq \varepsilon^d \# J_b^\varepsilon + (2c)^d \left\langle \rho^{d-2} \right\rangle^{\frac{2}{d-2}} \varepsilon^d \sum_{z_j \in J_b^\varepsilon} \rho_j{}^{d-2}. \tag{2.31}$$

If we now argue that

$$\lim_{\varepsilon \to 0} \varepsilon^d \# J_b^\varepsilon = 0, \tag{2.32}$$

then the limit (2.26) yields immediatly from lemma 2.10 applied in the right hand side of (2.31). Indeed, We have $\delta < \frac{2}{d-2}$ and $\varepsilon^{\frac{d}{d-2}} \rho_j \geq \varepsilon^{1+\delta}$, then one has $1 \leq \varepsilon^{2-\delta(d-2)} \rho_j$. It follows

$$\varepsilon^d \# J_b^\varepsilon = \varepsilon^d \sum_{z_j \in J_b^\varepsilon} \leq \varepsilon^{2-\delta(d-2)} \varepsilon^d \sum_{z_j \in \mathbb{Z}^d \cap \frac{1}{\varepsilon} D} \rho_j^{d-2}.$$

Since $2-\delta(d-2)>1$ and $\varepsilon^d \sum_{z_j\in\mathbb{Z}^d\cap\frac{1}{\varepsilon}D} \rho_j^{d-2}$ is bounded by lemma 2.9, we get

$$\lim_{\varepsilon\to 0} \varepsilon^{2-\delta(d-2)}\varepsilon^d \sum_{z_j\in\mathbb{Z}^d\cap\frac{1}{\varepsilon}D} \rho_j^{d-2}=0,$$

which implies (2.32). Therefore

$$\lim_{\varepsilon\to 0}\sum_{z_j\in I_b^\varepsilon} \varepsilon^d\rho_j^{d-2}=0,$$

thus (2.17) is established.

Step 3: Construction of H_g^ε. We define H_g^ε as follows

$$\begin{aligned} H_g^\varepsilon &= H^\varepsilon\backslash H_b^\varepsilon \\ &= \bigcup_{z_j\in n^\varepsilon} B_{\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j), \end{aligned}$$

where $n^\varepsilon=(\mathbb{Z}^d\cap\frac{1}{\varepsilon}D)\backslash I_b^\varepsilon$. Since $J_b^\varepsilon\subset I_b^\varepsilon$, then for $z_j\in n^\varepsilon$ and for the choice of $\delta\in\left(0,\frac{2}{d-2}\right)$, we have $\varepsilon^{\frac{d}{d-2}}\rho_j\le\varepsilon^{1+\delta}$. We choose $\varepsilon\le\varepsilon_0$ with ε_0 satisfies a such assumption in order to ensure that for $z_j\in n^\varepsilon$ and $z_i\in I_b^\varepsilon$, we have

$$B_{\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j)\subset Q_j^\varepsilon$$

and

$$\frac{\varepsilon}{2}\le dist(B_{\varepsilon^{\frac{d}{d-2}}\rho_j},\partial Q_i^\varepsilon),$$

which implies

$$\frac{\varepsilon}{2}\le dist(B_{\varepsilon^{\frac{d}{d-2}}\rho_j},D_b^\varepsilon).$$

Then

$$\varepsilon^{\frac{d}{d-2}}\rho_j\le\varepsilon^{1+\delta}<\frac{\varepsilon}{2}.$$

Let us now provee that

$$\lim_{\varepsilon\to 0}\varepsilon^d\#n^\varepsilon=|D|. \tag{2.33}$$

Indeed, we have by definition of n^ε

$$\lim_{\varepsilon\to 0}\varepsilon^d\#n^\varepsilon=\lim_{\varepsilon\to 0}\varepsilon^d\#(\mathbb{Z}^d\cap\frac{1}{\varepsilon}D)-\lim_{\varepsilon\to 0}\varepsilon^d\#I_b^\varepsilon.$$

By (2.26) we have $\lim_{\varepsilon\to 0}\varepsilon^d\#I_b^\varepsilon=0$. Then, by lemma 2.9 we have

$$\lim_{\varepsilon\to 0}\varepsilon^d\#n^\varepsilon=\lim_{\varepsilon\to 0}\varepsilon^d\#(\mathbb{Z}^d\cap\frac{1}{\varepsilon}D)=\left\langle\#(\mathbb{Z}^d\cap Q)\right\rangle|D|,$$

since $\langle \#(\mathbb{Z}^d \cap Q) \rangle = 1$,where Q is the unitary cube of $\mathbb{R}^d$ centered at the origin, then (2.33) yields immediatly.
■

Let us return to the construction of w^ε. We first fix δ and $\varepsilon_0(\delta)$ as in the previous lemma and we fix $\mathbb{P}$-almost every event $\omega \in \Omega$ such that we find $H_b^\varepsilon(\omega), H_g^\varepsilon(\omega)$ and $D_b^\varepsilon(\omega)$ as in lemma 2.3. We give the following proposition where the proof follows later.

Proposition 2.5 *We may set w^ε as follow*

$$w^\varepsilon(x) = w_1^\varepsilon(x) \wedge w_2^\varepsilon(x) = \min_{x\in D}(w_1^\varepsilon(x), w_2^\varepsilon(x)), \tag{2.34}$$

with $w_1, w_2 \in H^1(D)$ and such that

$$w_1^\varepsilon \equiv 1 \ \ in \ \ D\backslash D_b^\varepsilon, \qquad w_1^\varepsilon \equiv 0 \ \ in \ H_b^\varepsilon, \tag{2.35}$$

$$0 \le w_2^\varepsilon \le 1, \qquad w_2^\varepsilon \equiv 0 \ \ in \ \ D_b^\varepsilon, \quad w_2^\varepsilon \equiv 1 \ \ in \ \ H_g^\varepsilon, \tag{2.36}$$

with, in addition

$$w_1^\varepsilon \to 1 \quad strongly\ in\ H^1(D). \tag{2.37}$$

Moreover, the function w^ε satisfies the properties (H1),(H2) and (H3).

Before giving the proof of proposition 2.5, we show the following lemmas 2.6-2.7.

Lemma 2.6 *In the same setting of lemma 2.3, for every $\varepsilon \le \varepsilon_0$ there exists a function $w_1^\varepsilon \in H^1(D^\varepsilon)$ satisfies*

$$\begin{cases} w_1^\varepsilon = 1 & in \ \ D\backslash D_b^\varepsilon \\ w_1^\varepsilon = 0 & in \ H_b^\varepsilon \\ w_1^\varepsilon \to 1 & strongly\ in\ H^1(D). \end{cases} \tag{2.38}$$

Proof. By the result (2.17) and definition of capacity we can define a function $\tilde{w}_1^\varepsilon \in H_0^1(D_b^\varepsilon)$ which satisfies

$$\begin{cases} -\Delta \tilde{w}_1^\varepsilon = 0 & \text{in} \ \ D_b^\varepsilon\backslash H_b^\varepsilon, \\ \tilde{w}_1^\varepsilon = 1 & \text{in} \ H_b^\varepsilon, \\ \tilde{w}_1^\varepsilon = 0 & \text{in} \ D\backslash D_b^\varepsilon \end{cases} \tag{2.39}$$

where

$$cap(H_b^\varepsilon, D_b^\varepsilon) = \int_{D_b^\varepsilon} |\nabla \tilde{w}_1^\varepsilon|^2 .$$

So let

$$w_1^\varepsilon = \begin{cases} 1 - \tilde{w}_1^\varepsilon & \text{in} \ D_b^\varepsilon, \\ 1 & \text{in} \ D\backslash D_b^\varepsilon. \end{cases}$$

Since $\tilde{w}_1^\varepsilon \in H_0^1(D_b^\varepsilon)$ we see that $w_1^\varepsilon|_{D_b^\varepsilon} \in H^1(D_b^\varepsilon)$ and $w_1^\varepsilon|_{D\backslash D_b^\varepsilon} \in H^1(D\backslash D_b^\varepsilon)$. In the other hand, we have by (2.39) $\tilde{w}_1^\varepsilon = 0$ in ∂D_b^ε,so $w_1^\varepsilon = 1$ in ∂D_b^ε. We also have $w_1^\varepsilon = 1$ in $D\backslash D_b^\varepsilon$. Then, w_1^ε is continuous in D, so that by proposition A.1.12 we get $w_1^\varepsilon \in H^1(D)$. We also have $\tilde{w}_1^\varepsilon = 1$ in H_b^ε, then $w_1^\varepsilon = 0$ in H_b^ε. Thus w_1^ε satisfies the two first properties of (2.38). Let us check the last property (2.38). The Poincaré's inequality gives

$$\|1 - w_1^\varepsilon\|^2_{H^1(D)} = \|\tilde{w}_1^\varepsilon\|^2_{H_0^1(D_b^\varepsilon)} \leq \alpha \|\nabla \tilde{w}_1^\varepsilon\|^2_{\left(L^2(D_b^\varepsilon)\right)^d} = \alpha cap(H_b^\varepsilon, D_b^\varepsilon)$$

where $\alpha > 0$ is a positive constant depend only on the measure of D_b^ε. We apply the result (2.17) of lemma 2.3 in (**??**), yields $w_1^\varepsilon \to 1$ strongly in $H^1(D)$. ■

We now give in the following result the construction of w_2^ε.

Lemma 2.7 *Under the hypotheses of Lemma 2.3, there exists a function $w_2^\varepsilon \in H^1(D^\varepsilon)$ such that*

$$0 \leq w_2^\varepsilon \leq 1, \qquad w_2^\varepsilon = 1 \text{ in } D_b^\varepsilon, \qquad w_2^\varepsilon = 0 \text{ in } H_g^\varepsilon. \tag{2.40}$$

Furthermore, w_2^ε satisfies the properties (P2) and (P3) of Lemma 1.2.

Proof. First, we put

$$w_2^\varepsilon = 1 \text{ on } D_b^\varepsilon.$$

For the definition of w_2^ε in $D\backslash D_b^\varepsilon$ which contains only the holes H_b^ε of disjoint balls, each striclty contained in the concentric cube Q_i^ε of size ε, we construct w_2^ε explicitly as done in the first chapter. For each $z_i \in n^\varepsilon$ with $n^\varepsilon = (\mathbb{Z}^d \cap \frac{1}{\varepsilon}D)\backslash I_b^\varepsilon$, we write

$$T_i^\varepsilon = B_{\varepsilon^{\frac{d}{d-2}}\rho_i}(\varepsilon z_i), \qquad B_i^\varepsilon = B_{\frac{\varepsilon}{2}}(\varepsilon z_i),$$

we define also

$$w_2^\varepsilon = 1 - \sum_{z_i \in n^\varepsilon} w_2^{\varepsilon,i}, \tag{2.41}$$

where each $w_2^{\varepsilon,i}$ is a solution of the following problem

$$\begin{cases} -\Delta w_2^{\varepsilon,i} = 0 & \text{in } B_i^\varepsilon \backslash T_i^\varepsilon, \\ w_2^{\varepsilon,i} = 1 & \text{in } T_i^\varepsilon, \\ w_2^{\varepsilon,i} = 0 & \text{in } D\backslash B_i^\varepsilon. \end{cases} \tag{2.42}$$

We can easily compute $w_2^{\varepsilon,i}$ in polar coordinates in the annulus $B_i^\varepsilon \backslash T_i^\varepsilon$ as done in lemma 1.2 taking $\varepsilon^{\frac{d}{d-2}}\rho_i < r = |x - \varepsilon z_i| < \frac{\varepsilon}{2}$ for $x \in \mathbb{R}^d$ and εz_i the center of T_i^ε, we get

$$\begin{cases} w_2^{\varepsilon,i}(x) = \frac{|x-\varepsilon z_i|^{-(d-2)} - \left(\frac{\varepsilon}{2}\right)^{-(d-2)}}{\varepsilon^{-d}\rho_i^{-(d-2)} - \left(\frac{\varepsilon}{2}\right)^{-(d-2)}} & \text{in } B_i^\varepsilon \backslash T_i^\varepsilon, \\ w_2^{\varepsilon,i} = 1 & \text{in } T_i^\varepsilon, \\ w_2^{\varepsilon,i} = 0 & \text{in } D \backslash B_i^\varepsilon. \end{cases} \tag{2.43}$$

Now, we show that w_2^ε satisfies the properties (2.40). Using the maximum principle (See proposition A.2.5), we get

$$0 \leq w_2^{\varepsilon,i}(x) \leq 1.$$

Since $w_2^{\varepsilon,i}$ has a disjoint supports then one has

$$0 \leq w_2^\varepsilon = 1 - \sum_{z_i \in n^\varepsilon} w_2^{\varepsilon,i} \leq 1.$$

Since $w_2^{\varepsilon,i} = 1$ in T_i^ε and $w_2^{\varepsilon,i}$ has disjoint supports, we obtain

$$w_2^\varepsilon = 1 - \sum_{z_j \in n^\varepsilon} w_2^{\varepsilon,j} = 1 - w_2^{\varepsilon,i} = 0 \text{ in all } T_i^\varepsilon.$$

The function w_2^ε belongs to $H_0^1(D)$. Indeed, by definition of $w_2^{\varepsilon,i}$ in (2.43) we can observe that the functions $w_2^{\varepsilon,i}$are continuous and H_0^1 by parts, hence by proposition A.1.12 the functions $w_2^{\varepsilon,i}$ belongs to $H_0^1(B_i^\varepsilon)$ for each $z_i \in n^\varepsilon$. Since the functions $w_2^{\varepsilon,i}$ has essentially disjoint supports, then

$$\sum_{z_i \in n^\varepsilon} w_2^{\varepsilon,i} = w_2^{\varepsilon,j} \quad \text{in } B_j^\varepsilon. \tag{2.44}$$

we also have for every $z_i \in n^\varepsilon$,

$$w_2^{\varepsilon,i} = 0 \quad \text{in} \quad \partial B_i^\varepsilon. \tag{2.45}$$

Then using again the proposition A.1.12 (See appendix A), we can conclude that $\sum_{z_i \in n^\varepsilon} w_2^{\varepsilon,i} \in H_0^1(D \backslash D_b^\varepsilon)$. Extending $\sum_{z_i \in n^\varepsilon} w_2^{\varepsilon,i}$ by 0 in D_b^ε we get

$$\sum_{z_i \in n^\varepsilon} w_2^{\varepsilon,i} \in H_0^1(D).$$

Finally, we find

$$w_2^\varepsilon = 1 - \sum_{z_i \in n^\varepsilon} w_2^{\varepsilon,i} \in H^1(D), \qquad w_2^\varepsilon = 1 \quad \text{in } D_b^\varepsilon.$$

Therefore, the function w_2^ε satisfies the property (2.40). Let us now show that w_2^ε satisfies the properties (H2), to do that we follow the same steps as in the periodic case. We have

$$\begin{aligned}\|\nabla w_2^\varepsilon\|^2_{(L^2(D))^d} &= \sum_{z_i\in n^\varepsilon}\int_{B_i^\varepsilon}\sum_{j=1}^{d}\left|\partial_{x_j} w_2^{\varepsilon,i}(x)\right|^2 dx\\ &= \sum_{z_i\in n^\varepsilon}\frac{(d-2)\sigma_d}{\left(\varepsilon^{-d}\rho_i^{-(d-2)}\right)-\left(\frac{\varepsilon}{2}\right)^{-(d-2)}}\\ &= \sum_{z_i\in n^\varepsilon}\frac{(d-2)\sigma_d\varepsilon^d\rho_i^{(d-2)}}{1-\left(\frac{\varepsilon}{2}\right)^{-(d-2)}\varepsilon^d\rho_i^{(d-2)}}\\ &\leq \beta(d)\sum_{z_i\in\mathbb{Z}^d\cap\frac{1}{\varepsilon}D}\varepsilon^d\rho_i^{(d-2)},\end{aligned}\tag{2.46}$$

where $\beta(d)>0$ is a strictly positive constant. By Lemma 2.9 applied on the right hand side of the last inequality of (2.46), we have almost surely

$$\limsup_{\varepsilon\to 0}\|\nabla w_2^\varepsilon\|^2_{(L^2(D))^d}\leq\beta(d)\left\langle\rho^{d-2}\right\rangle|D|.\tag{2.47}$$

Since $1-w_2^\varepsilon\in H_0^1(D\backslash D_b^\varepsilon)$ and $1-w_2^\varepsilon=0$ in D_b^ε, we can apply Poincaré's inequality: one has for ε small enough,

$$\|1-w_2^\varepsilon\|^2_{H_0^1(D)}\leq C_\varepsilon^2\beta(d)\,\|\nabla w_2^\varepsilon\|^2_{L^2(D)}\leq C_\varepsilon^2\beta(d)\left\langle\rho^{d-2}\right\rangle|D|,$$

where $C_\varepsilon>0$ is the Poincaré's constant, since B_i^ε is of diameter $n=\frac{\varepsilon}{2}<\varepsilon$ an estimation of Poincaré's constants, one has

$$C_\varepsilon\leq\frac{\varepsilon}{2}<\varepsilon,$$

then

$$\|1-w_2^\varepsilon\|^2_{L^2(D)}\leq\varepsilon^2\beta(d)\left\langle\rho^{d-2}\right\rangle|D|.$$

Sending ε to 0, one has

$$w_2^\varepsilon\to 1 \text{ strongly in } L^2(D).$$

This latter result implies that $1-w_2^\varepsilon$ is bounded in $L^2(D)$, we have also by (2.47) $\nabla(1-w_2^\varepsilon)$ is bounded in $L^2(D)$, thus $1-w_2^\varepsilon$ is bounded in $H_0^1(D)$. Using Eberlein-S̃muljan theorem one has up to a subsequence

$$w_2^\varepsilon\rightharpoonup 1 \qquad \text{weakly in } H^1(D)\tag{2.48}$$

and thus (H2) established for w_2^ε. We now argue that w_2^ε satisfies the property (H3), to do that the first step is to decompose $-\Delta w_2^\varepsilon$ as done in the first chapter, we get:

$$-\Delta w_2^\varepsilon = \mu^\varepsilon - \gamma^\varepsilon, \tag{2.49}$$

where

$$\mu^\varepsilon = \sum_{z_i \in n^\varepsilon} \frac{\partial w_2^\varepsilon}{\partial \upsilon_{ext}}\bigg|_{\partial B_i^\varepsilon} \delta^\varepsilon_{\partial B_i^\varepsilon},\ \gamma^\varepsilon = \sum_{z_i \in n^\varepsilon} \frac{\partial w_2^\varepsilon}{\partial \upsilon_{ext}}\bigg|_{\partial T_i^\varepsilon} \delta^\varepsilon_{\partial T_i^\varepsilon},$$

where υ_{ext} is the outward unit normal of ∂B_i^ε. Next, we prove that we need only to argue for v_ε and v defined as in (H3) the following result

$$\langle \mu^\varepsilon, v^\varepsilon \rangle_{H^{-1}(D), H_0^1(D)} \to C_0 \int_D v. \tag{2.50}$$

For $\varphi \in \mathcal{D}(D)$, we have by Green formula

$$\begin{aligned}
\langle -\Delta w_2^\varepsilon, \varphi \rangle_{\mathcal{D}'(D),\mathcal{D}(D)} &= \sum_{z_i \in n^\varepsilon} \int_{B_i^\varepsilon} \nabla w_2^\varepsilon \nabla \varphi = \sum_{z_i \in n^\varepsilon} \int_{B_i^\varepsilon \setminus T_i^\varepsilon} \nabla w_2^\varepsilon \nabla \varphi \\
&= \sum_{z_i \in n^\varepsilon} \langle -\Delta w_2^\varepsilon, \varphi \rangle_{\mathcal{D}'(B_i^\varepsilon \setminus T_i^\varepsilon), \mathcal{D}(B_i^\varepsilon \setminus T_i^\varepsilon)} \\
&\quad + \sum_{z_i \in n^\varepsilon} \int_{\partial B_i^\varepsilon} \varphi \nabla w_2^\varepsilon \cdot \upsilon_{ext} ds + \sum_{z_i \in n^\varepsilon} \int_{\partial T_i^\varepsilon} \varphi \nabla w_2^\varepsilon \cdot n_{ext} ds,
\end{aligned}$$

where n_{ext} is the outward unit normal of ∂T_i^ε. Since we have $-\Delta w_2^\varepsilon = 0$ in $B_i^\varepsilon \setminus T_i^\varepsilon$, it follows

$$\langle -\Delta w_2^\varepsilon, \varphi \rangle_{\mathcal{D}'(D),\mathcal{D}(D)} = \sum_{z_i \in n^\varepsilon} \int_{\partial B_i^\varepsilon} \frac{\partial w_2^\varepsilon}{\partial \upsilon_{ext}} \delta^\varepsilon_{\partial B_i^\varepsilon} \varphi ds + \sum_{z_i \in n^\varepsilon} \int_{\partial T_i^\varepsilon} \frac{\partial w_2^\varepsilon}{\partial n_{ext}} \delta^\varepsilon_{\partial T_i^\varepsilon} \varphi ds.$$

Since we have $n_{ext} = -\upsilon_{ext}$, we get (2.49) immediatly. We return now to the proof of

$$\langle -\Delta w_2^\varepsilon, v^\varepsilon \rangle_{H^{-1}(D), H_0^1(D)} \to C_0 \int_D v, \tag{2.51}$$

for v^ε and C_0 given as in lemma 1.2, since $v^\varepsilon = 0$ in all T_i^ε then to get (2.51) we need only to prove (2.50).

The second step is to arguing that it suffices to prove (2.51) for truncated process $\left(\mathbb{Z}^d, \mathcal{R}^M\right)$ with $M \in \mathbb{N}$ and $\mathcal{R}^M$ as defined above (2.15). In what follow, we denote by $w_{2,M}^\varepsilon$ and μ_M^ε introduced as the analogues of w_2^ε and μ^ε for the truncated marks, we denote also $C_{0,M} = (d-2)\sigma_d \left\langle \rho^{d-2} \mathbf{1}_{\rho \leq M} \right\rangle$. We have also

$$|C_0 - C_{0,M}| = \left| (d-2)\sigma_d \left\langle \rho^{d-2} - \rho^{d-2} \mathbf{1}_{\rho \leq M} \right\rangle \right| = (d-2)\sigma_d \left\langle \rho^{d-2} \mathbf{1}_{\rho \geq M} \right\rangle.$$

Then, we have

$$\left|\langle -\Delta w_2^\varepsilon, v^\varepsilon\rangle_{H^{-1}(D),H_0^1(D)} - C_0\int_D v\right| = \left|\int_D \nabla w_2^\varepsilon \nabla v^\varepsilon + \int_D \nabla w_{2,M}^\varepsilon \nabla v^\varepsilon - \int_D \nabla w_{2,M}^\varepsilon \nabla v^\varepsilon - C_0\int_D v\right|.$$

Using Green's formula and (**??**), it holds

$$\begin{aligned}\left|\langle -\Delta w_2^\varepsilon, v^\varepsilon\rangle_{H^{-1}(D),H_0^1(D)} - C_0\int_D v\right| &\leq \left|\int_D \nabla(w_2^\varepsilon - w_{2,M}^\varepsilon)\nabla v^\varepsilon\right| \\ &\quad + \left|\int_D \nabla w_{2,M}^\varepsilon \nabla v^\varepsilon - (d-2)\sigma_d\left\langle \rho^{d-2}\mathbf{1}_{\rho\geq M}\right\rangle \int_D v - C_{0,M}\int_D v\right| \\ &\leq \left|\int_D \nabla(w_2^\varepsilon - w_{2,M}^\varepsilon)\nabla v^\varepsilon\right| \\ &\quad + \left|\langle -\Delta w_{2,M}^\varepsilon, v^\varepsilon\rangle_{H^{-1}(D),H_0^1(D)} - C_{0,M}\int_D v\right| \\ &\quad + (d-2)\sigma_d\left\langle \rho^{d-2}\mathbf{1}_{\rho\geq M}\right\rangle \int_D |v|\,.\end{aligned}$$

Then, using Cauchy-Schwartz inequality we get

$$\begin{aligned}\left|\langle -\Delta w_2^\varepsilon, v^\varepsilon\rangle_{H^{-1}(D),H_0^1(D)} - C_0\int_D v\right| &\leq \left\|\nabla(w_2^\varepsilon - w_{2,M}^\varepsilon)\right\|_{L^2(D)} \|\nabla v^\varepsilon\|_{(L^2(D))^d} \\ &\quad + \left|\langle \mu_M^\varepsilon, v^\varepsilon\rangle_{H^{-1}(D),H_0^1(D)} - C_{0,M}\int_D v\right| \\ &\quad + (d-2)\sigma_d\left\langle \rho^{d-2}\mathbf{1}_{\rho\geq M}\right\rangle \|v\|_{L^1(D)}\,.\end{aligned}$$

We have also same as (2.46)

$$\begin{aligned}\left\|\nabla(w_2^\varepsilon - w_{2,M}^\varepsilon)\right\|_{(L^2(D))^d} &= \sum_{z_i\in n^\varepsilon}\int_{B_i^\varepsilon}\left|(\nabla w_2^{\varepsilon,i} - \nabla w_{2,M}^{\varepsilon,i})(x)\right|^2 dx \\ &= \sum_{z_i\in n^\varepsilon}\frac{(d-2)^2}{\left(\left(\varepsilon^{-d}\rho_i^{-(d-2)}\mathbf{1}_{\rho_i\geq M}\right) - \left(\frac{\varepsilon}{2}\right)^{-(d-2)}\right)^2}\int_{B_i^\varepsilon}\frac{1}{|x-\varepsilon z_i|^{2(d-1)}}dx \\ &\leq \beta(d)\sum_{z_i\in\mathbb{Z}^d\cap\frac{1}{\varepsilon}D}\varepsilon^d\rho_i^{d-2}\mathbf{1}_{\rho_i\geq M},\end{aligned}$$

for a positive constant $\beta(d) > 0$ which depend only on d. Thanks to lemma 2.9, one has

$$\lim_{\varepsilon\to 0}\sup\left\|\nabla(w_2^\varepsilon - w_{2,M}^\varepsilon)\right\|_{(L^2(D))^d} \leq \beta(d)\left\langle \rho^{d-2}\mathbf{1}_{\rho\geq M}\right\rangle |D|\,.$$

Since $v^\varepsilon \rightharpoonup v$ in $H_0^1(D)$, then there exist a constant $C > 0$ such that

$$\|\nabla v^\varepsilon\|_{(L^2(D))^d} \leq C.$$

Then, one has

$$\begin{aligned}\limsup_{\varepsilon\to 0}\left|\langle -\Delta w^{\varepsilon}, v^{\varepsilon}\rangle_{H^{-1}(D),H_0^1(D)} - C_0\int_D v\right| &\leq \limsup_{\varepsilon\to 0}\left|\langle \mu_M^{\varepsilon}, v^{\varepsilon}\rangle_{H^{-1}(D),H_0^1(D)} - C_{0,M}\int_D v\right| \\ &\quad + \left\langle \rho^{d-2}\mathbf{1}_{\rho\geq M}\right\rangle (d-2)\sigma_d \left\|v\right\|_{L^1(D)} \\ &\quad + C'\left\langle \rho^{d-2}\mathbf{1}_{\rho\geq M}\right\rangle,\end{aligned} \tag{2.52}$$

where $C'>0$ is a striclty positive constant depend only on d. Since v^{ε} is bounded in $L^2(D)$ (v^{ε} converges weakly in $H_0^1(D)$ then v^{ε} is bounded in $H_0^1(D)$, hence v^{ε} is bounded in $L^2(D)$) using the embedding of $L^2(D)$ in $L^1(D)$ we can conclude that

$$\left\|v\right\|_{L^1(D)} < +\infty.$$

Sending $M\uparrow +\infty$, this latter result and the assumption (2.15) allows to

$$\left\langle \rho^{d-2}\mathbf{1}_{\rho\geq M}\right\rangle \left((d-2)\sigma_d \left\|v\right\|_{L^1(D)} + C'\right) \to 0.$$

So we need only to prove (2.50) for truncated process $\left(\mathbb{Z}^d, \mathcal{R}^M\right)$. The third step which is the last, is to prove for any fixed $M\in\mathbb{N}$ that we have

$$\limsup_{\varepsilon\to 0}\left|\langle \mu_M^{\varepsilon}, v^{\varepsilon}\rangle_{H^{-1}(D),H_0^1(D)} - C_{0,M}\int_D v\right| = 0. \tag{2.53}$$

Indeed, we have

$$\begin{aligned}\mu_M^{\varepsilon} &= \sum_{z_i\in n^{\varepsilon}} \left.\frac{\partial w_2^{\varepsilon}}{\partial \upsilon_{ext}}\right|_{\partial B_i^{\varepsilon}} \delta^{\varepsilon}_{\partial B_i^{\varepsilon}} \\ &= \sum_{z_i\in n^{\varepsilon}}\sum_{k=1}^{d} \frac{d-2}{\left(\varepsilon^{-d}\rho^{-(d-2)}\right) - \left(\frac{\varepsilon}{2}\right)^{-(d-2)}} \left.\frac{x^k - \varepsilon z_i^k}{\left|x-\varepsilon z_i\right|^{d-1}}\right|_{\partial B_i^{\varepsilon}} \upsilon_{ext}^k \delta^{\varepsilon}_{\partial B_i^{\varepsilon}}.\end{aligned}$$

Taking $\upsilon_{ext} = \left(\upsilon_{ext}^1, .., \upsilon_{ext}^d\right) = \sum_{k=1}^{d} e_k$ where $(e_1, .., e_d)$ is canonical basis of $\mathbb{R}^d$, then one has

$$\begin{aligned}\mu^{\varepsilon} &= \sum_{z_i\in n^{\varepsilon}} \frac{(d-2)\left(\frac{\varepsilon}{2}\right)^{-(d-1)}}{\left(\varepsilon^{-d}\rho_{i,M}^{-(d-2)}\right) - \left(\frac{\varepsilon}{2}\right)^{-(d-2)}} \delta^{\varepsilon}_{\partial B_i^{\varepsilon}} \\ &= \sum_{z_i\in n^{\varepsilon}} \frac{2^{d-1}(d-2)\left(\rho_{i,M}\right)^{-(d-2)}}{1 - 2^{d-2}\varepsilon^2\left(\rho_{i,M}\right)^{(d-2)}} \varepsilon \delta^{\varepsilon}_{\partial B_i^{\varepsilon}}.\end{aligned}$$

Since $\rho_{i,M} \leq M$, it suffices to prove

$$\tilde{\mu}_M^\varepsilon = \sum_{z_i \in n^\varepsilon} 2^{d-1}(d-2)\left(\rho_{i,M}\right)^{-(d-2)} \varepsilon \delta_{\partial B_i^\varepsilon}^\varepsilon \longrightarrow C_{0,M} \quad \text{strongly in } W^{-1,\infty}(D). \tag{2.54}$$

To show (2.54), we argue for a fixed $M \in \mathbb{N}$ that

$$\tilde{\mu}_M^\varepsilon - \eta_M^\varepsilon \to 0 \text{ strongly in } W^{-1,\infty}(D), \tag{2.55}$$

with

$$\eta_M^\varepsilon = \sum_{z_i \in \mathbb{Z}^d \cap \frac{1}{\varepsilon} D} 2^d (d-2) d \rho_{M,i}^{d-2} \mathbf{1}_{B_i^\varepsilon}$$

and that

$$\eta_M^\varepsilon \to C_{0,M} \;\; \text{strongly in } W^{-1,\infty}(D). \tag{2.56}$$

Let us begin by proving (2.55). We define the following auxiliary problems

$$\begin{cases} -\Delta q_{i,M}^\varepsilon = 2^d(d-2)d\rho_{i,M}^{d-2} & \text{in } B_i^\varepsilon, \\ \frac{\partial q_{i,M}^\varepsilon}{\partial v_{ext}} = 2^{d-1}(d-2)\rho_{i,M}^{d-2}\varepsilon & \text{on } \partial B_i^\varepsilon, \end{cases} \tag{2.57}$$

we have

$$q_{i,M}^\varepsilon = 2^{d-1}(d-2)\rho_{i,M}^{d-2}\left(|x - \varepsilon z_i|^2 - \left(\frac{\varepsilon}{2}\right)^2\right).$$

Indeed, since we have for $0 < r = |x - \varepsilon z_i| < \frac{\varepsilon}{2}, x \in \mathbb{R}^d$

$$\frac{1}{r^{d-1}} \partial_r \left(r^{d-1} \partial_r q_{i,M}^\varepsilon(r)\right) = 2^d(d-2)d\rho_{i,M}^{d-2},$$

so we integrate over $[0, r]$ for a variable s we get

$$q_{i,M}^\varepsilon(r) = 2^{d-1}(d-2)\rho_{i,M}^{d-2} r^2 + c,$$

where $c \in \mathbb{R}$. In particular for $q^\varepsilon(\frac{\varepsilon}{2}) = 0$, one has

$$q_{i,M}^\varepsilon(r) = 2^{d-1}(d-2)\rho_{i,M}^{d-2}(r^2 - \left(\frac{\varepsilon}{2}\right)^2),$$

so for $0 < r = |x - \varepsilon z_i| < \frac{\varepsilon}{2}$, we have

$$q_{i,M}^\varepsilon(r) = 2^{d-1}(d-2)\rho_{i,M}^{d-2}(|x - \varepsilon z_i|^2 - \left(\frac{\varepsilon}{2}\right)^2).$$

We have

$$\partial_{x_k} q_{i,M}^{\varepsilon}(x) = 2^d (d-2) \rho_{i,M}^{d-2} \left(x^k - \varepsilon z_i^k \right).$$

So

$$\begin{aligned} \left\| \nabla q_{i,M}^{\varepsilon}(x) \right\|_{L^\infty(B_i^\varepsilon)} &= \sup_{x \in B_i^\varepsilon} \sum_{k=1}^{d} \left| \partial_{x_k} q_{i,M}^{\varepsilon}(x) \right| = \sup_{x \in B_i^\varepsilon} \sum_{k=1}^{d} \left| 2^d (d-2) \rho_{i,M}^{d-2} \left(x^k - \varepsilon z_i^k \right) \right| \\ &\leq 2^{d-1} (d-2) \rho_{i,M}^{d-2} \varepsilon. \end{aligned}$$

Since $\rho_{i,M}^{d-2} \leq M$, one has

$$\left\| \nabla q_{i,M}^{\varepsilon} \right\|_{\left(L^\infty(B_i^\varepsilon)\right)^d} \leq 2^{d-1} (d-2) M \varepsilon.$$

Then

$$\nabla q_{i,M}^{\varepsilon} \to 0 \ \text{ strongly in } \ \left(L^\infty(B_i^\varepsilon)\right)^d. \tag{2.58}$$

In the other hand, since $q_{i,M}^{\varepsilon}(x) = 0$ in ∂B_i^ε, we may extend $q_{i,M}^{\varepsilon}$ by 0 outside B_i^ε then we can use the Poincaré's inequality, we obtain

$$\left\| q_{i,M}^{\varepsilon} \right\|_{L^\infty(B_i^\varepsilon)} \leq K \left\| \nabla q_{i,M}^{\varepsilon}(x) \right\|_{\left(L^\infty(B_i^\varepsilon)\right)^d},$$

and conclude that

$$q_{i,M}^{\varepsilon} \to 0 \ \text{ strongly in } \ L^\infty(B_i^\varepsilon), \tag{2.59}$$

by (2.58) and (2.59), one has

$$q_M^{\varepsilon} = \sum_{z_i \in \mathbb{Z}^d \cap \frac{1}{\varepsilon} D} q_{i,M}^{\varepsilon} \to 0 \ \text{ strongly in } \ W^{1,\infty}(\mathbb{R}^d). \tag{2.60}$$

For $\varphi \in \mathcal{D}(D)$, we have

$$\begin{aligned} \langle \eta_M^\varepsilon - \tilde{\mu}_M^\varepsilon, \varphi \rangle_{\mathcal{D}'(D),\mathcal{D}(D)} &= \sum_{z_i \in \mathbb{Z}^d \cap \frac{1}{\varepsilon} D} \int_{B_i^\varepsilon} 2^d (d-2) d \rho_{i,M}^{d-2} \varphi(x) dx \\ &\quad - \sum_{z_i \in n^\varepsilon} \int_{\partial B_i^\varepsilon} 2^{d-1} (d-2) \rho_{i,M}^{d-2} \varepsilon \varphi(x) ds \\ &= \sum_{z_i \in \left(\mathbb{Z}^d \cap \frac{1}{\varepsilon} D\right) \setminus n^\varepsilon} \int_{B_i^\varepsilon} 2^d (d-2) d \rho_{i,M}^{d-2} \varphi(x) dx \\ &\quad + \sum_{z_i \in n^\varepsilon} \Big(\int_{B_i^\varepsilon} 2^d (d-2) d \rho_{i,M}^{d-2} \varphi(x) dx \\ &\quad - \int_{\partial B_i^\varepsilon} 2^{d-1} (d-2) \rho_{i,M}^{d-2} \varepsilon \varphi(x) ds \Big). \end{aligned}$$

Using (2.57), we obtain

$$\begin{aligned}\langle\eta_M^\varepsilon-\tilde\mu_M^\varepsilon,\varphi\rangle_{\mathcal{D}'(D),\mathcal{D}(D)} &= \sum_{z_i\in\left(\mathbb{Z}^d\cap\frac{1}{\varepsilon}D\right)\setminus n^\varepsilon}\int_{B_i^\varepsilon}2^d(d-2)d\rho_{i,M}^{d-2}\varphi(x)dx\\ &\quad+\sum_{z_i\in n^\varepsilon}\left\langle-\Delta q_{i,M}^\varepsilon,\varphi\right\rangle_{\mathcal{D}'(B_i^\varepsilon),\mathcal{D}(B_i^\varepsilon)}-\left\langle\frac{\partial q_{i,M}^\varepsilon}{\partial\upsilon_{ext}},\varphi\right\rangle_{\mathcal{D}'(B_i^\varepsilon),\mathcal{D}(B_i^\varepsilon)}.\end{aligned}$$

Using Green formula, one has

$$\begin{aligned}\langle\eta_M^\varepsilon-\tilde\mu_M^\varepsilon,\varphi\rangle_{\mathcal{D}'(D),\mathcal{D}(D)} &= \sum_{z_i\in\left(\mathbb{Z}^d\cap\frac{1}{\varepsilon}D\right)\setminus n^\varepsilon}\int_{B_i^\varepsilon}2^d(d-2)d\rho_{i,M}^{d-2}\varphi(x)dx\\ &\quad+\sum_{z_i\in\mathbb{Z}^d\cap\frac{1}{\varepsilon}D}\int_{B_i^\varepsilon}\nabla q_{i,M}^\varepsilon(x)\nabla\varphi(x)dx,\\ &= \sum_{z_i\in\left(\mathbb{Z}^d\cap\frac{1}{\varepsilon}D\right)\setminus n^\varepsilon}\int_{B_i^\varepsilon}2^d(d-2)d\rho_{i,M}^{d-2}\varphi(x)dx\\ &\quad+\int_D\nabla q_M^\varepsilon(x)\nabla\varphi(x)dx.\end{aligned}$$

Since $\varphi=0$ in ∂D, it follows

$$\langle\eta_M^\varepsilon-\tilde\mu_M^\varepsilon,\varphi\rangle_{\mathcal{D}'(D),\mathcal{D}(D)}=\langle-\Delta q_M^\varepsilon,\varphi\rangle_{\mathcal{D}'(D),\mathcal{D}(D)}+\langle R_M^\varepsilon,\varphi\rangle_{\mathcal{D}'(D),\mathcal{D}(D)},$$

with

$$\langle R_M^\varepsilon,\varphi\rangle_{\mathcal{D}'(D),\mathcal{D}(D)}=\sum_{z_i\in\left(\mathbb{Z}^d\cap\frac{1}{\varepsilon}D\right)\setminus n^\varepsilon}\int_{B_i^\varepsilon}2^d(d-2)d\rho_{i,M}^{d-2}\varphi(x)dx.$$

Therefore

$$\eta_M^\varepsilon-\tilde\mu_M^\varepsilon=-\Delta q_M^\varepsilon+R_M^\varepsilon\quad\text{in}\quad\mathcal{D}'(D),$$

and more precisely in $W^{-1,\infty}(D)$ (This latter is concluded from the caracterization of $W^{-1,\infty}(D)$ with $\varphi_0=R_M^\varepsilon$ and $\varphi_i=\frac{\partial q_M^\varepsilon}{\partial x_i}$)

We have by (2.60)

$$\begin{aligned}\left|\langle-\Delta q_M^\varepsilon,\varphi\rangle_{W^{-1,\infty}(D),W_0^{1,1}(D)}\right| &= \int_D|\nabla q_M^\varepsilon\nabla\varphi|\\ &= \sum_{z_i\in\mathbb{Z}^d\cap\frac{1}{\varepsilon}D}\int_{B_i^\varepsilon}\left|\nabla q_{i,M}^\varepsilon\nabla\varphi\right|\\ &\le \sum_{z_i\in\mathbb{Z}^d\cap\frac{1}{\varepsilon}D}\left\|q_{i,M}^\varepsilon\right\|_{W^{1,\infty}(D)}\|\varphi\|_{W_0^{1,1}(D)}\to 0.\end{aligned}$$

To prove (2.55), it suffices to show that $R_M^\varepsilon \overset{*}{\rightharpoonup} 0$ weakly-* in $L^\infty(D)$. Indeed, Since R_M^ε is bounded, we need only to test R_M^ε with $\varphi \in \mathcal{C}_c^1(D)$ ($\mathcal{C}_c^1(D)$ is the space of continuously differentiable functions with compact support)(This is concluded from Hahn-Banach theorem applied to the continous linear form $T(\varphi) = \int_D R_M^\varepsilon \varphi$ defined for every $\varphi \in \mathcal{C}_c^1(D)$ which is dense in $L^1(D)$). Then we have

$$\begin{aligned} \left|(R_M^\varepsilon, \varphi)_{L^\infty(D),L^1(D)}\right| &= \sum_{z_i \in \left(\mathbb{Z}^d \cap \frac{1}{\varepsilon}D\right) \setminus n^\varepsilon} \left| \int_{B_i^\varepsilon} 2^d(d-2)d\rho_{i,M}^{d-2}\varphi \right| \\ &\leq 2^d(d-2)d \sum_{z_i \in \left(\mathbb{Z}^d \cap \frac{1}{\varepsilon}D\right) \setminus n^\varepsilon} \rho_{i,M}^{d-2} \int_{B_i^\varepsilon} |\varphi| \,. \end{aligned}$$

Since φ is bounded in $L^\infty(D)$ and $B_i^\varepsilon \subset B_\varepsilon$ with B_ε is a ball with radius ε, then it follows by Hölder's inequality

$$\left|(R_M^\varepsilon, \varphi)_{L^\infty(D),L^1(D)}\right| \leq 2^d(d-2)d \left\|\varphi\right\|_{L^\infty(D)} \varepsilon^d \sum_{z_i \in \left(\mathbb{Z}^d \cap \frac{1}{\varepsilon}D\right) \setminus n^\varepsilon} \rho_{i,M}^{d-2},$$

To apply lemma 2.10, we remark by (2.18) of lemma 2.3 and (2.120) of lemma 2.9 that we have

$$\lim_{\varepsilon \to 0} \varepsilon^d \#(\left(\mathbb{Z}^d \cap \frac{1}{\varepsilon}D\right) \setminus n^\varepsilon) = \lim_{\varepsilon \to 0} \varepsilon^d \# \left(\mathbb{Z}^d \cap \frac{1}{\varepsilon}D\right) - \lim_{\varepsilon \to 0} \varepsilon^d \# n^\varepsilon = 0$$

and we can conclude $R_M^\varepsilon \overset{*}{\rightharpoonup} 0$ weakly-* in $L^\infty(D)$. Thus by proposition A.1.8 we have for φ belongs to $W_0^{1,1}(D)$

$$\begin{aligned} \|R_M^\varepsilon\|_{W^{-1,\infty}(D)} &= \sup_{\|\varphi\|_{W_0^{1,1}(D)}=1} \left|(R_M^\varepsilon, \varphi)_{W^{-1,\infty}(D),W_0^{1,1}(D)}\right| \\ &= \sup_{\|\varphi\|_{W_0^{1,1}(D)}=1} \left|(R_M^\varepsilon, \varphi)_{L^\infty(D),L^1(D)}\right| \to 0, \end{aligned}$$

then R_M^ε goes to 0 strongly in $W^{-1,\infty}(D)$ and this yields (2.55). It remains to show (2.56). By caracterization of $W^{-1,\infty}(D)$ and definition of η_M^ε, it sufficient to prove only

$$\eta_M^\varepsilon \overset{*}{\rightharpoonup} C_{0,M} \text{ weakly-* in } L^\infty(D). \tag{2.61}$$

Since η_M^ε is bounded, then we test only for $\varphi \in \mathcal{C}_c^1(D)$. We have

$$(\eta_M^\varepsilon, \varphi)_{H^{-1}(D),H_0^1(D)} = \sum_{z_i \in \mathbb{Z}^d \cap \frac{1}{\varepsilon}D} 2^d(d-2)d\rho_{i,M}^{d-2} \int_{B_i^\varepsilon} \varphi,$$

applying lemma 2.11, one has

$$(\eta_M^\varepsilon, \varphi)_{H^{-1}(D), H_0^1(D)} \to (d-2)\sigma_d \left\langle \rho^{d-2} \mathbf{1}_{\rho \leq M} \right\rangle \int_D \varphi.$$

Then the proof of (2.61) is established, hence (2.56) holds true. ■

We return to the proof of proposition 2.5

Proof. We return to our main goal, and argue that the function w^ε defined in proposition 2.5 is $H^1(D)$ and satisfies (H1), (H2) and (H3). We starts with (H1), we have by definition $w_1^\varepsilon = 0$ in H_b^ε and $w_2^\varepsilon = 1$ in $H_b^\varepsilon \subseteq D_b^\varepsilon$, then

$$w^\varepsilon = w_1^\varepsilon \wedge w_2^\varepsilon = w_1^\varepsilon = 0 \quad \text{in } H_b^\varepsilon,$$

we have also $w_1^\varepsilon = 1$ in $H_b^\varepsilon \subseteq D \backslash D_b^\varepsilon$ and $w_1^\varepsilon = 0$ in H_g^ε, then

$$w^\varepsilon = w_1^\varepsilon \wedge w_2^\varepsilon = w_2^\varepsilon = 0 \quad \text{in } H_g^\varepsilon.$$

So the first property (H1) is satisfied. Let us prove that w^ε belongs to $H^1(D)$, we have by definition of w_1^ε and w_2^ε

$$\begin{cases} w^\varepsilon|_{D \backslash D_b^\varepsilon} = w_2^\varepsilon \in H^1\left(D \backslash D_b^\varepsilon\right), \\ w^\varepsilon|_{D_b^\varepsilon} = w_1^\varepsilon \in H^1\left(D_b^\varepsilon\right). \end{cases}$$

We have also

$$w^\varepsilon|_{\partial\left(D \backslash D_b^\varepsilon\right)} = w^\varepsilon|_{\partial D_b^\varepsilon} = 1.$$

Then we can use the proposition A.1.12 (See Appendix A) to conclude that $w^\varepsilon \in H^1(D)$. We pass to (H2), we have for every function $v \in H^1(D)$

$$\begin{aligned} \langle w^\varepsilon, v \rangle_{H^1(D)} &= \langle w^\varepsilon, v \rangle_{L^2(D)} + \langle \nabla w^\varepsilon, \nabla v \rangle_{(L^2(D))^d} \\ &= \langle w_1^\varepsilon, v \rangle_{L^2(D_b^\varepsilon)} + \langle \nabla w_1^\varepsilon, \nabla v \rangle_{\left(L^2(D_b^\varepsilon)\right)^d} \\ &\quad + \langle w_2^\varepsilon, v \rangle_{L^2(D \backslash D_b^\varepsilon)} + \langle \nabla w_2^\varepsilon, \nabla v \rangle_{\left(L^2(D \backslash D_b^\varepsilon)\right)^d}. \end{aligned} \tag{2.62}$$

Since w_2^ε satisfies the property (H2) and w_1^ε converge to 1 strongly in $H^1(D)$ hence weakly in $H^1(D)$, then

$$\langle w^\varepsilon, v \rangle_{H^1(D)} \to \langle 1, v \rangle_{H^1(D)}.$$

Thus, the property (H2) is established for w^ε. Now, we prove that (H3) is satisfied for w^ε but first of all we need to argue that it sufficient to prove (H3) only for w_2^ε. Indeed, let $v^\varepsilon \in H_0^1(D)$ such that v^ε vanishes in the holes H^ε and v^ε converge weakly to v in $H_0^1(D)$. By definition of w_1^ε, w_2^ε in lemma 2.6 and lemma 2.7, we have ∇w_1^ε and ∇w_2^ε has disjoint supports where

$$\operatorname{supp}(\nabla w_1^\varepsilon) \subseteq D_b^\varepsilon \backslash H_b^\varepsilon, \qquad \operatorname{supp}(\nabla w_2^\varepsilon) \subseteq D\backslash \left(D_b^\varepsilon \cup H_g^\varepsilon\right).$$

Then, one has

$$\begin{aligned} \langle -\Delta w^\varepsilon, v^\varepsilon\rangle_{H^{-1}(D),H_0^1(D)} &= \langle -\Delta w_1^\varepsilon, v^\varepsilon\rangle_{H^{-1}(D_b^\varepsilon\backslash H_b^\varepsilon),H_0^1(D_b^\varepsilon\backslash H_b^\varepsilon)} \\ &\quad + \langle -\Delta w_2^\varepsilon, v^\varepsilon\rangle_{H^{-1}(D\backslash D_b^\varepsilon),H_0^1(D\backslash D_b^\varepsilon)} \\ &= \int_{D_b^\varepsilon\backslash H_b^\varepsilon} \nabla w_1^\varepsilon \nabla v^\varepsilon \\ &\quad + \langle -\Delta w_2^\varepsilon, v^\varepsilon\rangle_{H^{-1}(D\backslash D_b^\varepsilon),H_0^1(D\backslash D_b^\varepsilon)}. \end{aligned}$$

Using lemma 2.6, one has

$$\int_{D_b^\varepsilon\backslash H_b^\varepsilon} \nabla w_1^\varepsilon \nabla v^\varepsilon \to 0.$$

We have also by lemma 2.7

$$\langle -\Delta w_2^\varepsilon, v^\varepsilon\rangle_{H^{-1}(D\backslash D_b^\varepsilon),H_0^1(D\backslash D_b^\varepsilon)} \to C_0 \int_D v, \tag{2.63}$$

where C_0 defined as in theorem 2.2. Then (H3) is satisfied for w^ε. ■

2.2.2 Case(b): General case

Let $(\Phi, \mathcal{R})$ be a marked point process defined as in of theorem 2.2. We give the following lemma which is similar to Lemma 2.3 where we can use it for the proof of Lemma 2.2.

Lemma 2.8 *There exist an $\varepsilon_0 = \varepsilon_0(d)$ and a family of random variables $\{r_\varepsilon\}_{\varepsilon>0} \subseteq \mathbb{R}^+$ such that for $\mathbb{P}-$almost every $\omega \in \Omega$*

$$\lim_{\varepsilon\to 0} r_\varepsilon(\omega) = 0, \tag{2.64}$$

and for any $\varepsilon \le \varepsilon_0$ there exist $H_g^\varepsilon(\omega), H_b^\varepsilon(\omega), D_b^\varepsilon(\omega) \subseteq \mathbb{R}^d$. such that

$$\begin{aligned} &H^\varepsilon(\omega) = H_g^\varepsilon(\omega) \cup H_b^\varepsilon(\omega), \quad H_b^\varepsilon(\omega) \subseteq D_b^\varepsilon(\omega), \\ &dist(H_g^\varepsilon(\omega), D_b^\varepsilon(\omega)) \ge \tfrac{\varepsilon r_\varepsilon}{2}, \end{aligned}$$

when

$$\lim_{\varepsilon\to 0} cap(H_b^\varepsilon(\omega), D_b^\varepsilon(\omega)) = 0. \tag{2.65}$$

Moreover, $H_g^\varepsilon(\omega)$ may be written as the following union of disjoint balls centered in $n^\varepsilon(\omega) \subseteq \Phi\left(\frac{1}{\varepsilon}D\right)$:

$$\begin{aligned} H_g^\varepsilon(\omega) &= \bigcup_{z_j \in n^\varepsilon} B_{\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j), \\ \min_{z_i \neq z_j \in n^\varepsilon} \varepsilon\,|z_i - z_j| &\geq 2r_\varepsilon\varepsilon, \qquad \varepsilon^{\frac{d}{d-2}}\rho_j \leq \frac{\varepsilon r_\varepsilon}{2}, \qquad \lim_{\varepsilon\to 0}\varepsilon^d \#(n^\varepsilon) = \langle N(Q)\rangle\,|D|\,. \end{aligned} \tag{2.66}$$

Furthermore, if for $\delta > 0$ the process Φ_δ is defined as in (2.14), then

$$\lim_{\varepsilon\to 0}\varepsilon^d \#\left(\{z_i \in \Phi^\varepsilon_{2\delta}(D)(\omega) : dist(z_i, D_b^\varepsilon) \leq \varepsilon\delta\}\right) = 0. \tag{2.67}$$

Proof. The proof of this lemma is divided in five steps: First, we construct the random variables $\{r_\varepsilon\}_{\varepsilon>0}$ for a fixed $\alpha \in \left(0, \frac{2}{d-2}\right)$, we write

$$r_\varepsilon = \left(\varepsilon^{\frac{d}{d-2}} \max_{z_j\in\Phi^\varepsilon(D)} \rho_j\right)^{\frac{1}{d}} \vee \varepsilon^{\frac{\alpha}{4}} = \max\left\{\left(\varepsilon^{\frac{d}{d-2}} \max_{z_j\in\Phi^\varepsilon(D)} \rho_j\right)^{\frac{1}{d}}, \varepsilon^{\frac{\alpha}{4}}\right\}. \tag{2.68}$$

We can show that r_ε satisfies (2.64). Indeed, For F^ε a subset of $\Phi^\varepsilon(D)$ defined as

$$F^\varepsilon = \left\{z_j \in \Phi^\varepsilon(D) : \varepsilon^{\frac{d}{d-2}}\rho_j \geq \varepsilon\right\}.$$

If $F^\varepsilon = \emptyset$, then for $z_j \in \Phi^\varepsilon(D)$ the corresponding radii satisfies

$$\varepsilon^{\frac{1}{d-2}} \max_{z_j\in\Phi^\varepsilon(D)} \rho_j^{\frac{1}{d}} \leq \varepsilon^{\frac{1}{d}}.$$

Since $r_\varepsilon \geq 0$ we have for every $\varepsilon > 0$

$$\lim_{\varepsilon\to 0} r_\varepsilon \leq \lim_{\varepsilon\to 0}\varepsilon^{\frac{1}{d}} \vee \varepsilon^{\frac{\alpha}{4}} = 0.$$

If $F^\varepsilon \neq \emptyset$, we get

$$\varepsilon^d \max_{z_j\in\Phi^\varepsilon(D)} \rho_j^{d-2} = \varepsilon^d \max_{z_j\in F^\varepsilon} \rho_j^{d-2} \leq \varepsilon^d \sum_{z_j\in F^\varepsilon} \rho_j^{d-2},$$

then, one has

$$\lim_{\varepsilon\to 0} r_\varepsilon \leq \lim_{\varepsilon\to 0}\left(\left(\varepsilon^d \sum_{z_j\in F^\varepsilon}\rho_j^{d-2}\right) \vee \varepsilon^{\frac{\alpha}{4}}\right).$$

So to get (2.64) immediatly applaying lemma 2.10, it's sufficient to claim that

$$\lim_{\varepsilon\to 0}\varepsilon^d \# F^\varepsilon = 0. \tag{2.69}$$

Indeed, for $z_j \in F^\varepsilon$ the corresponding radii ρ_j satisfies $1 \leq \varepsilon^2 \rho_j^{d-2}$, then one has

$$\varepsilon^d \# F^\varepsilon = \varepsilon^d \sum_{z_j \in F^\varepsilon} \leq \varepsilon^d \varepsilon^2 \sum_{z_j \in \Phi^\varepsilon(D)} \rho_j^{d-2}.$$

So applying lemma 2.9, (2.69) yields true and the proof of (2.64) is complete. The second step is about the construction of $H_b^\varepsilon(\omega)$ and its safety layer $D_b^\varepsilon(\omega)$. Equipped with the definition of r_ε defined as above (2.68) and denote by $\eta_\varepsilon = r_\varepsilon \varepsilon$. In this step we will give the set of the centers of bad balls denoted I_b^ε as a union of three sets, the first one is denoted by J_b^ε and contains the point of $\Phi^\varepsilon(D)$ where the corresponding radii are too large, then we put

$$J_b^\varepsilon = \left\{ z_j \in \Phi^\varepsilon(D) : \varepsilon^{\frac{d}{d-2}} \rho_j \geq \frac{\eta_\varepsilon}{2} \right\}. \tag{2.70}$$

The second set of points contains the centers generating the balls too close to each other, we indeed set

$$K_b^\varepsilon = \Phi^\varepsilon(D) \backslash \left(\Phi_{2r_\varepsilon}^\varepsilon(D) \cup J_b^\varepsilon \right), \tag{2.71}$$

where $\Phi_{2r_\varepsilon}^\varepsilon(D)$ is defined as in (2.14). Similarly to the periodic case, we define

$$\tilde{H}_b^\varepsilon = \bigcup_{z_j \in J_b^\varepsilon} B_{2\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j).$$

The third set contains the centers of balls might be close to $\tilde{H}_b^\varepsilon$: We denote

$$\tilde{I}_b^\varepsilon = \left\{ z_j \in \Phi^\varepsilon(D) \backslash (K_b^\varepsilon \cup J_b^\varepsilon) : \tilde{H}_b^\varepsilon \cap B_{\eta_\varepsilon}(\varepsilon z_j) \neq \emptyset \right\}. \tag{2.72}$$

Finally, we put

$$I_b^\varepsilon = J_b^\varepsilon \cup K_b^\varepsilon \cup \tilde{I}_b^\varepsilon, \tag{2.73}$$

$$H_b^\varepsilon = \bigcup_{z_j \in I_b^\varepsilon} B_{\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j), \quad D_b^\varepsilon = \bigcup_{z_j \in I_b^\varepsilon} B_{2\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j), \quad H_g^\varepsilon = H^\varepsilon \backslash H_b^\varepsilon. \tag{2.74}$$

In the third step, we prove (2.65). By the sub-additivity of capacity and definitions (2.74) we compute as in the simplest case

$$\begin{aligned} cap(H_b^\varepsilon, H_b^\varepsilon) &= \sum_{z_j \in I_b^\varepsilon} cap(B_{\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j), D_b^\varepsilon) \\ &\leq \sum_{z_j \in I_b^\varepsilon} cap(B_{\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j), B_{2\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j)) \\ &\leq \sum_{z_j \in I_b^\varepsilon} \varepsilon^d \rho_j^{d-2}. \end{aligned}$$

The proof is concluded from lemma 2.10 if we argue that

$$\lim_{\varepsilon\to 0} \varepsilon^d \# I_b^\varepsilon = 0. \tag{2.75}$$

Indeed, by definition of I_b^ε in (2.73) it sufficient to prove (2.75) only for the sets $J_b^\varepsilon, K_b^\varepsilon$ and $\tilde{I}_b^\varepsilon$. We start with

$$\lim_{\varepsilon\to 0} \varepsilon^d \# J_b^\varepsilon = 0. \tag{2.76}$$

We have by definition of J_b^ε in (2.70)

$$1 \leq \varepsilon^2 (r_\varepsilon)^{-(d-2)} 2^{d-2} \rho_j^{d-2} \quad \text{for} \quad z_j \in J_b^\varepsilon,$$

then one has

$$\begin{aligned} \varepsilon^d \# J_b^\varepsilon &= \varepsilon^d \sum_{z_j \in J_b^\varepsilon} \\ &\leq \varepsilon^2 (r_\varepsilon)^{-(d-2)} 2^{d-2} \varepsilon^d \sum_{z_j \in \Phi^\varepsilon(D)} \rho_j^{d-2}. \end{aligned} \tag{2.77}$$

We have also by definition of r_ε

$$\begin{aligned} r_\varepsilon^{-(d-2)} &\leq \left(\varepsilon^{\frac{d}{d-2}} \max_{z_j \in \Phi^\varepsilon(D)} \rho_j \right)^{\frac{-(d-2)}{d}} \wedge \varepsilon^{\frac{-\alpha(d-2)}{4}} \\ &\leq \varepsilon^{-\alpha(d-2)}. \end{aligned}$$

Substituting this last result in (2.77), we get

$$\varepsilon^d \# J_b^\varepsilon \leq \varepsilon^{2-\alpha(d-2)} 2^{d-2} \varepsilon^d \sum_{z_j \in \Phi^\varepsilon(D)} \rho_j^{d-2}.$$

By lemma 2.9 we have

$$\lim_{\varepsilon\to 0} \varepsilon^d \sum_{z_j \in \Phi^\varepsilon(D)} \rho_j^{d-2} = \langle N(Q) \rangle |D| \left\langle \rho^{d-2} \right\rangle < +\infty,$$

then since $2-\alpha(d-2) > 1$, (2.76) is established., For a sequence $\{\delta_k\}_{k\in\mathbb{N}} \subseteq \mathbb{R}^{+*}$ with $\delta_k \to 0$ when $k \to +\infty$ if we suppose also that $N_{\delta_k}^\varepsilon(D) \leq N_{r_\varepsilon}^\varepsilon(D)$, we get

$$\limsup_{\varepsilon\to 0} \varepsilon^d \# K_b^\varepsilon \leq \limsup_{\varepsilon\to 0} \varepsilon^d (N^\varepsilon(D) - N_{r_\varepsilon}^\varepsilon(D)) \leq \limsup_{\varepsilon\to 0} \varepsilon^d (N^\varepsilon(D) - N_{\delta_k}^\varepsilon(D)). \tag{2.78}$$

We can apply (2.122) of lemma 2.9 for the right hand side of (2.78), one has

$$\limsup_{\varepsilon\to 0} \varepsilon^d \# K_b^\varepsilon \leq \langle N(Q) - N_{\delta_k}(Q)\rangle |D| ,$$

for Q is a unitary cube. Sending $\delta_k \to 0$ and applying (2.122) of lemma 2.9, we obtain

$$\lim_{\varepsilon\to 0} \varepsilon^d \# K_b^\varepsilon = 0. \tag{2.79}$$

It remains to prove

$$\lim_{\varepsilon\to 0} \varepsilon^d \# \tilde{I}_b^\varepsilon = 0. \tag{2.80}$$

By definitions (2.70), (2.71) and (2.72), we have for $z_i \in \Phi^\varepsilon(D) \backslash (K_b^\varepsilon \cup J_b^\varepsilon)$

$$\min_{\substack{z_j, z_i \in \Phi^\varepsilon(D) \\ z_i \neq z_j}} \varepsilon |z_i - z_j| \quad \geq \quad 2\eta_\varepsilon, \tag{2.81}$$

$$\varepsilon^{\frac{d}{d-2}} \rho_i \quad < \quad \frac{\eta_\varepsilon}{2}. \tag{2.82}$$

Since the balls of $\tilde{I}_b^\varepsilon$ have radii satisfies (2.82) and centers satisfies (2.81), then the balls $\left\{B_{\eta_\varepsilon}(\varepsilon z_i)\right\}_{z_i \in \tilde{I}_b^\varepsilon}$ are disjoints. So one has

$$\varepsilon^d \# \tilde{I}_b^\varepsilon = \varepsilon^d \sum_{z_i \in \tilde{I}_b^\varepsilon} \leq \varepsilon^d \sum_{z_i \in \tilde{I}_b^\varepsilon} \frac{\Gamma(\frac{d}{2}+1)}{\eta_\varepsilon^d \pi^d} \left|B_{\eta_\varepsilon}(\varepsilon z_i)\right| = r_\varepsilon^{-d} \sum_{z_i \in \tilde{I}_b^\varepsilon} \frac{\Gamma(\frac{d}{2}+1)}{\pi^d} \left|B_{\eta_\varepsilon}(\varepsilon z_i)\right| ,$$

with $\left|B_{\eta_\varepsilon}(\varepsilon z_i)\right| = \frac{\eta_\varepsilon^d \pi^d}{\Gamma(\frac{d}{2}+1)}$ and Γ is the gamma function defined as generalization of the factorial function for non integer value. We have for any $z_i \in \tilde{I}_b^\varepsilon$ there exists $c = c(d)$ and $z_j \in J_b^\varepsilon$ such that

$$B_{\eta_\varepsilon}(\varepsilon z_i) \subseteq B_{c\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j),$$

then one has

$$\begin{aligned} \varepsilon^d \# \tilde{I}_b^\varepsilon &\leq r_\varepsilon^{-d} \sum_{z_i \in J_b^\varepsilon} \frac{\Gamma(\frac{d}{2}+1)}{\pi^d} \left| B_{c\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j) \right| \\ &\leq r_\varepsilon^{-d} \sum_{z_i \in J_b^\varepsilon} C_1 \left(\varepsilon^{\frac{d}{d-2}} \rho_j\right)^d, \end{aligned}$$

with $C_1 > 0$ is a constant depend only on d. We have also

$$(\varepsilon^{\frac{d}{d-2}}\rho_j)^d = \varepsilon^{\frac{d^2-2d+2d}{d-2}} {\rho_j}^{d-2} {\rho_j}^2 \leq (\varepsilon^{\frac{d}{d-2}} \max_{z_j \in \mathbb{Z}^d \cap \frac{1}{\varepsilon} D} \rho_j)^2 \varepsilon^d {\rho_j}^{d-2},$$

then, we obtain

$$\varepsilon^d \# \tilde{I}_b^\varepsilon \le r_\varepsilon^{-d} C_1 (\varepsilon^{\frac{d}{d-2}} \max_{z_j \in \mathbb{Z}^d \cap \frac{1}{\varepsilon} D} \rho_j)^2 \sum_{z_i \in J_b^\varepsilon} \varepsilon^d {\rho_j}^{d-2}. \tag{2.83}$$

In the other hand, we have by definition of r_ε

$$r_\varepsilon^{-d} \le \left(\varepsilon^{\frac{d}{d-2}} \max_{z_j \in \Phi^\varepsilon(D)} \rho_j \right)^{-1} \wedge \varepsilon^{\frac{-\alpha d}{4}} \le \left(\varepsilon^{\frac{d}{d-2}} \max_{z_j \in \Phi^\varepsilon(D)} \rho_j \right)^{-1},$$

it follows

$$\begin{aligned} r_\varepsilon^{-d} C_1 (\varepsilon^{\frac{d}{d-2}} \max_{z_j \in \mathbb{Z}^d \cap \frac{1}{\varepsilon} D} \rho_j)^2 &\le C_1 (\varepsilon^{\frac{d}{d-2}} \max_{z_j \in \mathbb{Z}^d \cap \frac{1}{\varepsilon} D} \rho_j) \\ &\le C_1 (\varepsilon^d \sum_{z_j \in \mathbb{Z}^d \cap \frac{1}{\varepsilon} D} \rho_j^{d-2})^{\frac{1}{d-2}}. \end{aligned}$$

For ε small enough, we can apply lemma 2.9 then we get

$$r_\varepsilon^{-d} C_1 (\varepsilon^{\frac{d}{d-2}} \max_{z_j \in \mathbb{Z}^d \cap \frac{1}{\varepsilon} D} \rho_j)^2 \le C_1 \left(\langle N(Q) \rangle |D| \left\langle \rho^{d-2} \right\rangle \right)^{\frac{1}{d-2}}, \tag{2.84}$$

substituting (2.84) in (2.83) one has

$$\varepsilon^d \# \tilde{I}_b^\varepsilon \le C_1 \left(\langle N(Q) \rangle |D| \left\langle \rho^{d-2} \right\rangle \right)^{\frac{1}{d-2}} \sum_{z_i \in J_b^\varepsilon} \varepsilon^d {\rho_j}^{d-2}.$$

Since we have proved (2.76), we can apply lemma 2.9, thus (2.80) is established. Finally, we get

$$\lim_{\varepsilon \to 0} \varepsilon^d \# I_b^\varepsilon = \lim_{\varepsilon \to 0} \varepsilon^d \# J_b^\varepsilon + \lim_{\varepsilon \to 0} \varepsilon^d \# K_b^\varepsilon + \lim_{\varepsilon \to 0} \varepsilon^d \# \tilde{I}_b^\varepsilon = 0.$$

The fourth step is to contruct the set of good holes H_g^ε which satisfies (2.66). We can set $n^\varepsilon = \Phi^\varepsilon(D) \backslash I_b^\varepsilon$ and define H_g^ε as follows

$$H_g^\varepsilon = \bigcup_{z_i \in n^\varepsilon} B_{\varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i).$$

Let us prove that

$$dist \left(H_g^\varepsilon, D_b^\varepsilon \right) \ge \frac{\eta_\varepsilon}{2}. \tag{2.85}$$

Since for $z_i \in n^\varepsilon$, we have $z_i \notin K_b^\varepsilon \cup J_b^\varepsilon$ then the properties (2.81) and (2.82) are satisfies then one has

$$dis(B_{\varepsilon^{\frac{d}{d-2}} \rho_i}(\varepsilon z_i), B_{\eta_\varepsilon}(\varepsilon z_i)) \ge \frac{\eta_\varepsilon}{2}.$$

So to claim (2.85), it sufficient to prove that for $z_i \in n^\varepsilon$ and $z_j \in I_b^\varepsilon$ we have

$$B_{2\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j) \cap B_{\eta_\varepsilon}(\varepsilon z_i) = \emptyset. \tag{2.86}$$

Indeed, If $z_j \in J_b^\varepsilon$ then $z_j \notin \tilde{I}_b^\varepsilon$, we have also $z_i \notin \tilde{I}_b^\varepsilon$ then by definition of $\tilde{I}_b^\varepsilon$ in (2.72), (2.86) is established. Now, if $z_j \in K_b^\varepsilon \cup \tilde{I}_b^\varepsilon$ then

$$2\varepsilon^{\frac{d}{d-2}}\rho_j < \eta_\varepsilon, \qquad \min_{z_i \in n^\varepsilon} \varepsilon\,|z_i - z_j| \geq 2\eta_\varepsilon,$$

then, we can conclude (2.86) and finally this yields (2.85). We now prove the properties (2.66). For the first, by definition of n^ε for any $z_i, z_j \in n^\varepsilon$ with $z_i \neq z_j$ we have $z_i, z_j \notin K_b^\varepsilon$, Then we get

$$\min_{z_i \in n^\varepsilon} \varepsilon\,|z_i - z_j| \geq 2\eta_\varepsilon. \tag{2.87}$$

The second result follows from the definition of n^ε. For $z_j \in n^\varepsilon$ we have $z_j \notin I_b^\varepsilon$ then one has

$$2\varepsilon^{\frac{d}{d-2}}\rho_j \leq \eta_\varepsilon,$$

we have also

$$\lim_{\varepsilon \to 0} \varepsilon^d \# n^\varepsilon = \lim_{\varepsilon \to 0} \varepsilon^d \left(\#\Phi^\varepsilon(D) - \# I_b^\varepsilon\right).$$

Using (2.75), one has

$$\lim_{\varepsilon \to 0} \varepsilon^d \# n^\varepsilon = \lim_{\varepsilon \to 0} \varepsilon^d \#\Phi^\varepsilon(D).$$

Using the result (2.2) in lemma 2.9, we get

$$\lim_{\varepsilon \to 0} \varepsilon^d \# n^\varepsilon = \langle N\left(Q\right)\rangle\,|D|\,, \tag{2.88}$$

where Q is the unitary cube. The last step is to prove (2.67), to do that we first set

$$Y_\delta^\varepsilon = \left\{z_i \in \Phi_{2\delta}^\varepsilon(D) : dist\left(z_i, D_b^\varepsilon\right) \leq \delta\varepsilon\right\}. \tag{2.89}$$

We have

$$\begin{aligned} Y^\varepsilon_\delta &\subseteq \{z_i \in n^\varepsilon \cup I^\varepsilon_b : dist\,(\varepsilon z_i, D^\varepsilon_b) \leq \delta\varepsilon\} \\ &\subseteq I^\varepsilon_b \cup \left\{ z_i \in n^\varepsilon : dist\left(\varepsilon z_i, \bigcup_{z_j \in J^\varepsilon_b} B_{2\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j)\right) \leq \delta\varepsilon \right\} \\ &\quad \cup \left\{ z_i \in n^\varepsilon \cap \Phi^\varepsilon_{2\delta}(D) : dist\left(\varepsilon z_i, \bigcup_{z_j \in \tilde{I}^\varepsilon_b \cup K^\varepsilon_b} B_{2\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j)\right) \leq \delta\varepsilon \right\}. \end{aligned}$$

We denote

$$E^\varepsilon = \left\{ z_i \in n^\varepsilon : dist\left(\varepsilon z_i, \bigcup_{z_j \in J^\varepsilon_b} B_{2\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j)\right) \leq \delta\varepsilon \right\} \tag{2.90}$$

and

$$C^\varepsilon = \left\{ z_i \in n^\varepsilon \cap \Phi^\varepsilon_{2\delta}(D) : dist\left(\varepsilon z_i, \bigcup_{z_j \in \tilde{I}^\varepsilon_b \cup K^\varepsilon_b} B_{2\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j)\right) \leq \delta\varepsilon \right\}, \tag{2.91}$$

to argue (2.67) we show that

$$\lim_{\varepsilon\to 0} \varepsilon^d \# I^\varepsilon_b = \lim_{\varepsilon\to 0} \varepsilon^d \# E^\varepsilon = \lim_{\varepsilon\to 0} \varepsilon^d \# C^\varepsilon = 0.$$

Indeed, the first result is concluded from (2.75). We pass to

$$\lim_{\varepsilon\to 0} \varepsilon^d \# E^\varepsilon = 0. \tag{2.92}$$

We may choose $\varepsilon_0 = \varepsilon_0(d)$ such that for all $\varepsilon \leq \varepsilon_0$ the property (2.64) is satisfied and $\varepsilon r_\varepsilon \leq \varepsilon\delta$. For $z_i \in E^\varepsilon$ there exists $z_j \in J^\varepsilon_b$ where z_i, z_j satisfies the following properties

$$\begin{cases} B_{\eta_\varepsilon}(\varepsilon z_i) \subseteq B_{\varepsilon\delta}(\varepsilon z_i), \\ 2\varepsilon^{\frac{d}{d-2}}\rho_j \geq \eta_\varepsilon, \\ dist(\varepsilon z_i, \partial B_{2\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j)) \leq \delta\varepsilon. \end{cases}$$

Then, we can remark that

$$B_{\eta_\varepsilon}(\varepsilon z_i) \subseteq B_{2\delta\varepsilon + 2\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j).$$

Using $1 \leq \delta r_\varepsilon^{-1}$, we get

$$2\delta\varepsilon \leq \frac{2\delta}{r_\varepsilon} 2\varepsilon^{\frac{d}{d-2}}\rho_j, \qquad 2\varepsilon^{\frac{d}{d-2}}\rho_j \leq \frac{2\delta}{r_\varepsilon}\varepsilon^{\frac{d}{d-2}}\rho_j,$$

then, one has

$$B_{\eta_\varepsilon}(\varepsilon z_i) \subseteq B_{2\delta\varepsilon+2\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j) \subseteq B_{6\delta r_\varepsilon^{-1}\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j). \tag{2.93}$$

So, we obtain

$$\begin{aligned}
\varepsilon^d \# E^\varepsilon &= r_\varepsilon^{-d}\eta_\varepsilon^d \# E^\varepsilon = r_\varepsilon^{-d}\eta_\varepsilon^d \sum_{z_i \in n^\varepsilon} \frac{\Gamma(\frac{d}{2}+1)}{\pi^d \eta_\varepsilon^d} \left| B_{\eta_\varepsilon}(\varepsilon z_i) \right| \\
&\leq \left(\frac{\Gamma(\frac{d}{2}+1)}{\pi^d} \right) r_\varepsilon^{-d} \sum_{z_i \in J_b^\varepsilon} \left| B_{6\delta r_\varepsilon^{-1}\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j) \right| \\
&\leq \left(\frac{\Gamma(\frac{d}{2}+1)}{\pi^d} \right) \delta^d r_\varepsilon^{-2d} \sum_{z_i \in J_b^\varepsilon} \left(\varepsilon^{\frac{d}{d-2}} \rho_j \right)^d \leq \left(\frac{\Gamma(\frac{d}{2}+1)}{\pi^d} \right) \delta^d r_\varepsilon^{-2d} \sum_{z_i \in J_b^\varepsilon} \varepsilon^{\frac{d^2-2d+2d}{d-2}} \rho_j^{d-2} \rho_j^2,
\end{aligned}$$

with Γ is the Gamma function. Since by definition of r_ε we have

$$r_\varepsilon^{-2d} \leq \varepsilon^{\frac{-2d}{d-2}} \max \rho_j^{-2},$$

one has

$$\varepsilon^d \# E^\varepsilon \leq \left(\frac{\Gamma(\frac{d}{2}+1)}{\pi^d} \right) \delta^d \varepsilon^d \sum_{z_i \in J_b^\varepsilon} \rho_j^{d-2}.$$

Using (2.76) and lemma 2.10 we get out (2.92). Now, we claim the last result

$$\lim_{\varepsilon \to 0} \varepsilon^d \# C^\varepsilon = 0, \tag{2.94}$$

we show that the set C^ε is empty for ε small enough. We have by definition of $\tilde{I}_b^\varepsilon$ and K_b^ε, if $z_i \in n_\varepsilon$ satisfies

$$dist\left(\varepsilon z_i, \bigcup_{z_j \in \tilde{I}_b^\varepsilon \cup K_b^\varepsilon} B_{2\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j) \right) \leq \delta\varepsilon,$$

then, there exists a $z_j \in \tilde{I}_b^\varepsilon \cup K_b^\varepsilon$ with $2\varepsilon^{\frac{d}{d-2}}\rho_j < r^\varepsilon \varepsilon \leq \delta\varepsilon$ such that

$$\varepsilon \left| z_i - z_j \right| \leq dist\left(\varepsilon z_i, \partial B_{2\varepsilon^{\frac{d}{d-2}}\rho_j}(\varepsilon z_j) \right) + r^\varepsilon \varepsilon \leq 2\delta\varepsilon,$$

this implies $C^\varepsilon \subseteq \Phi^\varepsilon(D) \backslash \Phi_{2\delta}^\varepsilon(D)$ and thus by definition C^ε is empty. Hence (2.94) is established. ■

We now return to the proof of lemma 2.2 in the general case.

Proof. Equiping with the sets $H_g^\varepsilon, H_b^\varepsilon$ and D_b^ε constructed as in lemma 2.8, the construction of w^ε follows the same steps as in the periodic case where we take $w^\varepsilon = w_1^\varepsilon \wedge w_2^\varepsilon$ with w_1^ε and w_2^ε defined as the same as in lemma 2.6 and 2.7 respectivly for the simplest case with $H_g^\varepsilon, H_b^\varepsilon$ and D_b^ε as in lemma 2.8. The only change here is due to the construction of w_2^ε under the setting of lemma 2.8. Indeed, we set

$$w_2^\varepsilon = 1 \quad \text{in } D_b^\varepsilon,$$

then it remains to construct w_2^ε only in $D \backslash D_b^\varepsilon$. For each $z_j \in n^\varepsilon$ with n^ε being the set of centers of the particles in H_g^ε, we denote by d_j^ε the random variables

$$d_j^\varepsilon = \min\left\{ dist(\varepsilon z_j, D_b^\varepsilon), \frac{1}{2} \min_{z_i \neq z_j \in n^\varepsilon} \varepsilon \left| z_i - z_j \right|, \varepsilon \right\}.$$

We have by (2.66) and (2.85)

$$dist(\varepsilon z_j, D_b^\varepsilon) \geq \eta_\varepsilon, \qquad \frac{1}{2} \min_{i \neq j} \varepsilon \left| z_i - z_j \right| \geq \eta_\varepsilon,$$

then we can remark that $d_j^\varepsilon \geq \eta_\varepsilon$ where $\eta_\varepsilon = \varepsilon r_\varepsilon$ and r_ε defined as in lemma 2.8. So we define the sets for $z_j \in n^\varepsilon$

$$T_j^\varepsilon = B_{\varepsilon^{\frac{d}{d-2}} \rho_j}(\varepsilon z_j), \qquad B_j^\varepsilon = B_{d_j^\varepsilon}(\varepsilon z_j),$$

and consider the functions $w_2^{\varepsilon,j}$ as in lemma 2.8, solving

$$\begin{cases} -\Delta w_2^{\varepsilon,j} = 0 & \text{in } B_j^\varepsilon \backslash T_j^\varepsilon, \\ 1 & \text{in } T_j^\varepsilon, \\ 0 & \text{in } D \backslash B_j^\varepsilon, \end{cases} \tag{2.95}$$

then taking $\varepsilon^{\frac{d}{d-2}} \rho_j < r_\varepsilon = |x - \varepsilon z_j| < d_j^\varepsilon$ with εz_j is the center of T_j^ε and $x \in \mathbb{R}^d$. The function $w_2^{\varepsilon,j}$ is defined as follow

$$w_2^{\varepsilon,j} = \begin{cases} \dfrac{|x-\varepsilon z_j|^{-(d-2)} - (d_j^\varepsilon)^{-(d-2)}}{\varepsilon^{-d} \rho_j^{-(d-2)} - (d_j^\varepsilon)^{-(d-2)}} & \text{in } B_j^\varepsilon \backslash T_j^\varepsilon, \\ 1 & \text{in } T_j^\varepsilon, \\ 0 & \text{in } D \backslash B_j^\varepsilon. \end{cases}$$

By definition of d_j^ε, we have

$$d_j^\varepsilon \geq 2\varepsilon^{\frac{d}{d-2}} \rho_j. \tag{2.96}$$

Indeed, by definition of n^ε and (2.66) for $z_j \in n^\varepsilon$ the corresponding radii satisfies

$$2\varepsilon^{\frac{d}{d-2}} \rho_j \leq \varepsilon r_\varepsilon \leq \min\left\{ \frac{1}{2} \min_{i \neq j} \varepsilon \left| z_i - z_j \right|, \varepsilon \right\}.$$

The definition of T_j^ε and (2.85) gives

$$\frac{\varepsilon r_\varepsilon}{2} \leq dist(T_j^\varepsilon, D_b^\varepsilon),$$

then

$$2\varepsilon^{\frac{d}{d-2}}\rho_j \leq \varepsilon^{\frac{d}{d-2}}\rho_j + \frac{\varepsilon r_\varepsilon}{2} \leq \varepsilon^{\frac{d}{d-2}}\rho_j + dist(T_j^\varepsilon, D_b^\varepsilon) \leq dist(\varepsilon z_j, D_b^\varepsilon).$$

Thus (2.96) yields. The previous result (2.96) argue that the functions $w_2^{\varepsilon,j}$ have disjoint supports and same for $\nabla w_2^{\varepsilon,j}$. Then, we can set

$$w_2^\varepsilon = 1 - \sum_{z_j \in n^\varepsilon} w_2^{\varepsilon,j}$$

and show that the function w_2^ε satisfies the properties (2.40), by definition of $w_2^{\varepsilon,j}$ in T_j^ε and since the functions $w_2^{\varepsilon,j}$ has disjoint supports then we can conclude that w_2^ε vanishes in H_g^ε. We can argue also that

$$0 \leq w_2^\varepsilon \leq 1,$$

as in the periodic case using the maximum principle. Thus the properties (2.40) are satisfied under the setting of lemma 2.8. By definition of w_2^ε in D_b^ε and $D\backslash D_b^\varepsilon$ we can easly conclude that $w_2^\varepsilon|_{D_b^\varepsilon} \in H^1(D_b^\varepsilon)$ and $w_2^\varepsilon|_{D\backslash D_b^\varepsilon} \in H^1(D\backslash D_b^\varepsilon)$, we can remark also that the function is continuous in the whole set D then applaying proposition A.1.12 (See Appendix A) we get w_2^ε belongs to $H^1(D)$. Let us return to the properties of w^ε and show that w^ε satisfies (H1), (H2) and (H3): We starts with (H1), we have by definiton $w_1^\varepsilon = 0$ in H_b^ε, and $w_2^\varepsilon = 1$ in $H_b^\varepsilon \subseteq D_b^\varepsilon$, then

$$w^\varepsilon = w_1^\varepsilon \wedge w_2^\varepsilon = w_1^\varepsilon = 0 \quad \text{ in } H_b^\varepsilon,$$

we have also $w_1^\varepsilon = 1$ in $H_b^\varepsilon \subseteq D\backslash D_b^\varepsilon$, and $w_1^\varepsilon = 0$ in H_g^ε, then

$$w^\varepsilon = w_1^\varepsilon \wedge w_2^\varepsilon = w_2^\varepsilon = 0 \quad \text{ in } H_g^\varepsilon,$$

then (H1) is satisfied. Same prove as in the periodic case, it sufficient to prove (H2) and (H3) only for w_2^ε, we begins with (H2) we have by definition of w_2^ε for $x \in \mathbb{R}^d$ where $\varepsilon^{\frac{d}{d-2}}\rho_j < |x - \varepsilon z_j| < d_j^\varepsilon$

$$\begin{aligned} \partial_{x_i} w_2^\varepsilon(x) &= -\sum_{z_i \in n^\varepsilon} \partial_{x_i} w_2^{\varepsilon,j}(x) \\ &= \sum_{z_j \in n^\varepsilon} \frac{(d-2)}{\left(\varepsilon^{-d}\rho_j^{-(d-2)}\right) - \left(d_j^\varepsilon\right)^{-(d-2)}} \frac{(x^i - \varepsilon z_j^i)}{|x - \varepsilon z_j|^d}, \end{aligned} \tag{2.97}$$

then

$$\begin{aligned}
\|\nabla w_2^\varepsilon\|_{(L^2(D))^d}^2 &= \sum_{z_j\in n^\varepsilon}\int_{B_j^\varepsilon}\left|\nabla w_2^{\varepsilon,j}(x)\right|^2 dx \\
&= \sum_{z_j\in n^\varepsilon}\int_{B_j^\varepsilon}\sum_{j=1}^{d}\left|\partial_{x_i} w_2^{\varepsilon,j}(x)\right|^2 dx \\
&= \sum_{z_j\in n^\varepsilon}\frac{(d-2)^2}{\left(\left(\varepsilon^{-d}\rho_j^{-(d-2)}\right)-\left(d_j^\varepsilon\right)^{-(d-2)}\right)^2}\int_{B_j^\varepsilon}\frac{1}{|x-\varepsilon z_j|^{2(d-1)}}dx,
\end{aligned}$$

we obtain

$$\begin{aligned}
\|\nabla w_2^\varepsilon\|_{(L^2(D))^d}^2 &= \sum_{z_j\in n^\varepsilon}\frac{(d-2)\sigma_d}{\left(\varepsilon^{-d}\rho_j^{-(d-2)}\right)-\left(d_j^\varepsilon\right)^{-(d-2)}} \\
&= \sum_{z_j\in n^\varepsilon}\frac{(d-2)\sigma_d\varepsilon^d\rho_j^{(d-2)}}{1-\left(d_j^\varepsilon\right)^{-(d-2)}\varepsilon^d\rho_j^{(d-2)}} \\
&\leq C(d)\sum_{z_j\in\Phi^\varepsilon\cap\frac{1}{\varepsilon}D}\varepsilon^d\rho_j^{(d-2)}, \qquad (2.98)
\end{aligned}$$

where $C(d)>0$ is a constant and σ_d is the $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$. Using (2.121) of lemma 2.9, one has

$$\|\nabla w_2^\varepsilon\|_{(L^2(D))^d}^2 \leq K, \qquad (2.99)$$

where $K = C(d)\langle N(Q)\rangle\left\langle\rho^{d-2}\right\rangle|D|>0$ and Q is the unitary cube of $\mathbb{R}^d$.
Since $1-w_2^\varepsilon=0$ in ∂D, then we can apply Poincaré's inequality one has

$$\|1-w_2^\varepsilon\|_{H^1(D)}^2 \leq \alpha\,\|\nabla w_2^\varepsilon\|_{(L^2(D))^d}^2 \leq \alpha K,$$

where $\alpha>0$ is a Poincaré constant. By Eberlein-Šmuljan theorem; up to a subsequence, we have almost surely

$$1-w_2^\varepsilon \rightharpoonup w \text{ weakly in } H^1(D),$$

it follows by Rellich-Kondrachov theorem

$$1-w_2^\varepsilon \to w \text{ strongly in } L^2(D).$$

Let us prove that $w=0$, to do that we need to prove the equivalent result

$$w_2^\varepsilon \rightharpoonup 1 \text{ weakly in } H^1(D),$$

only for the truncated processes $\left(n_M^\varepsilon, \{\rho_{j,M}\}_{z_j \in n^\varepsilon}\right)$. We take

$$n_M^\varepsilon = \left\{ z_j \in n^\varepsilon : d_j^\varepsilon \geq \frac{\varepsilon}{M} \right\}, \qquad \rho_{j,M} = \rho_j \wedge M = \min\{\rho_j, M\}$$

and

$$H_g^{\varepsilon,M} = \bigcup_{z_j \in n_M^\varepsilon} B_{\varepsilon^{\frac{d}{d-2}} \rho_{j,M}}(\varepsilon z_j), \qquad D^{\varepsilon,M} = D \backslash \left(H_g^{\varepsilon,M} \cup H_b^\varepsilon\right)$$

and $w_2^{\varepsilon,M}$ defined for the truncated process as w_2^ε. Let us prove first that $1 - w_2^\varepsilon$ converges strongly to 0 in $L^2(D)$. Indeed, we have by triangular inequality

$$\begin{aligned} \lim_{\varepsilon\to 0}\sup \|1 - w_2^\varepsilon\| \quad &\leq \quad \lim_{M\to+\infty}\sup \lim_{\varepsilon\to 0}\sup \left\| w_2^\varepsilon - w_2^{\varepsilon,M} \right\|_{L^2(D)} \\ &\quad + \lim_{M\to+\infty}\sup \lim_{\varepsilon\to 0}\sup \left\| 1 - w_2^{\varepsilon,M} \right\|_{L^2(D)}. \end{aligned} \tag{2.100}$$

To show the second right hand side of (2.100), we first remark that

$$1 - w_2^{\varepsilon,M} = 0 \text{ in } \mathbb{R}^d \backslash \bigcup_{z_j \in n_M^\varepsilon} B_j^\varepsilon,$$

then, the Poincaré's inequality gives

$$\begin{aligned} \left\| 1 - w_2^{\varepsilon,M} \right\|_{L^2(D)}^2 \quad &= \quad \sum_{z_j \in n^\varepsilon} \left\| 1 - w_2^{\varepsilon,j} \right\|_{L^2(B_j^\varepsilon)}^2 \leq \sum_{z_j \in n^\varepsilon} \left\| 1 - w_2^{\varepsilon,j} \right\|_{L^2(B_j^\varepsilon)}^2 \\ &\leq \quad m^2 \left\| \nabla w_2^\varepsilon \right\|_{L^2(D)}^2 \leq m^2 K, \end{aligned}$$

where m is a Poincaré's constant and K is a strictly positive constant defined as in (2.99). Since for every $z_j \in n^\varepsilon$, $d_j^\varepsilon \leq \varepsilon$ then we get $m \leq \varepsilon$. Hence

$$\left\| 1 - w_2^{\varepsilon,M} \right\|_{L^2(D)}^2 \leq \varepsilon^2 K.$$

Sending ε to 0 we get

$$w_2^{\varepsilon,M} \to 0 \text{ strongly in } L^2(D). \tag{2.101}$$

Now, it remains to prove

$$\lim_{M\to+\infty}\sup \lim_{\varepsilon\to 0}\sup \left\| w_2^\varepsilon - w_2^{\varepsilon,M} \right\|_{L^2(D)} = 0. \tag{2.102}$$

The definition of $w_2^{\varepsilon,M}$ above gives

$$w_2^{\varepsilon,M} = \begin{cases} 1 - \sum_{z_j \in n_M^\varepsilon} w_2^{\varepsilon,M,j} & \text{in } \bigcup_{z_j \in n_M^\varepsilon} B_j^\varepsilon, \\ 1 & \text{in } \mathbb{R}^d \backslash \bigcup_{z_j \in n_M^\varepsilon} B_j^\varepsilon, \end{cases} \tag{2.103}$$

where each function $w_2^{\varepsilon,M,j}$solving (2.95) with $B_j^\varepsilon = B_{d_j^\varepsilon}(\varepsilon z_j)$ and $\rho_{j,M} \leq M$. By definition of $w_2^{\varepsilon,M}$, we have

$$w_2^\varepsilon - w_2^{\varepsilon,M} = \begin{cases} 0 & \text{in } \bigcup_{z_j \in n_M^\varepsilon} B_j^\varepsilon, \\ w_2^\varepsilon - 1 & \text{in } \bigcup_{z_j \in n^\varepsilon \backslash n_M^\varepsilon} B_j^\varepsilon, \\ 0 & \text{in } \mathbb{R}^d \backslash \bigcup_{z_j \in n^\varepsilon} B_j^\varepsilon. \end{cases} \tag{2.104}$$

So by Poincaré's inequality to show (2.102) it's sufficient to prove only

$$\lim_{M \to +\infty} \sup \lim_{\varepsilon \to 0} \sup \left\| \nabla \left(w_2^\varepsilon - w_2^{\varepsilon,M} \right) \right\|_{(L^2(D))^d} = 0. \tag{2.105}$$

Using (2.104) one has

$$\begin{aligned} \left\| \nabla \left(w_2^\varepsilon - w_2^{\varepsilon,M} \right) \right\|^2_{(L^2(D))^d} &= \sum_{z_j \in n^\varepsilon} \left\| \nabla \left(w_2^{\varepsilon,j} - w_2^{\varepsilon,M,j} \right) \right\|^2_{(L^2(D))^d} \\ &= \sum_{z_j \in n^\varepsilon} \left\| \nabla w_2^{\varepsilon,j} \right\|^2_{(L^2(D))^d} 1_{\rho_j \geq M} \mathbf{1}_{d_j^\varepsilon \geq M^{-1}\varepsilon} \\ &\quad + \sum_{z_j \in n^\varepsilon} \left\| \nabla w_2^{\varepsilon,j} \right\|^2_{(L^2(D))^d} \mathbf{1}_{d_j^\varepsilon \leq \frac{\varepsilon}{M}}. \end{aligned} \tag{2.106}$$

Let us prove that the first right hand side of (2.106) vanishes in the limit using (2.97) and $d_j^\varepsilon \geq M^{-1}\varepsilon$, we get

$$\begin{aligned} \sum_{z_j \in n^\varepsilon} \left\| \nabla w_2^{\varepsilon,j} \right\|^2_{(L^2(D))^d} 1_{\rho_j \geq M} \mathbf{1}_{d_j^\varepsilon \geq M^{-1}\varepsilon} &\leq \sum_{z_j \in n^\varepsilon} \frac{(d-2)\sigma_d \varepsilon^d \rho_j^{(d-2)}}{1 - \left(d_j^\varepsilon\right)^{-(d-2)} \varepsilon^d \rho_j^{(d-2)}} 1_{\rho_j \geq M} \mathbf{1}_{d_j^\varepsilon \geq M^{-1}\varepsilon} \\ &\leq \sum_{z_j \in n^\varepsilon} \frac{(d-2)\sigma_d \varepsilon^d \rho_j^{(d-2)}}{1 - M^{d-2} \varepsilon^2 \rho_j^{d-2}} 1_{\rho_j \geq M} \mathbf{1}_{d_j^\varepsilon \geq M^{-1}\varepsilon} \\ &\leq (d-2)\sigma_d \sum_{z_j \in n^\varepsilon} \varepsilon^d \rho_j^{(d-2)} 1_{\rho_j \geq M}. \end{aligned}$$

Applaying lemma 2.9 to the process $\left(\Phi, \left\{\rho_j^{(d-2)} 1_{\rho_j \geq M}\right\}_{z_j \in \Phi}\right)$, one has

$$\lim_{\varepsilon \to 0} \sum_{z_j \in n^\varepsilon} \left\|\nabla w^{\varepsilon,j}\right\|^2_{(L^2(D))^d} 1_{\rho_j \geq M} \mathbf{1}_{d_j^\varepsilon \geq M^{-1}\varepsilon} \leq (d-2)\sigma_d \left\langle \rho^{(d-2)} 1_{\rho \geq M} \right\rangle \langle N(Q) \rangle |D|,$$

where Q is a unitary cube. Sending $M \to +\infty$, we get

$$\lim_{M \to +\infty} \sup \lim_{\varepsilon \to 0} \sum_{z_j \in n^\varepsilon} \left\|\nabla w_2^{\varepsilon,j}\right\|^2_{(L^2(D))^d} 1_{\rho_j \geq M} \mathbf{1}_{d_j^\varepsilon \geq M^{-1}\varepsilon} = 0. \tag{2.107}$$

In the other hand, we have

$$\begin{aligned} \sum_{z_j \in n^\varepsilon} \left\|\nabla w_2^{\varepsilon,j}\right\|^2_{(L^2(D))^d} \mathbf{1}_{d_j \leq \frac{\varepsilon}{M}} &= \sum_{z_j \in n^\varepsilon} \frac{(d-2)\sigma_d \varepsilon^d \rho_j^{(d-2)}}{1 - \left(d_j^\varepsilon\right)^{-(d-2)} \varepsilon^d \rho_j^{(d-2)}} \mathbf{1}_{d_j^\varepsilon \leq \frac{\varepsilon}{M}} \\ &\leq \sum_{z_j \in n^\varepsilon} (d-2)\sigma_d \varepsilon^d \rho_j^{(d-2)} \mathbf{1}_{d_j^\varepsilon \leq \frac{\varepsilon}{M}}. \end{aligned}$$

Since $d_j^\varepsilon \leq \frac{\varepsilon}{M}$, then

$$\min\left\{dist(\varepsilon z_j, D_b^\varepsilon), \frac{1}{2}\min_{i \neq j} \varepsilon |z_i - z_j|, \varepsilon\right\} \leq \frac{\varepsilon}{M},$$

it follows

$$\begin{cases} dist(\varepsilon z_j, D_b^\varepsilon) \leq \frac{\varepsilon}{M}, \text{ or} \\ \min_{i \neq j} \varepsilon |z_i - z_j| \leq \frac{2\varepsilon}{M}. \end{cases}$$

So, we can writte

$$\begin{cases} z_j \in I_M^\varepsilon = \left\{z_j \in n^\varepsilon \cap \Phi^\varepsilon_{2M^{-1}}(D),\ dist(z_j, D_b^\varepsilon) \leq \frac{\varepsilon}{M}\right\}, \quad \text{or} \\ z_j \in \Phi^\varepsilon(D) \backslash \Phi^\varepsilon_{2M^{-1}}(D). \end{cases}$$

Then, one has

$$\begin{aligned} \lim_{\varepsilon \to 0} \sup \sum_{z_j \in n^\varepsilon} \left\|\nabla w_2^{\varepsilon,j}\right\|^2_{(L^2(D))^d} \mathbf{1}_{d_j \leq \frac{\varepsilon}{M}} &\leq \lim_{\varepsilon \to 0} \sup \sum_{z_j \in \Phi^\varepsilon(D) \backslash \Phi^\varepsilon_{2M^{-1}}(D)} (d-2)\sigma_d \varepsilon^d \rho_j^{(d-2)} \\ &\quad + \lim_{\varepsilon \to 0} \sup \sum_{z_j \in I_M^\varepsilon} (d-2)\sigma_d \varepsilon^d \rho_j^{(d-2)}. \end{aligned}$$

By (2.68) of lemma 2.8, for $\delta = \frac{1}{M}$ we have

$$\lim_{\varepsilon \to 0} \varepsilon^d \# I_M^\varepsilon = 0,$$

then we can apply lemma 2.10, we get

$$\lim_{\varepsilon\to 0}\sup \sum_{z_j\in I_M^\varepsilon}(d-2)\sigma_d\varepsilon^d\rho_j^{(d-2)}=0.$$

On the other hand, applying lemma 2.9 for the process Φ and $\Phi_{\frac{2}{M}}$ we obtain

$$\lim_{\varepsilon\to 0}\sup \sum_{z_j\in\Phi^\varepsilon(D)\setminus\Phi^\varepsilon_{2M^{-1}}(D)}(d-2)\sigma_d\varepsilon^d\rho_j^{(d-2)}=(d-2)\sigma_d\left\langle N(Q)-N_{2M^{-1}}(Q)\right\rangle\left\langle\rho^{(d-2)}\right\rangle|D|\,.$$

Since we have

$$\lim_{M\to 0}\left\langle N(Q)-N_{2M^{-1}}(Q)\right\rangle=0,$$

we get

$$\lim_{M\to+\infty}\lim_{\varepsilon\to 0}\sup \sum_{z_i\in\Phi^\varepsilon(D)\setminus\Phi^\varepsilon_{2M^{-1}}(D)}(d-2)\sigma_d\varepsilon^d\rho_j^{(d-2)}=0. \tag{2.108}$$

By (2.107) and (2.108) we conclude immediatly (2.105).

It remains to prove (H3). First, we show that it sufficient to prove (H3) for truncated sequences $\left\{w_2^{\varepsilon,M}\right\}_{\varepsilon>0}$ for a fixed $M\in\mathbb{N}$. Namely

$$\left(-\Delta w_2^{\varepsilon,M},v_\varepsilon\right)_{H^{-1}(D),H_0^1(D)}\to C_{0,M}\int_D v, \tag{2.109}$$

with $C_{0,M}=(d-2)\sigma_d\left\langle N_{2M^{-1}}(Q)\right\rangle\left\langle\rho_M^{d-2}\right\rangle$. Indeed, we have by Cauchy-Schwarz inequality and for v_ε, v defined as in (H3)

$$\begin{aligned}\left|(-\Delta w_2^\varepsilon,v_\varepsilon)_{H^{-1}(D),H_0^1(D)}-C_0\int_D v\right| \;&\leq\; \left|\left(-\Delta\left(w_2^\varepsilon-w_2^{\varepsilon,M}\right),v_\varepsilon\right)_{H^{-1}(D),H_0^1(D)}\right|\\ &\quad+\left|(C_0-C_{0,M})\int_D v\right|\\ &\quad+\left|\left(-\Delta w_2^{\varepsilon,M},v_\varepsilon\right)_{H^{-1}(D),H_0^1(D)}-C_{0,M}\int_D v\right|.\end{aligned}$$

Using Green's formula, one has

$$\begin{aligned}\left|\left(-\Delta\left(w_2^\varepsilon-w_2^{\varepsilon,M}\right),v_\varepsilon\right)_{H^{-1}(D),H_0^1(D)}\right| \;&=\; \left|\int_D\nabla\left(w_2^\varepsilon-w_2^{\varepsilon,M}\right)\nabla v_\varepsilon\right|\\ &\leq\; \left(\int_D\left|\nabla\left(w_2^\varepsilon-w_2^{\varepsilon,M}\right)\right|^2\right)^{\frac12}\left(\int_D|\nabla v_\varepsilon|^2\right)^{\frac12}.\end{aligned}$$

Using (2.105) and since $v_\varepsilon \in H^1(D)$, then

$$\lim_{M\to+\infty}\limsup_{\varepsilon\to 0}\left|\left(-\Delta\left(w_2^\varepsilon - w_2^{\varepsilon,M}\right), v_\varepsilon\right)_{H^{-1}(D),H_0^1(D)}\right| = 0. \tag{2.110}$$

We have also

$$\left|(C_0 - C_{0,M})\int_D v\right| \le (d-2)\sigma_d \left\langle N(Q) - N_{2M^{-1}}(Q)\right\rangle \left\langle \rho^{d-2} - \rho_M^{d-2}\right\rangle \|v\|_{L^1(D)},$$

Using (2.122) of lemma 2.9 for $\delta = M^{-1}$ and by assumption (2.10) one has

$$\lim_{M\to+\infty}\left|(C_0 - C_{0,M})\int_D v\right| = 0. \tag{2.111}$$

Then, by (2.110) and (2.111) we conclude that we need only to prove (2.109). The proof of (2.109) follows the same lines of the third step of the proof of (H3) for the periodic case for w_2^ε. We just put here the changes in the proof, we arguing as that case we prove only that

$$\eta_M^\varepsilon = \sum_{z_j\in n_M^\varepsilon} d(d-2)\rho_{j,M}^{d-2}\frac{\varepsilon^d}{\left(d_j^\varepsilon\right)^d}\mathbf{1}_{B_j^\varepsilon} \overset{*}{\rightharpoonup} C_{0,M} \quad \text{in } L^\infty(D). \tag{2.112}$$

The factor $\frac{\varepsilon^d}{d_j^\varepsilon}$ in this latter is due to the fact that the balls B_j^ε have radii d_j^ε instead of $\frac{\varepsilon}{2}$. Since $\rho_{j,M} \le M$ and $\frac{\varepsilon}{d_j^\varepsilon} \le M$, we have

$$\|\eta_M^\varepsilon\|_{L^\infty(D)} = \sum_{z_j\in n_M^\varepsilon} d(d-2)\rho_{j,M}^{d-2}\frac{\varepsilon^d}{\left(d_j^\varepsilon\right)^d} \le M^{d(d-2)} d(d-2)\,\#\left(n_M^\varepsilon\right) < \infty.$$

Then, since η_M^ε is bounded in $L^\infty(D)$, and the density of $\mathcal{C}_0^1(D)$ in $L^1(D)$ by Hahn-Banach corollary (Corollary A.2.1 See appendix A) applied to the continuous linear form T defined by

$$T(\varphi) = \int_D \eta_M^\varepsilon \varphi \quad \text{for } \varphi \in \mathcal{C}_0^1(D),$$

it's sufficient to test η_M^ε only for $\zeta \in \mathcal{C}_0^1(D)$ ($\mathcal{C}_0^1(D)$ is the space of functions of classe $\mathcal{C}^1$ with compact support in D). To prove (2.112), we define

$$\tilde{\eta}_M^\varepsilon = \sum_{z_j\in \Phi_{\frac{2}{M}}(D)} d(d-2)\rho_{j,M}^{d-2}\frac{\varepsilon^d}{\left(d_j^\varepsilon\right)^d}\mathbf{1}_{B_j^\varepsilon} \tag{2.113}$$

and prove that for $\zeta \in \mathcal{C}_0^1(D)$

$$\int_D \left(\tilde{\eta}_M^\varepsilon - \eta_M^\varepsilon\right)\zeta \to 0 \tag{2.114}$$

and

$$\int_D \tilde{\eta}_M^\varepsilon \zeta \to C_{0,M} \int_D \zeta. \tag{2.115}$$

Indeed, we have

$$\begin{aligned}
\int_D (\tilde{\eta}_M^\varepsilon - \eta_M^\varepsilon)\, \zeta &= \sum_{z_j \in \Phi^\varepsilon_{2M^{-1}}(D) \backslash n_M^\varepsilon} d(d-2) \rho_{j,M}^{d-2} \frac{\varepsilon^d}{\left(d_j^\varepsilon\right)^d} \int_{B_j^\varepsilon} |\zeta| \\
&\leq d(d-2) M^{d-2} \sum_{z_j \in \Phi^\varepsilon_{2M^{-1}}(D) \backslash n_M^\varepsilon} \int_{B_\varepsilon(\varepsilon z_j)} |\zeta| \\
&\leq M^{d-2} \left\| \zeta \right\|_{L^\infty(D)} \varepsilon^d \# \left(\left\{ z_j \in \Phi^\varepsilon_{2M^{-1}}(D) : d_j^\varepsilon \leq \frac{\varepsilon}{M} \right\} \right).
\end{aligned}$$

Applying (2.68) of lemma 2.8, (2.114) yields immediatly. In other hand, applying lemma 2.11 for $\left(\Phi^\varepsilon_{2M^{-1}}, \left\{ \rho_{j,M}^{d-2} \right\} \right)$, one has almost surely

$$\sum_{z_j \in \Phi^\varepsilon_{2M^{-1}}(D) \backslash n_M^\varepsilon} d(d-2) \rho_{j,M}^{d-2} \frac{\varepsilon^d}{\left(d_j^\varepsilon\right)^d} \int_{B_j^\varepsilon} \zeta \to \frac{\sigma_d}{d} \left\langle N(Q) \right\rangle \left\langle \rho_M^{d-2} \right\rangle \int_D \zeta,$$

then (2.115) holds. ■

2.2.3 Proof of theorem 2.2

In this subsection, we give the proof of theorem 2.2 using lemma 2.2 similarly as the first chapter. Indeed, let $\omega \in \Omega$ be fixed for which the function $\{w^\varepsilon(\omega, .)\}_{\varepsilon > 0}$ of lemma 2.2 exist and satisfy hypotheses (H1), (H2) and (H3). Taking $v = u_\varepsilon$ in (2.9), we get

$$\int_{D^\varepsilon(\omega)} |\nabla u_\varepsilon|^2 = \langle f, u_\varepsilon \rangle_{H^{-1}(D^\varepsilon(\omega)), H_0^1(D^\varepsilon(\omega))},$$

then,

$$\begin{aligned}
\int_D |\nabla \tilde{u}_\varepsilon|^2 &= \int_{D^\varepsilon(\omega)} |\nabla u_\varepsilon|^2 = \langle f, u_\varepsilon \rangle_{H^{-1}(D^\varepsilon(\omega)), H_0^1(D^\varepsilon(\omega))} \\
&= \langle f, \tilde{u}_\varepsilon \rangle_{H^{-1}(D), H_0^1(D)} \\
&\leq \|f\|_{H^{-1}(D)} \|\tilde{u}_\varepsilon\|_{H_0^1(D)}.
\end{aligned} \tag{2.116}$$

Poincaré's inequality gives

$$\|\tilde{u}_\varepsilon\|_{H_0^1(D)} \leq C \|f\|_{H^{-1}(D)} < +\infty,$$

which a constant $C > 0$ that depends only on the domain D.

Then by Eberlein-$\tilde{\text{S}}$muljan theorem up to a subsequence which may depend on ω, one has

$$\tilde{u}_\varepsilon \rightharpoonup u_h \text{ weakly in } H_0^1(D) \text{ when } \varepsilon \to 0^+. \tag{2.117}$$

Let us show that u_h solves (2.11), for a fixed test function $\varphi \in \mathcal{D}(D)$ and since (H1) yields for w^ε then $\varphi w^\varepsilon \in H_0^1(D)$. we can substitute φw^ε in (2.9) we get

$$\int_D \varphi \nabla \tilde{u}_\varepsilon \nabla w^\varepsilon + \int_D w^\varepsilon \nabla \tilde{u}_\varepsilon \nabla \varphi = \langle f, \varphi w^\varepsilon \rangle_{H^{-1}(D), H_0^1(D)}. \tag{2.118}$$

By (H2), the right-hand side converges to

$$\langle f, \varphi w^\varepsilon \rangle_{H^{-1}(D), H_0^1(D)} \to \langle f, \varphi \rangle_{H^{-1}(D), H_0^1(D)}.$$

We now rewrite the left-hand side of (2.118) using Green formula

$$\begin{aligned} \int_D \varphi \nabla \tilde{u}_\varepsilon \nabla w^\varepsilon + \int_D w^\varepsilon \nabla \tilde{u}_\varepsilon \nabla \varphi &= \langle -\Delta w^\varepsilon, \varphi \tilde{u}_\varepsilon \rangle_{H^{-1}(D), H_0^1(D)} \\ &\quad - \int_D \tilde{u}_\varepsilon \nabla w^\varepsilon \nabla \varphi + \int_D w^\varepsilon \nabla \varphi \nabla \tilde{u}_\varepsilon. \end{aligned}$$

For the first term on the right-hand side, we use (H3) one has

$$\langle -\Delta w^\varepsilon, \varphi \tilde{u}_\varepsilon \rangle_{H^{-1}(D), H_0^1(D)} \to C_0 \int_D u_h \varphi.$$

For the second, by (2.117) and (H2)

$$\int_D \tilde{u}_\varepsilon \nabla w^\varepsilon \nabla \varphi \to 0.$$

Using (H2) and (2.117) yields

$$\int_D w^\varepsilon \nabla \varphi \nabla \tilde{u}_\varepsilon \to \int_D \nabla \varphi \nabla u_h.$$

These results gives

$$\int_D \nabla \varphi \nabla u_h + C_0 \int_D u_h \varphi = \langle f, \varphi \rangle_{H^{-1}(D), H_0^1(D)}.$$

We use Green formula again, we obtain

$$\langle -\Delta u_h + C_0 u_h, \varphi \rangle_{\mathcal{D}'(D), \mathcal{D}(D)} = \langle f, \varphi \rangle_{\mathcal{D}'(D), \mathcal{D}(D)},$$

then

$$-\Delta u_h + C_0 u_h = f \quad \text{ in } \mathcal{D}'(D).$$

Let us show the uniqueness of u_h. if u_1 and u_2 two solutions of (2.11) then they satisfy for $\varphi \in \mathcal{D}(D)$

$$\int_D \nabla\varphi\nabla u_1 + C_0 \int_D u_1\varphi = \langle f, \varphi\rangle_{H^{-1}(D),H_0^1(D)},$$

and

$$\int_D \nabla\varphi\nabla u_2 + C_0 \int_D u_2\varphi = \langle f, \varphi\rangle_{H^{-1}(D),H_0^1(D)}.$$

The substruction gives

$$\int_D \nabla\varphi\nabla (u_1 - u_2) + C_0 \int_D (u_1 - u_2)\varphi = 0,$$

taking $\varphi = u_1 - u_2$, we get

$$\int_D |\nabla (u_1 - u_2)|^2 = -C_0 \int_D |u_1 - u_2|^2,$$

since $C_0 > 0$, then by Poincaré's inequality yields

$$u_1 = u_2.$$

Thus the uniqueness of u_h.

2.3 Auxiliary results

We define the marked point process (Φ, χ) where the process Φ satisfies the properties (2.2), (2.3) and (2.4) and the marks $\chi = \{X_i\}_{z_i\in\Phi}$ satisfying (2.5) and (2.6) with

$$\langle X\rangle = \int_0^{+\infty} xh_X(x)dx < +\infty. \tag{2.119}$$

with h_X is the density function of $X \in \chi$.

Lemma 2.9 *Let Q a unitary cube and let (Φ, χ) be a marked point process as introduced above. Then, for every bounded set $B \subseteq \mathbb{R}^d$ which is star shaped with respect to the origin, we have*

$$\lim_{\varepsilon\to 0} \varepsilon^d N^\varepsilon(B) = \langle N(Q)\rangle |B| \quad \textit{almost surely,} \tag{2.120}$$

and

$$\lim_{\varepsilon\to 0} \varepsilon^d \sum_{z_i\in\Phi^\varepsilon(B)} X_i = \langle N(Q)\rangle \langle X\rangle |B| \quad \textit{almost surely.} \tag{2.121}$$

Furthermore, for every $\delta < 0$ the process Φ_δ obtained from Φ as in 2.14 satisfies the analogue of (2.120), *(2.121) and*

$$\lim_{\varepsilon\to 0} \varepsilon^d \langle N_\delta(A)\rangle = \langle N(A)\rangle . \tag{2.122}$$

for every bounded set $A \subseteq \mathbb{R}^d$.

Proof. In order to simplify, we prove this lemma for

$$B = Q^R = \left]-\frac{R}{2}, \frac{R}{2}\right[,$$

i.e Q^R is a cube of size R centered at the origin and $\frac{1}{\varepsilon}B = Q^{\frac{R}{\varepsilon}}$. Let $\{Q_{z_i}\}_{z_i\in\mathbb{Z}^d}$ or $\{Q_i\}_{z_i\in\mathbb{Z}^d}$ the partition of $\mathbb{R}^d$ made of essentially disjoint unit cubes centered in the points of the lattice $\mathbb{Z}^d = \{z_i\}_{i\in\mathbb{N}}$. For all $\mu > 0$ and all ε small enough, we have

$$\varepsilon^d \sum_{z_i\in\Phi^\varepsilon(Q^R)} X_i = \varepsilon^d \sum_{z_i\in\mathbb{Z}^d} \mathbf{1}_{\Phi^\varepsilon(Q^R)} \sum_{z_j\in\Phi(Q_i)} X_j = \varepsilon^d \sum_{z_i\in\mathbb{Z}^d\cap Q^{\frac{R}{\varepsilon}}} \sum_{z_j\in\Phi(Q_i)} X_j,$$

where $\mathbf{1}_{\Phi^\varepsilon(Q^R)}$ is the characteristic function of the set $\Phi^\varepsilon(Q^R)$. Since $Q^R \subset Q^{R+\mu}$ we can write

$$\varepsilon^d \sum_{z_i\in\Phi^\varepsilon(Q^R)} X_i \le \varepsilon^d \sum_{z_i\in\mathbb{Z}^d\cap Q^{\frac{R+\mu}{\varepsilon}}} \sum_{z_j\in\Phi(Q_i)} X_j. \tag{2.123}$$

We can denote by Z_i the following sum

$$Z_i = \sum_{z_j\in\Phi(Q_i)} X_j. \tag{2.124}$$

By definition of $\Phi(Q_i)$, the cardinality of $\Phi(Q_i)$ is finite then since a finite sum of random variables is a random variable so for every $z_i \in \mathbb{Z}^d \cap Q^{\frac{R+\mu}{\varepsilon}}$, Z_i are random variables. In addition, the point process Φ is stationary then

$$\langle \#\Phi(Q_i)\rangle = \langle \#\Phi(Q)\rangle \text{ for any } z_i \in \mathbb{Z}^d,$$

hence

$$\langle Z_i\rangle = \left\langle \sum_{z_j\in\Phi(Q_i)} X_j \right\rangle = \langle N(Q)\rangle \langle X\rangle$$

and the random variables Z_i are identically distributed. We have also by the assumption (2.119) for every $z_i \in \mathbb{Z}^d \cap Q^{\frac{R+\mu}{\varepsilon}}$ that

$$\langle Z_i\rangle < +\infty.$$

In the other hand we have for every $z_i, z_j \in \mathbb{Z}^d \cap Q^{\frac{R+\mu}{\varepsilon}}$ with $i \neq j$

$$\begin{aligned}\left|\langle Z_i Z_j\rangle - \langle Z\rangle^2\right| &= \left|\left\langle \sum_{z_l\in\Phi(Q_i)} \sum_{z_k\in\Phi(Q_j)} X_l X_k \right\rangle - \langle N(Q)\rangle^2 \langle X\rangle^2\right| \\ &= \left|\langle X_i X_j\rangle \langle N(Q_i) N(Q_j)\rangle - \langle N(Q)\rangle^2 \langle X\rangle^2\right|,\end{aligned}$$

with $Z \in \{Z_i\}_{z_i\in\mathbb{Z}^d\cap Q^{\frac{R+\mu}{\varepsilon}}}$. We have for $x, y \in \mathbb{R}^+$ and by the assumption (2.6)

$$\begin{aligned}\langle X_i X_j\rangle &= \int_0^{+\infty}\int_0^{+\infty} xy h_{X_iX_j}(x,y)dxdy = \int_0^{+\infty}\int_0^{+\infty} xy h_{X_i}(x) h_{X_j}(y)\, dxdy \\ &+ \frac{c}{1+|z_i-z_j|^\gamma}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(1+x^p)(1+y^p)}dxdy,\end{aligned}$$

Since $xyh_{X_i}(x)h_{X_j}(y)$ and $\frac{1}{(1+x^p)(1+y^p)}$ are positive then we can apply Fubini's theorem (See appendix A)one has

$$\langle X_i X_j\rangle = \langle X\rangle^2 + \frac{C}{1+|z_i-z_j|^\gamma},$$

with

$$C = c\int_0^{+\infty}\frac{1}{(1+x^p)}\int_0^{+\infty}\frac{1}{(1+y^p)},$$

which is finite since $p > d-1$. Then we get

$$\begin{aligned}\left|\langle Z_i Z_j\rangle - \langle Z\rangle^2\right| &\leq \left|\langle X\rangle^2 \langle N(Q_i) N(Q_j)\rangle - \langle X\rangle^2 \langle N(Q)\rangle^2\right| \\ &+ \frac{C}{|z_i-z_j|^\gamma}\langle N(Q_i) N(Q_j)\rangle. \end{aligned} \tag{2.125}$$

By stationarity of Φ, we have that for any $i, j \in \mathbb{N}, i \neq j$

$$\langle N(Q_i) N(Q_j)\rangle = \langle N(Q_{i-j}) N(Q)\rangle,$$

so for $N(Q), N(Q_{i-j})$ two random variables, measurable with respect to $\mathcal{F}(Q)$ and $\mathcal{F}(Q_{i-j} = \tau_{z_i-z_j}Q)$ ($\mathcal{F}(Q)$ is the smallest σ-algebra which make the random variable $N(Q)$ measurable), there exists $C_1 < +\infty$, such that for $\gamma > 0$ and $|z_i - z_j| > diam(Q)$ (with $diam(Q)$ denotes the diameter of Q), we have

$$\begin{aligned}\left|\langle N(Q_i) N(Q_j)\rangle - \langle N(Q)\rangle^2\right| &\leq \frac{C_1}{1+(|z_i-z_j| - diam(Q))^\gamma}\left\langle N(Q)^2\right\rangle \\ &\leq \frac{C_1}{|z_i-z_j|^\gamma}\left\langle N(Q)^2\right\rangle.\end{aligned} \tag{2.126}$$

We thus insert this latter (2.126) into (2.125), we get

$$\left|\langle Z_i Z_j\rangle - \langle Z\rangle^2\right| \leq \frac{M}{|z_i - z_j|^\gamma}\left\langle N\left(Q\right)^2\right\rangle, \tag{2.127}$$

which a constant $M > 0$. So the conditions of lemma B.2.3 (See appendix B) are satisfied and then we can apply the strong law of large numbers for the sequence Z_i and conclude that we have for $\mu > 0$

$$\lim_{\varepsilon\to 0}\sup \varepsilon^d \sum_{z_i\in\Phi^\varepsilon(Q^R)} X_i \leq \langle N(Q)\rangle \langle X\rangle \left|Q^{R+\mu}\right|. \tag{2.128}$$

Arguing analogously for the lower limit by taking the following inequality

$$\varepsilon^d \sum_{z_i\in\Phi^\varepsilon(Q^R)} X_i \geq \varepsilon^d \sum_{z_i\in\mathbb{Z}^d\cap Q^{\frac{R-\mu}{\varepsilon}}} Z_i,$$

where $\mu > 0$ and Z_i is defined as (2.124). So since Z_i satisfies the assumption of lemma B.2.3, we can apply the strong law of large numbers to Z_i, it follows

$$\lim_{\varepsilon\to 0}\inf \varepsilon^d \sum_{z_i\in\Phi^\varepsilon(Q^R)} X_i \geq \langle N(Q)\rangle \langle X\rangle \left|Q^{R-\mu}\right|. \tag{2.129}$$

Thus by (2.128) and (2.129), the result (2.121) holds true. To prove (2.120), we take $X_i = 1$ for all $z_i \in \mathbb{Z}^d$ in (2.121) as follow

$$\varepsilon^d \sum_{z_i\in\Phi^\varepsilon(Q^R)} = \langle N(Q)\rangle \left|Q^R\right|,$$

and we get our result. For $\delta > 0$ be fixed, let us show (2.122). Indeed, for $\delta' > 0$ and δ small enough

$$\mathbb{P}\left(\left(N(B) - N^\delta\left(B\right)\right) > \delta'\right) = \mathbb{P}\left(\#\left\{x \in \Phi\cap B : \min_{\substack{y\in\Phi(\omega)\cap B\\ y\neq x}} |x-y| < \delta\right\} > \delta'\right) = 0.$$

then

$$N^\delta\left(B\right) \underset{\delta\to 0}{\to} N\left(B\right) \text{ almost surely.}$$

We have also $\Phi_\delta \cap B \subseteq \Phi\cap B$ then

$$N^\delta(B) = \#\Phi_\delta \cap B \leq N(B) = \#\Phi\cap B$$

and $\langle N(B)\rangle < +\infty$, it follows from the dominated convergence theorem (See appendix B) that

$$\lim_{\delta\to 0}\left\langle N^{\delta}(B)\right\rangle = \langle N(B)\rangle .$$

To show (2.120) and (2.121) for Φ_δ we may argue exactly as above for the original process Φ and apply the strong law of large numbers to the random variables

$$Z_i^{\delta} = \sum_{z_j\in\Phi_\delta^{\varepsilon}(Q_i)} X_j^{\delta}.$$

Since for each $z_i \in \mathbb{Z}^d$ we have $\Phi_\delta^{\varepsilon}(Q_i) \subseteq \Phi^{\varepsilon}(Q_i)$, then

$$0 \leq Z_i^{\delta} \leq Z_i.$$

So the only condition that remains to be proved for the collection $\left\{Z_i^{\delta}\right\}_{z_i\in\mathbb{Z}^d}$ is (2.125). By arguing same as (2.123), we have

$$\left|\langle Z_iZ_j\rangle - \langle Z\rangle^2\right| \leq \left|\langle X\rangle^2\left\langle N^{\delta}(Q_i)N^{\delta}(Q_j)\right\rangle - \left\langle N^{\delta}(Q)\right\rangle^2\langle X\rangle^2\right| + \frac{C}{|z_i - z_j|^{\gamma}}\left\langle N^{\delta}(Q_i)N^{\delta}(Q_j)\right\rangle ,$$

with Z a random variable take the same expectation with Z_j for every $z_i \in \mathbb{Z}^d$. So the only challenge here is to prove (2.126) for $N^{\delta}(Q_i)$ instead of $N(Q_i)$ for any $z_i \in \mathbb{Z}^d$. To do that, for every $x \in \mathbb{R}^d$, we define

$$d_x = \min_{\substack{y\in\Phi,\\ y\neq x}} |x-y| ,$$

which allows to write

$$N^{\delta}(Q) = \sum_{z_i\in\Phi_\delta\cap Q} 1 = \sum_{z_i\in\Phi\cap Q} \mathbf{1}_{d_x>\delta}(z_i),$$

and

$$N^{\delta}(Q) = \sum_{z_i\in\tau_{x_i}(\Phi\cap Q)} \mathbf{1}_{d_x>\delta}(z_i).$$

where $\tau_{x_i}((\Phi\cap Q))$ is the translation of $(\Phi\cap Q)$ to $(\Phi\cap Q_i)$ by the vector x_i. Since

$$\mathbf{1}_{d_x>\delta} = \mathbf{1}_{N(B_\delta(x)\backslash\{x\})=0},$$

where

$$B_\delta(x)\backslash\{x\} = \left\{y\in\mathbb{R}^d\backslash\{x\}, |x-y|\leq\delta\right\}.$$

It follows that, each $N^\delta(Q_i)$ is a measurable random variable with repect to $\mathcal{F}(B_\delta(Q_i))$ defined as in (2.4) with

$$B_\delta(Q_i) = \left\{y \in \mathbb{R}^d, dist(x, Q_i) \leq \delta\right\}.$$

Then, we can apply (2.4) as in (2.126) we get

$$\left|\left\langle N^\delta(Q_{i-j})N^\delta(Q)\right\rangle - \left\langle N^\delta(Q)\right\rangle^2\right| \leq \frac{C}{|z_i - z_j|}\left\langle N^\delta(Q)^2\right\rangle.$$

■

Lemma 2.10 *In the same setting of the previous lemma 2.9, let* $\{I_\varepsilon\}_{\varepsilon>0}$ *be a familly of collections of points such that* $I_\varepsilon \subseteq \Phi^\varepsilon(B)$ *and*

$$\lim_{\varepsilon\to 0} \varepsilon^d \# I_\varepsilon = 0 \qquad \textit{almost surely.} \tag{2.130}$$

Then,

$$\lim_{\varepsilon\to 0} \varepsilon^d \sum_{z_i\in I_\varepsilon} X_i = 0 \quad \textit{almost surely.}$$

Proof. Let $M \in \mathbb{N}$. We define for every $z_i \in \Phi$ the truncated marks $\{Y_i\}_{z_i\in\Phi}$ as follow

$$Y_i = X_i \mathbf{1}_{[M,\infty)} = \begin{cases} X_i & \text{if} \quad X_i \geq M, \\ 0 & \text{if} \quad X_i < M. \end{cases}$$

Since the original marks $\{X_i\}_{z_i\in\Phi}$ satisfies the assumptions (2.5) and (2.6) then the truncated marks $\{Y_i\}_{z_i\in\Phi} \subseteq \{X_i\}_{z_i\in\Phi}$ satisfies the same assumptions. Moreover, we have by lemma B.1.4 (See appendix B)

$$\begin{aligned} \langle Y_i\rangle &= \int_0^{+\infty} P(Y_i > y)dy = \int_0^{+\infty} P(X_i\mathbf{1}_{[M,\infty)} > y)dy = \int_0^{+\infty} P(X_i > y)\mathbf{1}_{y\leq M}dy = \int_0^{M} P(X_i > y)dy \\ &\leq \int_0^{+\infty} P(X_i > y)dy = \langle X\rangle < +\infty, \end{aligned}$$

then, we can apply lemma 2.9 to the point process Φ with truncated marks $\{Y_i\}_{z_i\in\Phi}$ to infer that almost surely

$$\varepsilon^d \sum_{z_i\in\Phi^\varepsilon(B)} Y_i \to \left\langle X\mathbf{1}_{[M,+\infty)}\right\rangle.$$

This yields

$$\begin{aligned}\limsup_{\varepsilon\to 0}\varepsilon^d\sum_{z_i\in I_\varepsilon}X_i &= \limsup_{\varepsilon\to 0}\varepsilon^d\sum_{z_i\in I_\varepsilon}X_i\mathbf{1}_{[0,M)}+\limsup_{\varepsilon\to 0}\varepsilon^d\sum_{z_i\in I_\varepsilon}X_i\mathbf{1}_{[M,+\infty)}\\ &\leq \limsup_{\varepsilon\to 0}\varepsilon^d\sum_{z_i\in I_\varepsilon}X_i\mathbf{1}_{[0,M)}+\left\langle X\mathbf{1}_{[M,+\infty)}\right\rangle\\ &\leq M\limsup_{\varepsilon\to 0}\varepsilon^d\sum_{z_i\in I_\varepsilon}+\left\langle X\mathbf{1}_{[M,+\infty)}\right\rangle\\ &\leq M\limsup_{\varepsilon\to 0}\varepsilon^d\# I_\varepsilon+\left\langle X\mathbf{1}_{[M,+\infty)}\right\rangle,\end{aligned}$$

by the assumption (2.130), we obtain

$$\limsup_{\varepsilon\to 0}\varepsilon^d\sum_{z_i\in I_\varepsilon}X_i\leq\left\langle X\mathbf{1}_{[M,+\infty)}\right\rangle=P(X\in[M,+\infty)).$$

We may take the limit $M\to+\infty$ and conclude that

$$\limsup_{\varepsilon\to 0}\varepsilon^d\sum_{z_i\in I_\varepsilon}X_i=0.$$

Since X_i are positive, then our main result

$$\lim_{\varepsilon\to 0}\varepsilon^d\sum_{z_i\in I_\varepsilon}X_i=0,$$

holds true. ■

Lemma 2.11 *In the same setting of lemma 2.9, let us assume that in addition the marks satisfy* $\left\langle X^2\right\rangle<+\infty$. *For* $z_i\in\Phi$ *and* $\varepsilon>0$, *let* $r_{i,\varepsilon}>0$, *and assume that there exists a constant* $C>0$ *such that for all* $z_i\in\Phi$ *and* $\varepsilon>0$

$$r_{i,\varepsilon}\leq C\varepsilon. \tag{2.131}$$

Then, almost surely, we have

$$\lim_{\varepsilon\to 0}\sum_{z_i\in\Phi^\varepsilon(B)}X_i\frac{\varepsilon^d}{r_{i,\varepsilon}^d}\int_{B_{r_{i,\varepsilon}}(\varepsilon z_i)}\zeta(x)dx=\frac{\sigma_d}{d}\left\langle N\left(Q\right)\right\rangle\left\langle X\right\rangle\int_B\zeta\left(x\right)dx, \tag{2.132}$$

for every $\zeta\in\mathcal{C}_0^1\left(B\right)$.

Proof. First, we show that it suffices to prove (2.132) for $r_{i,\varepsilon}=\varepsilon$ for all $z_i\in\Phi$ and $\varepsilon>0$. For $\zeta\in\mathcal{C}_0^1\left(B\right)$, we put for $x\in\mathbb{R}^d$ and εz_i the center of the balls $B_\varepsilon(\varepsilon z_i)$, $B_{r_{i,\varepsilon}}(\varepsilon z_i)$

$$r=|x-\varepsilon z_i|$$

with

$$\tilde{\zeta}(r) = \tilde{\zeta}(|x - \varepsilon z_i|) = \zeta(x).$$

Then, we get

$$\begin{aligned}\sum_{z_i\in\Phi^\varepsilon(B)}\left|\frac{\varepsilon^d}{r_{i,\varepsilon}^d}\int_{B_{r_{i,\varepsilon}}(\varepsilon z_i)}\zeta(x)dx - \int_{B_\varepsilon(\varepsilon z_i)}\zeta(x)\,dx\right| &= \sum_{z_i\in\Phi^\varepsilon(B)}\left|\frac{\varepsilon^d}{r_{i,\varepsilon}^d}\sigma_d\int_0^{r_{i,\varepsilon}}\tilde{\zeta}(r)r^{d-1}dr - \sigma_d\int_0^{\varepsilon}\tilde{\zeta}(r)r^{d-1}dr\right| \\ &= \sum_{z_i\in\Phi^\varepsilon(B)}\left|\sigma_d\int_0^{\varepsilon}\tilde{\zeta}(\frac{r_{i,\varepsilon}^d}{\varepsilon^d}r)r^{d-1}dr - \sigma_d\int_0^{\varepsilon}\tilde{\zeta}(r)r^{d-1}dr\right|,\end{aligned}$$

using mean value theorem and the assumption $r_{i,\varepsilon} \leq C\varepsilon$, we get almost surely

$$\limsup_{\varepsilon\to 0}\sum_{z_i\in\Phi^\varepsilon(B)}\left|\frac{\varepsilon^d}{r_{i,\varepsilon}^d}\int_{B_{r_{i,\varepsilon}}(\varepsilon z_i)}\zeta(x)dx - \int_{B_\varepsilon(\varepsilon z_i)}\zeta(x)\,dx\right| \leq \limsup_{\varepsilon\to 0} c(d)\varepsilon\left\|\nabla\zeta\right\|_{(L^\infty(B))^d}\varepsilon^d N^\varepsilon(B) = 0,$$

with $c(d)$ is a positive constant independant of ε. Since $\varepsilon^d N^\varepsilon(B)$ is bounded by lemma 2.9, thus it suffices to argue (2.132) only for $r_{i,\varepsilon} = \varepsilon$. Without loss of generality we assume that $r_{i,\varepsilon} = \varepsilon$ and $|B| = 1$. We can remark by density of countable subset of $W_0^{1,\infty}(B)$ in $\mathcal{C}_0^1(B)$ that it suffices to show (2.132) only for $\zeta \in W_0^{1,\infty}(B)$. Let $\zeta \in W_0^{1,\infty}(B)$, we begin by writing

$$\begin{aligned}\sum_{z_i\in\Phi^\varepsilon(B)} X_i\int_{B_\varepsilon(\varepsilon z_i)}\zeta(x)\,dx &= \sum_{z_i\in\Phi^\varepsilon(B)}(X_i - \langle X\rangle)\int_{B_\varepsilon(\varepsilon z_i)}\zeta(x)\,dx \\ &\quad + \langle X\rangle\sum_{z_i\in\Phi^\varepsilon(B)}\int_{B_\varepsilon(\varepsilon z_i)}\zeta(x)\,dx.\end{aligned}$$

then

$$\begin{aligned}\left|\sum_{z_i\in\Phi^\varepsilon(B)} X_i\int_{B_\varepsilon(\varepsilon z_i)}\zeta(x)\,dx - \frac{\sigma_d}{d}\langle N(Q)\rangle\langle X\rangle\int_B\zeta\right| &\leq \left|\sum_{z_i\in\Phi^\varepsilon(B)}(X_i - \langle X_i\rangle)\int_{B_\varepsilon(\varepsilon z_i)}\zeta(x)\,dx\right| \\ &\quad + \langle X\rangle\left|\sum_{z_i\in\Phi^\varepsilon(B)}\int_{B_\varepsilon(\varepsilon z_i)}\zeta(x)\,dx - \frac{\sigma_d}{d}\langle N(Q)\rangle\int_B\zeta\right|.\end{aligned} \tag{2.133}$$

Let $\{Q_i\}_{i\in\mathbb{N}}$ be a partition of $\mathbb{R}^d$ into essentialy disjoint unitary cubes and let $\{y_i\}_{i\in\mathbb{N}}$ the collection of their centers. We claim that if T_ε, $\tilde{T}_\varepsilon$, R_ε and $\tilde{R}_\varepsilon$ defined by

$$T_\varepsilon(\zeta) = \int_B\zeta, \qquad \tilde{T}_\varepsilon(\zeta) = \varepsilon^d\sum_{Q_i\cap\frac{1}{\varepsilon}B\neq\emptyset}\zeta(\varepsilon y_i),$$

$$R_\varepsilon(\zeta) = \sum_{z_i\in\Phi^\varepsilon(B)} \int_{B_\varepsilon(\varepsilon z_i)} \zeta(x)\,dx, \qquad \tilde{R}_\varepsilon(\zeta) = \varepsilon^d \frac{\sigma_d}{d} \sum_{Q_i\cap\frac{1}{\varepsilon}B\neq\emptyset} N(Q)\,\zeta(\varepsilon y_i).$$

then

$$\lim_{\varepsilon\to 0} \left| T_\varepsilon(\zeta) - \tilde{T}_\varepsilon(\zeta) \right| = 0, \qquad \lim_{\varepsilon\to 0} \left| R_\varepsilon(\zeta) - \tilde{R}_\varepsilon(\zeta) \right| = 0 \text{ almost surely.} \tag{2.134}$$

The first limit is a standard Riemann sum, we have

$$\begin{aligned} \lim_{\varepsilon\to 0} \left| T_\varepsilon(\zeta) - \tilde{T}_\varepsilon(\zeta) \right| &= \lim_{\varepsilon\to 0} \left| \int_B \zeta - \varepsilon^d \sum_{Q_i\cap\frac{1}{\varepsilon}B\neq\emptyset} \zeta(\varepsilon y_i) \right| = \lim_{\varepsilon\to 0} \left| \int_B \zeta - \sum_{Q_i\cap\frac{1}{\varepsilon}B\neq\emptyset} \zeta(\varepsilon y_i)\,|Q_i| \right| \\ &= 0, \end{aligned}$$

with $|Q_i|$ is the Lebesgue measure of Q_i. Let us argue the second limit of (2.134) for $\zeta \in W_0^{1,\infty}(B)$, we have

$$\left| R_\varepsilon(\zeta) - \tilde{R}_\varepsilon(\zeta) \right| = \left| \sum_{Q_i\cap\frac{1}{\varepsilon}B\neq\emptyset} \left(\sum_{z_i\in\Phi^\varepsilon(Q_i)} \int_{B_\varepsilon(\varepsilon z_i)} \zeta(x)\,dx - \varepsilon^d \frac{\sigma_d}{d} N(Q)\,\zeta(\varepsilon y_i) \right) \right|.$$

Since by change of coordinates we have

$$\sum_{z_i\in\Phi^\varepsilon(Q_i)} \int_{B_\varepsilon} dx = \varepsilon^d \frac{\sigma_d}{d} N(Q), \tag{2.135}$$

then by mean value theorem and (2.135) we can write

$$\begin{aligned} \left| R_\varepsilon(\zeta) - \tilde{R}_\varepsilon(\zeta) \right| &= \left| \sum_{Q_i\cap\frac{1}{\varepsilon}B\neq\emptyset} \left(\sum_{z_i\in\Phi^\varepsilon(Q_i)} \int_{B_\varepsilon(\varepsilon z_i)} \zeta(x) - \zeta(\varepsilon y_i) \right) \right| \\ &\leq 2\varepsilon \left\| \nabla\zeta \right\|_{L^\infty(B)} \varepsilon^d N^\varepsilon(B). \end{aligned}$$

Since by lemma 2.9 the term $\varepsilon^d N^\varepsilon(B)$ is finite, then the second limit follows immediatly.

So we can use these results to write

$$\begin{aligned} &\limsup_{\varepsilon\to 0} \left| \sum_{z_i\in\Phi^\varepsilon(B)} X_i \int_{B_\varepsilon(\varepsilon z_i)} \zeta(x)\,dx - \frac{\sigma_d}{d} \langle N(Q)\rangle \langle X\rangle \int_B \zeta \right| \\ \leq\; &\limsup_{\varepsilon\to 0} \left| \sum_{z_i\in\Phi^\varepsilon(B)} (X_i - \langle X\rangle) \int_{B_\varepsilon(\varepsilon z_i)} \zeta(x)\,dx \right| \\ &+ \limsup_{\varepsilon\to 0} \left| \varepsilon^d \langle X\rangle \frac{\sigma_d}{d} \sum_{Q_i\cap\frac{1}{\varepsilon}B\neq\emptyset} \int_{B_\varepsilon(\varepsilon z_i)} \zeta(\varepsilon y_i)\left(N(Q_i) - \langle N(Q)\rangle\right) \right|. \end{aligned} \tag{2.136}$$

It remains to show that the two terms of the right-hand side of (2.136) vanishes almost surely, before that we define

$$\begin{aligned} a_{i,\varepsilon} &= \int_{B_\varepsilon(\varepsilon z_i)} \zeta(x)\,dx, \qquad \tilde{X}_i = X_i - \langle X_i \rangle, \\ S_\varepsilon &= \sum_{z_i \in \Phi^\varepsilon(B)} a_{i,\varepsilon} X_i, \qquad \tilde{S}_\varepsilon = \sum_{z_i \in \Phi^\varepsilon(B)} a_{i,\varepsilon} \tilde{X}_i. \end{aligned}$$

We begin by proving the first right hand side of (2.136) which means $\tilde{S}_\varepsilon$ defined above vanishes in the limit. For $\delta > 0$, we estimate by Chebyshev's inequality, one has

$$\mathbb{P}\left(\tilde{S}_\varepsilon > \delta\right) \le \delta^{-2} \left\langle \tilde{S}_\varepsilon^2 \right\rangle. \tag{2.137}$$

We want to show that $\tilde{S}_\varepsilon$ converges to zero in probability, so we rewrite

$$\begin{aligned} \left\langle \tilde{S}_\varepsilon^2 \right\rangle &= \left\langle \sum_{z_i, z_k \in \Phi^\varepsilon(B)} a_{i,\varepsilon} a_{k,\varepsilon} \tilde{X}_i \tilde{X}_k \right\rangle \\ &= \sum_{\substack{Q_i \cap \frac{1}{\varepsilon} B \neq \emptyset \\ Q_j \cap \frac{1}{\varepsilon} B \neq \emptyset.}} \left\langle \left(\sum_{z_l \in \Phi^\varepsilon(Q_j)} a_{l,\varepsilon} \tilde{X}_l \right) \left(\sum_{z_k \in \Phi^\varepsilon(Q_i)} a_{k,\varepsilon} \tilde{X}_k \right) \right\rangle. \end{aligned} \tag{2.138}$$

We set

$$Y_i = \sum_{z_l \in \Phi^\varepsilon(Q_i)} a_{l,\varepsilon} \tilde{X}_l,$$

it follows

$$\left\langle \tilde{S}_\varepsilon^2 \right\rangle = \left\langle \sum_{\substack{Q_i \cap \frac{1}{\varepsilon} B \neq \emptyset \\ Q_j \cap \frac{1}{\varepsilon} B \neq \emptyset.}} Y_i Y_j \right\rangle = \sum_{Q_i \cap \frac{1}{\varepsilon} B \neq \emptyset} \left\langle Y_i^2 \right\rangle + \sum_{\substack{i \neq j, Q_i \cap \frac{1}{\varepsilon} B \neq \emptyset, \\ Q_j \cap \frac{1}{\varepsilon} B \neq \emptyset.}} \left\langle Y_i Y_j \right\rangle. \tag{2.139}$$

For the second right hand side of (2.139), we can write

$$\sum_{\substack{i \neq j, Q_i \cap \frac{1}{\varepsilon} B \neq \emptyset, \\ Q_j \cap \frac{1}{\varepsilon} B \neq \emptyset.}} \left\langle Y_i Y_j \right\rangle = \sum_{\substack{Q_i \cap \frac{1}{\varepsilon} B \neq \emptyset, \\ Q_j \cap \frac{1}{\varepsilon} B \neq \emptyset.}} \left(\left\langle \sum_{z_l \in \Phi^\varepsilon(Q_j), z_k \in \Phi^\varepsilon(Q_i)} a_{l,\varepsilon} a_{k,\varepsilon} \tilde{X}_l \tilde{X}_k \right\rangle \right). \tag{2.140}$$

Since for $\zeta \in W_0^{1,\infty}(B)$ we have

$$|a_{l,\varepsilon}| = \left| \int_{B_\varepsilon(\varepsilon z_i)} \zeta(x)\,dx \right| \le \|\zeta\|_{L^\infty(B)} |B_\varepsilon| = \varepsilon^d \|\zeta\|_{L^\infty(B)},$$

then we argue similarly as lemma 2.9 using the assumption (2.6), we get

$$\begin{aligned}\sum_{\substack{i\neq j, Q_i\cap\frac{1}{\varepsilon}B\neq\emptyset,\\ Q_j\cap\frac{1}{\varepsilon}B\neq\emptyset.}}\langle Y_iY_j\rangle &\leq \sum_{\substack{i\neq j, Q_i\cap\frac{1}{\varepsilon}B\neq\emptyset,\\ Q_j\cap\frac{1}{\varepsilon}B\neq\emptyset.}}\varepsilon^{2d}\left\|\zeta\right\|^2_{L^\infty(B)}\left\langle\sum_{z_l\in\Phi^\varepsilon(Q_j),z_k\in\Phi^\varepsilon(Q_i)}\tilde{X}_l\tilde{X}_k\right\rangle\\ &\leq \sum_{\substack{Q_i\cap\frac{1}{\varepsilon}B\neq\emptyset,\\ Q_j\cap\frac{1}{\varepsilon}B\neq\emptyset.}}\varepsilon^{2d}\left\|\zeta\right\|^2_{L^\infty(B)}\frac{c\left\langle N(Q_i)N\left(Q_j\right)\right\rangle}{\left|z_i-z_j\right|^\gamma},\end{aligned}$$

with $\gamma > d$. Adding and substructing the term $\frac{c\langle N(Q)\rangle^2}{|z_i-z_j|^\gamma}$ we get by assumption (2.4)

$$\sum_{\substack{i\neq j, Q_i\cap\frac{1}{\varepsilon}B\neq\emptyset,\\ Q_j\cap\frac{1}{\varepsilon}B\neq\emptyset.}}\langle Y_iY_j\rangle \leq \sum_{\substack{Q_i\cap\frac{1}{\varepsilon}B\neq\emptyset,\\ Q_j\cap\frac{1}{\varepsilon}B\neq\emptyset.}}\varepsilon^{2d}\left\|\zeta\right\|^2_{L^\infty(B)}\frac{c\left\langle N(Q)^2\right\rangle}{\left|z_i-z_j\right|^\gamma}.$$

A similar estimation as the first limit in (2.134) for ε small enough gives

$$\sum_{\substack{i\neq j, Q_i\cap\frac{1}{\varepsilon}B\neq\emptyset\\ Q_j\cap\frac{1}{\varepsilon}B\neq\emptyset.}}\frac{1}{\left|z_i-z_j\right|^\gamma}=\frac{1}{\varepsilon^d}\int_B\varphi, \tag{2.141}$$

with φ defined as

$$\varphi\left(x\right)=\frac{1}{\left|x\right|^\gamma}\text{ for }x\neq 0_{\mathbb{R}^d},$$

and the assumption $\gamma > d$ gives

$$\frac{1}{\varepsilon^d}\int_B\varphi<+\infty.$$

It follows

$$\sum_{\substack{i\neq j, Q_i\cap\frac{1}{\varepsilon}B\neq\emptyset,\\ Q_j\cap\frac{1}{\varepsilon}B\neq\emptyset.}}\langle Y_iY_j\rangle \leq c\varepsilon^d\left\|\zeta\right\|^2_{L^\infty(B)}\left\langle N(Q)^2\right\rangle\int_B\varphi,$$

sending $\varepsilon\to 0$, we get under the assumption (2.3)

$$\sum_{\substack{i\neq j, Q_i\cap\frac{1}{\varepsilon}B\neq\emptyset,\\ Q_j\cap\frac{1}{\varepsilon}B\neq\emptyset.}}\langle Y_iY_j\rangle\to 0. \tag{2.142}$$

For the first term, we have

$$\sum_{Q_i\cap\frac{1}{\varepsilon}B\neq\emptyset}\left\langle Y_i^2\right\rangle \leq \sum_{\substack{Q_i\cap\frac{1}{\varepsilon}B\neq\emptyset\\ Q_j\cap\frac{1}{\varepsilon}B\neq\emptyset}} \varepsilon^{2d}\left\|\zeta\right\|^2_{L^\infty(B)}\left(\left\langle\sum_{z_i\in\Phi^\varepsilon(Q_i)}\tilde{X}_i^2\right\rangle+\left\langle\sum_{\substack{i\neq j,\\ z_l\in\Phi^\varepsilon(Q_i),z_k\in\Phi^\varepsilon(Q_j)}}\tilde{X}_k\tilde{X}_l\right\rangle\right)$$

$$\overset{(2.6)}{\leq} \sum_{Q\cap\frac{1}{\varepsilon}B\neq\emptyset}\varepsilon^{2d}\left\|\zeta\right\|^2_{L^\infty(B)}\left\langle N(Q)^2\right\rangle var(X) + \sum_{\substack{i\neq j,Q_i\cap\frac{1}{\varepsilon}B\neq\emptyset\\ Q_j\cap\frac{1}{\varepsilon}B\neq\emptyset.}}\varepsilon^{2d}\left\|\zeta\right\|^2_{L^\infty(B)}\frac{c\left\langle N(Q_i)N\left(Q_j\right)\right\rangle}{|z_i-z_j|^\gamma},$$

adding and substructing the term $\frac{c\langle N(Q)\rangle^2}{|z_i-z_j|^\gamma}$ and using the assumption (2.4)

$$\sum_{Q_i\cap\frac{1}{\varepsilon}B\neq\emptyset}\left\langle Y_i^2\right\rangle \leq \sum_{Q\cap\frac{1}{\varepsilon}B\neq\emptyset}\varepsilon^{2d}\left\|\zeta\right\|^2_{L^\infty(B)}\left\langle N(Q)^2\right\rangle var(X) + C\sum_{\substack{i\neq j,Q_i\cap\frac{1}{\varepsilon}B\neq\emptyset\\ Q_j\cap\frac{1}{\varepsilon}B\neq\emptyset.}}\varepsilon^{2d}\left\|\zeta\right\|^2_{L^\infty(B)}\frac{\left\langle N\left(Q\right)^2\right\rangle}{|z_i-z_j|^\gamma}.$$

A similar estimation as in (2.141) gives

$$\sum_{Q_i\cap\frac{1}{\varepsilon}B\neq\emptyset}\left\langle Y_i^2\right\rangle \leq \sum_{Q\cap\frac{1}{\varepsilon}B\neq\emptyset}\varepsilon^{2d}\left\|\zeta\right\|^2_{L^\infty(B)}\left\langle N(Q)^2\right\rangle var(X) + C\varepsilon^d\left\|\zeta\right\|^2_{L^\infty(B)}\left\langle N\left(Q\right)^2\right\rangle\int_B\varphi.$$

Since $\left\langle X^2\right\rangle$ and $\left\langle N(Q)^2\right\rangle$ are finite, then

$$\sum_{Q_i\cap\frac{1}{\varepsilon}B\neq\emptyset}\left\langle Y_i^2\right\rangle \to 0 \text{ when } \varepsilon\to 0. \tag{2.143}$$

Hence by (2.143), (2.142) and the assumption (2.3)

$$\left\langle\tilde{S}_\varepsilon^2\right\rangle \to 0 \text{ when } \varepsilon\to 0,$$

So

$$\tilde{S}_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0 \text{ in probability.}$$

Then, we can use the Borel-Cantelli's theorem B.1.10 (See appendix B) for the subsequence $\varepsilon_n=\frac{1}{n}$ with $n\in\mathbb{N}$ we get

$$\lim_{n\to+\infty}\tilde{S}_{\varepsilon_n}=0 \text{ almost surely.}$$

For the second right hand side of (2.136) we argue in a similar way as above, we denote

$$\tilde{Q}_i = Q_i - \langle Q \rangle, \qquad I_\varepsilon = \varepsilon^d \langle X \rangle \frac{\sigma_d}{d} \sum_{Q_i \cap \frac{1}{\varepsilon} B \neq \emptyset} \zeta(\varepsilon y_i) \tilde{Q}_i.$$

We estimate by Chebyshev's inequality for each $\delta > 0$, we get

$$\mathbb{P}(I_\varepsilon > \delta) \leq \delta^{-2} \left\langle I_\varepsilon^2 \right\rangle$$

and we write

$$\begin{aligned}
\left\langle I_\varepsilon^2 \right\rangle &= \varepsilon^{2d} \langle X \rangle^2 \frac{\sigma_d^2}{d^2} \sum_{\substack{Q_i \cap \frac{1}{\varepsilon} B \neq \emptyset \\ Q_j \cap \frac{1}{\varepsilon} B \neq \emptyset}} \left\langle \zeta(\varepsilon y_i) \zeta(\varepsilon y_j) \tilde{Q}_i \tilde{Q}_j \right\rangle \\
&\leq \varepsilon^{2d} \langle X \rangle^2 \|\zeta\|_{L^\infty(B)}^2 \frac{\sigma_d^2}{d^2} \sum_{\substack{Q_i \cap \frac{1}{\varepsilon} B \neq \emptyset \\ Q_j \cap \frac{1}{\varepsilon} B \neq \emptyset}} \left(N(Q_i) N(Q_j) - \langle N(Q) \rangle^2 \right) \\
&\overset{(2.4)}{\leq} \varepsilon^{2d} \langle X \rangle^2 \|\zeta\|_{L^\infty(B)}^2 \frac{\sigma_d^2}{d^2} \sum_{\substack{Q_i \cap \frac{1}{\varepsilon} B \neq \emptyset \\ Q_j \cap \frac{1}{\varepsilon} B \neq \emptyset}} \frac{C \left\langle N(Q)^2 \right\rangle}{|z_i - z_j|^\gamma},
\end{aligned} \tag{2.144}$$

since by definition of Riemann sum we have

$$\sum_{\substack{i \neq j, Q_i \cap \frac{1}{\varepsilon} B \neq \emptyset \\ Q_j \cap \frac{1}{\varepsilon} B \neq \emptyset}} \frac{C \left\langle N(Q)^2 \right\rangle}{|z_i - z_j|} \leq \frac{1}{\varepsilon^d} \int_B C \left\langle N(Q)^2 \right\rangle \varphi,$$

with

$$\varphi(x) = \frac{1}{|x|^\gamma} \text{ for } x \neq 0_{\mathbb{R}^d},$$

thanks to the assumption $\gamma > d$ we have

$$\int_B C \left\langle N(Q)^2 \right\rangle \varphi < +\infty.$$

Then we substitute in (2.144) we get

$$\left\langle I_\varepsilon^2 \right\rangle \leq \varepsilon^d \langle X \rangle^2 \|\zeta\|_{L^\infty(B)}^2 \frac{\sigma_d^2}{d^2} \int_B C \left\langle N(Q)^2 \right\rangle \varphi \to 0 \text{ when } \varepsilon \to 0,$$

it follows

$$I_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} 0 \text{ in probability.}$$

Applaying the Borel-Cantelli theorem B.1.10 for a subsequence $\varepsilon_n = \frac{1}{n}$ with $n \in \mathbb{N}$, we get

$$\lim_{n\to+\infty} I_{\varepsilon_n} = 0 \quad \text{almost surely.}$$

So for a subsequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ we have

$$\lim_{n\to+\infty} \sum_{z_i\in\Phi\left(\frac{1}{\varepsilon_n}B\right)} X_i \int_{B_\varepsilon(\varepsilon z_i)} \zeta(x)\,dx = \frac{\sigma_d}{d}\langle N(Q)\rangle\langle X\rangle \int_B \zeta(x)\,dx \text{ almost surely.} \tag{2.145}$$

To extend (2.145) to any sequence $\varepsilon_j \to 0$, we fix first the following notation

$$\underline{\varepsilon} = \left(\left\lfloor\frac{1}{\varepsilon}\right\rfloor + 1\right)^{-1}, \qquad \overline{\varepsilon} = \left(\left\lfloor\frac{1}{\varepsilon}\right\rfloor\right)^{-1}.$$

Note that $\underline{\varepsilon}^{-1}, \overline{\varepsilon}^{-1} \in \mathbb{N}$ and $\underline{\varepsilon} \le \varepsilon \le \overline{\varepsilon}$. We write $\zeta = \zeta^+ + \zeta^-$ then we can remark by linearity of the integral that it suffices to consider the positive functions which allows to keep with the case $a_{i,\varepsilon} \ge 0$.
For $\varepsilon_j \to 0$, by definition of $\underline{\varepsilon}_j$ we can estimate

$$\begin{aligned} S_{\varepsilon_j} &= \sum_{z_i\in\Phi^{\varepsilon_j}(B)} a_{i,\varepsilon_j} X_i \le S_{\underline{\varepsilon}_j} + \sum_{i=1}^{N^{\varepsilon_j}(B)} \left| a_{i,\varepsilon_j} - a_{\underline{\varepsilon}_j} \right| X_i \\ &\le S_{\underline{\varepsilon}_j} + \max_{i=1,..,N^{\varepsilon_j}(B)} \left| a_{i,\varepsilon_j} - a_{\underline{\varepsilon}_j} \right| \sum_{i=1}^{N^{\varepsilon_j}(B)} X_i. \end{aligned} \tag{2.146}$$

We can claim that we have almost surely

$$\lim_{\varepsilon\to 0} \frac{\max_{i\le N^{\overline{\varepsilon}}(B)} |a_{i,\varepsilon} - a_{i,\overline{\varepsilon}}|}{\overline{\varepsilon}^d} = \lim_{\varepsilon\to 0} \frac{\max_{i\le N^{\underline{\varepsilon}}(B)} |a_{i,\varepsilon} - a_{i,\underline{\varepsilon}}|}{\underline{\varepsilon}^d} = 0. \tag{2.147}$$

We first show that if (2.147) is true, we can conclude the lemma immediatly. We have

$$S_{\varepsilon_j} \le S_{\underline{\varepsilon}_j} + \frac{\max_{i=1,..,N^{\varepsilon_j}(B)} \left| a_{i,\varepsilon_j} - a_{i,\underline{\varepsilon}_j} \right|}{\underline{\varepsilon}_j^d} \underline{\varepsilon}_j^d \sum_{i=1}^{N^{\varepsilon_j}(B)} X_i, \tag{2.148}$$

from lemma 2.9 we have $\underline{\varepsilon}_j^d \sum_{i=1}^{N^{\varepsilon_j}(B)} X_i$ is bounded for ε small enough, by (2.147) the second right hand side of (2.148) vanishes in the limit. For the first we can use the result (2.145), we get

$$\lim_{\varepsilon_j\to 0}\sup S_{\varepsilon_j} \le \frac{\sigma_d}{d}\langle N(Q)\rangle\langle X\rangle \int_B \zeta(x)\,dx.$$

We may argue similarly, we have

$$S_{\varepsilon_j} \geq S_{\overline{\varepsilon}_j} + \frac{\max_{i=1,..,N^{\varepsilon_j}(B)} \left|a_{i,\varepsilon_j} - a_{i,\overline{\varepsilon}_j}\right|}{\overline{\varepsilon}^d} \overline{\varepsilon}^d \sum_{i=1}^{N^{\varepsilon_j}(B)} X_i.$$

Using (2.147), (2.145) and lemma 2.9 we get

$$\liminf_{\varepsilon_j \to 0} S_{\varepsilon_j} \geq \frac{\sigma_d}{d} \langle N(Q) \rangle \langle X \rangle \int_B \zeta(x)\, dx.$$

Then, our main result holds true.

Now, it remains to argue (2.147), we have for $\zeta \in W_0^{1,\infty}(B)$ and for every $z_i \in B$ and $\varepsilon_1 \leq \varepsilon_2$ the following estimation

$$\begin{aligned} |a_{i,\varepsilon_1} - a_{i,\varepsilon_2}| \;\; &\leq \;\; \int_{B_{\varepsilon_1}(0)} |\zeta(x + \varepsilon_1 z_i) - \zeta(x + \varepsilon_2 z_i)|\, dx \\ &\quad + \int_{B_{\varepsilon_2}(0) \setminus B_{\varepsilon_1}(0)} \zeta(x + \varepsilon_2 z_i)\, dx, \end{aligned}$$

using mean value theorem one has

$$|a_{i,\varepsilon_1} - a_{i,\varepsilon_2}| \leq \|\nabla \zeta\|_{(L^\infty(B))^d} |\varepsilon_2 - \varepsilon_1| |z_i| \varepsilon_1^d + \|\zeta\|_{L^\infty(B)} \left(\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^d - 1 \right) \varepsilon_1^d.$$

Since we have $N^{\varepsilon_2}(B) \leq N^{\varepsilon_1}(B)$ and thus $i \leq N^{\varepsilon_2}(B)$ we have that $|z_i| \leq \varepsilon_2^{-1}$ and

$$|a_{i,\varepsilon_1} - a_{i,\varepsilon_2}| \leq \|\zeta\|_{W^{1,\infty}(B)} \left(\left(1 - \frac{\varepsilon_1}{\varepsilon_2}\right) + \left(\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^d - 1 \right) \right) \varepsilon_1^d.$$

We choose $\varepsilon_1 = \varepsilon, \varepsilon_2 = \overline{\varepsilon}$ this yields

$$|a_{i,\varepsilon_1} - a_{i,\varepsilon_2}| \leq \|\zeta\|_{W^{1,\infty}(B)} \left(\varepsilon + \left(\frac{1}{1-\varepsilon}\right)^d - 1 \right) \overline{\varepsilon}^d.$$

and thus the first limit in (2.147) holds. ■

Appendix A: Some preliminaries on functional analysis

In all what follows D is an open bounded set of $\mathbb{R}^d, d \geq 2$.

A.1 $\mathbf{L}^p$ spaces and Sobolev spaces

Definition A.1.1 [4] We denote by $L^1(D)$ the space of real-valued measurable functions u defined in D that satisfies

$$\int_D |u(x)|\, dx < +\infty.$$

Then, we set

$$L^1(D) = \left\{ u : D \to \mathbb{R} \text{ measurable such that } \int_D |u(x)|\, dx < +\infty \right\},$$

For $1 < p < \infty$, we set

$$L^p(D) = \left\{ u : D \to \mathbb{R} \text{ measurable such that } |u|^P \in L^1(D) \right\},$$

We define $L^\infty(D)$ as a space of essentially bounded measurable fonctions mesurables i.e

$$\exists C \geq 0 : |u(x)| \leq C \text{ almost every } x \in D.$$

Then, we set

$$L^1(D) = \{ u : D \to \mathbb{R} \text{ measurable } : \exists C \geq 0 : |u(x)| \leq C \text{ almost every } x \in D \},$$

Proposition A.1.2 [4] Equipped $L^p(D), 1 \leq p < \infty$ with the norm

$$\|u\|_{L^p(D)} = \left(\int_D |u(x)|\, dx \right)^{\frac{1}{p}}, \quad \forall u \in L^p(D),$$

and $L^\infty(D)$ with

$$\|u\|_{L^\infty(D)} = \inf \{ C : |u(x)| \leq C \text{ almost every } x \in D \}.$$

$L^p(D), 1 \leq p \leq \infty$ is a Banach space. Moreover, $L^2(D)$ endowed with

$$\int_D u(x)v(x)dx, \quad \forall u, v \in L^2(D),$$

is Hilbert space.

Theorem A.1.3 (**An embedding theorem for L^p spaces**) [2] Suppose that $|D| = \int_D dx < +\infty$ and $1 \leq p \leq q \leq +\infty$. If $u \in L^q(D)$, then $u \in L^p(D)$ and

$$\|u\|_{L^p(D)} \leq (|D|)^{\frac{1}{p}-\frac{1}{q}} \|u\|_{L^q(D)}.$$

Hence

$$L^q(D) \hookrightarrow L^p(D).$$

Theorem A.1.4 (**Hölder's inequality**) [4] Let $f \in L^p(D)$ and $g \in L^q(D)$ with $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then, $f \cdot g \in L^1(D)$ and

$$\int_D |f \cdot g| \leq \|f\|_{L^p(D)} \|g\|_{L^q(D)}.$$

Remark A.1.5 Hölder's inequality for $L^2(D)$ is just well-knows as Cauchy-Schwarz inequality.

Definition A.1.6 [4] Let $m \geq 1$ be an integer. For $1 \leq p \leq \infty$ we define Sobolev spaces denoted by $W^{m,p}(D)$ as follow

$$W^{m,p}(D) = \left\{ u \in L^p(D) : \begin{array}{l} \forall \alpha \in \mathbb{N}^N \text{ avec } |\alpha| \leq m, \exists g_\alpha \in L^p(D) \text{ telle que} \\ \int_\Omega u D^\alpha \varphi = (-1)^\alpha \int_\Omega g_\alpha \varphi, \ \forall \varphi \in \mathcal{D}(D) \end{array} \right\},$$

where $D^\alpha u = g_\alpha, |\alpha| = \sum_{i=1}^{N} \alpha_i$ and $\mathcal{D}(D)$ is the space of infinitely differentiable functions $\phi : D \to \mathbb{R}$ with compact support.

We define $W_0^{m,p}(D)$ as the closure of $\mathcal{D}(D)$ in $W^{m,p}(D)$, i.e

$$W_0^{m,p}(D) = \overline{\mathcal{D}(D)}^{W^{m,p}(D)}.$$

For $p = 2$, we denote by $H^m(D)$ and $H_0^m(D)$ the spaces $W^{m,2}(D)$ et $W_0^{m,2}(D)$ respectivly.

Theorem A.1.7 [4] Equipped $W^{m,p}(D)$ with the norm

$$\|u\|_{W^{m,p}(\Omega)} = \sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_{L^p(D)}, \quad \forall 1 \leq p \leq +\infty,$$

$W^{m,p}(D)$ is a Banach space.

For $p=2$, $H^m(D)$ is a Hilbert space with respect to the scalar product

$$\sum_{0\leq|\alpha|\leq m}\int_D D^\alpha u(x)D^\alpha v(x)dx.$$

est un espace de Hilbert. Moreover, $W^{m,p}(D)$ is reflexif for $1<p<\infty$ and separable for $1\leq p<+\infty$.

The product φu of a smooth function $\varphi\in\mathcal{D}(D)$ and $u\in W^{m,p}(D)$ belongs to $W_0^{m,p}(D)$.

Proposition A.1.8 (**Caracterization of dual space of** $W_0^{1,p}(D)$) [2] For $1\leq p<\infty$, and $L\in W^{-1,q}(D)$ with $q=\frac{p-1}{p}$, there exist $\varphi_0,\varphi_1,...,\varphi_d\in L^q(D)$ such that

$$L(u)=\int_D\varphi_0 u+\sum_{i=1}^d\int_D\frac{\partial u}{\partial x_i}\varphi_i,\qquad\forall u\in W_0^{1,p}(D),$$

Moreover,

$$\|L\|_{W^{-1,q}(D)}=\inf\left(\|(\varphi_0,\varphi_1,...,\varphi_d)\|_{(L^q(D))^{d+1}}\right).$$

Corollary A.1.9 (**Poincaré inequality**) [1] If D is a bounded open set of $\mathbb{R}^d$. There exist a positive constant $C=C(D,p)$ with $1\leq p<\infty$ such that

$$\|u\|_{L^p(D)}\leq C\|\nabla u\|_{(L^p(D))^d},\qquad\forall u\in W_0^{1,p}(D).$$

Theorem A.1.10 (**Rellich-Kondrachov**) [4] Suppose that D is bounded, and ∂D is $\mathcal{C}^1$. We have

$$\text{If } p<N \text{ then } W^{1,p}(\Omega)\hookrightarrow L^q(\Omega),\forall q\in[1,p^*[\text{ where }\frac{1}{p^*}=\frac{1}{p}-\frac{1}{N},$$

$$\text{If } p=N \text{ then } W^{1,p}(\Omega)\hookrightarrow L^q(\Omega),\forall q\in[1,\infty[,$$

$$\text{If } p>N \text{ then } W^{1,p}(\Omega)\hookrightarrow C(\overline{\Omega}),$$

with compact embedding.

Theorem A.1.11 (Green formula) [1] Suppose that D be an open bounded regular set of classe C^1. If u and v are functions of $H^1(\Omega)$, they satisfy

$$\int_\Omega u(x)\frac{\partial v}{\partial x_i}(x)dx=-\int_\Omega v(x)\frac{\partial u}{\partial x_i}(x)dx+\int_{\partial\Omega}u(x)v(x)n_i dx,$$

where $n=(n_i)_{1\leq i\leq N}$ is the outward unit normal to $\partial\Omega$.

Proposition A.1.12 [1] Suppose that D is an open bounded regular set of classe C^1. Let $(w_i)_{1\leq i\leq k}$ be a regular partition of D, that is each w_i is a regular open set of classe C^1, $w_i \cap w_j = \emptyset$ if $i \neq j$ and $\overline{\Omega} = \bigcup_{1\leq i\leq k} \overline{w_i}$. Let u be a fonction whose restriction to each w_i, $u|_{w_i} = u_i$ belongs to $H^1(w_i)$. If u is continous over $\overline{\Omega}$, then u belongs to $H^1(\Omega)$.

Theorem A.1.13 (Gagliardo-Nirenberg-Sobolev inequality) [2] Assume that $1 \leq p < d$ and that ∂D is $\mathcal{C}^1$. $u \in W^{1,p}(D)$ then $u \in L^{p^*}(D)$ with $p^* = \frac{dp}{d-p}$. and we have the estimation

$$\|u\|_{L^{p^*}(D)} \leq C \|u\|_{W^{1,p}(D)},$$

the constant C depending only on p, d, and D.

If we consider the case $p = d$, then we the continuous embedding of $W^{1,d}(D)$ in $L^q(D)$ with $q \in [d, +\infty)$.

A.2 Functional analysis results

Corollary A.2.1 [4] (**Hahn-Banach**)Let G be a subset of a Banach space E, G' and E' the dual space of E and $g : G \longrightarrow \mathbb{R}$ is a continuous linear function of norm

$$\|g\|_{G'} = \sup_{\substack{\|x\|_G \leq 1 \\ x\in G}} |g(x)|.$$

Then there exists $f \in E'$ that extends g and such that$\|g\|_{G'} = \|f\|_{E'}$.

Theorem A.2.2 (Lax-Milgram) [4] Let a a bilineair form defined in $H \times H$ and satisfies

$$\begin{array}{ll} \exists M > 0,\ |a(u,v)| \leq M \|u\|_H \|v\|_H & \forall u, v \in H \\ \exists \alpha > 0,\ a(u,v) \geq \alpha \|u\|_H^2 & \forall u \in H \end{array},$$

with H a Hilbert space. Then, for every $\psi \in H'$, there exist a unique $u \in H$ such that

$$a(u,v) = \langle \psi, v \rangle_{H',H}, \quad \forall v \in H.$$

Definition A.2.3 (**Weak and weak-star convergence**) [4] Let E be a banach space.

A sequence $(u_n)_{n\in\mathbb{N}} \subseteq E$ is said to converge strongly to an element $u \in E$, if

$$\|u_n\|_E \to \|u\|_E,$$

where $\|.\|_E$ is a norm defined in E.

A sequence $(u_n)_{n\in\mathbb{N}} \subseteq E$ is said to converge weakly to an element $u \in E$ and we write

$$u_n \rightharpoonup u \text{ weakly in } E.$$

if

$$\langle g, u_n \rangle_{E',E} \to \langle g, u \rangle_{E',E} \text{ for every } g \in E',$$

where E' is the daul space of E.

A sequence $(g_n)_{n\in\mathbb{N}} \subseteq E'$ is said to converge weakly-star to $g \in E'$ and we write

$$g_n \overset{*}{\rightharpoonup} g \text{ weakly-star in } E'.$$

Theorem A.2.4 (**Eberlein-S̃muljan**) [4]

a. Let E be a reflexif Banach space and $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in E. Then, there exist a subsequence $(u_{n_k})_{n_k\in\mathbb{N}}$ converge weakly to a limit $u \in E$.

b. If E is a separable Banach space and $(g_n)_{n\in\mathbb{N}}$ a bounded sequence in E' with E' is the dual of E. Then, there exist a subsequence $(g_{n_k})_{n_k\in\mathbb{N}}$ converge weakly-star to a limit $g \in E'$.

Theorem A.2.5 (Maximum principle [9]) Let $u \in C^2(D) \cap C\left(\overline{D}\right)$ such that

$$-\Delta u = 0.$$

Then

$$\min_{\partial D} u \leq u(x) \leq \max_{\partial D} u \text{ for } x \in D.$$

A.3 Additional definitions and results

Definition A.3.1 (**Periodic functions**) [6] Let $Y =]0, l_1[\times ... \times]0, l_d[$ be a cell in $\mathbb{R}^d$ and u is a function defined in almost everywhere in $\mathbb{R}^d$. the fonction u est said to be $Y-$periodic if

$$u(x + kl_ie_i) = f(x) \quad \forall k \in \mathbb{Z},\ \forall i = 1, .., d,$$

where $(e_i)_{1\leq i\leq d}$ is the canonical basis of $\mathbb{R}^d$.

Proposition A.3.2 (See D. Cioranescu and Murat [6]) Let $1 \leq p \leq +\infty$ and f be a $Y-$ periodic function in $L^p(Y)$ where $Y =]0, l_1[\times ... \times]0, l_d[$ be a cell in $\mathbb{R}^d$ Set

$$f_\varepsilon(x) = f(\frac{x}{\varepsilon}) \quad \text{almost everywhere in } \mathbb{R}^d.$$

Then, if $p < \infty$, as $\varepsilon \to 0$

$$f_\varepsilon \rightharpoonup M_Y(f) = \frac{1}{|\mathrm{Y}|} \int_Y f(x)\, dx \text{ weakly in } L^p(\omega),$$

for any open subset ω of $\mathbb{R}^d$.

If $p = +\infty$, one has

$$f_\varepsilon \rightharpoonup M_Y(f) = \frac{1}{|Y|}\int_Y f(x)\,dx \text{ weakly* in } L^\infty\left(\mathbb{R}^d\right).$$

Theorem A.3.3 (Fubini's theorem) [12] Suppose that $f(x,y)$ is a non-negative measurable function on $\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}=\mathbb{R}^d$.Then, for almost every $x\in\mathbb{R}^{d_1}$ and $y\in\mathbb{R}^{d_2}$

1. The slice $f^y(x) := f(x,y)$ is measurable on $\mathbb{R}^d$.

2. the function defined by $\int_{\mathbb{R}^{d_1}} f^y(x)\,dx$ is measurable on $\mathbb{R}^{d_2}$.

3.the slice $f_x(y) := f(x,y)$ is measurable on $\mathbb{R}^{d_2}$.

4. the function defined by $\int_{\mathbb{R}^{d_2}} f^y(x)\,dx$ is measurable on $\mathbb{R}^{d_1}$.

Moreover,

$$\int_{\mathbb{R}^{d_2}}\left(\int_{\mathbb{R}^{d_1}} f(x,y)\,dx\right)dy = \int_{\mathbb{R}^d} f = \int_{\mathbb{R}^{d_1}}\left(\int_{\mathbb{R}^{d_2}} f(x,y)\,dy\right)dx.$$

Appendix B : Some basic facts on stochastic analysis

We denote $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is the set of outcomes, $\mathcal{F}$ is a set of events and $\mathbb{P}$: $\mathcal{F} \rightarrow [0,1]$ is a function that assigns probabilities to events.

B.1 Some probability results

Definition B.1.1 [7] Let $(S, \mathcal{S})$ an aribtrary measurable space. A map $X : \Omega \rightarrow S$ is said to be a measurable map from $(\Omega, \mathcal{F})$ to $(S, \mathcal{S})$ if

$$X^{-1}(B) = \{\omega : X(\omega) \in B\} \in \mathcal{F} \text{ for all } B \in \mathcal{S}.$$

If $(S, \mathcal{S}) = \left(\mathbb{R}^d, \mathcal{R}^d\right)$ and $d \geq 2$ then X is called a random vector, if $d = 1$, X is called a random variable, or random vector for short.

Definition B.1.2 [7] The distribution function of a random variable X is the function F defined as

$$F(x) = \mathbb{P}(X \leq x),$$

defined for every $x \in (-\infty, +\infty)$.

When the distribution function F has the form

$$F(x) = \int_{-\infty}^{x} f(y)\, dy.$$

we say that X has a density function f.

Definition B.1.3 [7] The expectation of a random variable with density function f is defined by

$$\langle X \rangle = \int_{-\infty}^{+\infty} x f(x)\, dx.$$

Generaly, the expectation defined as the integration over the set of probability Ω with respect to the probability measure $\mathbb{P}$, we write

$$\langle X \rangle = \int_{\Omega} X(\omega)\, d\mathbb{P}(\omega),$$

We define the $n^{th}-$moments of a random variable X with $n \in \mathbb{N}^*$ as follow

$$\langle X^n \rangle = \int_{-\infty}^{+\infty} x^n f(x)\, dx.$$

We define also the variance $var(X)$ of X as follow

$$\begin{aligned} var(X) &= \left\langle (X - \langle X \rangle)^2 \right\rangle \\ &= \left\langle X^2 \right\rangle - \langle X \rangle^2 . \end{aligned}$$

Lemma B.1.4 [7] Let X a random variable. If $X \geq 0$, and $p > 0$, then

$$\langle X^p \rangle = \int_{0}^{+\infty} p y^{p-1} \mathbb{P}(X > y) dy.$$

Theorem B.1.5 (Chebychev inequality) [7] Let X a random variable, for any $a > 0$, we have

$$a^2 \mathbb{P}(|X| \geq a) \leq \left\langle X^2 \right\rangle .$$

Theorem B.1.6 (Dominated convergence theorem) Let X_n is a sequence of random variables. X, Y two random variables. If X_n converges to X almost surely, $|X_n| \leq Y$ and $\langle Y \rangle < +\infty$ then

$$\langle X_n \rangle \to \langle X \rangle .$$

Definition B.1.7 (The joint distribution and density) [11]

The joint distribution $F_{X,Y}$ of two random variables X and Y the probability of the event

$$\{X \leq x, Y \leq y\},$$

with $x, y \in (-\infty, +\infty)$.

The joint density function of X and Y is defined as follow

$$f_{X,Y}(x, y) = \frac{\partial^2 F_{X,Y}(x, y)}{\partial x \partial y}.$$

Definition B.1.8 (Independance) [11] Two random variables X and Y is said to be independent if for every $A, B \in \mathcal{F}$, the events $(X \in A)$ and $(Y \in B)$ are independent that is, if

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\,\mathbb{P}(Y \in B)\,.$$

applaying this latter for the events $(X \leq x)$ and $(Y \leq y)$ for the real numbers x and y,then

$$F_{X,Y}(x, y) = F_X(x)\,F_Y(y)\,.$$

hence

$$f_{X,Y} = f_X f_Y.$$

In what follow E is an arbitrary complete separable metric space, $\mathcal{B}(E)$ the $\sigma-$field of its Borel sets.

Theorem B.1.9 (Borel-Cantelli I) [7] Let $A_i, i \in \mathbb{N}^*$ a sequence of subsets of Ω, if

$$\sum_{i=1}^{\infty} \mathbb{P}(X \geq i) < \infty.$$

Then

$$\mathbb{P}(X \geq i \text{ i.o}) = 0.$$

(i.o infinitely often which means $\mathbb{P}(X_i > i \text{ i.o}) = \mathbb{P}\left(\limsup_i \ (X_i > i)\right)$ with $\limsup_i (X_i > i) = \bigcap_{i\geq 0} \bigcup_{k\geq i} (X_i > i)$ and $(X_i > i) = \{\omega \in \Omega : X_i(\omega) > i\}$

Theorem B.1.10 (Borel-Cantelli II) [7] Let X_n a sequence of random variables, $X_n \to X$ in probability if and only if for every sequence $X_{n(m)}$ there is a further subsequence that converges almost surely to X.

Definition B.1.11 (Convergence types) [11] A sequence X_n of random variables is convergent to a random variable X in probabaility if for every $\varepsilon > 0$

$$\mathbb{P}(|X_n - X| > \varepsilon) \to 0 \text{ when } n \to +\infty.$$

We said that X_n converge almost surely (this type of convergence called almost everywhere in measure theory) if for every $\varepsilon > 0$ we have

$$\mathbb{P}(|X_n - X| > \varepsilon \text{ a.e}) = 0.$$

B.2 Stochastic processes

Definition B.2.1 [8]

We denote by $\mu : \mathcal{B}(E) \longrightarrow \mathbb{R}^+$ the Borel measure which is said to be boundedly finite if $\mu(A) < +\infty$ for every bounded Borel set $A \in \mathcal{B}(E)$,

a. $\mathcal{M}_E$ the space of all boundedly finite measure on $\mathcal{B}(E)$.

b. $\mathcal{N}_E$ is the space of all boundedly finite integer-valued measures $N \in \mathcal{M}_E$, called counting measures for short.

c. $\mathcal{N}_E^*$ is the family of all simple counting measures, consisting of all those elements of $\mathcal{N}_E$ for which $N(\{x\}) = 0$, or 1 (all $x \in E$).

d. $\mathcal{N}_{E\times\mathcal{K}}^g$ is the family of all boundedly finite counting measures defined on the product $\mathcal{B}(E \times \mathcal{K})$, where $\mathcal{K}$ is a complete separable metric space of marks, subject to the additional requirement that the ground measure N_g defined by

$$N_g(A) = N(A \times \mathcal{K}) \text{ for all } A \in \mathcal{B}(E).$$

is boundedly finite simple counting measure; i.e $N_g \in \mathcal{N}_E^*$.

Definition.B.2.2 [8]

a. A random measure ξ on the space E is a measurable mapping from $(\Omega, \mathcal{F}, \mathbb{P})$ into $(\mathcal{M}_E, \mathcal{B}(\mathcal{M}_E))$.

b. A point process N on E is a measurable mapping from $(\Omega, \mathcal{F}, \mathbb{P})$ into $(\mathcal{N}_E, \mathcal{B}(\mathcal{N}_E))$.

c. A point process is simple when $\mathbb{P}(N \in \mathcal{N}_E^*)$ =1.

d. A marked point process on E with marks in $\mathcal{K}$ is a point process N on $\mathcal{B}(E \times \mathcal{K})$ for which $\mathbb{P}(N \in \mathcal{N}_{E\times\mathcal{K}}^g)$ =1, its ground process is given by $N(.) = N(. \times \mathcal{K})$.

Lemma B.2.3 (**Strong law of large numbers for sums of random variables with correlations**) [10] Let $\{x_i\}_{i\in\mathbb{N}} = \mathbb{Z}^d$, and let $\{X_i\}_{i\in\mathbb{N}}$ be identically distributed random variables with $X_i \geq 0$ and X is a random variable takes the same properties as X_i, for each $i \in \mathbb{N}$ such that $\langle X \rangle < +\infty$. Let us assume that for every $i, j \in \mathbb{N}$ with $i \neq j$

$$\left| \langle X_i X_j \rangle - \langle X \rangle^2 \right| < \frac{C}{|x_i - x_j|^\gamma} \quad \gamma > d.$$

Then, for every bounded Borel set $B \subseteq \mathbb{R}^d$ which is star-shaped with respect to the origin, we have

$$\lim_{\varepsilon\to 0} \varepsilon^d \sum_{x_i \in \mathbb{Z}^d \cap \frac{1}{\varepsilon} D} X_i = X \quad \text{almost surely.}$$

Conclusion

In this thesis we have used the homogenization theory to study a Dirichlet problem with Laplace operator in a bounded domain, perforated by spherical holes using the oscillating test function method. We have treated two examples of perforated domain. In the begining, we have focused on the case where the holes are distributed periodically and which have a critical size, we have introduced some hyptheses on holes in order to obtain in the limit the Laplace operator with an additional term and this where the charm of the problem lies. For the second example we have treated a perforated domain with random number of balls, assuming that the centers of the balls are generated according to a stationary point process and the radii are random variables with short-range correlations. In addition, we have recovered in the homogenized limit an averaged analogue of strange term obtained as in the first case under a minimal assumption on the size of the holes.

Bibliography

[1] G. Alaire, Numerical analysis and optimization, introduction to mathematical modelling and numerical simulation, Oxford University Press (2007)

[2] R. A. Adams, J. F. Fournier, Sobolev spaces, 2^{nd} Edition, pure and applied mathematics series, Elsevier Ltd (2003)

[3] A. Baernstein II. Symmetrization in analysis, New mathematical monographs, Combridge university press, 1.15 (2019).

[4] H. Brézis, functional analysis, Sobolev spaces and Partial Differential Equations, Univertext, Springer (2011).

[5] D. Cioranescu and F. Murat, Un terme étrange venu d'ailleurs II, Nonlinear partial differential equations and their applications, Collège de France Seminar, Vol. III (Paris, 1980/1981), Res. Notes in Math. (1982),70, 154-178, 425-426. Pitman, Boston, Mass.

[6] D. Cioranescu, P. Donato, An introduction to homogenization, Oxford University Press (1999).

[7] R. Durret, Proability:Theory and examples, Combridge University Press New York, NY, USA (2010).

[8] D.J. Daley and D. Vere-Jones, An introduction to the theory of point processes, Vol.II: General theory and structures, Probability and its applications, Springer-Verlag New York (2003).

[9] L. C. Evans, Partial Differential Equation, Graduate studies in mathematics 19, 2^{nd} Edition, American Mathematical Socioty (2010).

[10] A. Giunti, R. Höfer, and J.J.L. Velazquez, Homogenization for the Poisson equation in randomly perforated domains under minimal assumptions on the size of the holes, to appear in Comm .in PDEs, arXiv preprint 1803.10214(2018).

[11] A. Papoulis, Probability, random variables and stochastic processes, third edition, McGraw-Hill Inc (1991).

[12] E. Stein and R. Shakarchi, Real Analysis: Measure Theory, Integration, and Hilbert Spaces, Princeton University Press, Priceton and Oxford, 2005

[13] L. Tartar, Cours Peccot, collège de France (1977). Partialy written in F. Murat & L. Tartar, H-convergence, translated in the present volume.