Friedrich-Alexander-Universität
Naturwissenschaftliche Fakultät

Friedrich-Alexander-Universität
DYNAMICS, CONTROL, MACHINE LEARNING AND NUMERICS

THE TURNPIKE PHENOMENON IN GAS NETWORKS

Christopher Sorg Chair in Dynamics, Control, and Numerics – AvH Professorship
FAU Erlangen-Nürnberg

Martin Gugat Chair in Dynamics, Control, and Numerics – AvH Professorship
FAU Erlangen-Nürnberg

Introduction

Consider the control of a process on a time interval

$$[0,T]$$

for large T. In many long-time-horizon optimal control problems, the optimal strategy spends most of the time near a *stationary* optimum and deviates only briefly at the beginning and the end. Consider

$$\dot{x} = f(x,u), \qquad J_T(u) = \int_0^T \ell(x(t),u(t))\,\mathrm{d}t + \phi(x(T)).$$

The (associated) **static** problem

$$\min_{x,u}\ \ell(x,u) \quad \text{s.t.} \quad f(x,u) = 0$$

has an optimal pair $(\bar{x},\bar{u})$. Under standard dissipativity assumptions, an optimal solution (x^*,u^*) of the dynamic problem satisfies the (informal) (exponential) turnpike estimate

$$\|x^*(t)-\bar{x}\| + \|u^*(t)-\bar{u}\| \ \le\ C\big(e^{-\alpha t} + e^{-\alpha(T-t)}\big), \qquad t\in[0,T],$$

i.e., the trajectory quickly approaches $(\bar{x},\bar{u})$, remains close for most of the horizon, and departs only near $t=0$ and $t=T$ to meet boundary conditions. As T increases, the boundary layers stay $O(1)$ in length while the central plateau extends. In the gas-pipe example, this appears as a nearly constant valve setting for most of the interval with short start-up and shut-down transients.

Modelling

For the sake of simplicity we restrict the problem to a single pipe. Continuum mechanical models for a gas flow in a single pipe are typically of the following type

$$\begin{aligned} \partial_t\rho + \partial_x(\rho v) &= 0,\\ \partial_t(\rho v) + \partial_x(p+\rho v^2) &= -\frac{\lambda}{2D}\rho v|v| - g\rho\partial_x h, \end{aligned} \tag{1}$$

called isothermal Euler equations, where p,ρ,v denote the pressure, density and velocity of the gas, $\lambda, D, h(x)$ are the pipe friction coefficient, diameter and height and g is the gravitational constant. These equations describe the behavior of compressible, inviscid fluids, like many gases.

Infinite Dimensional Port-Hamiltonian Form

We can rewrite (1) in the following way. Denote for short $\mathbb{I} := [0,L]$, where $L>0$ is the length of the gas pipeline and $\mathbb{T} := [0,T]$ for the time interval.Take as state space $\mathbb{S} = H^1(\mathbb{I},\mathbb{R}^2)$ and consider a state $z = (\rho, v) \in \mathbb{S}$, where we assume to have $\min_{x\in\mathbb{I}}\rho(x) > 0$. An infinite-dimensional port-Hamiltonian system for a state $z:\mathbb{T}\to\mathbb{S}$ is given by

$$\partial_t z(t) = \Big(\mathcal{J}(z(t)) - \mathcal{R}(z(t))\Big)\mathcal{H}'(z(t)) + \mathcal{B}(z(t))\,u, \tag{2}$$

using slightly abuse of notation for the time t, with initial condition $z(0) = z_0 \in \mathbb{S}$, and control input $u:\mathbb{T}\to\mathbb{U}$. Here

- $\mathcal{H}\in C^1(\mathbb{S},\mathbb{R})$ is the corresponding Hamiltonian (where the derivative here is meant in the Fréchet sense and $\mathcal{H}'(z)\in\mathbb{S}^*$),
- $\mathcal{J}\in L(\mathbb{S},\mathbb{S}^*)$ is the structure operator, which is skew-adjoint ($\mathcal{J}(z) = -\mathcal{J}(z)^*$),
- $\mathcal{R}\in L(\mathbb{S},\mathbb{S}^*)$ is the dissipation operator, which is self-adjoint and positive semidefinite ($\mathcal{R}(z) = \mathcal{R}(z)^* \succeq 0$),
- $\mathcal{B}\in L(\mathbb{U},\mathbb{S}^*)$ is the input operator.

Here, $\mathbb{U}\subseteq H^1(\mathbb{I},\mathbb{R}^2)$.

Semi-Discrete Port-Hamiltonian Form

To derive the finite-dimensional version of the so called weak formulation of (2), we use now the classical Galerkin-method. We will write $\mu := \rho v$ for the momentum in the following. Let $\mathbb{I}$ be partitioned into n subintervals $\mathbb{I}_1,\dots,\mathbb{I}_n$, which is a equidistant mesh $\{\mathbb{I}_h\}_h$ with grid length $h>0$. We introduce the finite-dimensional spaces

$$\begin{aligned} \mathbb{Q}_h &:= \mathbb{P}_0(\mathbb{I}_h) \text{ (for approximating } \rho),\\ \mathbb{R}_h &:= \mathbb{P}_1(\mathbb{I}_h)\cap H^1(0,L) \text{ (for approximating } \mu), \end{aligned}$$

where $\mathbb{P}_0(\mathbb{I}_h)$ and $\mathbb{P}_1(\mathbb{I}_h)$ denote the spaces of piecewise constant and piecewise linear functions over $\mathbb{I}_h$. Then we can represent the discrete approximations as

$$\rho_h(x,t) = \sum_{j=1}^n c_{\rho,j}(t)\,q_j(x), \qquad \mu_h(x,t) = \sum_{k=1}^m c_{\mu,k}(t)\,r_k(x),$$

where $q_1,\dots,q_n$ is a basis for $\mathbb{Q}_h$ and $r_1,\dots,r_m$ is a basis for $\mathbb{R}_h$. One ends up at

$$E(\mathbf{c})\,\dot{\mathbf{c}}(t) = J\,\eta(\mathbf{c}) - R(\mathbf{c})\,\eta(\mathbf{c}) + B\,u, \tag{3}$$

subject to the algebraic constraint

$$y = B(\mathbf{c})^\top\eta(\mathbf{c}), \tag{4}$$

where c, u and y are the state, input and output of the system respectively. Here we assume to have

$$E,J,R\in C(\mathbb{R}^d,\mathbb{R}^{d\times d}),\ \eta\in C(\mathbb{R}^d,\mathbb{R}^d),\ B\in C(\mathbb{R}^d,\mathbb{R}^{d\times 2}),$$

where $d := n+m$. The Hamiltonian $\mathcal{H}\in C^1(\mathbb{R}^d,\mathbb{R})$ is chosen, such that it is bounded from below along any solution of (3)–(4) satisfying

$$\frac{d}{d\mathbf{c}}\mathcal{H}(\mathbf{c}) = E(\mathbf{c})^\top\eta(\mathbf{c})$$

for each $\mathbf{c}\in\mathbb{R}^d$. As shown in [1], one also gets the discrete power balance equation

$$\frac{d}{dt}\mathcal{H}(\mathbf{c}) = -\eta(\mathbf{c})^\top R(\mathbf{c})\eta(\mathbf{c}) + y^\top u. \tag{5}$$

Optimal Control Problem

Now we are able to formulate our optimal control problem. We consider as set of admissable controls u in $\mathbb{A}\subseteq L^\infty(\mathbb{T},\mathbb{R}^2)$ and aim to solve

$$\left.\begin{aligned} &\min_{u\in\mathbb{A}}\ C_T(u) := \int_{\mathbb{T}} y^\top u\,dt\\ &\text{subject to (3) and (4)}\\ &\mathbf{c}(0) = c_0\in K,\quad \mathbf{c}(T) = c_T\in K. \end{aligned}\right\} \tag{OCP}$$

Here, $K\subseteq\mathbb{R}^{n+m}$ is a compact set. Further we will from now on assume, that (OCP) has an optimal solution u^* with optimal trajectory c^*. Using (5), we can rewrite the cost functional $C_T(u)$ as

$$C_T(u) = \mathcal{H}(c_T) - \mathcal{H}(c_0) + \int_{\mathbb{T}}\|R^{1/2}\eta\|^2\,dt$$

and therefore

$$\mathcal{H}(c_T) - \mathcal{H}(c_0) = \int_{\mathbb{T}} y^\top u - \|R^{1/2}\eta\|^2\,dt. \tag{6}$$

Trim Manifold

Taking a closer look at (6), we see, that any optimal trajectory c^* will have to spend most of the time close to the set

$$\mathcal{M} := \left\{\mathbf{c}\in\mathbb{R}^d \mid R^{1/2}(\mathbf{c})\eta(\mathbf{c}) = 0\right\}. \tag{7}$$

Finding turnpike properties w.r.t. this set $\mathcal{M}$ in general is a big challenge. Following e.g. [2], one typically chooses suitable assumptions, such that $\mathcal{M}$ will be a smooth submanifold of $\mathbb{R}^d$ for the setting of a gas pipeline.

Measure-Turnpike Property

There are many possible turnpike properties, where we will for simplicity formulate here the probably simplest one. Consider the control problem eq. (OCP). Let $\mathcal{M}\subseteq\mathbb{R}^d$ be a non-empty Borel manifold. The optimal control problem is said to exhibit the *measure-turnpike property* (with respect to $\mathcal{M}$), if for all compact sets $K\subseteq\mathbb{R}^d$ and every $\epsilon>0$, there exists a constant $C_{K,\epsilon}>0$, such that for all $T>0$ all optimal trajectories c^* of eq. (OCP) satisfy

$$\mathcal{L}\Big(\{t\in[0,T] \mid \operatorname{dist}(c^*(t),\mathcal{M}) > \epsilon\}\Big) \le C_{K,\epsilon}$$

for all $c_0, c_T\in K$, where $\mathcal{L}$ denotes the Lebesgue measure.

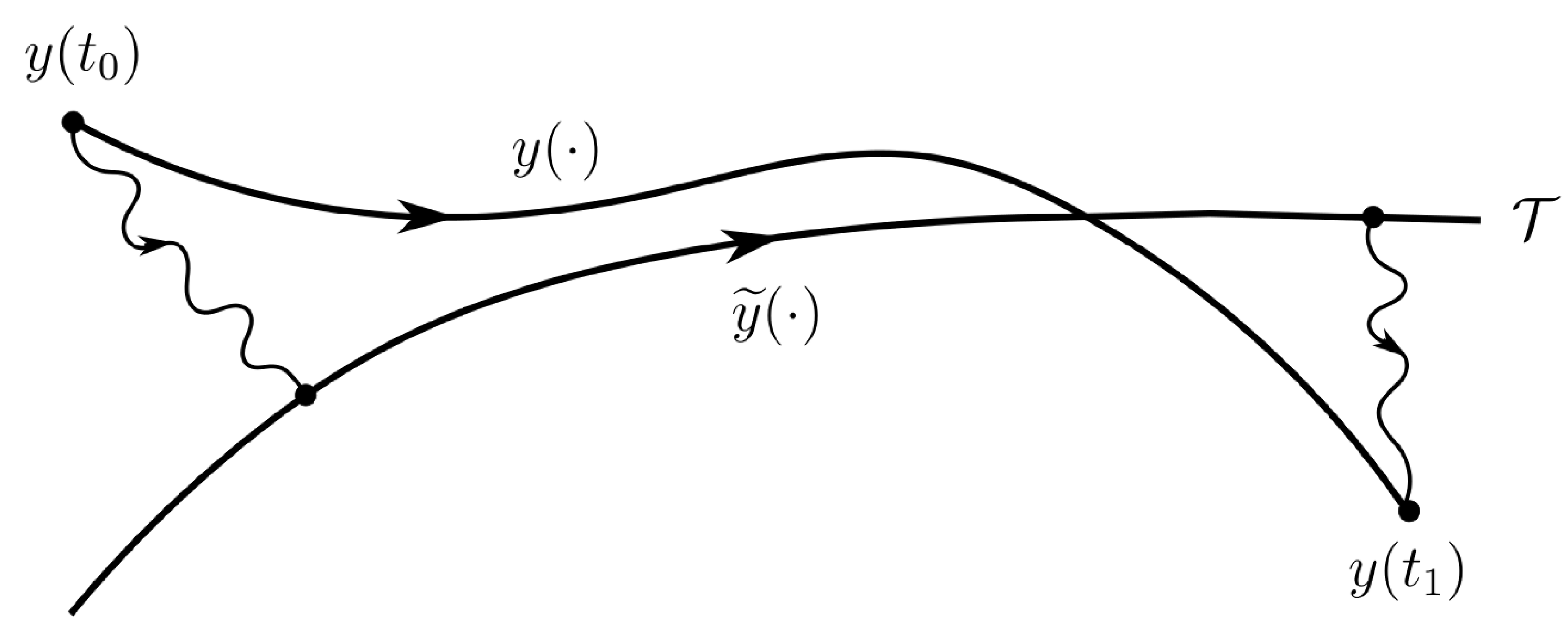

Figure: Illustration of the measure-turnpike property: The optimal trajectory y and addmissible trajectory $\tilde{y}$ remain along the turnpike set $\mathcal{T}$ as long as possible, see [3].

Strict Dissipativity

Apart from suitable structure and regularity assumptions, a typically used assumption is a property called strict dissipativity. These are then sufficient (often even necessary) assumptions to ensure the measure-turnpike property for (OCP). For those, that are a bit familiar with Lyapunov theory: Intuively, this is using the Hamiltonian $\mathcal{H}$ as a Lyapunov function. More precisely it is defined as follows.

Consider eq. (OCP) together with a non-empty manifold $\mathcal{M}\subseteq\mathbb{R}^d$. Problem (OCP) is then called dissipative with respect to $\mathcal{M}$, if there exists a function $Q:\mathbb{R}^d\to[0,\infty[$, that is bounded on compact sets and a function $\alpha:[0,\infty[\to[0,\infty[$ with $\alpha(0)=0$, which is continuous and strictly increasing, such that all optimal controls u^* and associated trajectories c^* satisfy the dissipation inequality

$$Q(c_T) - Q(c_0) \le \int_0^T \ell(c^*,u^*) - \alpha(\operatorname{dist}(c^*,\mathcal{M}))\,dt, \tag{8}$$

for all $T>0$, where ℓ is a continuous function.

Selected publications

[1] Faulwasser, T., Flaßkamp, K., Ober-Blöbaum, S., Schaller, M., Worthmann, K. (2022). **Manifold turnpikes, trims, and symmetries.** Mathematics of Control, Signals, and Systems, 34(4), 759-788.

[2] Karsai, A. (2024). **Manifold turnpikes of nonlinear port-Hamiltonian descriptor systems under minimal energy supply.** Mathematics of Control, Signals, and Systems, 36(3), 707-728.

[3] Trélat, E., Zhang, C. (2018). **Integral and measure-turnpike properties for infinite-dimensional optimal control systems.** Mathematics of Control, Signals, and Systems, 30(1), 3.

[4] Faulwasser, T., Maschke, B., Philipp, F., Schaller, M., Worthmann, K. (2022). **Optimal control of port-Hamiltonian descriptor systems with minimal energy supply.** SIAM Journal on Control and Optimization, 60(4), 2132-2158.

Department MATHEMATIK

Unterstützt von / Supported by
Alexander von Humboldt Stiftung/Foundation