CONTROL OF REACTION-DIFFUSION MODELS IN BIOLOGY AND SOCIAL SCIENCES

DOMÈNEC RUIZ-BALET AND ENRIQUE ZUAZUA

Abstract. These lecture notes address the controllability under state constraints of reaction-diffusion equations arising in socio-biological contexts. We restrict our study to scalar equations with monostable and bistable nonlinearities.

The uncontrolled models describing, for instance, population dynamics, concentrations of chemicals, temperatures, etc., intrinsically preserve pointwise bounds of the states that represent a proportion, volume-fraction, or density. This is guaranteed, in the absence of control, by the maximum or comparison principle.

We focus on the classical controllability problem, in which one aims to drive the system to a final target, for instance, a steady-state. In this context the state is required to preserve, in the presence of controls, the pointwise bounds of the uncontrolled dynamics.

The presence of constraints introduces significant added complexity for the control process. They may force the needed control-time to be large enough or even make some natural targets to be unreachable, due to the presence of barriers that the controlled trajectories might not be able to overcome.

We develop and present a general strategy to analyze these problems. We show how the combination of the various intrinsic qualitative properties of the systems' dynamics and, in particular, the use of traveling waves and steady-states' paths, can be employed to build controls driving the system to the desired target.

We also show how, depending on the value of the Allee parameter and on the size of the domain in which the process evolves, some natural targets might become unreachable. This is consistent with empirical observations in the context of endangered minoritized languages and species at risk of extinction.

Further recent extensions are presented, and open problems are settled. All the discussions are complemented with numerical simulations to illustrate the main methods and results.

Keywords Reaction-diffusion, control, paths of steady-states, constraints, phase plane, traveling waves, comparison principle, Mathematical Biology.

Domènec Ruiz-Balet
Department of Mathematics,
Imperial College, SW7 2AZ London, United Kingdom
email: domenec.ruizi@uam.es

Enrique Zuazua
Chair in Dynamics, Control and Numerics / Alexander von Humboldt-Professorship,
Department of Data Science,
Friedrich-Alexander-Universität Erlangen-Nürnberg
91058 Erlangen, Germany

and

Chair of Computational Mathematics,
Fundación Deusto
Av. de las Universidades, 24,
48007 Bilbao, Basque Country, Spain

and

Departamento de Matemáticas,
Universidad Autónoma de Madrid, 28049 Madrid, Spain
email: enrique.zuazua@fau.de.

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1. Introduction

1.1. **Motivation.** These lecture notes concern the control of reaction-diffusion equations with state constraints. Many quantities whose evolution is modeled by reaction-diffusion equations (such as those arising chemical reactions or biological populations) are positive, simply because they describe some density, concentration, distribution functions, volume fractions, etc. They can also be subject to bilateral bounds, for instance, when considering population fractions. The maximum or comparison principle for parabolic equations plays an important role in this context since it guarantees that the modeled dynamics fulfill these unilateral or bilateral bounds.

In control theory, the general goal is to interact with the system to achieve a specific purpose. One of the prototypical problems is the controllability one: by the choice of an appropriate control, one aims to drive the dynamics from an initial condition to a target configuration in a given time horizon. Other objectives can also be pursued, such as feedback stabilization (see [18, Part 3]). The problem of controllability becomes challenging in the presence of constraints, something that is unavoidable when dealing with the applications above, where the state is intrinsically and naturally constrained to fulfill some pointwise bounds.

In this work, we present the main challenges arising when dealing with controllability for reaction-diffusion equations with state constraints, some of the existing main results, and the techniques needed to handle them.

There is, by now, a fertile literature on the controllability of parabolic equations: [27,30,31,35,36,41,64], and references therein. Roughly, using Fourier series techniques and Carleman inequalities, parabolic equations and systems can be controlled in an arbitrarily small time, due to the intrinsic infinite velocity of propagation. However, most of the existing results do not guarantee that state constraints requirements, such as the positivity of solutions, are met. The study of controllability with state constraints for parabolic partial differential equations is a much more recent research topic [69, 71, 72, 84, 93, 98, 106]. The presence of constraints requires the development of new methods for controllability. The staircase or quasi-static control method, [20], [93], which consists of controlling the system keeping the trajectory in a neighborhood of a path of steady-states connecting the initial and the final datum, is, by now, the most useful tool to achieve controllability under constraints. But this method requires the time-horizon to be long enough, something that, as we shall see, is in fact necessary to meet the constraints. Indeed, in the presence of state constraints, controllability is only met after a minimal controllability or waiting time, a fact that is not observed in the unconstrained setting.

Another relevant difficulty arising in the context of constrained controllability is the appearance of barrier states, limiting the constrained dynamics, independently of the controls chosen. A barrier is a nontrivial steady-state that prevents any controlled trajectory from reaching specific targets due to the comparison principle. The staircase method cannot overcome these barriers, i.e. the paths of steady-states cannot provide a way to cross them.

We will explain how to build paths of steady-states, preserving the constraints, and therefore limited by the barrier functions, allowing to reach the full ensemble of reachable steady-states. Our study relies on the phase-plane analysis of the dynamical system associated with the elliptic equation that steady-states satisfy, initiated in [98].

In these lecture notes, we mainly focus on the one-dimensional case. However, the methods can also be applied to several space dimensions and different nonlinearities. We will also briefly present the main multi-dimensional versions in [106], those in [84] involving spatially heterogeneous drifts, and the result that can be achieved through the Allee interaction [114], which can also be combined with a boundary control.

1.2. **Organization of the lecture notes.** The lecture notes are organized as follows.

(1) First of all, we complement this introduction by discussing different applications in which the control problems addressed in the present manuscript arise and by a short description of various types of control problems.
(2) In Section 2, we present a model in which the control problem requires the fulfillment of state constraints.
(3) In Section 3, we review some of the most important properties of parabolic and elliptic equations that will be employed along the paper.
(4) In Section 4, firstly, we discuss the well-posedness of the control problem and the controllability of parabolic equations. We also present the staircase method following [93].
(5) Section 5 is devoted to analyzing the existence of nontrivial solutions of the elliptic problem (1.2). Furthermore, depending on the measure of the domain, we discuss the possible existence of barrier functions.
(6) Section 6 gathers graphical illustrations of the energy functional associated with the elliptic problem (1.2).
(7) Section 7 is devoted to the construction of admissible paths of steady-states fulfilling the constraints. We shall mainly focus on the problem of driving the system towards the intermediate constant steady-state θ for the bistable nonlinearity. First, we introduce the strategy used in [98], and later, we extend the reasoning to build paths of symmetric (even with respect to the center point of the space-interval) steady-states. We discuss the length of the maximal path depending on the size of the domain.
(8) In Section 8 we summarize the results of the previous sections.
(9) In Section 9 some numerical simulations of the control process under consideration are presented.
(10) Most of the results presented in the paper can be extended to several space dimensions, [106], and to models involving spatially heterogeneous drift terms, [84]. These extensions are presented in Section 10, where we also describe how the boundary control and the Allee-control introduced in [114] can be combined.
(11) In the last Section 11, we present some open problems and perspectives for future research.

1.3. **A model problem.** Let us first introduce the problem of the boundary control of some of the most common $1-d$ reaction-diffusion equations. This problem entails the core phenomenology arising in the context of constrained controllability.

Let $L>0$ and consider the system

$$\begin{cases} \partial_t v - \partial_{xx} v = f(v) & (x,t) \in (0,L)\times(0,T), \\ v(0,t) = a_1(t) \quad v(L,t) = a_2(t) & t\in(0,T), \\ 0 \leq v(x,0) \leq 1, & x\in(0,L), \end{cases} \tag{1.1}$$

where $f \in C^2$ satisfies $f(0)=f(1)=0$. We will mainly consider two types of nonlinearities f, (see Figure 1.3), namely:

- **Monostable**: when $f'(0)>0$ and $f'(1)<0$ with $f(s)>0$ for $s\in(0,1)$ (see Figure 1.1 right). A prototypical example of monostable nonlinearity is $f(v)=v(1-v)$.
- **Bistable**: there exists $\theta\in(0,1)$ such that $f(\theta)=0$, $f'(\theta)>0$ and $f'(0)<0$ $f'(1)<0$ with $f(s)<0$ in $(0,\theta)$ and $f(s)>0$ in $(\theta,0)$ (see Figure 1.1 left). A typical example of bistable nonlinearity is $f(v)=v(1-v)(v-\theta)$.

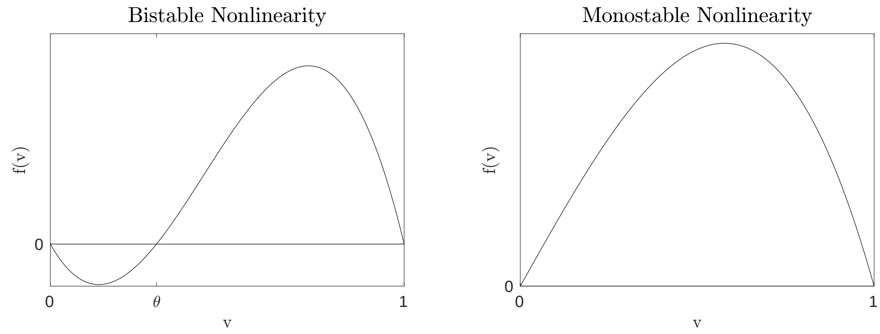

FIGURE 1.1. Left (resp. Right): A bistable (resp. monostable) nonlinearity.

Note that $v \equiv 1$ is a constant steady-state with $a_1 = a_2 = 1$, $v \equiv 0$ with $a_1 = a_2 = 0$ and $v \equiv \theta$, in the case of the bistable nonlinearities, with $a_1 = a_2 = \theta$.

In this model the state v typically represents a proportion or a population density. For this reason, we impose natural constraints: $0 \leq v(x,t) \leq 1$. The boundary controls $a_j(t)$, $j = 1, 2$, are therefore constrained by the same bounds, and, as we shall see, this limits significantly the possibility of controlling the system.

Most of these notes will be devoted to discussing whether the system can be driven, by the action of suitable controls, to reach these specific steady targets in a way that the constraints $0 \leq v \leq 1$ are preserved.

The classical methodology for controlling semilinear equations (e.g., [36]), based on using the linear controllability properties with careful estimates on the cost of control and fixed point arguments, do not apply since, typically, the controls obtained in this way will violate the constraints. Therefore, ad-hoc techniques, taking into account and exploiting the nonlinear dynamics of the system, will be developed to analyze the controllability properties under these constraints. One of the difficulties that we shall encounter is that barrier functions may arise as nontrivial solutions to the steady-state equation:

$$\begin{cases} -\partial_{xx} v = f(v) & x \in (0, L), \\ 0 \leq v(x, 0) \leq 1, & x \in (0, L), \\ v(0) = 0, \quad v(L) = 0 \end{cases} \tag{1.2}$$

with null boundary conditions/controls. Obviously, the trivial constant solution $v \equiv 0$, is a steady-state solution of the system. The existence of nontrivial solutions to (1.2), v_b, depends, basically, on the length of L. Due to the comparison principle, when such a nontrivial barrier v_b exists, the final target $v = 0$ will not be reachable when the dynamics departs from an initial configuration above v_b, since, whatever the controls are, being non-negative, the solution will always remain above this barrier. This is is an important warning since it indicates that the control results we might expect will vary depending on L.

This fact has a clear interpretation in applications. For instance, it is well-known that the survival of species is related to the size of the available territory [104] (see also [8]). In other words, if the area in which the population lives and evolves is too small, it will likely tend to extinction, while in larger domains, survival will be more feasible. This is due to the fact that, even if individuals on the vicinity of the boundary will tend to die, the reproduction rate inside the domain will suffice to compensate for this loss, assuring the overall survival of the species.

This can be understood in mathematical terms by the stability properties of the trivial steady-state $v \equiv 0$, that will attract, or not, the whole ensemble of initial data within the bounds $0 \leq v \leq 1$, depending on the length of L: when L is small enough, all initial data will be attracted towards $v \equiv 0$, while for L large, because of the barrier effects mentioned above, some initial data will evolve remaining always above v_b, avoiding convergence towards $v \equiv 0$.

Of course, the long-time stability properties of the system and the nature of the set of steady-state solutions are intimately related. For instance, from [78] (see also [122]), we know that bounded solutions of one-dimensional reaction-diffusion equations converge to steady-states. This classical result, combined with the instability of the null steady-state for large domains, is a way to understand the existence of a nontrivial solution to (1.2), a fact that can be directly explained in the elliptic context using the theory of critical points from the Calculus of Variations.

1.4. **Bibliographical notes on applications.** Reaction-diffusion systems frequently appear in natural sciences and applications, in a diversity of contexts that we will briefle describe. As we will also see, control problems for these models can adopt different forms and formulations, depending on the application context.

- **Population dynamics and spatial ecology**. Kolmogorov [60] proposed a model in ecology in which a population diffuses in space on the whole real line and grows in a nonlinear manner, discovering the existence of traveling wave solutions of the form $U(x-ct)$, with a given profile $U(x)$ and a travel velocity c. This pioneering work gave rise to a vast literature on reaction-diffusion, and spatial ecology [38]. For more general diffusion models in ecology, we refer to [5, 6, 17, 40, 85, 88, 92]. We also refer to [11] for an empirical finding of traveling waves in ecology.
- **Chemical reactions** [92, 109]. In most cases, models in this context are systems and not scalar equations as those analyzed here, enjoying a much richer dynamics. Turing patterns [117] is one of the paradigmatic phenomena, which emerges when there is a large contrast in the diffusivity constants of the various equations constituting the system. Alan Turing's model was originally proposed for morphogenesis, and it has been proven experimentally to be successful [89].
- **Magnetic systems** in material science: Reaction-diffusion equations have been also employed to understand the dynamical evolution of spins in magnetic materials. In [23], a reaction-diffusion system for the evolution of the magnetization of the material is derived from the microscopic stochastic dynamics of spins [45].
- **Evolutionary game theory**: In this field, one seeks to understand how players change their strategies depending on those of the other players [22, 54, 56, 91, 118]. When considering the possible spatial diffusion of the players, reaction-diffusion equations arise naturally, [53, 55].
- **Neuroscience:** traveling wave phenomena also arise when modeling nerve impulses [28].
- **Linguistics**: parabolic models may also be employed to analyze language shift, [102].

1.5. **Introduction to control problems.** Control problems arising in these fields can be presented in a diversity of forms and allow for various mathematical formulations. Here we briefly present some of them.

- **Interior Control**: This type of control action finds applications, for example, in parabolic equations arising in heat processes. Consider a bar of length L and a sub-interval $\omega \subset (0,L)$ where a heater/cooler is placed. The evolution of the temperature of the bar is governed by the system:
$$\begin{cases} \partial_t v - \partial_{xx} v = \chi_\omega a, \\ v(0,t) = v(L,t) = 0, \\ v(\cdot, t=0) = v_0. \end{cases}$$
Here v_0 stands for the initial temperature distribution, while the temperature at the boundary is fixed. The heating control is modeled by $a = a(x,t)$ which acts locally in ω, but aiming a global effect.

  Several types of control problems can be considered. The controllability towards a steady-state (in particular, when $v \equiv 0$, the so-called null-control problem) has been extensively studied in the literature: [31, 32]. But the case where state constraints are imposed has been much less studied and only more recently: [72, 76].
- *Multiplicative and Bilinear Control*: In this case the control does not enter on the system as a linear right hand side source term but rather as a potential $a(x,t)$ multiplying the state itself:
$$\begin{cases} \partial_t v - \partial_{xx} v = av, \\ v(0,t) = v(L,t) = 0, \\ v(\cdot, t=0) = v_0. \end{cases}$$
Of course, in this case, the impact of the control $a(x,t)$ on the system is much weaker. In particular, if the initial datum $v_0 \equiv 0$ is the trivial one, then the solution will also be $v \equiv 0$ for all time. Therefore at the final time, $t = T$, only the final target $v(\cdot, T) \equiv 0$ will be reached, regardless of the value of the control. We refer to [15, 25, 58, 59] and to [24, 50, 75, 79–83, 86] for the analysis of bilinear control on population dynamics systems.

- *Allee Control*: In [114], adopting a micro-macro modeling perspective (as in [23, 26]), the role of the Allee threshold parameter θ of the bistable nonlinearity as an effective control of the system is justified. In practical applications, for instance, on the regulation of the propagation of invasive mosquito species, this Allee threshold can be regulated by releasing sterile mosquitoes. This leads to a model of the form
$$\begin{cases} \partial_t v - \partial_{xx} v = v(v - \theta(t))(1 - v) & x \in \mathbb{R}, \\ v(\cdot, t = 0) = v_0 \in L^\infty(\Omega; [0, 1]), \end{cases}$$
where $\theta = \theta(t)$ plays the role of control.
- *Boundary Control*: This action is another prototypical way of possible interaction in control systems. Boundary control mechanisms are closely related to interior controls acting on a neighborhood of the boundary. This can be rigorously justified through the classical extension-restriction argument, [110]. The corresponding control system then takes the form
$$\begin{cases} \partial_t v - \partial_{xx} v = 0, \\ v(0, t) = a_1(t), \quad v(L, t) = a_2(t), \\ v(\cdot, t = 0) = v_0, \end{cases}$$
where $a_j = a_j(t)$, $j = 1, 2$ play the role of controls. Most of these lecture notes will be devoted to considering boundary control problems.

In all these contexts, control problems can be formulated in different forms. One may distinguish, in particular, the following goals and issues.

(1) Controllability problems, [18, 116, 126]: given an initial datum and a specific target, to find a control function that drives the system to the target in a given time-horizon.
(2) Stabilization, [18, Part III]: Given an unstable equilibrium configuration, can we find a control function in a feedback form (depending on the state) that stabilizes the systems towards this equilibrium?
(3) Optimal control. This problem can be formulated as a minimization one, for instance in a least-squares context. One seeks to minimize a cost functional depending on the state and the control, typically, to lead the system towards a neighborhood of a reference trajectory. Optimal control problems for parabolic problems have been widely considered, in particular, in [14, 65, 107, 115].

2. Parabolic models

In this section, we derive the model that will be considered and explain the relevance of constraints for the associated control problems.

2.1. **ODE versus PDE modelling.** The first Ordinary Differential Equation (ODE) considered in population dynamics exhibits an exponential growth of the population $P = P(t) > 0$:
$$\frac{P'}{P} = \beta,$$
Verhulst [120] noticed that the competition for limited resources among individuals of the same population provides a more accurate model and leads to an upper threshold of the population growth:
$$P' = \beta P \left(1 - \frac{P}{\kappa}\right),$$
where κ is the capacity of the environment.

ODEs can be adapted to the Partial Differential Equation (PDE) setting to model the movement and invasion of species. Assuming that the diffusion is homogeneous and that the resources are space invariant, the equation can be formulated as follows:
$$\begin{cases} \partial_t v - \partial_{xx} v = \beta v(1 - v) & (x, t) \in (0, L) \times (0, T), \\ \partial_x v = 0 & (x, t) \in \{0, L\} \times (0, T), \\ v(\cdot, t = 0) = v_0 \in L^\infty(\Omega; \mathbb{R}^+), \end{cases}$$
for $\beta > 0$. Note that in the equation above, for the sake of simplicity, we have assumed $\kappa = 1$ and the diffusivity constant $\mu = 1$. As it will be seen in the next section, the solution of the system above is nonnegative for all times. This is coherent with the fact of v being a density of population. Null

Neumann type boundary conditions are adopted to represent the fact that no population flow through the boundary is allowed.

The constraint on the state $v \leq 1$ is relevant when the quantity under consideration is a proportion and can be integrated into the above system or for more general nonlinearities f such that, $f(0) = f(1) = 0$ with $f'(0) > 0$, $f'(1) < 0$ and $f \geq 0$ in $(0,1)$.

2.2. **The bistable model.** Let $\Omega = (0, L) \subset \mathbb{R}$ and assume that two populations represented by non-negative functions V and W, interact in a nonlinear fashion while preserving the quantity $V + W$, and diffusing in space. We will discuss later possible contexts in which such a situation might occur. The corresponding model reads:

$$\begin{cases} \partial_t V - \partial_{xx} V = F(V,W) & (x,t) \in (0,L) \times (0,T), \\ \partial_t W - \partial_{xx} W = -F(V,W) & (x,t) \in (0,L) \times (0,T), \\ \partial_x V = \partial_x W = 0 & (x,t) \in \{0,L\} \times (0,T), \\ V(\cdot, t=0) = V_0 \in L^\infty((0,L);\mathbb{R}^+), & \\ W(\cdot, t=0) = W_0 \in L^\infty((0,L);\mathbb{R}^+), & \end{cases} \tag{2.1}$$

where F is a Lipschitz continuous function satisfying $F(V,0) = F(0,W) = 0$ for every $V, W \in \mathbb{R}$.

We first look for a simplified approximation of (2.1) involving one single equation. Note that

$$P := V + W,$$

is positive and satisfies:

$$\begin{cases} \partial_t P - \partial_{xx} P = 0 & (x,t) \in (0,L) \times (0,T), \\ \partial_x P = 0 & (x,t) \in \{0,L\} \times (0,T), \\ P(\cdot, t=0) = V_0 + W_0, & \end{cases} \tag{2.2}$$

whose long time asymptotic is given by the constant

$$\lim_{t\to\infty} P(t,x) = \frac{1}{L}\int_0^L [V(x,0) + W(x,0)]dx.$$

It is therefore natural, for simplification purposes, to assume that P is actually constant. This allows to define $v := V/P$ as the proportion of the V type population in the whole population P. Then (2.1) reduces to:

$$\begin{cases} \partial_t v - \partial_{xx} v = f(v) & (x,t) \in (0,L) \times (0,T), \\ \partial_x v = 0 & (x,t) \in \{0,L\} \times (0,T), \\ v(\cdot, t=0) = v_0 \in L^\infty((0,L);[0,1]). & \end{cases}$$

for a suitable f that can be easily derived out of F.

In this setting, the bilateral constraints $0 \leq v(x,t) \leq 1$ arise naturally for all (x,t). The zeros of the original nonlinearity F guarantee that $f(0) = f(1) = 0$. Then, by the comparison principle, when the initial datum v_0 satisfies these bilateral bounds, they are guaranteed to hold for all (x,t) (see next section, Section 3).

A nonlinearity $f \in C^1$ is said to be bistable if:

$$f(0) = f(\theta) = f(1) = 0, \quad f'(0) < 0, \quad f'(1) < 0, \quad f'(\theta) > 0,$$
$$f(s) < 0 \text{ for } s \in (0,\theta), \quad f(s) > 0 \text{ for } s \in (\theta,1).$$

As an example, observe that if

$$F(V,W) = \frac{VW}{(V+W)^2}\left((1-\theta)V + \theta W\right)$$

for $\theta \in (0,1)$, the corresponding f takes the form

$$f(v) = v(1-v)(v-\theta)$$

which has such a bistable structure. This model has special interest, in particular, in game theory as described below:

- *Game theoretical interpretation.* This nonlinearity arises naturally in the context of evolutionary game theory in coordinated games.

  The choice of languages in a social context is a prototypical example. Indeed, assume two bilingual individuals meet each other, and they establish a conversation in one of the two languages they master. The possible different preferences for languages, can be modeled on a payoff matrix.

  In this context, the replicator dynamics is a nonlinear ODE that models the players' change of strategy in time. Players continuously play the game, and they evaluate their success while comparing it with the average success of all players. The replicator dynamics for a two-strategy game is represented by:
$$\frac{d}{dt}v = v\left((1,0)A\begin{pmatrix} v \\ 1-v \end{pmatrix} - (v, 1-v)A\begin{pmatrix} v \\ 1-v \end{pmatrix}\right)$$
  where $A \in M_2(\mathbb{R})$ is a payoff matrix.

  In this model, v is the proportion of players playing the first strategy, and $1-v$ that of those playing the remaining one. The first term on the right-hand side evaluates the success of the first strategy, while the second one is the players' average success. If the first strategy has less success than the average, the proportion of players playing the first strategy will diminish in favor of the second strategy. The payoff matrix of a coordination two strategy game is
$$A = \begin{pmatrix} (1-\theta) & 0 \\ 0 & \theta \end{pmatrix},$$
  that leads to the nonlinearity $f(v) = v(1-v)(v-\theta)$.

  In this game, we see that if $\theta < 1/2$, players prefer to play the first strategy rather than the second one. However, the preference does not uniquely determine the behavior of the system. One can see for instance that both consensus configuration $v = 0$ and $v = 1$ are stable. Indeed, $f'(0) < 0$ and $f'(1) < 0$, and this assures that the less wanted strategy is stable if enough players are currently playing it.

  For further details and other possible models we cite [22, 53, 54, 54, 55, 91].
- *Biological interpretation.*

  In a biological context, according to the so-called Allee effect (see [112]), when the population in a given habitat is lower than a given threshold θ, they cannot survive.

  A game-theoretical approach can also be adopted in this context. Assume the population presents two distinct genes. More successful genes will be transferred to the next generations, while the less successful ones will disappear. In this situation, the replicator dynamics applies.

  In ecology, spatial heterogeneities are omnipresent. Note that when A depends on x like in
$$A(x) = \begin{pmatrix} (1-\theta(x)) & 0 \\ 0 & \theta(x) \end{pmatrix},$$
  $\theta(x)$ determines which strategy is having an advantage, depending on the spatial location x. So that, if $0 < s < \theta(x)$, then $f < 0$, but $f > 0$ if $\theta(x) < s < 1$. This leads to a model in which the nonlinearity depends on x. In ecological systems, one may also consider situations in which one gene is favorable in dry environments, and the other one in humid habitats, for instance. This case will be briefly discussed later on.

  One may also consider different diffusivities for each of the genes, where the reduction to a scalar case might not be possible.
- *Chemical reactions.* A chemical reaction in which the number of reactants, constituted by particles of different classes, can also be represented by similar systems, [92, Chapter 1].

2.3. **Boundary control.** In these lecture notes we will mainly consider the situation in which the control acts on the boundary. The control can enter through the Neumann boundary condition, for instance, regulating the flux:
$$\begin{cases} \partial_t v - \partial_{xx} v = f(v) & (x,t) \in (0,L) \times (0,T), \\ \dfrac{\partial v}{\partial \nu} = b(x,t) & (x,t) \in \{0,L\} \times (0,T), \\ 0 \leq v(x,0) \leq 1 & x \in (0,L). \end{cases}$$
But, obviously, a trajectory with Neumann control can also be understood as being submitted to a Dirichlet one, with controls that, actually, coincide with the trace of the obtained solution. Therefore, equivalently, we may consider the same problems with Dirichlet controls that are somehow easier to

handle. In particular in the Dirichlet context, constraints on the state are simply reflected as constraints on controls, thanks to the maximum or comparison principle.

Indeed, given the Neumann control $b(x,t)$, and the corresponding solution $v = v(x,t)$, its Dirichlet trace $a(x,t)$ can be understood as a Dirichlet control, leading to the equivalent control system

$$\begin{cases} \partial_t v - \partial_{xx} v = f(v) & (x,t) \in (0,L) \times (0,T), \\ v = a(x,t) & (x,t) \in \{0,L\} \times (0,T), \\ 0 \leq v(x,0) \leq 1 & x \in (0,L). \end{cases} \tag{2.3}$$

Of course the constraints on v, solution of the Neumann problem, are automatically transferred to the Dirichlet controls

$$0 \leq a(x,t) \leq 1 \quad \text{for any } (x,t) \in \{0,L\} \times [0,T].$$

On the other hand, as we shall see in the subsequent section, by the comparison principle, the constraints on the controls a suffice for them to hold for the solution v of (2.3) as well.

Thus, in the context of Dirichlet controls, imposing bilateral bounds on solutions is equivalent to imposing them on the controls.

3. Review of some results on parabolic equations

In this section, we gather some classical results on semilinear parabolic equations that are useful for control purposes. We will mainly expose results concerning

(1) convergence to steady-states,
(2) comparison pinciples,
(3) traveling waves.

The primary tool to control reaction-diffusion equations with constraints to steady targets is the staircase method, which uses paths of steady-states. The stability of these steady-states plays also an important role since it allows to extend the ensemble of the controllable data. But, as mentioned above, the existence of nontrivial steady-states can also be a fundamental obstruction for control, due to the comparison principle.

This section gathers the main technical results needed to develop these principles. In particular, we will employ the comparison principle between the solution of the reaction-diffusion equation and a section of a traveling wave, that corresponds to its restriction to the spatial domain under consideration.

3.1. Convergence to steady-states.

Consider the following one-dimensional semilinear heat equation

$$\begin{cases} \partial_t v = \partial_x \left(a(x)\partial_x v\right) + f(x,v) & (x,t) \in (0,L) \times (0,T) \\ v(x,0) = v_0(x) & x \in (0,L), \\ v(0,t) = \beta_0, \quad v(L,t) = \beta_L & t \in (0,T), \end{cases} \tag{3.1}$$

β_0 and β_L being constants independent of t.

Let $t_e(v_0)$ be the maximum time of existence of the solution of (3.1). For general nonlinearities, solutions of (3.1) may blow-up in finite time.

Let $v_0 \in C([0,L];\mathbb{R})$ and suppose that the solution is global, so that the maximum time of definition of the solution is $+\infty$: $t_e(v_0) = \infty$. The ω-limit set of the solutions can be defined as the set of accumulation points of the trajectory $v(\cdot,t)$ in $C^1([0,L])$ as $t \to \infty$. It is by now well-known that any bounded trajectory converges to a steady-state. The result actually holds for more general boundary conditions [77] (see also [122] for an earlier reference).

Theorem 3.1 (Matano, Theorems A and B from [78]). *The ω-limit set of any function $v_0 \in C([0,L];\mathbb{R})$ contains at most one element.*

For any initial data $v_0 \in C([0,L];\mathbb{R})$, one and only one of the following three properties holds:

(1) *The solution blows up in finite time, $t_e(v_0) < \infty$, and $\lim_{t\to t_e(v_0)} \|v(t)\|_{L^\infty([0,L];\mathbb{R})} = \infty$.*
(2) *The solution is global, $t_e(v_0) = \infty$, grows as $t \to \infty$, and $\lim_{t\to t_e(v_0)} \|v(t)\|_{L^\infty([0,L];\mathbb{R})} = \infty$.*
(3) *The solution converges to a solution of the elliptic problem.*

$$\begin{cases} \partial_x \left(a(x)\partial_x v\right) + f(x,v) = 0 & x \in (0,L), \\ v(0) = \beta_0, \quad v(L) = \beta_L, \end{cases}$$

in the $C^1([0,L];\mathbb{R}) \cup C^2((0,L);\mathbb{R})$ *topology.*

In higher dimensions, the theorem above is no longer true, as shown in [95,96]. However, if the nonlinearity is analytic, one can ensure the convergence to steady-states thanks to the Łojasiewicz gradient inequality [57,111] (see also [48,68]).

3.2. **Comparison results.** The comparison principle enables us to gain further understanding on the dynamics of the trajectories of the PDE under consideration, [13,39,103], [38, Chapter 5]. We present simultaneously parabolic and elliptic comparison principles. We state them in the one dimensional context; although they also hold in several space dimensions.

Definition 3.2 (Parabolic sub- and supersolutions)**.** Consider the elliptic operator:

$$\mathcal{L} := \partial_{xx} + k(x)\partial_x$$

where $k:(0,L)\to\mathbb{R}$ is a smooth function. Let $f:\mathbb{R}\to\mathbb{R}$ be a smooth function and $h:\{0,L\}\times(0,T)\to\mathbb{R}$. Consider the parabolic problem:

$$\begin{cases} \partial_t v - \mathcal{L}v = f(v) & (x,t)\in(0,L)\times(0,T) \\ v(t,x) = h(x,t) & (x,t)\in\{0,L\}\times(0,T) \\ v(0,x) = v_0(x) & x\in(0,L). \end{cases} \tag{3.2}$$

A subsolution $\underline{v}$ of (3.2) satisfies:

$$\begin{cases} \partial_t \underline{v} - \mathcal{L}\underline{v} \leq f(\underline{v}) & (x,t)\in(0,L)\times(0,T) \\ \underline{v}(t,x) \leq h(t,x) & (x,t)\in\{0,L\}\times(0,T) \\ \underline{v}(0,x) \leq v_0(x) & x\in(0,L). \end{cases}$$

A supersolution $\overline{v}$ of (3.2) satisfies:

$$\begin{cases} \partial_t \overline{v} - \mathcal{L}\overline{v} \geq f(\overline{v}) & (x,t)\in(0,L)\times(0,T) \\ \overline{v}(t,x) \geq h(t,x) & (x,t)\in\{0,L\}\times(0,T) \\ \overline{v}(0,x) \geq v_0(x) & x\in(0,L). \end{cases}$$

Theorem 3.3 (Parabolic comparison principle [13])**.** *If $\underline{v}$ is a subsolution (respectively $\overline{v}$ a supersolution) to* (3.2) *and v is a solution such that $v\geq\underline{v}$ (respectively $v\leq\overline{v}$) on $(x,t)\in\{0,L\}\times[0,T)\bigcup(0,L)\times\{0\}$ then $v\geq\underline{v}$ ($v\leq\overline{v}$) in $(x,t)\in(0,L)\times(0,T)$.*

In the elliptic context the comparison principle reads as follows. Let $f:[0,L]\times\mathbb{R}\longrightarrow\mathbb{R}$ be a locally Lipschitz function and consider the equation:

$$\begin{cases} -\mathcal{L}v(x) = f(x,v(x)) & x\in(0,L), \\ v(x) = 0 & x\in\{0,L\}. \end{cases} \tag{3.3}$$

Definition 3.4 (Elliptic sub- and supersolutions)**.** We say that $\underline{v}$ (respectively $\overline{v}$) is a subsolution (respectively a supersolution) if $\underline{v}$ (respectively $\overline{v}$) belongs to $C^0([0,L])\cup C^2((0,L))$ and verifies that:

$$\begin{cases} -\mathcal{L}\underline{v}(x) \leq f(x,\underline{v}(x)) & x\in(0,L), \\ \underline{v}\leq 0 & x\in\{0,L\}, \end{cases}$$

or, respectively,

$$\begin{cases} -\mathcal{L}\overline{v}(x) \geq f(x,\overline{v}(x)) & x\in(0,L), \\ \overline{v}\geq 0 & x\in\{0,L\}. \end{cases}$$

Theorem 3.5 (Elliptic comparison principle, Theorem 5.17 in [63])**.** *Assume that there exist a subsolution $\underline{v}$ and a supersolution $\overline{v}$ of* (3.3) *such that $\underline{v}\leq\overline{v}$.*

Then (3.3) *admits a minimal solution $\underline{v}_*$ and a maximal (possibly equal) solution $\overline{v}^*$ such that, $\underline{v}\leq\underline{v}_*\leq\overline{v}^*\leq\overline{v}$ and there exist no solution u between $\underline{v}$ and $\overline{v}$ such that at a certain point $x\in\Omega$ it satisfies either $v(x)\leq\underline{v}_*(x)$ or $v(x)\geq\overline{v}^*(x)$*

As mentioned above, solutions of reaction-diffusion systems can blow up in finite time for certain initial data and specific nonlinearities growing super linearly at infinity, [29, Chapter 9, pp 547-550]. The following corollary guarantees the stability of the system if the initial data lies in $0\leq v_0\leq 1$.

Corollary 3.6 (Stability)**.** *Assume that* $0 \le v_0 \le 1$ *and* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *is locally Lipschitz continuous with* $f(0) = f(1) = 0$*. Then, the solution of the problem*

$$\begin{cases} \partial_t v = \partial_{xx} v + f(v) & (x,t) \in (0,L) \times (0,T), \\ v(x,t) = a(x) \in [0,1] & (x,t) \in \{0,L\} \times (0,T), \\ v(x,0) = v_0(x) & x \in (0,L), \end{cases}$$

is defined for all positive time and

$$0 \le v(x,t) \le 1.$$

Proof. The existence and uniqueness of solutions follows from classical methods, existence and uniqueness for the linear problem and the application of a fixed point method for the nonlinear equation [29, Chapter 9].

The comparison principle applies as well, observing that the functions $\overline{v}, \underline{v} : (0,L) \times \mathbb{R}^+ \to \mathbb{R}$ defined as $\overline{v}(x,t) = 1$ and $\underline{v}(x,t) = 0$ are a supersolution and a subsolution, respectively.

□

3.3. **Traveling waves.** We are considering Dirichlet boundary controls in both extremes of the domain: $x = 0, L$. This is, in fact, equivalent to not imposing boundary conditions at all, and simply reading-off the boundary traces of the solutions, provided they fulfill the bilateral bounds. Note that this argument is actually the one that allowed us to show the equivalence between the Neumann and the Dirichlet control problems in the presence of constraints. Observe, however, that this equivalence is no longer true when the control acts only on one extreme of the boundary, for instance. This case requires a specific treatment, depending on the boundary conditions under consideration.

Considering the Dirichlet boundary control problem from this perspective, i.e. ignoring the boundary conditions under the sole condition that the solution satisfies $0 \le v(x,t) \le 1$, allows for instance considering the Cauchy problem in the whole real line, and the restrictions of its solutions to the domain $(0,L)$ under consideration.

A particular relevant example of trajectories defined in the whole real line are the so-called traveling waves, [38, 60] for the Cauchy problem:

$$\begin{cases} \partial_t v - \partial_{xx} v = f(v) & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ v(\cdot, t=0) = v_0 \in L^\infty(\mathbb{R}). \end{cases} \tag{3.4}$$

Definition 3.7 (Traveling waves)**.** A traveling wave solution to (3.4) is a solution of the form $v(t,x) = U(x - ct)$ with $c \in \mathbb{R}$ being the wave speed and $U = U(s)$ its profile.

The existence of such functions was discovered by Kolmogorov [60] and, since then, they have been exhaustively studied.

Note that the C^2 profile U such that $U(+\infty) = 0$ and $U(-\infty) = 1$ defines a traveling wave solution iff

$$-cU'(s) + U''(s) = f(U(s)) \qquad s \in \mathbb{R} \tag{3.5}$$

where $s = x - ct$. From (3.5) by multiplying by $U'(s)$ and integrating over $\mathbb{R}$ one observes,

$$-c \int_{\mathbb{R}} (U'(s))^2 ds + \int_{\mathbb{R}} U''(s) U'(s) ds = \int_{\mathbb{R}} f(U(s)) U'(s) ds.$$

This gives an implicit definition of the velocity of propagation of the profile. Indeed, using the fact that $U'(\pm\infty) = 0$, one can see that the above expression is reduced to:

$$c = \frac{F(1)}{\int_{\mathbb{R}} (U'(s))^2 ds}, \tag{3.6}$$

where F is the primitive of f.

Equation (3.6) gives, in particular, the direction of the traveling wave. Indeed, if $F(1) > 0$, the traveling wave is moving to the right, so that eventually, as t increases, the value 1 invades the whole real line, while, if $F(1) = 0$, the profile defines a steady-state solution. The existence of traveling waves for bistable nonlinearities can be proved, for instance, using phase-plane techniques, understanding (3.5) as a dynamical system and looking for a trajectory that connects $(U(-\infty) = 1, U'(-\infty) = 0)$ with $(U(-\infty) = 0, U'(-\infty) = 0)$.

The next theorem guarantees the existence of traveling waves for bistable nonlinearities:

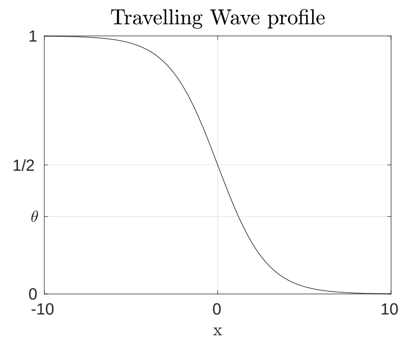

FIGURE 3.1. Profile $U(x)$ for the traveling wave for the cubic nonlinearity $f(u) = u(1-u)(u-\theta)$ restricted in the interval $[-10, 10]$. This profile U is independent of the value of θ.

Theorem 3.8 (Traveling waves for the bistable nonlinearity, Theorem 4.9 in [92], Theorem 4.15 [38])**.** *Assume that:*

$$f(0) = 0, \quad f'(0) < 0, \quad f(\theta) = 0, \quad f(1) = 0, \quad f'(1) < 0$$
$$f(v) < 0 \quad \textit{for} \quad 0 < v < \theta, \quad f(v) > 0 \quad \textit{for} \quad \theta < v < 1$$

Then, there exists a unique traveling wave $(c^*, v = U(x - c * t))$ of (3.4) with a decreasing profile U and satisfying:

$$c^* > 0 \textit{ for } F(1) > 0, \quad c^* < 0 \textit{ for } F(1) < 0, \quad c^* = 0 \textit{ for } F(1) = 0$$

where $F(s) = \int_0^s f(v)dv$.

For the cubic nonlinearity $f(u) = u(1-u)(u-\theta)$ the profile $U(x)$ has an explicit expression

$$U(x) = \frac{\exp\{-x/\sqrt{2}\}}{1 + \exp\{-x/\sqrt{2}\}},$$

shown in Figure 3.1, with traveling speed

$$c^* = \sqrt{2}\left(\frac{1}{2} - \theta\right).$$

Traveling wave solutions connect the steady-state 0 with the steady-state 1. They can act as attractors for the dynamical system. Indeed, there is a wide class of initial data that exponentially converges to a traveling wave as $t \to +\infty$.

Theorem 3.9 (Theorem 4.16 in [38])**.** *Assume that f bistable. Then if the initial data of (3.4), v_0 satisfies:*

$$\limsup_{x\to-\infty} v_0(x) < \theta \qquad \liminf_{x\to+\infty} v_0(x) > \theta,$$

there exist constants $C > 0$, $\mu > 0$ (independent of x) and $x_0 \in \mathbb{R}$ such that:

$$|u(x,t) - U(x - ct - x_0)| < Ce^{-\mu t}.$$

Assuming that f is monostable:

$$f(0) = f(1) = 0, \quad f(s) > 0 \quad s \in (0,1); \quad f'(0) > 0, \quad f'(1) < 0,$$

there is no uniqueness of the traveling waves

Theorem 3.10 (Theorem 4.15 in [38])**.** *Let f be monostable. Then there exist $c^* > 0$ such that:*

- *there exist a traveling wave solution with $U(-\infty) = 1$, $U(+\infty) = 0$ if $c \geq c^*$,*
- *if $c < c^*$ there does not exist any traveling wave.*

In this case there exist infinitely many traveling waves. This issue has been extensively analysed in some specific models like the Fisher-KPP equation below, [60]:

$$\begin{cases} \partial_t v - \partial_{xx} v = rv(1-v) & (x,t) \in \mathbb{R} \times \mathbb{R}^+ \\ 0 \leq v(x,0) \leq 1 & x \in \mathbb{R}. \end{cases} \tag{3.7}$$

Concerning the stability of traveling wave solutions in the monostable case there is a specific traveling wave that enjoys stability properties:

Theorem 3.11 ([60]). *Let $f : \mathbb{R} \to \mathbb{R}$ be monostable, fulfilling $f'(0) \geq f'(s)$ for all $0 \leq s \leq 1$ and let $v = v(x,t)$ be the solution of*

$$\begin{cases} \partial_t v - \partial_{xx} v = f(v) & (x,t) \in \mathbb{R} \times \mathbb{R}^+ \\ v(\cdot, t=0) = v_0 \end{cases}$$

where

$$v_0(x) = \begin{cases} 0 & for \quad x > 0 \\ 1 & for \quad x < 0. \end{cases}$$

Then there exist a function $\psi \in C^1$ such that

$$|u(x,t) - U(x - c^* t - \psi(t))| \to 0 \qquad as \qquad t \to \infty \tag{3.8}$$

uniformly in $x \in \mathbb{R}$ and:

$$\lim_{t \to \infty} \psi'(t) = 0$$

We refer to [108], [105], [37], among others, for further results on the asymptotic stability of solutions.

Traveling waves play a very interesting role in the context of control since its restriction to the bounded domain $[0, L]$ yields a trajectory linking an arc near the steady-state 0 with another one near 1 (or the reverse, depending in the sense of the propagation). Traveling waves can also naturally be used to construct sub- and super-solutions for controlled trajectories.

4. Well-posedness of the control problem and controllability

4.1. **Comments on the well-posedness.** Let us first discuss the well-posedness of the main control problem considered in these lecture notes, namely the boundary control of the scalar reaction-diffusion equation:

$$\begin{cases} \partial_t v - \partial_{xx} v = f(v) & (x,t) \in (0,L) \times (0,T), \\ v(0,t) = a_1(t), \quad v(L,t) = a_2(t) & t \in (0,T), \\ v(x,0) = v_0(x), \quad v_0 \in L^\infty((0,L);[0,1]) \end{cases} \tag{4.1}$$

where $a_i \in L^\infty((0,T);\mathbb{R})$ for $i = 1,2$ are control functions.

Splitting the solution into two subproblems, $v = w + y$, as in [93], where

$$\begin{cases} \partial_t w - \partial_{xx} w = 0 & (x,t) \in (0,L) \times (0,T), \\ w(0,t) = a_1(t), \quad w(L,t) = a_2(t) & t \in (0,T), \\ w(x,0) = 0, \end{cases} \tag{4.2}$$

and

$$\begin{cases} \partial_t y - \partial_{xx} y = f(y + w) & (x,t) \in (0,L) \times (0,T), \\ y(0,t) = 0, \quad y(L,t) = 0 & t \in (0,T), \\ y(x,0) = v_0(x), \quad v_0 \in L^\infty((0,L);[0,1]), \end{cases} \tag{4.3}$$

the existence and uniqueness of a weak solution can be shown. In the presence of boundary control weak solutions are defined as follows:

Definition 4.1 (Weak solution). Consider the space:

$$\mathcal{T} := \{\varphi \in C^\infty([0,T] \times [0,L]) : \quad \varphi(\cdot, T) = 0, \quad \varphi(x,t) = 0 \quad (x,t) \in \{0, L\} \times [0,T]\}.$$

For $v_0 \in L^\infty([0,L];[0,1])$ and for $a_i \in L^\infty((0,T))$ for $i = 1,2$,

$$v \in C^0((0,T), H^{-1}((0,L))) \cap L^\infty((0,L) \times (0,T))$$

is a weak solution of system (4.1) if for every $\varphi \in \mathcal{T}$ one has that:

$$\int_0^T \int_0^L v\big(-\partial_t \varphi - \partial_{xx} \varphi\big) - f(v)\varphi dx dt = \int_0^L v_0 \varphi(x,0) dx + \int_0^T a_1(t) \partial_x \varphi(0,t) - a_2(t) \partial_x \varphi(L,t) dt.$$

Note that, in the construction above, solving the non-homogeneous boundary-value problem (4.2), due to the low regularity of the boundary controls, requires to first introduce the notion of solution in the sense of transposition [66, Ch.13].

Let us introduce the following notation:

$$\Omega = (0, L), \quad Q = \Omega \times (0, T), \quad \Sigma = \partial\Omega \times (0, T)$$

and the adjoint problem

$$\begin{cases} -\partial_t p - \partial_{xx} p = \phi & (x,t) \in (0,L) \times (0,T), \\ p(0,t) = 0, \quad p(L,t) = 0 & t \in (0,T), \\ p(x,T) = 0 \end{cases} \tag{4.4}$$

for $\phi \in L^2((0,T); L^2((0,L)))$.

The well-posedness of the adjoint equation (4.4) can be addressed by classical methods, [29, Chapter 7 pp. 378]. In fact, under the inversion of the time variable ($t \to -t$) this system becomes an homogeneous Dirichlet problem for the forward heat equations. The solution then belongs to:

$$p \in L^2((0,T); H_0^1(0,L)), \quad \frac{d}{dt} p \in L^2((0,T); H^{-1}(0,L)).$$

On the other hand, multiplying (4.2) by ϕ and integrate over Q:

$$\begin{aligned} \int_0^T \int_0^L w\phi dx dt &= \int_0^T \int_0^L w(-\partial_t p - \partial_{xx} p) dx dt = \int_0^T w \frac{\partial p}{\partial \nu}\Big|_{x=0}^{x=L} dt \\ &= \int_0^T a_2(t)\partial_x p(L,t) - a_1(t)\partial_x p(0,t) dt \\ &= \langle a, \Lambda\phi \rangle_{L^2(\Sigma)}, \end{aligned}$$

where $a = (a_1, a_2)$. The map $\Lambda : L^2(Q) \to L^2((0,T); \mathbb{R}^2)$, $\Lambda(\phi) = (-\partial_x p(0,t), \partial_x p(L,t))$, such that p solves (4.4), is linear and continuous. A transposition solution w of (4.2) is a distribution w satisfying the following relationship for every $\phi \in L^2(Q)$:

$$\int_0^T \int_0^L w\phi = \langle a, \Lambda\phi \rangle_{L^2(\Sigma)}. \tag{4.5}$$

By duality, the existence and uniqueness of the transposition solution holds. For details see [66, Chapter 4, Sections 8, 12.3, 13 and Section 15 Example 1] and [65, page 202].

The well-posedness of problem (4.3) can then be achieved as an application of Banach Fixed-point [93] to

$$\begin{cases} \partial_t y - \partial_{xx} y = f(\xi + w) & (x,t) \in (0,L) \times (0,T) \\ y(0,t) = y(L,t) = 0 & t \in (0,T) \\ y(x,0) = v_0(x), \quad v_0 \in L^\infty((0,L); [0,1]). \end{cases} \tag{4.6}$$

The solution $\psi(\xi)$ of (4.6), defines a map $\psi : B_R \longrightarrow B_R$ where $B_R \subset L^\infty(Q)$ is a ball of radius R.

Using the variations of constants formula with the semigroup generated by the heat equation, a contraction map is defined for T small enough, leading to local existence and uniqueness. When the nonlinearity is assumed to be globally Lipschitz, the solution is globally defined in time. Blow up may occur when the nonlinearity is superlinear at infinity.

4.2. **Null Controllability of linear problems.** We present some of the main features of the null controllability problem for scalar parabolic equations, the main arguments employed in its analysis, and the bibliographical references.

The first method to deal with the controllability of the heat equation in one dimension is based on the use of biorthogonal functions, [31].

On the other hand, the extension-restriction principle allows to transform the boundary control problem into an interior control one. The argument consists of extending the system to a larger domain in which the control acts in its interior but outside the original domain. Once the solution in the extended domain is controlled, the restriction to the original domain leads to a controlled trajectory with Dirichlet boundary controls.

Here we shall only consider scalar equations. For extensions to parabolic systems we refer to [1, 33, 46, 51, 62].

The original domain $\Omega = (0, L)$ is extended to $\Omega_E = (-1, L+1)$. Let $\omega \subset (-1, 0)$ (see Figure 4.1) and consider the following linear parabolic control problem:

$$\begin{cases} \partial_t v - \partial_{xx} v - b(x,t) v = \chi_\omega h & (x,t) \in \Omega_E \times (0,T), \\ v(-1,t) = 0, \quad v(L+1,t) = 0 & t \in (0,T), \\ v(x,0) = v_0(x), \quad v_0 \in L^\infty(\Omega_E; [0,1]) & x \in \Omega_E, \end{cases} \tag{4.7}$$

where the initial datum is a smooth extension by zero of compact support of v_0 and $b \in L^2((0,T), L^2(\Omega_E))$.

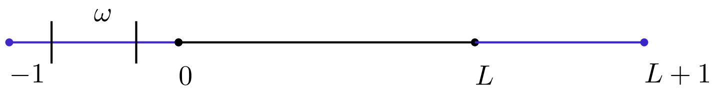

FIGURE 4.1. Original domain $\Omega = [0, L]$, extended domain $\Omega_E = (-1, L+1)$ and the control region ω.

We now consider the extended adjoint equation:

$$\begin{cases} \partial_t p + \partial_{xx} p + b(x,t) p = 0 & (x,t) \in \Omega_E \times (0,T), \\ p(-1,t) = 0, \quad p(L+1,t) = 0 & t \in (0,T), \\ p(x,T) = p^T(x) & x \in \Omega_E. \end{cases} \tag{4.8}$$

Integrating by parts we get

$$0 = \int_{\Omega_E} v(T) p(T) dx - \int_{\Omega_E} v(0) p(0) dx - \int_0^T \int_{\Omega_E} v \left(\partial_t p + \partial_{xx} p + bp) \right) dx dt - \int_0^T \int_\omega hp dx dt.$$

Hence,

$$0 = \int_{\Omega_E} v(T) p(T) dx - \int_{\Omega_E} v(0) p(0) dx - \int_0^T \int_\omega hp dx dt.$$

A control $\overline{h}$ driving the solution v to the null state, i.e. $v(T) \equiv 0$, is characterized by the duality identity:

$$- \int_{\Omega_E} v(0) p(0) dx - \int_0^T \int_\omega \overline{h} p dx dt = 0$$

for all p^T such that the solution of (4.8) satisfies $p \in L^2(\omega \times (0,T))$. Observe that the condition $p \in L^2(\omega \times (0,T))$ does not imply that $p^T \in L^2(\Omega_E)$. In fact, p^T belongs to a much larger space, [36].

The control $\overline{h}$ fulfilling the above identity can be obtained minimizing the functional

$$J : H \longrightarrow \mathbb{R}, \quad p^T \longrightarrow J(p^T) = \int_0^T \int_\omega p^2 dx dt + \int_{\Omega_E} p(0) v(0) dx.$$

where,

$$H := \left\{ p^T \text{ such that } p \text{ solves (4.8) and } \int_0^T \int_\omega p^2 dx dt < +\infty \right\},$$

The Euler-Lagrange equation satisfied by the minimizer $p^{*,T}$ of J is given by

$$DJ(p^{*,T})[\xi^T] = \int_0^T \int_\omega \xi p^* dx dt + \int_{\Omega_E} \xi(0) v(0) dx = 0.$$

This leads to the desired control $\overline{h} = p^*$, which is of minimal L^2-norm among all possible controls.

In order to show that J has a minimizer we observe that J is continuous and convex. Its coercivity is equivalent to the so-called observability inequality for the adjoint system:

$$\|p(0)\|^2_{L^2(\Omega_E)} \leq C \int_0^T \int_\omega p^2 dx dt, \quad \forall p^T \in H. \tag{4.9}$$

This kind of inequalities for parabolic equations in one and several space dimensions was proved using Carleman inequalities in [41] (see also [34]).

This leads to the control of the system (4.7) such that $v(x,T;h)=0$. Taking its restriction to the original domain $(0,L)$ we find the boundary controls

$$a(0,t)=v(0,t;h) \quad a(L,t)=v(L,t;h)$$

leading to the null control of the original system

$$\begin{cases} \partial_t v - \partial_{xx} v - b(x,t)v = 0 & (x,t)\in\Omega\times(0,T) \\ v(0,t)=a(0,t), \quad v(L,t)=a(L,t) & t\in(0,T) \\ v(x,0)=v_0, & x\in\Omega \end{cases}$$

so that

$$v(x,T)=0, \, \forall x\in(0,L).$$

Note however that these arguments do not guarantee that the controls and/or the controlled states fulfill the bilateral constraints.

4.3. **Null controllability for the semilinear problem and further comments.** Once the linear control problem has been solved, the semilinear one can be addressed using either Schauder's or Kakutani's fixed point [27]. This fixed point argument was originally employed for the semilinear wave equation, see for instance [123, 124] and references therein (see also [61]).

The fixed point argument can be implemented as follows. Given $\xi\in L^2(Q)$, we introduce the bounded potential b_ξ

$$b_\xi(x,t)=\begin{cases} \dfrac{f(\xi(x,t))}{\xi(x,t)} & \text{if } \xi(x,t)\neq 0, \\ f'(0) & \text{otherwise.} \end{cases}.$$

Then, one considers the linear controlled problem

$$\begin{cases} \partial_t v - \partial_{xx} v - b_\xi(x,t)v = \chi_\omega h_\xi & (x,t)\in\Omega_E\times(0,T), \\ v(-1,t)=v(L+1,t)=0 & t\in(0,T), \\ v(x,0)=v_0, \quad v(x,T)=0 & x\in\Omega_E. \end{cases} \tag{4.10}$$

Applying the linear methods above, this system can be controlled, and in this way we can define the map $\psi: L^2(Q)\to L^2(Q)$ defined as $\xi\to\psi(\xi)=v$ solution of (4.10). This map turns out to be continuous and compact. Under suitable growth conditions on the nonlinearity, in particular, when f is globally Lipschitz, it can be shown that this map is invariant in a sufficiently large ball. This allows to apply Schauder fixed point. The fixed point corresponds to a controlled trajectory for the nonlinear system.

There is, by now, an extensive literature in linear and semilinear parabolic control problems. We refer to [35, 36] for some of the earliest works and to [64, 70] for alternative methods.

As mentioned above, these arguments do not yield estimates in the controls and controlled states allowing to assure that the bilateral constraints are fulfilled. A careful analysis of the Carleman inequalities allows to obtain estimates on the cost of control of the form

$$\|a\|_{L^2((0,T),\mathbb{R}^2)} \leq \exp\left\{C\left(\frac{1}{T}+\|b\|_{L^\infty}T+\|b\|_{L^\infty}^{2/3}\right)\right\}\|v_0\|_{L^2(-1,L+1)}.$$

As expected, the size of controls increases exponentially when the time horizon T tends to zero leading to oscillations on controls and states that are incompatible with the bilateral constraints considered in these lecture notes (see Figure 4.2).

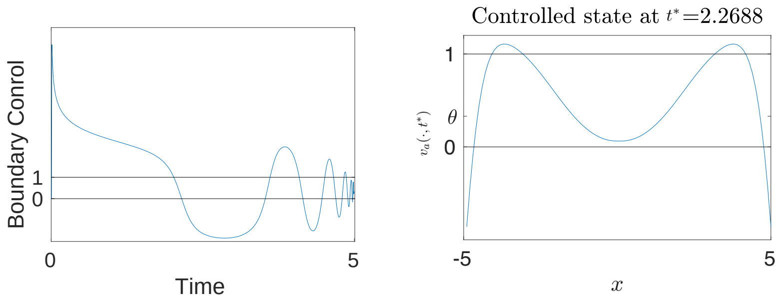

FIGURE 4.2. Left: Control function $a(0,t)$ steering the semilinear equation with cubic bistable nonlinearity $f(s) = s(1-s)(s-\theta)$ to the steady-state $w \equiv \theta$ in time $T = 5$. Right: Snapshot at time $t = 2.2688$ of the controlled trajectory $v_a(\cdot,t)$ violating the constraints $0 \le v \le 1$.

Important further developments are needed in order to understand the controllability of these systems under bilateral constraints of the form $0 \le v \le 1$.

4.4. **Constrained controllability results.** In this subsection, we present the main Theorem given in [93], which ensures that under certain assumptions, and, in particular, for long time-horizons, the problem of high amplitude oscillations observed in Figure 4.2 can be avoided and bilateral constraints can be assured. However, this will require a much more detailed analysis of the dynamics of the system and, in particular, to build paths of steady-states linking one steady-state to another.

Consider

$$\begin{cases} \partial_t v - \partial_{xx} v = f(v) & (x,t) \in (0,L) \times (0,T), \\ v = a(x,t) & (x,t) \in \{0,L\} \times (0,T), \\ v(0,x) = v_0(x) \in [0,1] & x \in (0,L). \end{cases} \tag{4.11}$$

We say that v_1 and v_0 are path-connected steady-states if there exists a continuous function from $[0,1]$ to the set of admissible steady-states S endowed with the L^∞ topology, $\gamma : [0,1] \to S$, such that $\gamma(0) = v_0$ and $\gamma(1) = v_1$. Denote by $\overline{v}^s := \gamma(s)$.

Theorem 4.2 (Theorem 1.2 in [93]). *Let v_0 and v_1 be path-connected admissible steady-states. Assume there exists $\nu > 0$ such that:*

$$\nu \le \overline{v}^s(x) \le 1 - \nu \quad \textit{for } x \in \{0, L\}, \tag{4.12}$$

for any $s \in [0,1]$. Then, if T is large enough, there exist a control function $a \in L^\infty\left((0,T);[0,1]^2\right)$ such that the problem (4.11), with initial datum v_0 and control function $a(t)$ at $x = 0, L$, admits unique solution verifying $v(\cdot,T) = v_1$.

Figure 4.3 shows the strategy qualitatively. If one has a connected path of steady-states, one can use local controllability to control sequentially along elements in the path in short time intervals by preserving the constraints. The strategy is based on using an L^∞ bound on the control in terms of the L^∞ norm of the difference between the initial datum and the target in a given time interval, for instance of unit length. This allows the identification of a finite number of intermediate steady-states along the path and applying local controllability from one to another recurrently without breaking the constraints. This strategy, by construction, requires a large time, of the order of the number of intermediate steps required, while the controllability of the linear and semilinear heat equation, in the absence of constraints, can be achieved in arbitrary small time.

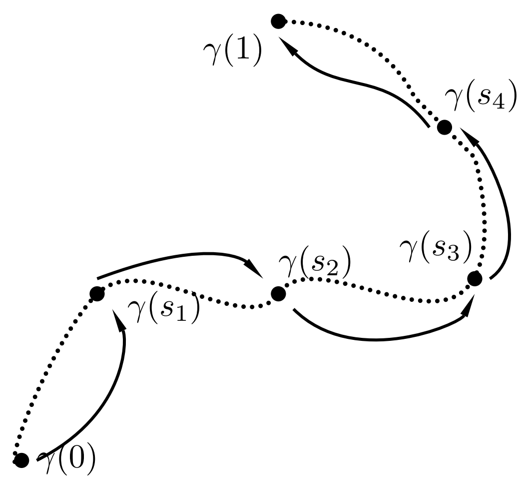

FIGURE 4.3. Qualitative representation of the staircase strategy.

4.5. **Minimal controllability time.** Let us consider the problem (4.11) where f is globally Lipschitz. As mentioned before, in absence state constraints, the controllability problem to some steady-state can be achieved in arbitrary small time [36]. But the cost of controlling the system becomes exponentially large as the control time-horizon shrinks

In the presence of positivity constraints, there is a minimal controllability time for the linear heat equation (see [72] and [69] for systems). The same occurs for the semilinear heat equation (see [93]). Here, we will provide a schematic proof of the existence of a minimal control time under bilateral bounds.

Theorem 4.3 (Positivity of the minimal controllability time for the semilinear equation [93])**.** *Let us consider a steady-state target $\overline{v}$:*

$$\begin{cases} -\overline{v}_{xx} = f(\overline{v}) & x \in (0, L), \\ 0 < \overline{v} < 1 & x \in (0, L), \\ \overline{v}(0) = a_1, \quad \overline{v}(L) = a_2. \end{cases}$$

Furthermore, consider the boundary control problem (4.11) *with target function $\overline{v}$ and $v_0 \neq \overline{v}$. Then the controllability cannot be achieved in arbitrary small time if the control function satisfies* $0 \leq a \leq 1$.

Proof. The result is an application of the comparison principle.

Let $\overline{v}$ be an admissible target steady-state. Assume that the admissible initial condition v_0 is different from the target. Then, necessarily, there exists an open interval $I \subset (0, L)$ such that either

$$\begin{cases} (A) & \overline{v} < v_0 \quad x \in I, \\ (B) & \overline{v} > v_0 \quad x \in I. \end{cases}$$

(A) For any control strategy a, one has that by the comparison principle

$$v(t; v_0, a) \geq v(t; v_0, a = 0)$$

where $v(t; v_0, a)$ is the solution of (4.11) with control a. Moreover, there exists a nonnegative test function $\phi \in H_0^1(0, L)$ such that:

$$\int_I (v_0 - \overline{v})\phi dx > 0$$

Since the solution of (4.11) is continuous with values in $H^{-1}(0, L)$, there exists $T^* > 0$ for which

$$\int_I (v(t; v_0, a = 0) - \overline{v})\phi > 0 \quad t \in (0, T^*).$$

Hence,

$$\int_I (v(t; v_0, a) - \overline{v})\phi > \int_I (v(t; v_0, a = 0) - \overline{v}) > 0 \quad t \in (0, T^*).$$

Therefore, for any admissible control a, controllability cannot hold before T^*.

(B) The same principle applies, but by employing the control $a = 1$ in the comparison argument.

□

Remark 4.4. The result, i.e. the need of a large enough time of control, is still valid under unilateral constraints. However, it requires employing duality techniques and using the continuity in time of the normal derivative of the adjoint equation and choosing a suitable profile as a final datum (see [93]).

5. Barriers and multiplicity of steady-states

5.1. **Barriers.** One of the main novel features of the systems under consideration is a fundamental lack of controllability, consequence of the comparison principle, when state-constraints are present. We will see that this fact is related, in particular, to the existence of nontrivial elliptic solutions, a fact that depends on the length of the domain. These nontrivial solutions, by the comparison principle, will impede specific trajectories to reach the prescribed target.

The comparison principle, presented in Subsection 3.2, ensures that a solution of the elliptic equation:

$$\begin{cases} -\partial_{xx} w = f(w) & x \in (0,L), \\ 0 < w(x) < 1 & x \in (0,L), \\ w(0) = w(L) = 0, \end{cases} \tag{5.1}$$

will always be below the solution of the following parabolic problem

$$\begin{cases} \partial_t v - \partial_{xx} v = f(v) & (x,t) \in (0,L) \times (0,T), \\ v(x,t) = a(x,t) \geq 0 & (x,t) \in \{0,L\} \times (0,T), \\ v(x,0) \geq w(x). \end{cases}$$

This shows that $w(x)$ constitutes an intrinsic obstruction to the controllability for the initial data above w, $v(x,0) \geq w(x)$, $x \in (0,L)$, to targets taking values below w on a subinterval, since the solution, for any admissible control satisfies necessarily

$$w(x) \leq v(x,t).$$

Hence, any target v_T satisfying $v_T(x) < w(x)$ in a sub-interval I will not be reachable from initial data above the nontrivial steady state $v(x,0) \geq w(x)$, $x \in (0,L)$. For this reason, we say that $w(x)$ acts as a barrier.

Recall that the control a is acting on the boundary and therefore is submitted to the same constraints $0 \leq a(x,t) \leq 1$.

As we have seen, the existence of solutions with null Dirichlet boundary conditions (5.1) is an impediment for the controllability to 0 (see Figure 5.1). The existence of such solution depends on the length of the domain L and it can be understood intuitively in the application context. Roughly, the population reproduces in the interior of a domain where the process evolves while vanishing at the boundary. These two phenomena compete. If the domain is large enough, the reproduction inside can compensate for individuals' loss through the boundary.

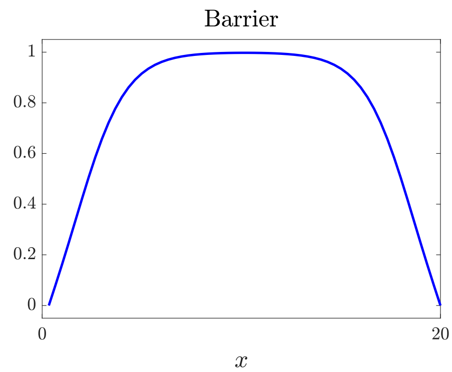

Figure 5.1. A barrier function for the semilinear heat equation with cubic nonlinearity $f(u) = u(1-u)(u-1/3)$ for $L = 20$.

Note that nontrivial solutions with a boundary value 1 will have the same effect when the aim is that the system reaches the state $v \equiv 1$.

The following sections are devoted to studying the existence and nonexistence of such nontrivial steady-state solutions. We will restrict the study to the one-dimensional case, even though the results also hold in several space dimensions by the same techniques.

5.2. **Rescaling.** Consider the space interval $(0, L)$ and the evolution equation

$$\begin{cases} \partial_t v - \partial_{xx} v = f(v) & (x,t) \in (0,L) \times (0,T), \\ v(x,t) = a(x,t) & (x,t) \in \{0,L\} \times (0,T), \\ v(\cdot, t=0) = v_0 \in L^\infty((0,L),[0,1]). \end{cases} \tag{5.2}$$

Equation (5.2) can be rescaled to be considered in the space interval $(0, 1)$, through the space-time transformation $s(x) = x/L$, $\tau(t) = t/L^2$. By setting $v(L^2\tau, Ls) = u(\tau, s)$ the problem reads:

$$\begin{cases} u_\tau - \partial_{ss} u = L^2 f(u) & (x,t) \in (0,1) \times (0,T), \\ u(x,t) = a(x,t) & (x,t) \in \{0,1\} \times (0,T), \\ u(\cdot, t=0) = u_0 \in L^\infty((0,1),[0,1]). \end{cases} \tag{5.3}$$

Abusing of notation, in the sequel we will continue using (x, t) instead of (s, τ) and we denote $\lambda := L^2$.

In [67] the existence of positive solutions for the semilinear elliptic problem (5.3) and its multiplicity is studied. We collect here the results of the work [67] that can be applied in our steady-state problem (5.4).

Consider the boundary value problem in $(0, 1)$:

$$\begin{cases} -\partial_{xx} u = \lambda f(u) & x \in (0,1), \\ 0 < u < 1 & x \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{5.4}$$

5.3. **Variational formulation.** The weak formulation of the boundary value problem:

$$\begin{cases} -\partial_{xx} v = \lambda f(x,v) & x \in (0,1), \\ v(0) = v(1) = 0, \end{cases} \tag{5.5}$$

is:

$$v \in H_0^1(0,1) : \quad \int_0^1 [v_x h_x - \lambda f(x,v) h] dx = 0 \qquad \forall h \in H_0^1(0,1),$$

which corresponds to the critical points of the energy functional:

$$I : H_0^1(0,1) \longrightarrow \mathbb{R}; \quad v \longrightarrow I[v] := \int_0^1 \left(\frac{1}{2} v_x^2 - \lambda F(x,v) \right) dx,$$

where $F(x, v) = \int_0^{v(x)} f(x, s) ds$.

Lemma 5.1 (Coercivity of I)**.** *Assume that:*

$$\lambda \limsup_{|s| \to \infty} \frac{f(x,s)}{s} < \lambda_1(0,1), \tag{5.6}$$

where $\lambda_1(0, 1) = \pi^2$ is the first eigenvalue of the Dirichlet Laplacian. Then I is coercive.

Proof. For simplicity we will consider only the case in which $s \to +\infty$. The case $s \to -\infty$ follows similarly. Since

$$\lambda \limsup_{s \to \infty} \frac{f(x,s)}{s} < \lambda_1(0,1),$$

we know that there exist $R > 0$ such that

$$\lambda \frac{f(x,s)}{s} < \lambda_1 \quad \forall s \geq R.$$

Using also the fact that $f(x, 0) = 0$ we can write:

$$\int_0^v \lambda f(s) ds = \int_0^R \lambda f(s) ds + \int_R^v \lambda \frac{f(s)}{s} s ds \leq \int_0^R \lambda f(s) ds + \frac{p}{2} \left(u^2 - R^2\right) \leq \frac{p}{2} u^2 + C(R, f, \lambda)$$

for a certain $p < \lambda_1((0,L))$. So

$$\begin{aligned} I[u] &= \int_0^1 \frac{1}{2}u_x^2 - F(u)dx \\ &\geq \int_0^1 \frac{1}{2}u_x^2 - \frac{p}{2}u^2 - C(R,f,\lambda)dx \geq \frac{1}{2}\int_0^1 \left(1 - \frac{p}{\lambda_1(0,1)}\right) u_x^2 - C(R,f,\lambda)dx \\ &\geq \frac{1}{2}\left(1 - \frac{p}{\lambda_1(0,1)}\right)\int_0^1 u_x^2 dx - C(R,f,\lambda) \geq c\|u\|^2_{H_0^1(0,1)} - C(R,f,\lambda) \end{aligned}$$

where $c > 0$ because $p < \lambda_1((0,1))$. So I is coercive.

□

Theorem 5.2 (Existence of a minimizer)**.** *Under the assumptions of Lemma 5.1, I has a minimizer.*

Proof. Note that the Lagrangian $L(p,z,x) = \frac{1}{2}|p|^2 - F(z)$ is the additive superposition of a convex term, with respect to the variable p, and a reminder that will lead to a compact functional perturbation. Therefore I is weakly lower-semicontinuous (Theorem 1 Ch.8 pp.468 in [29]) and the functional has a minimizer $u \in H_0^1((0,L))$. □

Remark 5.3. The hypothesis (5.6) is only needed for proving that the functional is coercive. For the monostable and bistable nonlinearities, since $f(1) = 0$, one can extend f by zero for $s \geq 1$.

We are interested on solutions taking values in $[0,1]$. Thus, the nonlinearity f (monostable or bistable) is extended naturally by setting it to be constant before $s = 0$ ($f(x,s) = 0$ for $s \leq 0$) and after $s = 1$ ($f(x,s) = 0$ for $s \geq 1$). After redefining f (if needed) by

$$\tilde{f}(s) := \begin{cases} f(s) & \text{if } s \geq 0, \\ 0 & \text{if } s < 0, \\ 0 & \text{if } s > 1, \end{cases}$$

we define the functional:

$$\begin{aligned} I : H_0^1((0,1)) &\longrightarrow \mathbb{R} \\ v &\longrightarrow I[v] := \int_0^1 \left[\frac{1}{2}v_x^2 - \lambda\tilde{F}(v)\right] dx \end{aligned}$$

where $\tilde{F}(v) = \lambda \int_0^v \tilde{f}(s)ds$

Obviously, $I[0] = 0$.

Thanks to the redefinition of f, we ensure that whenever $v \equiv 0$ is not a minimizer, the minimizer satisfies $0 \geq v \geq 1$ and the Euler-Lagrange equations:

$$\begin{cases} -\partial_{xx}v = \lambda\tilde{f}(v) & x \in (0,1) \\ 0 < v < 1 & x \in (0,1) \\ v(0) = v(1) = 0. \end{cases}$$

Indeed, if the minimizer would not satisfy $0 \geq v \geq 1$, then one would reach a contradiction. Suppose that v takes values outside $[0,1]$, then the function

$$\underline{v} = \begin{cases} v(x) & \text{if } v(x) \in [0,1] \\ 1 & \text{if } v(x) > 1 \\ 0 & \text{if } v(x) < 0 \end{cases}$$

would satisfy $\int_0^1 \underline{v}_x^2 < \int_0^1 v_x^2$, while the nonlinear part of the energy containing $F(v)$ would remain the same. However, by the strong maximum principle $\underline{v}$ cannot be a solution of the elliptic problem since it is incompatible with v being constant on a subinterval. Therefore the minimizer must satisfy $0 < v < 1$.
Since $0 < v < 1$ in $(0,L)$, $\tilde{f} = f$ in this range and therefore it is a solution of our original problem, with the nonlinearity f.

Definition 5.4. We define λ^* (analogously $(L^*)^2 = \lambda^*$) as the infimum value $\lambda \in \mathbb{R}^+$ for which there is a solution of:

$$\begin{cases} -\partial_{xx} v = \lambda f(v) & x \in (0,1), \\ 0 < v < 1 & x \in (0,1), \\ v(0) = v(1) = 0. \end{cases} \tag{5.7}$$

Note that, because of the constraint $0 < v < 1$ in $(0,1)$, v is guaranteed to be a nontrivial solution.

Analogously, for the bistable nonlinearity, we will denote by λ^*_θ (or L^*_θ) the infimum value on $\lambda \in \mathbb{R}^+$ (or $L \in \mathbb{R}^+$) for which a nontrivial solution with boundary value θ exists.

It will be proven later on (Theorem 5.6 and Theorem 5.14) that for monostable and certain bistable nonlinearities $\lambda^* < +\infty$

The following classical result [67] makes use of subsolutions to prove that, if for a certain $\overline{\lambda}$ there exists a nontrivial solution to (5.7), then, for any $\lambda > \overline{\lambda}$ there will exist a nontrivial solution.

Proposition 5.5. *Assume that $\lambda^* < +\infty$. For every $\lambda > \lambda^*$ there exist a nontrivial solution to* (5.7)

Proof. We will use the fact that $\lambda = L^2$, and prove the result making use of different domains. More precisely, assuming that there exists a nontrivial solution of the elliptic problem for a certain $L_1 > 0$, we will see that there exists a nontrivial solution for any $L > L_1$. This is equivalent to saying that if there exists a nontrivial solution for $\overline{\lambda}$, there will also exist a nontrivial solution for any $\lambda > \overline{\lambda}$.

Let us therefore assume that there exists a solution to:

$$\begin{cases} -\partial_{xx} v_1 = f(v_1) & x \in (0,L_1), \\ 0 < v_1 < 1 & x \in (0,L_1), \\ v_1(0) = v_1(L_1) = 0. \end{cases}$$

To prove that for any $L > L_1$ there exists a nontrivial solution, one can construct a subsolution of:

$$\begin{cases} -\partial_{xx} v = f(v) & x \in (0,L), \\ 0 < v < 1 & x \in (0,L), \\ v(0) = v(L) = 0, \end{cases}$$

using v_1. Indeed, one can extend v_1 by zero in (L_1, L),

$$\tilde{v}(x) = \begin{cases} v_1(x) & \text{if } x \in (0,L_1), \\ 0 & \text{if } x \in (L_1,L). \end{cases}$$

Then $\tilde{v}$ is a weak subsolution for the problem in $(0,L)$. On the other hand, note that 1 is always a supersolution, therefore, there exists a nontrivial solution for the problem in $(0,L)$ and therefore for any $\lambda > \lambda^*$. $\square$

5.4. **Monostable nonlinearity.**

5.4.1. *Null Dirichlet condition.* Using the variational structure of the problem, we will be able to give estimates from above for λ^*, the minimal value for which the non-trivial steady state solution exists.

The following Theorem was proven in [10, Theorem II.1], however, the alternative proof given below uses a variational argument.

Theorem 5.6 (An upper bound of λ^*)**.** [1] *Assume that*

$$\lambda \lim_{s\to 0^+} \frac{f(x,s)}{s} > \lambda_1(0,1) \ x \in [0,1]$$

and

$$\lambda \overline{\lim_{s\to\infty}} \frac{f(x,s)}{s} < \lambda_1((0,1)) \quad \forall x \in [0,1]$$

Then, for all $\lambda > \frac{\lambda_1(0,1)}{\min_{x\in[0,1]} \partial_s f(0,x)}$ there exist a solution of the problem (5.4)*. Therefore:*

$$\lambda^* \leq \frac{\lambda_1(0,1)}{\min_{x\in[0,1]} \partial_s f(0,x)}.$$

[1] Here we provide a weaker version of the original theorem in [10] and a proof using a variational argument

Proof. First of all, note that a minimizer exists by Theorem 5.2. The constant function $v = 1$ is not a possible candidate because of the null Dirichlet boundary conditions. The main idea of the proof is simple: It suffices to prove the existence of a non-trivial global minimizer for I taking values in $0 \le v \le 1$.
Let e_1 be the first eigenfunction of the operator $A = (-\partial_{xx})$ with null Dirichlet boundary conditions. We know that this function is positive. Set $v = \epsilon e_1$ for $\epsilon > 0$ to be chosen later on.
By hypothesis,

$$\lambda \lim_{s \to 0^+} \frac{f(x,s)}{s} > \lambda_1(0,1) \ x \in [0,1].$$

This means that there exists $r \in \mathbb{R}^+$ such that

$$\frac{\lambda f(x,s)}{s} > \lambda_1(0,1) \quad \forall s \in [0,r].$$

Choose ϵ small enough such that $\epsilon e_1 < r$. Evaluating the functional we obtain:

$$I[\epsilon e_1] = \int_0^1 \frac{\epsilon^2}{2}\partial_x e_1^2 - \int_0^{\epsilon e_1} \lambda \frac{f(x,s)}{s} sdsdx \le \int_0^1 \frac{\epsilon^2}{2}\partial_x e_1^2 - p\frac{\epsilon^2 e_1^2}{2}dx = \frac{\epsilon^2}{2}\int_0^1 (\lambda_1(0,1) - p)\, e_1^2 dx < 0$$

for some $p > \lambda_1((0,1))$.
This shows that $v \equiv 0$ is not a minimizer. Accordingly, a global non-trivial minimizer exists. □

The following Theorem gives a lower bound on λ for the existence of a nontrivial solution in the case in which the nonlinearities are concave and twice differentiable.

Theorem 5.7 (A lower bound of λ^*)**.** *Let f be twice differentiable such that $f'(0) > 0$ and concave, i.e. $f''(s) \le 0$ in $s \in [0,1]$. Then if:*

$$\lambda < \frac{\lambda_1(0,1)}{f'(0)}$$

there cannot be any positive solution of (5.4)*, Therefore,*

$$\lambda^* \ge \frac{\lambda_1(0,1)}{f'(0)} > 0.$$

Proof. Multiply the equation by v and integrate over the domain and integrate by parts. We obtain

$$\int_0^1 v_x^2 - \lambda \int_0^1 f(v)v = 0.$$

By the Poincaré inequality,

$$\int_0^1 \lambda_1(0,1)v^2 - \lambda f(v)v \le 0.$$

Now consider the Taylor formula for f:

$$f(v) = f(0) + f'(0)v + \int_0^v f''(s)(v-s)dt$$

Due to the fact that $f(0) = 0$ we end up with

$$\int_0^1 (\lambda_1(0,1) - \lambda f'(0))\, v^2 - \lambda v \int_0^v f''(s)(v-s)dsdx \le 0.$$

Since $v > 0$ we have that $v - s \ge 0$. Moreover, by assumption $f''(s) \le 0$, and we obtain that the second term is nonnegative. We can conclude that a necessary condition for the existence of a positive solution is:

$$\int_0^1 (\lambda_1(0,1) - \lambda f'(0))\, v^2 \le 0.$$

The proof is complete.

□

Remark 5.8 (Space dependent nonlinearity)*.* With a subtle change in the proof of Theorem 5.7, one can see that the same result holds for the following problem:

$$\begin{cases} -\partial_{xx} v = \lambda f(v,x) & x \in (0,1) \\ v > 0 & x \in (0,1) \\ v(0) = v(L) = 0, \end{cases} \tag{5.8}$$

where $f(v,x)$ is twice differentiable with respect to v and concave with respect to v. Following the same argument one arrives at the following condition

$$\int_0^1 (\lambda_1(0,1) - \lambda f'(0,x))\, v^2 \leq 0.$$

If we assume that

$$0 < \min_{x\in[0,L]} \partial_s f(0,x) \leq \partial_s f(0,x) \leq \max_{x\in[0,L]} \partial_s f(0,x) < +\infty$$

we obtain the lower bound λ^*:

$$\lambda^* \geq \frac{\lambda_1(0,1)}{\max_{x\in[0,1]} \partial_s f(0,x)} > 0.$$

Theorem 5.9 (Monostable concave nonlinearities)**.** *When f is monostable, concave and does not depend on x we have:*

$$\lambda^* = \frac{\lambda_1(0,1)}{f'(0)} > 0.$$

When f monostable and concave but depending on x, we have

$$\frac{\lambda_1(0,1)}{\max_{x\in[0,1]} \partial_s f(0,x)} \leq \lambda^* \leq \frac{\lambda_1(0,1)}{\min_{x\in[0,1]} \partial_s f(0,x)}.$$

Remark 5.10 (Uniqueness of positive solutions for concave nonlinearities). When f is concave and a positive solution exists, it is unique [9, 67].

Remark 5.11 (Non uniqueness of positive solutions for non concave nonlinearities). When f is not concave uniqueness of the nontrivial positive solution might not hold. Assuming that $f'(0) > 0$, if $\lambda^* < \lambda_1(0,1)/f'(0)$, we have that for all λ such that $\lambda^* < \lambda < \lambda_1(0,1)$ there exists a second positive solution. This is proven using topological degree arguments [67].

5.4.2. *Phase Portrait.* In one dimension the elliptic equation

$$-\partial_{xx} v = f(v),$$

can be interpreted as a dynamical system:

$$\frac{d}{dx}\begin{pmatrix} v \\ v_x \end{pmatrix} = \begin{pmatrix} v_x \\ -f(v) \end{pmatrix}. \tag{5.9}$$

For the monostable nonlinearity, we notice that $(1,0)$ is a topological saddle for the nonlinear system, and $(0,0)$ is a center for the linearized system. The differential matrix is:

$$DF(v,v_x) = \begin{pmatrix} 0 & 1 \\ -\partial_v f(v) & 0 \end{pmatrix}.$$

Since, by definition of monostable nonlinearity, $\partial_v f(v)\big|_{v=1} < 0$ and $\partial_v f(v)\big|_{v=0} > 0$, we have that

$$DF(1,0) = \begin{pmatrix} 0 & 1 \\ -\partial_v f(v)\big|_{v=1} > 0 & 0 \end{pmatrix}, \qquad DF(0,0) = \begin{pmatrix} 0 & 1 \\ -\partial_v f(v)\big|_{v=0} < 0 & 0 \end{pmatrix}.$$

By symmetry with respect to the horizontal axis, we can also conclude that $(0,0)$ is a center for the nonlinear system. Moreover, by the first integral of the system, we know that the separatrix of the saddle is given by:

$$v_x = \pm\sqrt{2(F(1) - F(v))}.$$

The following Figure 5.2 is a representation of the phase portrait in the monostable case

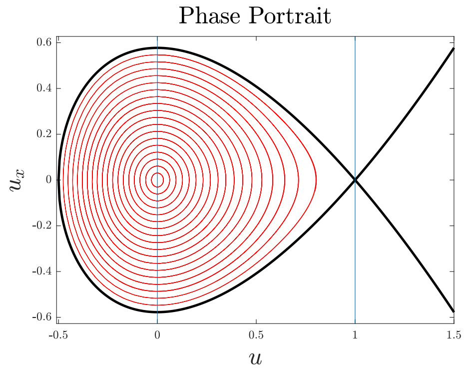

FIGURE 5.2. Separatrix of the system and phase portrait inside the region it determines, $f(s) = s(1-s)$.

5.4.3. *Bifurcation diagram.* In Figure 5.3 (left), we give a qualitative representation of the bifurcation diagram for a concave nonlinearity, while in Figure 5.3 (right) for non-concave one. In the horizontal axis, the parameter $\lambda = L^2$ is represented, and in the vertical one, the infinity norm of the nontrivial solution, when it exists (for further examples, see [67]).

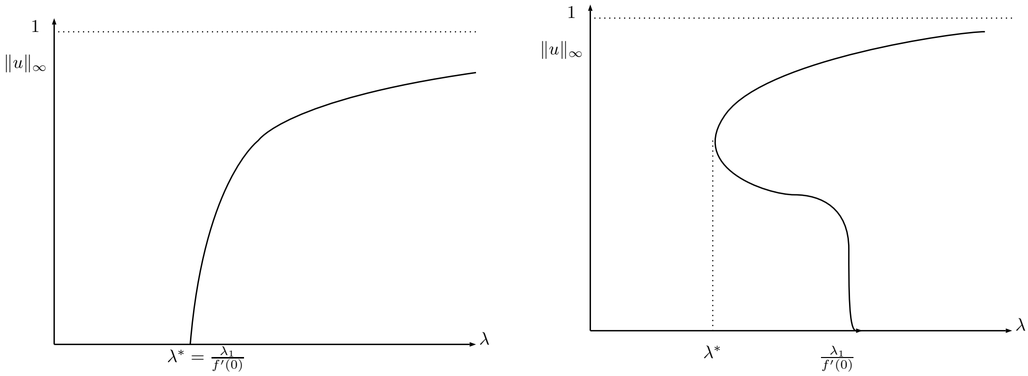

FIGURE 5.3. Qualitative bifurcation diagram for the stationary solutions. The black curve represents a nontrivial solution. At the left, the diagram for convave monostable nonlinearities is plotted and, at the right, for a general monostable nonlinearity [67].

5.4.4. *Dirichlet condition equal to* $\equiv 1$. As mentioned before, a nontrivial solution with boundary value $\equiv 1$ would have a similar blocking effect. However such a solution does not exist.

To prove the nonexistence of a solution to the problem:

$$\begin{cases} -\partial_{xx} v = \lambda f(v) & x \in (0,1), \\ 0 < v < 1 & x \in (0,1), \\ v(0) = v(L) = 1, \end{cases} \tag{5.10}$$

we will show that for any initial datum between 0 and 1 the solution of the parabolic problem:

$$\begin{cases} \partial_t u - \partial_{xx} u = \lambda f(u) & (x,t) \in (0,1) \times \mathbb{R}^+, \\ 0 < u < 1 & (x,t) \in (0,1) \times \mathbb{R}^+, \\ u(0,t) = u(1,t) = 1 & t \in \mathbb{R}^+, \\ 0 \le u(x,0) \le 1 & x \in (0,1), \end{cases}$$

goes asymptotically to the constant solution $v = 1$ as $t \to +\infty$.

For monostable nonlinearities,

$$V(t) := \int_0^1 [u - 1 - \log(u)] \, dx$$

plays the role of a Lyapunov functional ([97])

$$\frac{d}{dt}V(t) = -\int_0^1 \frac{u_x^2}{u^2}dx - \int_0^1 \lambda f(u)\frac{1-u}{u}dx \leq 0$$

Remark 5.12 (Comparison with traveling waves). Another way to check the lack of existence of nontrivial admissible steady-states other than $u \equiv 1$ with boundary contidion 1 is by using the comparison principle with the traveling wave solution for the Cauchy problem. Indeed we know that a decreasing traveling wave function exists for monostable nonlinearities [92, Ch.4 Th. 4.5 pp.67]. Since it is decreasing and connecting 0 and 1, for any initial data $u(x,0) > 0$ of the parabolic problem, we can choose a section of the traveling wave that is strictly under $u(x,0)$. Then, the boundary conditions of this section of the traveling wave will be below 1, constituting a subsolution of the parabolic problem with Dirichlet data equals to 1. This argument enables us to conclude that the solution of the parabolic problem will converge to 1 since the traveling waves converges to 1 uniformly in compact sets as $t \to \infty$.

5.5. **Bistable nonlinearity.**

5.5.1. *Null Dirichlet condition.* Now we turn our attention to bistable nonlinearities. The structure of the proofs will be similar to the monostable case.

Theorem 5.13 (A lower bound for λ^*)**.** *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ and assume that f is bounded uniformly with respect to x. Assume furthermore that $f(x,0) = 0$. Consider:*

$$\begin{cases} -\partial_{xx}v = \lambda f(x,v) & x \in (0,1), \\ 0 < v < 1 & x \in (0,1), \\ v(0) = v(1) = 0. \end{cases}$$

Then,

$$\frac{\lambda_1(0,1)}{\max_{(x,s)\in[0,1]^2}\frac{f(x,s)}{s}} \leq \lambda^*.$$

Proof.

$$\lambda_1(0,1)\int_0^1 v^2dx \leq \int_0^1 (\partial_x v)^2dx = \int_0^1 \lambda f(x,v)vdx \leq \int_0^1 \lambda P v^2 dx \tag{5.11}$$

with $P = \max_{(x,s)\in[0,1]^2} f(x,s)/s$. Note that if

$$\lambda P\int_0^1 v^2dx < \lambda_1(0,1)\int_0^1 v^2dx \tag{5.12}$$

we would violate (5.11). Therefore, for any $\lambda < \lambda(0,1)/P$ there cannot exist a nontrivial solution. Hence, we deduce the lower bound on λ^* in the statement of the Theorem.

□

Theorem 5.14 (An upper bound for λ^*)**.** *Assume that $f(0) = f(\theta) = f(1) = 0$, and that $f'(0) < 0$, $f'(1) < 1$, $f'(\theta) > 0$. Moreover consider $F(v) = \int_0^v f(s)ds$ and assume that $F(1) > 0$. Consider the space interval $[0,1]$. The following problem:*

$$\begin{cases} -\partial_{xx}u = \lambda f(u) & x \in (0,1), \\ 0 < u < 1 & x \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{5.13}$$

has a solution for every $\lambda \geq 8(F(1) - F(\theta))/F(1)^2$.

This fact, together with the lower bound in Theorem 5.13, leads to the following double bounds on λ^* :

$$\frac{\pi^2}{\max_{s\in[0,1]}\frac{f(s)}{s}} \leq \lambda^* \leq 8\frac{F(1)-F(\theta)}{F(1)^2}.$$

Proof. Obviously, $v \equiv 0$ is a trivial solution. On the other hand, the elliptic equation corresponds to the Euler-Lagrange equations of the functional

$$I[v] = \frac{1}{2}\int_0^1 v_x^2dx - \lambda\int_0^1 F(v)dx.$$

The strategy to find a non-trivial solution is similar than the one of Theorem 5.6. We construct a family of functions $v_\delta \in H_0^1(0,1)$ as follows (see Figure 5.4):

$$v_\delta(x) = \begin{cases} \dfrac{2}{\delta}x & \text{if } x \in \left[0, \dfrac{\delta}{2}\right] \\ 1 & \text{if } x \in \left(\dfrac{\delta}{2}, 1-\dfrac{\delta}{2}\right) \\ 1-\dfrac{2}{\delta}\left(x-\left(1-\dfrac{\delta}{2}\right)\right) & \text{if } x \in \left[1-\dfrac{\delta}{2}, 1\right]. \end{cases}$$

The functions v_δ take the constant value $= 1$ in an inner subinterval, they vanish at the boundary with linear transitions (recall that $F(1) > 0$) in intervals of length δ.

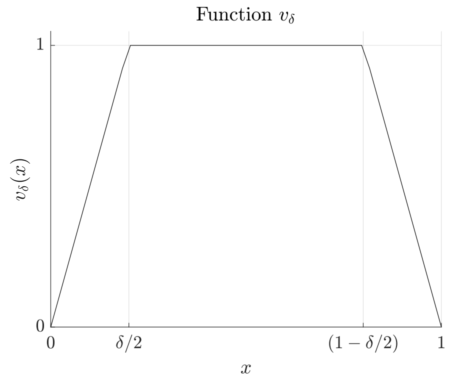

FIGURE 5.4. Function v_δ

Note that $v_\delta \in H_0^1(0,1)$ and

$$(\partial_x v_\delta)^2 = \begin{cases} 0 & \text{if } x \in \left(\dfrac{\delta}{2}, 1-\dfrac{\delta}{2}\right) \\ \dfrac{4}{\delta^2} & \text{if } x \in \left(0, \dfrac{\delta}{2}\right) \cup \left(1-\dfrac{\delta}{2}, 1\right). \end{cases}$$

We want to find a pair (λ, δ) for which:

$$I[v] = \int_0^1 \left[\frac{1}{2}|\partial_x v_\delta|^2 - \lambda \int_0^{v_\delta(x)} f(s)ds\right] dx < 0.$$

For doing so, first we choose $\delta > 0$ to be small enough such that:

$$\int_0^1 \int_0^{v_\delta(x)} f(s)dsdx > c > 0.$$

To analyze this integral we split it in two parts:

$$\begin{aligned} \int_0^1 \int_0^{v_\delta(x)} f(s)dsdx &= \int_{\frac{\delta}{2}}^{1-\frac{\delta}{2}} \int_0^{v_\delta(x)} f(s)dsdx + \int_{[0,1]\setminus\left(\frac{\delta}{2},1-\frac{\delta}{2}\right)} \int_0^1 f(s)dsdx \\ &\geq \int_{[0,1]\setminus\left(\frac{\delta}{2},1-\frac{\delta}{2}\right)} F(\theta)dx + F(1)(1-\delta) = F(1)(1-\delta) + F(\theta)\delta. \end{aligned}$$

So, it will suffice to require that:

$$0 < \delta < \frac{F(1)}{F(1)-F(\theta)}. \tag{5.14}$$

Once δ is fixed fulfilling (5.14), we choose λ large enough so that the space integral on $F(v_\delta(x))$ dominates the gradient component

$$\begin{aligned} I[v_\delta] &= \int_0^1 \left[\frac{1}{2}|\partial_x v_\delta|^2 - \lambda F(v_\delta(x))\right] dx \leq \frac{1}{2}\int_0^1 |\partial_x v_\delta|^2 dx - \lambda\left(F(1)(1-\delta) + \delta F(\theta)\right) \\ &= \int_{(0,1)\setminus\left(\frac{\delta}{2},1-\frac{\delta}{2}\right)} \frac{2}{\delta^2} dx - \lambda\left(F(1)(1-\delta) + \delta F(\theta)\right) = \frac{2}{\delta} - \lambda\left(F(1)(1-\delta) + \delta F(\theta)\right). \end{aligned}$$

So, the choice of λ

$$\lambda > \frac{2}{\delta\left(F(1)(1-\delta) + F(\theta)\delta\right)} \tag{5.15}$$

suffices.

Hence, any pair (λ, δ) satisfying both (5.14) and (5.15) will guarantee the existence of a nontrivial solution. It is then natural to analyze which is the choice of δ leading to the minimum upper bound in terms of λ? With this purpose, we choose δ in the interval $0 < \delta < F(1)/(F(1) - F(\theta))$ maximizing the denominator:

$$\delta\left(F(1)(1-\delta) + F(\theta)\delta\right).$$

We note that $-F(1) + F(\theta)$ is negative, and hence this constitutes a polynomial in δ that attains its maximum in:

$$\delta^* = \frac{F(1)}{2(F(1) - F(\theta))}$$

This optimal δ^* satisfies the requirement:

$$\delta^* = \frac{F(1)}{2(F(1) - F(\theta))} < \frac{F(1)}{F(1) - F(\theta)}.$$

So we have that

$$\lambda > \frac{8(F(1) - F(\theta))}{F(1)^2},$$

suffices to ensure existence of a nontrivial solution.

□

Remark 5.15. The strategy of the proof of Theorem 5.14 works also in the monostable case. In that case, when proving the upper bound of the integral of the primitive, $F(1) - F(\theta)$ will be replaced by $F(1)$, since the primitive in the monostable case is monotone.

Corollary 5.16 (Corollary)**.** *Let f by any $C^2(\mathbb{R};\mathbb{R})$ function satisfying:*

- $f(0) = f(\theta) = f(1) = 0$ *with* $0 < \theta < 1$
- *consider* $F(v) = \int_0^v f(s)ds$ *and suppose that* $F(1) > 0$
- $f'(0) < 0$, $f'(\theta) > 0$ *and* $f'(1) < 0$
- $f < 0$ *in* $(0, \theta)$ *and* $f > 0$ *in* $(\theta, 1)$

Then,

$$\pi^2 \leq 8\frac{F(1) - F(\theta)}{F(1)^2} \max_{s\in[0,1]} f'(s) := Q(f). \tag{5.16}$$

Proof. This is a corollary of Theorem 5.14

$$\pi^2 \leq 8\frac{F(1) - F(\theta)}{F(1)^2} \max_{s\in[0,1]} \frac{f(s)}{s} \leq 8\frac{F(1) - F(\theta)}{F(1)^2} \max_{s\in[0,1]} f'(s) \tag{5.17}$$

since, by the mean value theorem, $\max_{s\in[0,1]} \frac{f(s)}{s} \leq \max_{s\in[0,1]} f'(s)$. □

Remark 5.17 (Open question)*.* The statement in Corollary 5.16 has a very mild reminiscence of PDE theory, essentially because $\lambda_1 = \pi^2$ in $[0, 1]$. Inequality (5.17) seems to be a general functional inequality for bistable nonlinearities, unrelated to elliptic PDEs. It would be interesting to provide a PDE independent proof.

Remark 5.18 (Existence of positive solutions for nonlinearities depending on space). Similar arguments can be applied to show the existence of positive solutions with space-dependent nonlinearities:

$$\begin{cases} -\partial_{xx} v = f(x, v) & x \in (0,1), \\ 0 < v < 1 & x \in (0,1), \\ v(0) = v(1) = 0. \end{cases}$$

Assume that there exist smooth functions $\theta, K : [0,1] \longrightarrow \mathbb{R}^+$ such that:

- $\theta(x) < K(x)$ for all x
- $f(\theta(x), x) = 0$ for all x
- $f(K(x), x) = 0$ for all x
- $f(s, x) < 0$ for all $0 < s < \theta(x)$
- $f(s, x) > 0$ for all $\theta(x) < s < K(x)$
- There exist $x^* \in [0,1]$ such that $F(K(x^*), x^*) > 0$.

Then there is a finite $\lambda > 0$ for which the positive solution exists.

Remark 5.19 (Survival of the gene). The bio-physical interpretation of the previous result is the following: Consider a population with two traits 0 and 1. Each trait is advantageous only in part of the environment. If the set in which one trait is advantageous is large enough, the action of a boundary control will not be sufficient to annihilate it.

Remark 5.20 (Double blocking phenomenon). Setting different values of $\theta(x)$ above and below $1/2$ for the nonlinearity $f(y) = y(1-y)(y - \theta(x))$ one could apply these methods to prove the existence of both nontrivial steady-state solutions for boundary values $= 1$ and $= 0$. By the comparison principle, this example leads to a double blocking phenomenon, already observed in [84].

Proposition 5.21 (Maximum of positive solutions). *Let f be bistable and v be a solution to:*

$$\begin{cases} -\partial_{xx} v = \lambda f(v) & x \in (0,1), \\ 0 < v < 1 & x \in (0,1), \\ v(0) = v(1) = 0. \end{cases}$$

Then the maximum of v in $[0,1]$ is above θ:

$$\max_{x \in [0,1]} v(x) > \theta.$$

Proof. The proof follows by contradiction. Assume that the maximum of v is lower or equal than θ. Then the energy estimate yields to a contradiction:

$$0 < \int_0^1 v_x^2 dx = \lambda \int_0^1 v f(v) < 0$$

where the strict inequality in the left-hand side comes from the assumption that the solution is not trivial, and the right-hand side one from the fact that f is negative in $(0, \theta)$.

□

5.5.2. *Dirichlet condition $\equiv \theta$.* Note that the results for the monostable nonlinearity apply for the existence of nontrivial solutions with boundary value θ, i.e. nontrivial solution to:

$$\begin{cases} -\partial_{xx} v = \lambda f(v) & x \in (0,1), \\ v > \theta & x \in (0,1), \\ v(0) = v(1) = \theta. \end{cases}$$

By hypothesis, we know that $f > 0$ in $(\theta, 1)$, $f'(\theta) > 0$, $f'(1) < 0$ and $f(\theta) = f(1) = 0$. Then f is monostable in $[\theta, 1]$, and $w = v - \theta$ satisfies the criteria of the previous theorems.

Corollary 5.22 (Estimates on λ_θ^* for bistable nonlinearities). *We have*

$$\frac{\pi^2}{\max_{s \in [0, 1-\theta]} \frac{f(s+\theta)}{s}} \le \lambda_\theta^* \le \min \left\{ \frac{2}{F(1) - F(\theta)}, \frac{\pi^2}{f'(\theta)} \right\}.$$

If f is convex in $(0, \theta)$ and concave in $(\theta, 1)$ then:

$$\lambda_\theta^* = \frac{\pi^2}{f'(\theta)}.$$

Other than the estimates above, in general one has the following result.

Proposition 5.23 (Ordering of the thresholds)**.** *When $F(1) > 0$ we have*

$$\lambda^*_\theta \leq \lambda^*.$$

When $\lambda > \lambda^*$, denoting by v_θ and v_0 the maximum steady-state nontrivial solutions fulfilling the constraints with Dirichlet values $= \theta$ and $= 0$ respectively, then:

$$v_\theta \geq v_0.$$

Proof. The result follows from the elliptic comparison principle, together with the fact that any nontrivial solution of the boundary value problem has its maximum above θ. □

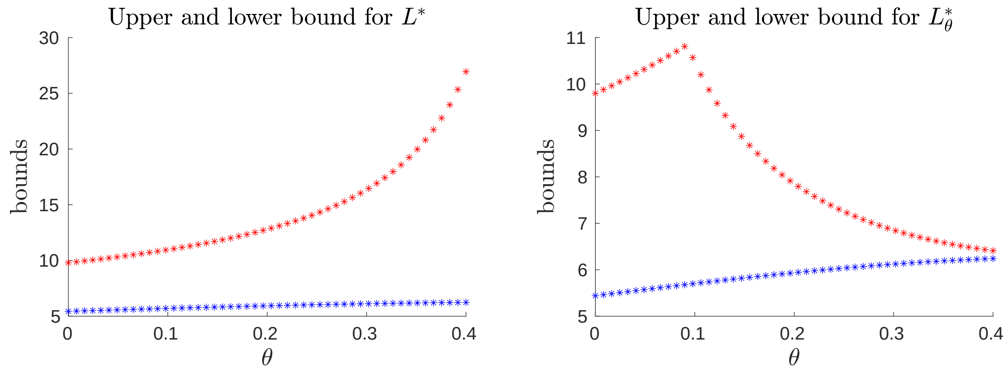

FIGURE 5.5. Bounds on L^* and L^*_θ for different values of θ in the nonlinearity $f(s) = s(1-s)(1-\theta)$. The red dots represent upper bounds and the blue dots lower bounds. Left: Bounds on L^*. Right: bounds on L^*_θ (Definition 5.4).

Remark 5.24. Recall that when $F(1) = 0$, i.e. when $\theta = 1/2$, there does not exist any nontrivial solution with boundary value 0, for this reason, in Figure 5.5, one observes that the upper bound on L^* goes to $+\infty$ as θ approaches $1/2$. Note that the lower bound proven for λ^* (Theorem 5.13) and λ^*_θ (Corollary 5.22) are independent of the sign of $F(1)$. They only depend on pointwise values of the nonlinearity f.

5.5.3. *Dirichlet boundary condition* $\equiv 1$. Consider

$$\begin{cases} -\partial_{xx} v = \lambda f(v) & x \in (0,1), \\ 0 < v < 1 & x \in (0,1), \\ v(0) = v(1) = 1 \end{cases} \tag{5.18}$$

As mentioned before, using comparison arguments with sections of traveling waves, we can prove that the solution of the parabolic model converges to 1 for any λ as $t \to +\infty$.

Proposition 5.25 (Convergence to 1)**.** *Consider $F(1) > 0$, for any interval $(0, L)$, the solution of the reaction diffusion system* (5.2) *with boundary value equal to* 1 *converges to $v = 1$.*

Proof. Consider

$$\begin{cases} \partial_t u - \partial_{xx} u = f(u) & (x,t) \in (0,L) \times (0,T), \\ u(x,t) = 1 & (x,t) \in \{0,L\} \times (0,T), \\ 0 < u(x,0) < 1 & x \in (0,L). \end{cases} \tag{5.19}$$

We know that this equation, when considered in the full real line, has a traveling wave solution whose profile, by Theorem 3.8, is a monotone function decreasing in the direction of the velocity of propagation. We can then use a segment or section of this traveling wave as a parabolic subsolution to our problem.

Indeed, since the traveling wave profile is monotone decreasing, we can choose a segment of the traveling wave below $u(x,0)$ in $[0,L]$. More precisely, let us denote by $TW(x)$ a traveling wave profile satisfying:

$$TW(x) \leq u(x,0) \quad \forall x \in [0,L].$$

The following problem:

$$
\begin{cases}
\partial_t u - \partial_{xx} u = f(u) & (x,t) \in (0,L) \times \mathbb{R}^+, \\
u(x,t) = TW(x-ct) & (x,t) \in \{0,L\} \times \mathbb{R}^+ \\
u(x,0) = TW(x) & x \in (0,L),
\end{cases}
\tag{5.20}
$$

leads to a solution $u(x,t) = TW(x-ct)$, which is a subsolution of (5.19). By the parabolic comparison principle, the solution of (5.19) will be above $u(x,t) = TW(x-ct)$ which converges to 1 as $t \to \infty$.

□

5.5.4. *Phase Portrait.* Here we study the ODE dynamics of system (5.9) for bistable nonlinearities. First of all we notice that the points $(0,0)$ (corresponding to the stationary solution $v \equiv 0$) and $(1,0)$ (corresponding to the stationary solution $v \equiv 1$) are saddles for all values of $\theta \in (0,1)$ since the nonlinearity fulfills:

$$
\frac{\partial}{\partial v} f(v)\big|_{v=0} < 0, \qquad \frac{\partial}{\partial v} f(v)\big|_{v=1} < 0, \qquad \frac{\partial}{\partial v} f(v)\big|_{v=\theta} > 0
$$

The matrices corresponding to the linearized systems around $(0,0)$ and $(1,0)$ are, respectively, as follows:

$$
\begin{pmatrix} 0 & 1 \\ -\frac{\partial}{\partial v} f(v)\big|_{v=0} & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ -\frac{\partial}{\partial v} f(v)\big|_{v=1} & 0 \end{pmatrix}
$$

They correspond to topological saddles, with real eigenvalues of opposite sign. On the other hand, the linearized system at $(\theta, 0)$ is a center since the eigenvalues of the matrix

$$
\begin{pmatrix} 0 & 1 \\ -\frac{\partial}{\partial v} f(v)\big|_{v=\theta} & 0 \end{pmatrix}
$$

lie in the imaginary axis. This is not enough to conclude the local behavior of the critical point since it is not hyperbolic. However, observing that the system is symmetric with respect to the horizontal axis, we have that $(\theta, 0)$ is a center for the nonlinear system.

On the other hand, the following functional plays the role of a first integral of the system

$$
E(v, v_x) = \frac{1}{2} v_x^2 + F(v)
$$

where $F(v) = \int_0^v f(s)ds$.

When $F(1) > 0$, the separatrix of the saddle in 0 is the same trajectory and encloses $(\theta, 0)$. Indeed, $E(0,0) = 0$, and the curves $v_x = \pm\sqrt{-2F(v)}$ lie below and above the horizontal axis, respectively. At the point $0 < \theta_1 < 1$, fulfilling $F(\theta_1) = 0$, these curves meet. Such point exists, since $F(1) > 0$ and $F(v) < 0$ for all $0 < v < \theta$.

When $F(1) = 0$ one has $F(v) < 0$ for all $0 < v < 1$ and the separatrix is split into two trajectories that connect 0 and 1 (the traveling wave profiles for $F(1) = 0$). We will call Γ the region in the phase-plane that the separatrix $(v_x)_{E=0}(v) = \pm\sqrt{-2F(v)}$ encloses.

Notice that Γ is included in $[0,1] \times \mathbb{R}$, which means that all arcs of any length inside Γ fulfill the constraints.

Implementing the same procedure for finding the separatrix that exits from $(1,0)$, we end up with the curves $(v_x)_{E=1}(v) = \pm\sqrt{2(F(1) - F(v))}$. At the vertical axis, $v = 0$, they take values $(v_x)_{E=1}(0) = \pm\sqrt{2F(1)}$.

Notice that, in the particular case of the cubic nonlinearity $f(v) = v(1-v)(v-\theta)$, as we increase θ towards $1/2$, $(v_x)_{E=1}(0)$ decreases until it reaches the value 0 for $\theta = 1/2$, while, at the same time θ_1 goes to 1.

Moreover, we can also find the separatrix outside the admissible domain $0 \le v \le 1$. We can see that

$$
(v_x)_{E=1} = \pm\sqrt{2(F(1) - F(v))}
$$

is well defined for $v > 1$ since $F(v)$ is a strictly decreasing function. So $-F(v)$ is increasing. This means that both separatrixes do not cross the horizontal axis anymore after $v = 1$. This is valid also for $F(1) = 0$.

Furthermore, the separatrix going out from 0 towards $-\infty$,

$$
(v_x)_{E=0} = \pm\sqrt{-2F(v)}
$$

is well defined for $v < 0$. Notice that this argument also holds for $F(1) = 0$.

See Figure 5.6 and 5.7 for the phase-portrait and the graphical representation of the separatrix.

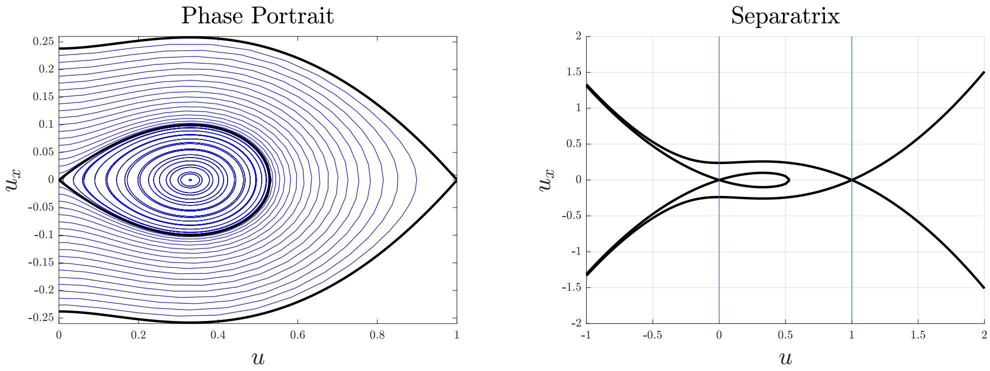

FIGURE 5.6. Phase portraits for $f(s) = s(1-s)(s-\theta)$, when $\theta < 1/2$ ($F(1) > 0$). Left/The phase plane: in black lines the stable and unstable manifolds of the points $(0,0)$ and $(1,0)$, and, in blue lines, the trajectories of the ODE system. Right: The separatrix, stable and unstable manifolds of the points $(0,0)$ and $(1,0)$ depicted outside of the admisible region $0 \leq u \leq 1$ (in blue).

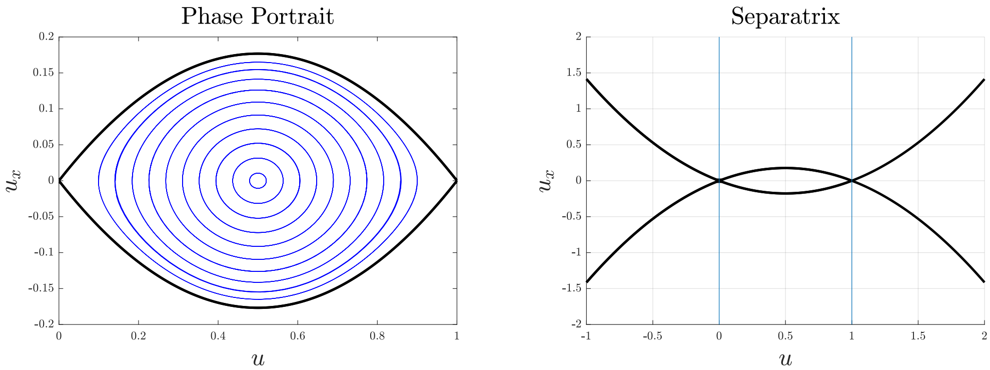

FIGURE 5.7. Phase portraits for $f(s) = s(1-s)(s-1/2)$ ($\theta = 1/2$, $F(1) = 0$). Left/The phase plane: in black lines the stable and unstable manifolds of the points $(0,0)$ and $(1,0)$, and, in blue lines, the trajectories of the ODE system. Right: The separatrix, stable and unstable manifolds of the points $(0,0)$ and $(1,0)$ depicted outside of the admisible region $0 \leq u \leq 1$ (in blue).

5.5.5. *An expression for L in the phase plane.* In the one-dimensional case, one can also obtain an expression for the length L in the phase portrait using the first integral of the system. We restrict our study to the curves that cross the vertical axis. The parameter $\alpha \in (0, F(1))$ is introduced.

Notice that any trajectory starting from the vertical axis is strictly increasing until it reaches its maximum. This means that it is a C^1 diffeomorphism from the time interval $[0, 1/2L(\alpha)]$ to its maximal point $[0, v_{max}(\alpha)]$. Let us denote this diffeomorphism (that depends on α) by $V : [0, 1/2L(\alpha)] \to [0, v_{max}(\alpha)]$. Then noticing that $v_{max}(\alpha) = V(L(\alpha)/2)$ and that $0 = V(0)$, we have

$$L(\alpha) = 2\int_0^{L(\alpha)/2} dz = 2\int_{V^{-1}(0)}^{V^{-1}(L(\alpha)/2)} dz = 2\int_0^{v_{max}(\alpha)} \left(V^{-1}\right)'(v)dv = 2\int_0^{v_{max}(\alpha)} \frac{1}{V'(V^{-1}(v))}dv.$$

We have $v_{max}(\alpha) = F^{-1}(\alpha)$ and $V' = \sqrt{2}\sqrt{\alpha - F(v)}$.

For the bistable nonlinearity, F is not globally invertible, but it is invertible for trajectories crossing the vertical axis and below the separatrix going out from 1. In fact,

$$F : [\theta_1, \sqrt{2F(1)}] \to [0, F(1)]$$

is monotone increasing and

$$L(\alpha) = \sqrt{2}\int_0^{F^{-1}(\alpha)} \frac{1}{\sqrt{\alpha - F(V(V^{-1}(v))}}dv = \sqrt{2}\int_0^{F^{-1}(\alpha)} \frac{1}{\sqrt{\alpha - F(v)}}dv$$

Changing the parametrization, L^* (Definition 5.4) can be written in the following way:

$$L^* = \inf_{\beta\in(\theta_1,1)} \sqrt{2}\int_0^{\beta} \frac{dv}{\sqrt{F(\beta) - F(v)}}$$

Using the aforementioned expression, in [98] the following threshold estimates are proved:

5.5.6. *Bifurcation diagrams.* In this subsection we describe the bifurcation diagrams for some bistable nonlinearities and plot the bounds obtained before.

The blue and the red lines in Figure 5.8 represent the nontrivial solutions for the Dirichlet boundary value problem, with boundary conditions $= 0$ and $= \theta$, respectively. For the blue curve, the vertical axis represents the L^∞-norm. For the red curve, the vertical axis represents the L^∞-norm when the curve is above θ, and $\theta - \|u-\theta\|_\infty$ when the curve is under θ. In this way, the minimal norm taken by the solution of the following equation is expressed

$$\begin{cases} -\partial_{xx}v = \lambda f(v) & x\in(0,1) \\ 0 < v < \theta & x\in(0,1) \\ v(0) = v(1) = \theta. \end{cases}$$

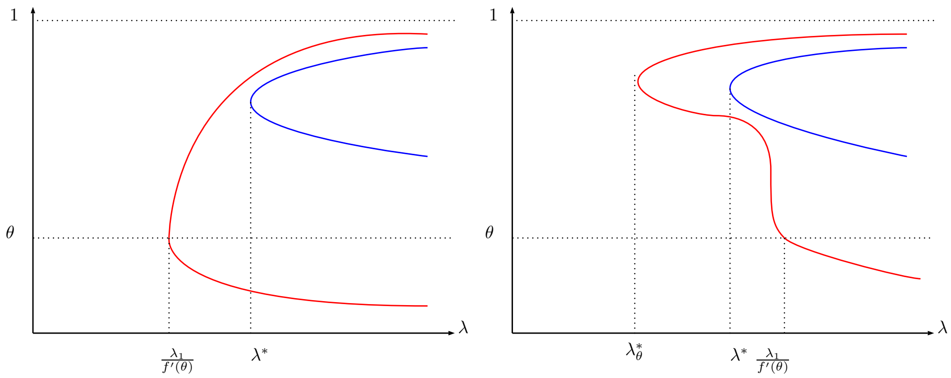

FIGURE 5.8. Left: Qualitative bifurcation diagram for the stationary solutions of a bistable nonlinearity that is convex in $(0,\theta)$ and concave in $(\theta,1)$. Richt: bifurcation dyagram for a general non-convex nonlinearity in $(0,\theta)$.

Remark 5.26 (Further bifurcations). Further bifurcation points for the boundary value θ can occur when increasing λ. Bifurcating solutions that oscillate around θ are not represented in the diagrams. Those solutions appear only after the nontrivial solution above θ and the nontrivial one below θ have emerged. This is due to the fact that oscillatory solutions of:

$$\begin{cases} -\partial_{xx}v = \lambda f(v) & x\in(0,1), \\ 0 < v < 1 & x\in(0,1), \\ v(0) = v(1) = \theta, \end{cases}$$

above/below θ in a smaller set than the original set $(0,1)$, satisfy

$$\begin{cases} -\partial_{xx}v = \lambda f(v) & x\in I, \\ \theta < v < 1 & x\in I, \\ v = \theta & x\in \partial I, \end{cases}$$

in a sub-interval $I\subset(0,1)$. Therefore, I should be large enough so that such solution exists.

Note that this reasoning can also be applied in several space dimensions, using the monotonicity of the eigenvalues with respect to domains scaling: If $D\subset\Omega$ then $\lambda_1(\Omega)\le\lambda_1(D)$ (see [49, Section 1.3.2])

Remark 5.27 (Harmonic Oscillator). Note that if we linearize the ODE dynamics associated with the elliptic problem around $(\theta, 0)$, one obtains the harmonic oscillator

$$\partial_{xx} v = -f'(\theta) v \tag{5.21}$$

or

$$\frac{d}{dx}\begin{pmatrix} v \\ v_x \end{pmatrix} = \begin{pmatrix} v_x \\ -f(\theta)v \end{pmatrix}.$$

We observe that $f'(\theta)$ corresponds to the frequency of the oscillations. This helps to heuristically understand whether nontrivial solutions close to $v \equiv \theta$ can exist or not.

The general solution of (5.21) is:

$$v(x) = A \sin\left(\sqrt{f'(\theta)}x\right) + B\cos\left(\sqrt{f'(\theta)}x\right).$$

Imposing Dirichlet conditions lead us to the choice of $B = 0$. The length $L = \pi/\sqrt{f'(\theta)}$ is critical since the stationary solution $v \equiv \theta$ becomes unstable in the first eigenfunction. Indeed, consider

$$\begin{cases} \partial_t u - \partial_{xx} u = f(u) & (x,t) \in (0,L) \times (0,T) \\ u(0) = u(L) = \theta \end{cases}$$

and linearize it around $v \equiv \theta$:

$$\begin{cases} \partial_t \tilde{u} - \partial_{xx} \tilde{u} = f'(\theta)\tilde{u} & (x,t) \in (0,L) \times (0,T), \\ \tilde{u}(0) = \tilde{u}(L) = 0. \end{cases} \tag{5.22}$$

The first eigenvalue of (5.22) is $\pi^2/L^2 - f'(\theta)$, which becomes unstable when $L^2 > \pi^2/f'(\theta)$.

The same reasoning can be applied for other bifurcations points.

6. Graphical representation of 1-D barriers

In this section, we aim to illustrate graphically the results obtained in the previous section in the context of the system

$$\begin{cases} -\partial_{xx} u = \lambda u(1-u)(u-\theta) & x \in (0,1) \\ u(0) = u(1) = a \\ 0 \le u \le 1 & x \in (0,1). \end{cases}$$

With the change of variables $v = u - a$, we aim to illustrate how the following energy functional changes

$$J_\lambda(v,a) = \int_0^1 \frac{1}{2} v_x^2 - \lambda \left\{ \int_0^{v(x)} f(s+a)ds \right\} dx,$$

depending on the parameter λ and the Dirichlet condition a.

For representing these energy functionals, we consider its projection onto the subspace V generated by the first and third eigenvector of the Dirichlet Laplacian:

$$e_1(x) := \sin(\pi x), \qquad e_3(x) := \sin(3\pi x),$$

which can be expressed as follows:

$$J_\lambda^{e_1,e_3}(\alpha,\beta,a) := J_\lambda(\alpha e_1 + \beta e_3, a) = \int_0^1 \frac{1}{2}\left|\alpha \partial_x e_1(x) + \beta \partial_x e_3(x)\right|^2 - \lambda \left\{ \int_0^{\alpha e_1 + \beta e_3} f(s+a)ds \right\} dx,$$

α and β being the coordinates in the e_1 and e_3 directions respectively.

In Figure 6.1, we observe how other critical points, aside of the constant steady-state 0, may appear for the Dirichlet condition $= 0$, as the measure of the domain $\lambda = L^2$ increases. Of course, these 2-d plots, although they illustrate the qualitative behavior of J, do not represent its full infinite-dimensional complexity.

Critical points can be local minima, saddle points or local maxima for the functional under consideration. It is worth noting that the functional J cannot have local maxima, since the PDE is a gradient system driven by the functional J. There is always a high frequency stable eigenmode for the linearization of the system around any steady-state. This is a consequence of the fact that the eigenvalues of the Laplacian tend to infinity, $\lambda_n \to +\infty$. This means, in particular, that every steady-state has a stable manifold associated to the semilinear parabolic dynamics, and that, accordingly, it cannot be a local maximum of J. Figure 6.2 represents how the functional evolves as λ changes. This helps understanding why the

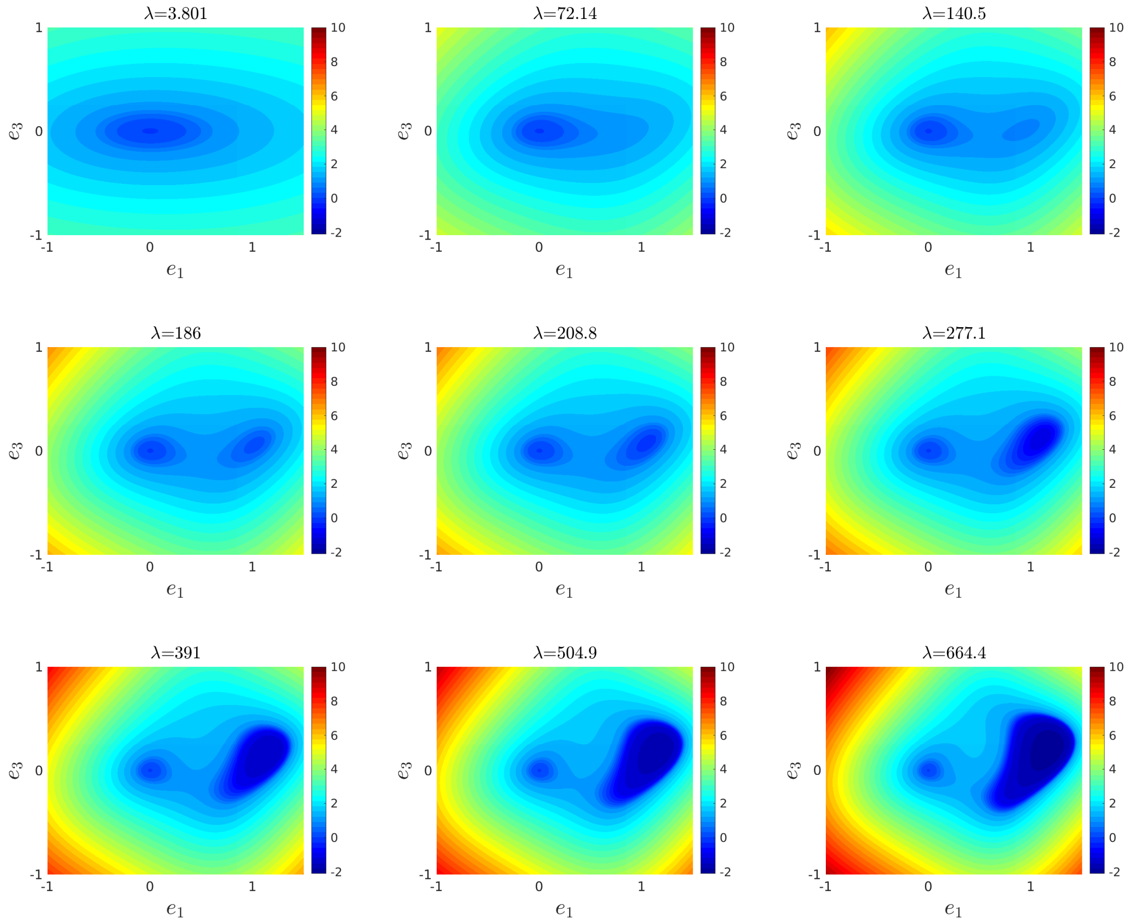

FIGURE 6.1. Functional $J_\lambda^{e_1,e_3}(\alpha,\beta,0)$ for different values of λ

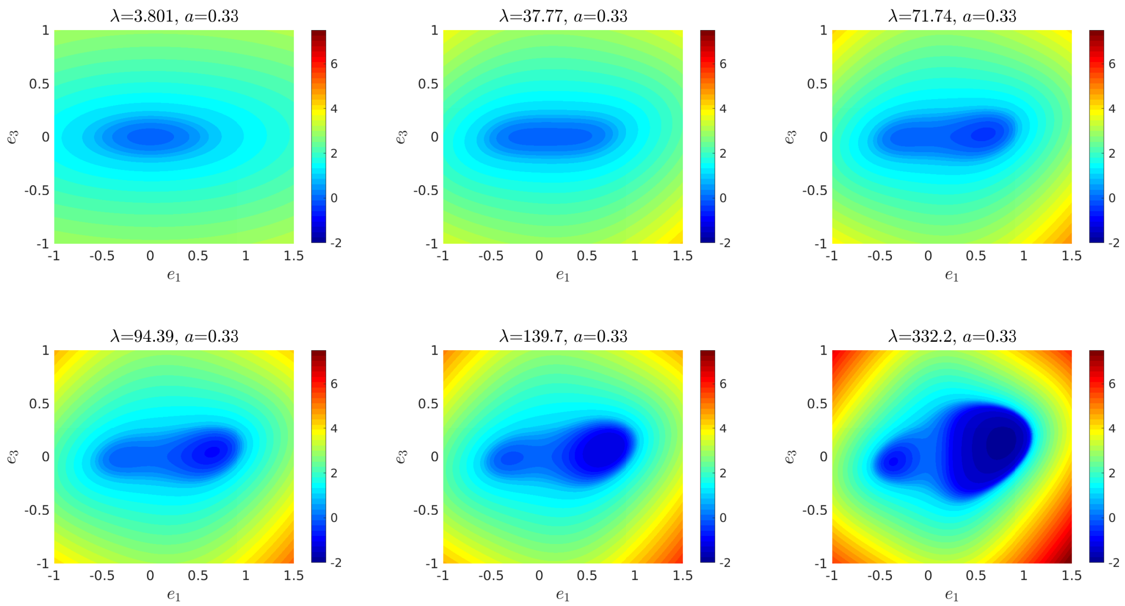

FIGURE 6.2. Graph of the functional $J_\lambda^{e_1,e_3}(\alpha,\beta,\theta=0.33)$ for different pairs (α,β) and values of λ

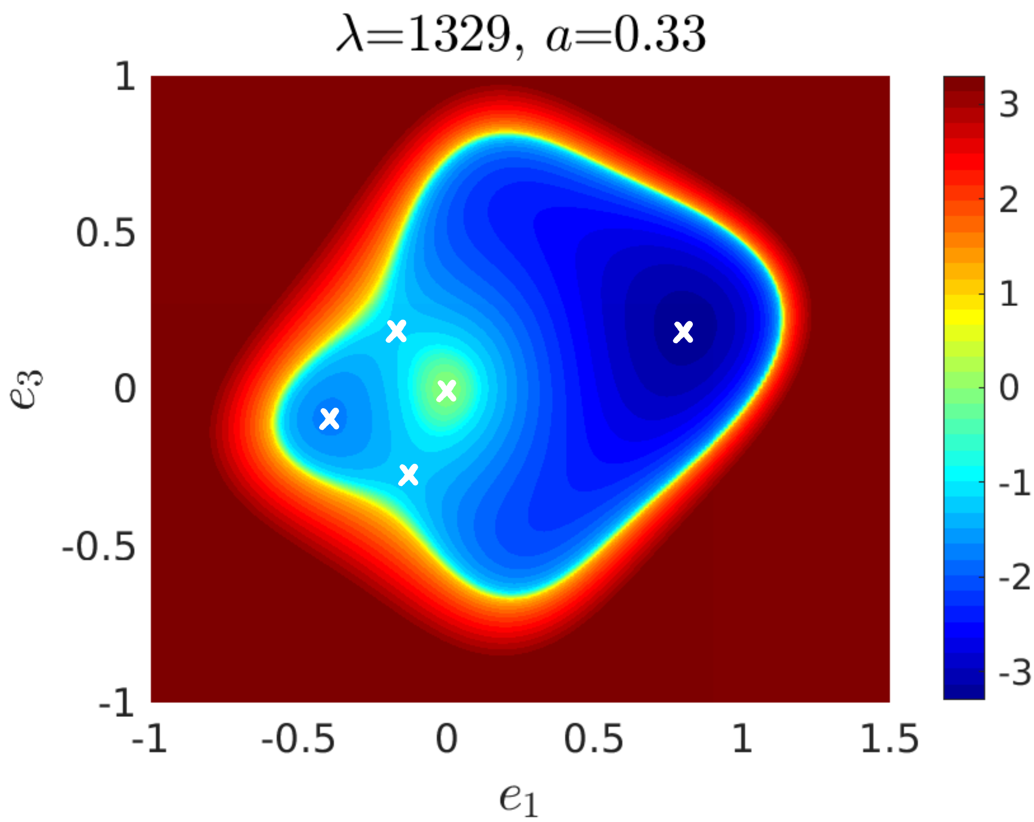

FIGURE 6.3. Energy functional $J_\lambda^{e_1,e_3}(\alpha,\beta,\theta)$. The values of the functional above 3 are represented as 3 in the picture. The white crosses indicate the critical points of the functional.

solution of

$$\begin{cases} -\partial_{xx}u = \lambda f(u) & x \in (0,1), \\ \theta < u < 1 & x \in (0,1), \\ u = \theta, \end{cases}$$

emerges for a smaller value of λ than the solution of:

$$\begin{cases} -\partial_{xx}u = \lambda f(u) & x \in (0,1), \\ 0 < u < \theta & x \in (0,1) \\ u = \theta. \end{cases} \tag{6.1}$$

Figure 6.3 exhibits two local minima, two mountain passes or saddles and one local maximum. Note, however, that we are representing the restriction of the functional in a low frequency subspace. This explains the presence of local maxima, even if the complete functional J does not have any.

In order to compute numerical approximations of the minimizer of the functional J, i.e.

$$\begin{cases} \displaystyle\min_{v \in H_0^1(0,1)\cap C} \int_0^1 \left[\frac{1}{2}v_x^2 - \lambda(F(v+a) - F(a))\right] dx \\ C := \{v \in L^\infty(0,1) \quad \text{s.t.} \quad -b_1 < v(x) < b_2, \quad b_1, b_2 \geq 0, \} \end{cases}$$

we employ a finite difference discretization of the functional and IpOpt [121]

In Figure 6.4, we plot different solutions, corresponding to various boundary conditions, and, in green, we represent a section of the nontrivial solution in the whole real line $\mathbb{R}$

$$\begin{cases} \partial_{xx} u = f(u) & x \in \mathbb{R}, \\ 0 < u < 1 & x \in \mathbb{R}, \end{cases}$$

which corresponds to the homoclinic orbit around $(0, 0)$ in the phase plane discussed in the previous section.

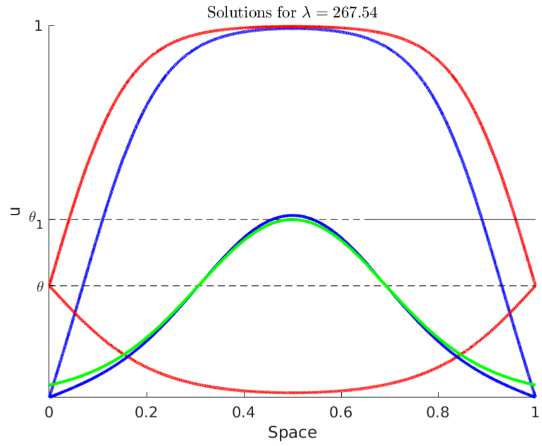

FIGURE 6.4. The blue and red lines corresponds to nontrivial solutions with Dirichlet conditions 0 and θ respectively. In green, a section of the ground state solution defined in the whole space $\mathbb{R}$.

7. Paths of steady-states and the control strategy

In this section, we will focus on the reaction-diffusion model with a bistable nonlinearity and its control to the constant steady-state θ, as a paradigm of controlling the dynamics towards an unstable equilibrium using the staircase method.

In the previous sections, we have analyzed and described the possible emergence of barrier steady-states and we have shown that, whenever such barriers exist, we cannot expect that all initial data can be driven to the final steady-states 0 or θ.

The steady-states 0 and 1 are linearly stable. This can be easily proved by linearization and the fact that $f'(1) < 0$ and $f'(0) < 0$. However, since the steady-states 0 and 1 also play the role, respectively, of a subsolution and a supersolution, all other solutions with controls constrained in $[0, 1]$, cannot reach them in finite time, but just at an exponential rate in infinite time.

In the case of the constant steady-state θ, being away from the limits of the constraints 0 and 1, in principle, there is no impediment for its reachability in a long enough finite time horizon, with suitable controls. However, different difficulties may arise. On one hand, it might be an unstable steady-state. On the other one, there may exist other non-constant solutions of the elliptic problem, taking the same boundary value θ on the boundary. For these reasons, understanding whether and how θ can be attained, without violating the bilateral constraints, is not immediate. For instance, when multiple elliptic solutions exist with boundary value θ, the trivial strategy of setting the boundary condition equal to θ to spontaneously attract the trajectory towards the constant stead-sate θ does not suffice.

In this section, we will analyze and describe how the system can be controlled to the steady-state θ, avoiding both instability and multiplicity issues.

The staircase method ensures that if there is a connected path of admissible steady-states in the interior of the admissibility region, the dynamics can be controlled from any initial configuration to any final one in the vicinity of that path. In fact, once the paths have been constructed, the control strategy consists of two phases:

(1) *Dynamic phase.* For the given initial datum to be controlled, to find a control function a steering the system from this initial configuration to an element of the path.
(2) *Quasistatic phase.* Application of the staircase method along the path to reach the target in large time.

Therefore we will now address the construction of admissible paths of steady-states for the one-dimensional problem under consideration.

We will split the discussion in two situations. When $F(1) > 0$ (remind that $F(t) = \int_0^t f(s)ds$), barriers can exist and one cannot have controllability to θ for all admissible initial data. When $F(1) = 0$, barriers do not exist and the controllability to θ holds for any admissible initial data, regardless of the stability of θ. The case $F(1) < 0$ can be treated as the case $F(1) > 0$, by reversing the roles of 0 and 1. Later on we will also analyze which states are path connected to the steady-state $= 0$ and the steady-state $= \theta$.

7.1. **The case** $F(1) > 0$**.** Recall that a bistable nonlinearity is such that $f < 0$ on $(0, \theta)$ and $f > 0$ on $(\theta, 1)$, with $f'(0) < 0$, $f'(1) < 0$, $f'(\theta) > 0$. In this subsection, we assume that the primitive, $F(u) = \int_0^u f(s)ds$, evaluated at 1 is positive, $F(1) > 0$. We denote by θ_1 the nontrivial value such that $F(\theta_1) = 0$. For the prototypical bistable nonlinearity, $f(u) = u(1-u)(u-\theta)$, one has that $F(1) = 0$ when $\theta = 1/2$ and $F(1) > 0$ when $\theta < 1/2$.

Hereafter we present the strategy in [98] for finding the connected path of steady-states for the constrained controllability of the one-dimensional problem

$$\begin{cases} \partial_t u - \partial_{xx} u = f(u) & (x,t) \in (0,L) \times (0,T), \\ u(0,t) = a_1(t) & t \in (0,T), \\ u(L,t) = a_2(t) & t \in (0,T), \\ 0 \le u(x,0) \le 1 & x \in (0,L), \end{cases}$$

using the phase-plane analysis.

More precisely, we look for a path of steady-states with fixed length, θ:

$$\begin{cases} -\partial_{xx} u = f(u) & x \in (0,L), \\ u(0) = a_1, \quad u(L) = a_2, \end{cases} \tag{7.1}$$

connecting to the stationary solution.

We emphasize that it might not always be possible to do this for any initial data in case barriers arise. Recall that for barrier steady-states, the maximum of the barrier solution is necessarily above θ. This implies, in particular that the initial data that are above the barrier cannot be controlled to the steady-state θ.

7.1.1. *The control strategy in [98].* The length of the space-domain, L, is chosen so that 0 is the only steady-state with null boundary conditions, i. e. $L < L^*$. In this way, choosing the null constant control for the parabolic problem, one approaches asymptotically this null steady-state as time increases. Afterwards, once the solution is close to 0, one implements a control strategy to control the trajectory to an element of the path connecting to the constant steady-state θ.

The following region of the phase-plane plays an important role in this construction.

Definition 7.1. Let Γ be the region in the phase-plane that the homoclinic orbit $(v_x)_{E=0}(u) = \pm\sqrt{-2F(v)}$ encloses i.e.

$$\Gamma := \left\{ (v, v_x) \in [0, \theta_1] \times \mathbb{R} : \quad -\sqrt{-2F(v)} \le v_x \le \sqrt{-2F(v)} \right\},$$

with $\theta_1 \in (0,1)$ such that $F(\theta_1) = 0$ (see Figure 7.1).

More specifically one proceeds in the following way:

(1) **Step 1. Stabilize to** 0. Set both controls at the boundary points $x = 0, L$ to be $= 0$ for the parabolic equation. For $L < L^*$ (see Definition 5.4) the solution will approach the stationary solution 0 in the L^∞ norm as time increases.
(2) **Step 2. Control to a steady-state in** Γ. Since the state converges to 0 as $t \to +\infty$, there exists a time t so that the maximum of the solution of the parabolic problem is strictly below

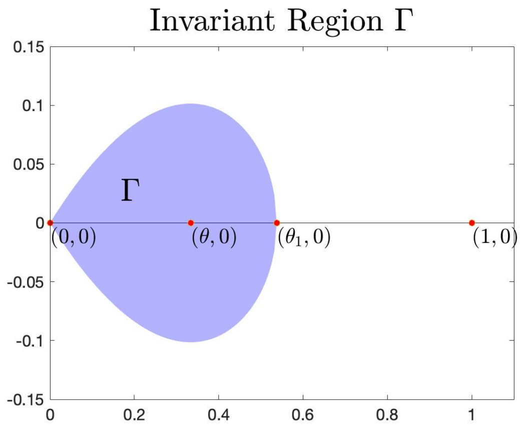

FIGURE 7.1. Invariant region Γ (in blue) for the nonlinearity $f(s) = s(1-s)(s-1/3)$.

the minimum solution of

$$\begin{cases} -\partial_{xx}v = f(v) & x \in (0,L) \\ v(0) = v(L) = \epsilon \\ 0 < v < \epsilon \end{cases}$$

for $0 < \epsilon < \theta$, that we denote as v_ϵ.

At that time we set the controls to take the value ϵ on the boundary. By Theorem 3.1([78]), we know that the parabolic solution is going to converge to a steady-state with boundary value ϵ. This steady limit takes values below ϵ and above 0. We keep the ϵ boundary-control long enough until the state is close enough to v_ϵ. Once the proximity is sufficient we can apply a local controllability result to reach the steady-state v_ϵ exactly in a time horizon of length 1. At this point it is important to guarantee that we do not violate the constraints. But this can be done in a time horizon of length one, provided the distance to v_ϵ is small enough (see [93, Lemma 8.3]).

(3) **Step 3. Construction of the path of steady-states towards θ.** Now we build the connected path of steady-states, connecting v_ϵ with the stationary solution θ, corresponding to the Dirichlet boundary control θ. To do this it is sufficient to build the steady-states corresponding to the Cauchy data at $x = 0$, interpolating those of v_ϵ and θ

$$\begin{cases} v^s(0) = (1-s)\epsilon + s\theta \\ \partial_x v^s(0) = \partial_x(1-s)v_\epsilon. \end{cases}$$

We then solve the corresponding Cauchy problems

$$\begin{cases} -\partial_{xx}v^s = f(v) \\ v^s(0) = (1-s)\epsilon + s\theta \\ \partial_x v^s(0) = \partial_x(1-s)v_\epsilon. \end{cases}$$

This path ends at θ by uniqueness of the solution of the ODE system:

$$\begin{cases} -\partial_{xx}v = f(v) \\ v(0) = \theta \\ \dfrac{\partial}{\partial x}v(0) = 0. \end{cases}$$

This gives the boundary controls corresponding to the path of steady-states

$$\begin{cases} u_1^s = v^s(0) \\ u_2^s = v^s(L). \end{cases}$$

Note however that their values at $x = L$ are not straightforward since they emerge out of the solutions of the Cauchy problems solved from $x = 0$ to $x = L$.

This path connects continuously the steady state v_ϵ with the steady state θ, due to the continuous dependence of the initial data of the ODE system. Furthermore, all elements of the path respect the bilateral bounds 0 and 1. This is a consequence of the following facts:

- There is an invariant region Γ such that $(0,0) \in \partial\Gamma$ and $(\theta,0) \in \Gamma$. Moreover, Γ is included in the admissible set of states and it is star-shaped with respect to $(\theta,0)$. This is a consequence of the fact that $f(s) < 0$ in $(0,\theta)$ and $f(s) > 0$ in $(0,\theta_1)$.
- In step 2, we reach a stationary curve that lies inside Γ. (Proof in Proposition 7.2)
- The control strategy in [98] is based on tracing a line between the Cauchy data in one extreme of the curve obtained in step 2 and those of the target (the point $(\theta,0)$). This line lies inside the invariant region Γ; hence, by solving the ODE problem from $x = 0$ up to $x = L$, we obtain a set of stationary solutions, that are guaranteed to lie in Γ and hence are admissible.

At this Step the following result is relevant:

Proposition 7.2. *Γ is an invariant region.*

Proof. Γ is enclosed by a homoclinic curve. By the uniqueness of solutions of the ODE, the result follows. □

Remark 7.3. Note that the convexity property of Γ is not needed. Indeed it suffices that there exists a continuous curve $l : [0,1] \to \mathbb{R}^2$ such that $l(s) \in \Gamma$ for all $s \in [0,1]$, connecting the point $l(0) = (v(0), \partial_x v(0))$ to $l(1) = (\theta, 0)$.

(4) **Step 4. Application of the staircase method** along the arc of steady-states generated before.

Figure 7.2 illustrates how the invariant region can be used to construct admissible continuous paths of steady-states, in particular, connecting the constant steady-states 0 and θ.

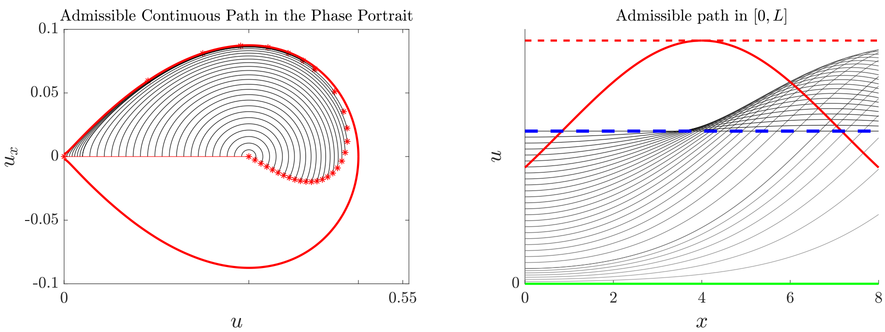

FIGURE 7.2. $L = 8 > L_\theta$, $\theta = 0.33$. Left: A path of steady-states of system (5.9) connecting the steady-states $(0,0)$ and $(\theta,0)$. In black the steady-states of (7.1) that are part of a continuous path connecting to the constant stationary solution θ are represented. The red crosses are the corresponding boundary controls. Right: The stationary path plotted in the space domain. In red the curve of maximum value in the invariant region Γ, and in green the initial condition and in blue the constant steady-state θ.

7.1.2. *Connected symmetric path.* Note that the arguments above employ a path of connected steady states in which the boundary values at $x = 0$ and $x = L$ do not coincide. Therefore the path of steady-states built does not guarantee that these states are symmetric with respect to $x = L/2$. This symmetry can be guaranteed by a different argument that we describe now.

As a consequence of this fact, the reaction-diffusion system can be controlled using the same applied control at $x = 0$ and $x = L$, so that the trajectory remains symmetric with respect to $x = L/2$ at all times, when the initial datum is symmetric (Note that the constant target θ is obviously symmetric as well). This improved construction allows for an easier extension to the multi-dimensional case.

This new construction is inspired on the symmetry of the phase plane dynamics (the symmetry of radial solutions of the Poisson equation in several space dimensions).

More precisely, we proceed as follows:

- Instead of imposing the boundary conditions at one extreme $x = 0$ (the Cauchy data for the ODE), as above, we impose the values at the middle point $x = L/2$. By symmetry, we know that it is a critical point of the solution of ODE, and hence it lies in the horizontal axis of the phase

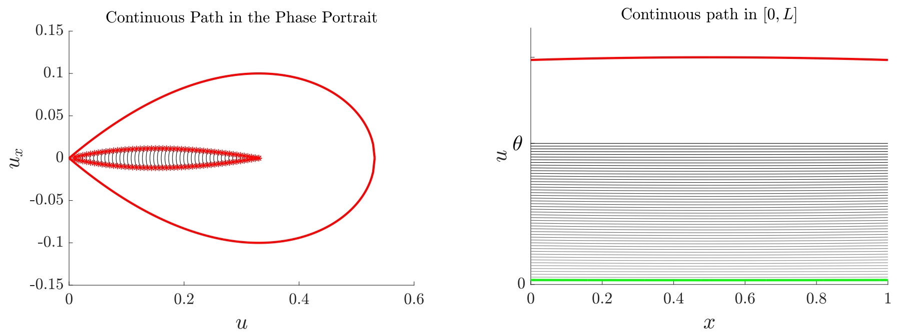

FIGURE 7.3. $L = 1$, $\theta = 0.33$. Representation of the "even strategy" in the phase portrait of system (5.9). Left: in black, the steady-states of (7.1) that constitute the connected path of stationary solutions, connecting to θ. In red, the values of the Dirichlet controls. Right: the stationary path in the space domain. The red curve represents the nontrivial elliptic solution in $\mathbb{R}$, that corresponds to the homoclinic orbit in the phase plane. In green the initial steady-state from which the path of steady-states departs.

portrait. We now solve the ODE from $x = L/2$ to $x = L$ (and backwards to $x = 0$) to extend the solution to the interval $x \in (0, L)$. The obtained set of steady-states takes then the boundary values

$$v^s(L) = v^s(0) = a^s \quad s \in [0, 1]$$

where a^s is the value of the solution at $x = L$, which is the same as that at $x = 0$.

More precisely, let

$$\Phi\left(x; \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, L/2\right)$$

be the solution of

$$\frac{d}{dx}\begin{pmatrix} v \\ v_x \end{pmatrix} = \begin{pmatrix} v_x \\ -f(v) \end{pmatrix}$$

with initial data at $x = L/2$ being $(v_1, v_2)^T$ and consider:

$$a^s := P_1 \Phi\left(L; \begin{pmatrix} (1-s)v_\epsilon(L/2) + s\theta \\ 0 \end{pmatrix}, L/2\right), \tag{7.2}$$

where P_1 is the projection on the first component, and v_ϵ is the even stationary solution to which we arrived after step 2 in the procedure above.

- a^s is continuous with respect to s due to continuous dependence on the initial data. This leads to a continuously connected set of solutions to the boundary value problem associated with a^s.
- Notice that the trajectories lie in Γ since the path $(v_\epsilon(L/2)(1-s) + s\theta)$ lies in the horizontal axis inside the invariant region Γ.

Remark 7.4. Notice that this procedure does not depend on the length L of the space interval. The only requirement is to be able to reach an even (with respect to $x = L/2$) stationary state inside the invariant region Γ. Therefore, for every $L > 0$, there exists a path connecting 0 with θ.

Figures 7.3 7.4 7.6 provide a graphical representation of the construction above for different values of L: $L < L^*_\theta$, $L^*_\theta < L < L^*$ and $L^* < L$. Figures 7.5 and 7.7 represent the value at the boundary of the continuous path of steady-states for different values of L. Figures 7.5 and 7.7 also represent the bifurcation diagrams depending on the parameter a. One observes, in particular, the number of steady-states emerging along the path.

In view of this construction, when the goal is to drive the system to the target θ, the key question becomes whether we may, out of the given initial configuration, drive the dynamics of the reaction-diffusion system into the set Γ, and this, sometimes, despite the possible presence of barrier functions.

7.2. **A non returning path.** The following result describes the admissible steady-states from which a connected path of steady-states can be constructed, connecting to the steady states θ and 0.

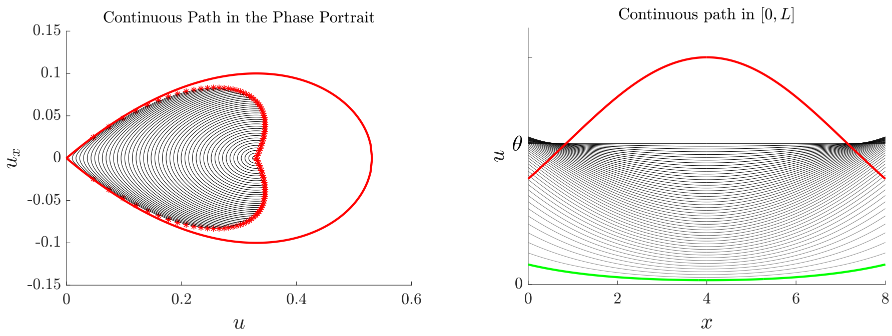

FIGURE 7.4. $L = 8$, $\theta = 0.33$. Representation of the "even strategy" in the phase portrait of system (5.9). Left: in black, the representation of the steady-states of (7.1) connecting to θ. In red, the values of the Dirichlet controls. Right: the plot of the stationary states along the space domain. The red curve represents the nontrivial elliptic solution in $\mathbb{R}$, i. e. the homoclinic orbit in the phase plane. In green the initial condition.

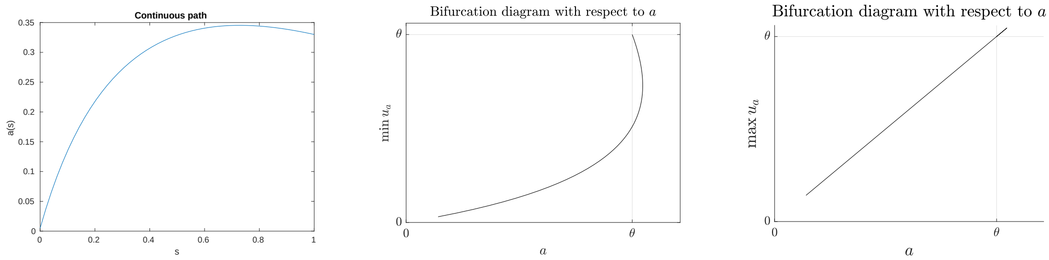

FIGURE 7.5. $L = 8$, $\theta = 0.33$. Left: Trace $a(s)$ of the connected path of the steady states in equation (7.2) as a function of the shooting parameter $0 \leq s \leq 1$. Middle: The minimum of the steady-states u_a depending on the boundary value a along the path. Note that for a given boundary value a, there might be two solutions associated with a inside the path. Right: The maximum of the steady-states u_a depending on the boundary value a along the path.

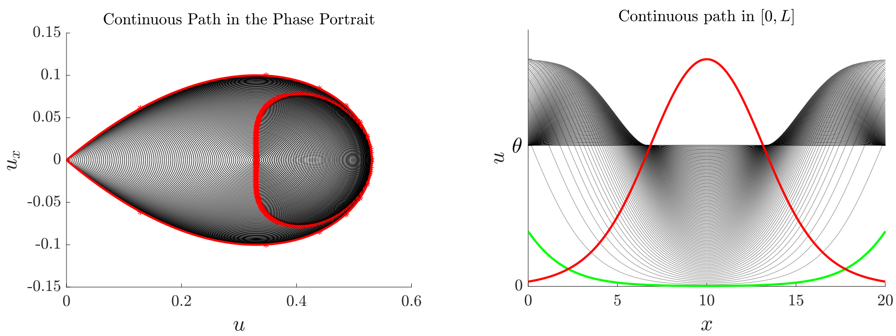

FIGURE 7.6. $L = 20$, $\theta = 0.33$. The "even strategy" is represented in the phase portrait of system (5.9). Left: in black, the plot of the steady-states of (7.1), constituting a connected path leading to the stationary solution θ. In red, the values of the corresponding boundary controls. Right: the stationary path plotted in the space domain. The red curve is the nontrivial elliptic solution in $\mathbb{R}$, corresponding to the homoclinic orbit in the phase plane. In green the steady-state v_ϵ from which the path departs.

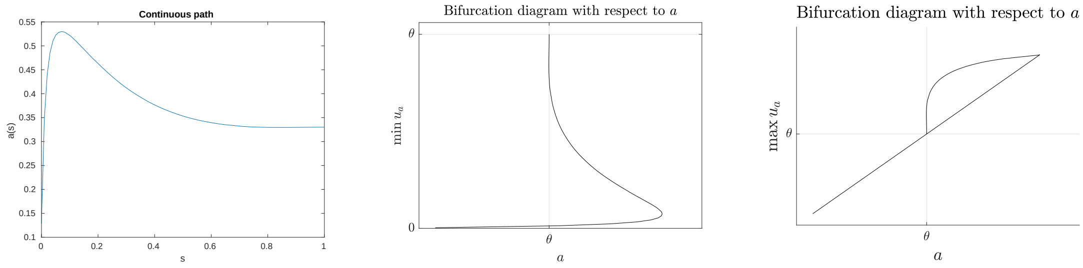

FIGURE 7.7. $L = 20$, $\theta = 0.33$. Left: trace of the connected path of the steady states $a(s)$ as a function of the shooting parameter $0 \leq s \leq 1$ in equation (7.2). Middle: The minimum of the steady-states u_a depending on the boundary value a along the path. Note that for a given boundary value a, there might be two solutions associated with a inside the path. Right: The maximum of the steady-states u_a depending on the boundary value a along the path.

Proposition 7.5. *Let w be an admissible steady-state. Consider $L > L^*$ and let $\underline{u}_L$ be the minimum non-trivial solution with respect to the L^∞ norm of the problem:*

$$\begin{cases} -\partial_{xx}\underline{u}_L = f(\underline{u}_L) & x \in (0, L), \\ 0 < \underline{u}_L < 1 & x \in (0, L), \\ \underline{u}_L(0) = \underline{u}_L(L) = 0. \end{cases}$$

If

$$\max_{x\in[0,L]} w(x) \leq \max_{x\in[0,L]} \underline{u}_L,$$

and

$$\frac{1}{2}w_x(0)^2 + F(w(0)) \leq F\left(\max_{x\in[0,L]} \underline{u}_L\right),$$

there are connected paths of steady-states, connecting w to θ and to 0, respectively.

Moreover, if w is symmetric, the second condition is not needed, and there is a path that is constituted by symmetric states with respect to $x = L/2$.

Proof. We present a brief sketch of the proof.

Let Λ denote the closed region between $\underline{u}_L$ and the vertical axis in the phase plane. Any point in Λ fulfills the conditions of Proposition 7.5.

From any given steady-state, we first generate a path towards a symmetric steady-state $w^*(x)$.

To do this, we use the trajectory in the phase portrait to which the initial steady-state belongs to (note that the first integral $w_x^2/2 + F(w)$ is preserved along the trajectory). Then, since $w_x(0)^2/2 + F(w(0)) \leq F(\max_{x\in[0,L]} \underline{u}_L)$, we know that $F(\max_{x\in[0,L]} w^*(x)) \leq F(\max_{x\in[0,L]} \underline{u}_L)$. We can then use the same argument as in the previous section (taking the Dirichlet and Neumann conditions in $L/2$) to build a connected path of steady-states that goes from w^* to $w \equiv \theta$. The admissibility of the path follows by the fact that $\underline{u}_L$ is the minimum solution of the elliptic problem and that the boundary of Γ is a homoclinic orbit (hence it is associated with a solution to the problem in the whole real line). This assures that any symmetric admissible trajectory in the phase plane within the interior of the set $\Lambda \setminus \Gamma$ will not attain the axis $u = 0$, since, otherwise, $\underline{u}_L$ would not be the minimum solution.

□

As explained before, when implementing the staircase strategy for control, given that one needs to apply local exact controllability results for heat-likes equations, and that this necessarily requires that controls and trajectories oscillate, the paths of steady-states considered cannot saturate the constraints.

Of course one may also build other paths of steady-states that saturate the constraints or even go beyond them. But these paths cannot be used when considering the control of the reaction-diffusion system under the given constraints.

Proposition 7.6. *When $L > L^*$ there is an admissible connected path of steady-states, continuous with respect to the L^∞ topology, connecting the stationary solution $u = 0$ with the minimal non-trivial solution of*

$$\begin{cases} -\partial_{xx} u = f(u) & x \in (0, L) \\ u(0) = u(L) = 0 & \\ 0 < u < 1 & x \in (0, L). \end{cases}$$

Proof. Consider the following ODE system, with Cauchy data at $x = L/2$ depending on the parameter $s \in [0, 1]$:

$$\begin{cases} y_x = v \\ y_{xx} = v_x = -f(y) \\ y(L/2) = s \\ y_x(L/2) = 0. \end{cases} \tag{7.3}$$

As before, we will use this system to construct a path of stationary steady-states. We start from $s = 0$. By the uniqueness of the ODE solutions, $y \equiv 0$ is the unique solution. Then, consider the family of solutions of (7.3) depending upon the parameter s that are defined on $x \in [0, L]$ (solving the ODE forward and backward in time). Let $s^* \in [0, 1]$ be such:

$$s^* = \min_{s \in [0,1]} \{\|y^s\|_{L^\infty((0,L))} \text{ s.t. } y^s \text{ solves (7.3), with } y^s(L) = y^s(0) = 0,\ \|y^s\|_{L^\infty((0,L))} \le 1\}.$$

This $s^* \in [0, 1]$ exists because $L > L^*$. The path of steady-states is:

$$\gamma : [0, s^*] \longrightarrow L^\infty(0, L) \quad s \longrightarrow y^s.$$

By the continuous dependence on the initial data, the path is continuous with respect to s with values in $L^\infty(0, L)$. Let

$$y' = F(y); \quad y(0) = y_0$$

and let

$$z' = F(z); \quad z(0) = z_0.$$

We have that:

$$\begin{aligned} \|y(x) - z(x)\|_\infty &= \left\| y_0 - z_0 + \int_0^x F(y(r)) - F(z(r)) dr \right\|_\infty \le \|y_0 - z_0\| + \int_0^x \|F(y(r)) - F(z(r))\|_\infty dr \\ &\le \|y_0 - z_0\| + \int_0^x \sup_{x \in \mathbb{R}^2 : \|x\| \le \max\{\|y\|_{L^\infty}, \|z\|_{L^\infty}\}} \|\nabla F\|_\infty \|y(r) - z(r)\| dr \\ &\le C(L, \|y\|_\infty, \|z\|_\infty) \|y_0 - z_0\|. \end{aligned}$$

□

Proposition 7.7. *When f bistable with $F(1) > 0$ and $L > L^*$, it does not exist any admissible control function bringing $\underline{u}_L$ to 0.*

Proof. By the comparison principle, and given that the controls remain non-negative, the solution will always remain above the non-trivial steady state $\underline{u}_L$ fulfilling the constraints. This makes it impossible that the controlled solution reaches the null steady-state. □

Remark 7.8. In Theorem 4.2, the assumption (4.12) that the path has to be uniformly away of the minimal and maximal values of the control is imposed.

In this section we have shown that, even when $L > L^*$, we can build a path of admissible steady-states outside Γ connecting $\underline{u}_L$ and 0 (not satisfying (4.12)). But we have also seen that this path cannot be used to control the dynamics while preserving the constraints. This shows that the assumption (4.12) is not of a technical nature, but rather a necessary one.

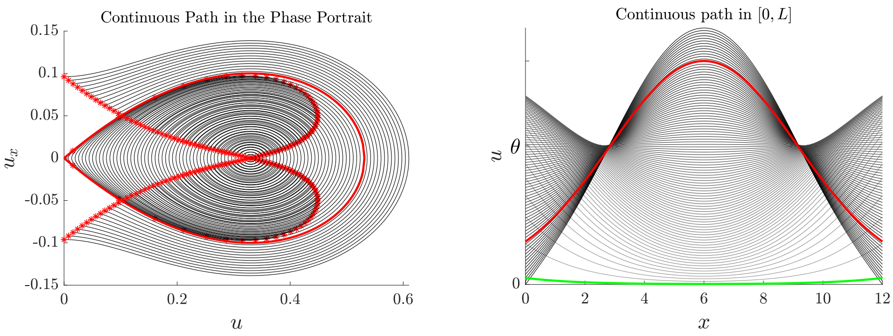

FIGURE 7.8. $f(s) = s(1-s)(s-\theta)$, $\theta = 0.33$, $L = 12$. Left: Connected path of steady-states from 0 to the minimal nontrivial solution $\underline{u}_L$ in the phase plane. This path has elements outside the invariant region Γ. Right: The connected path in the physical space.

7.3. **The case $F(1) = 0$: even and not even controls.** When $F(1) = 0$ (which, for the prototypical cubic nonlinearity, corresponds to $\theta = 1/2$), whatever the length L of the space-interval is, any even initial data of the reaction-diffusion system can be driven to any state on a connected path of steady-states connecting to 0, θ, and 1.

For instance, using the fact that, when $F(1) = 0$, the traveling wave profile connecting 0 to 1 is a stationary solution. Then the sections of the traveling wave profile to space-intervals of length L provide a path of steady-states depending continuously on the boundary data.

Hereafter we point out the main relevant aspects of the particular case $F(1) = 0$:

(1) The region Γ is defined by the two standing traveling waves connecting the points $(0, 0)$ and $(1, 0)$.
(2) By uniqueness of the ODE and the fact that the traveling wave connects the points $(0, 0)$ and $(1, 0)$, there cannot exist an admissible nontrivial solution with boundary values 0 or 1.
(3) Any symmetric admissible steady-state is inside Γ.

Remark 7.9 (A simpler dynamic strategy). In this case a simpler control strategy can be developed. Indeed, according to Theorem 3.1 ([78]), the convergence to 0 and to 1 in infinite time holds, since they are the only stationary solutions taking boundary values 0 and 1, respectively. Then, the path of admissible steady-states connecting any even solution to 1 or 0 is not needed in this case.

Proposition 7.10. *Let $L > 0$ be arbitrary, and assume that $F(1) = 0$. Then there exists a connected path of steady-states connecting* 0 *and* 1.

Proof. When $F(1) = 0$, the traveling wave profile v is a stationary solution in the whole real line $\mathbb{R}$, corresponding to the heteroclinic orbit in the phase portrait connecting 0 and 1:

$$\begin{cases} -\partial_{xx} v = f(v) & x \in \mathbb{R} \\ v(+\infty) = 0, \quad v(-\infty) = 1 \\ v(0) = 1/2. \end{cases}$$

The restriction of $v(\cdot + c)$ to $(0, L)$ is a stationary solution for any $c \in \mathbb{R}$, and constitutes a connected path of steady-states. $\square$

Note however, that this path is not uniformly away from 0 or 1, in other words, it does not satisfy the assumption (4.12) of Theorem 4.2. In particular, this means that we can only stabilize around these stationary solutions but never control in finite time. Moreover, observe that the parameterization of the path given ranges from $-\infty$ to $+\infty$.

Figures 7.9, 7.11 and 7.13 illustrate the paths connecting a steady-state to the stationary constant solutions θ, 1 and 0. Figures 7.10, 7.12 and 7.14 show its respective values at the boundary of the continuous paths of steady-states while Figures 7.10, 7.12 and 7.14 represent their bifurcation diagrams.

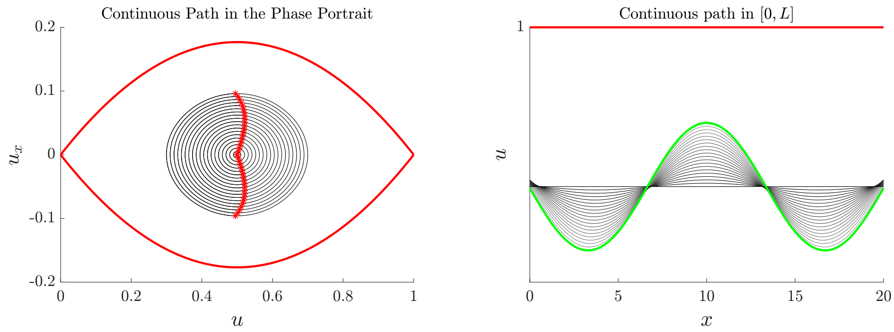

FIGURE 7.9. $L = 20$, $\theta = 0.5$. The even path of steady-states from an initial condition to the stationary solution θ of system (5.9). Left: In black, the steady-states of (7.1), constituting a connected path of stationary solutions leading to θ. In red, the corresponding values of the controls. Right: the stationary path plotted in the space domain, the green curve being the initial steady-state.

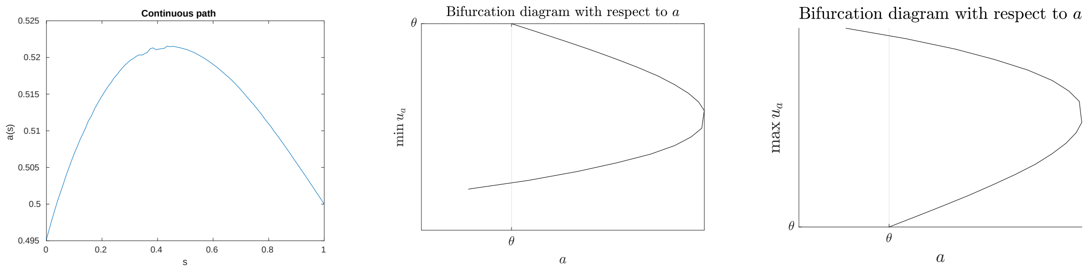

FIGURE 7.10. Graph associated with the path shown in Figure 7.9. Left: Connected path of steady-states, in the horizontal axis the parameter $s \in [0, 1]$ in the vertical axis the boundary value. Center: The minimum value of u_a, as a function of the boundary value a. Right: The maximum value of u_a, as a function of the boundary value a.

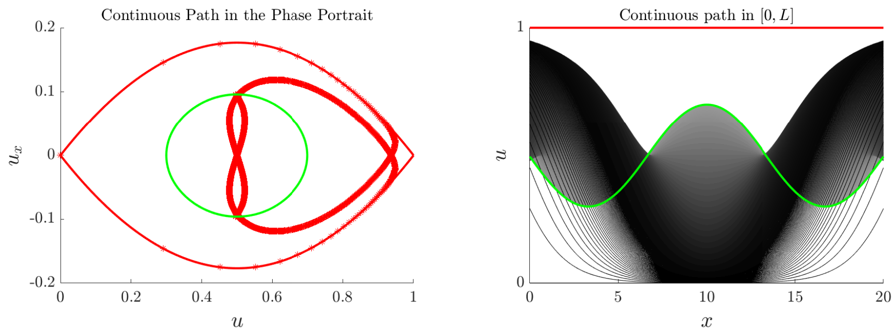

FIGURE 7.11. $L = 20$, $\theta = 0.5$. Phase portrait representation of the path of even steady-states of system (5.9) connecting the initial condition with the constant steady-state 0. Left: In black, the path of steady-states of (7.1) connecting to the stationary solution 0. In red, the values of the corresponding controls. Right: the stationary path plotted in the space domain, the green curve being the initial steady-state.

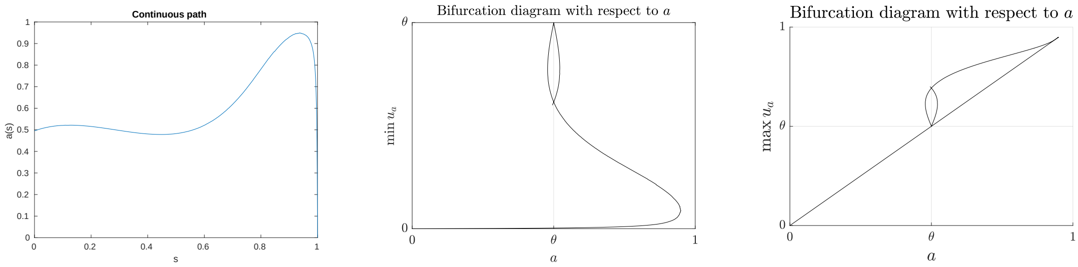

FIGURE 7.12. Graph associated with the path shown in Figure 7.11. Left: Boundary values of the connected path of steady-states as a function of the parameter $s \in [0, 1]$. Center: The minimum value of u_a, with respect to a, for the connected path of steady-states. Right: The maximum value of u_a, with respect to a.

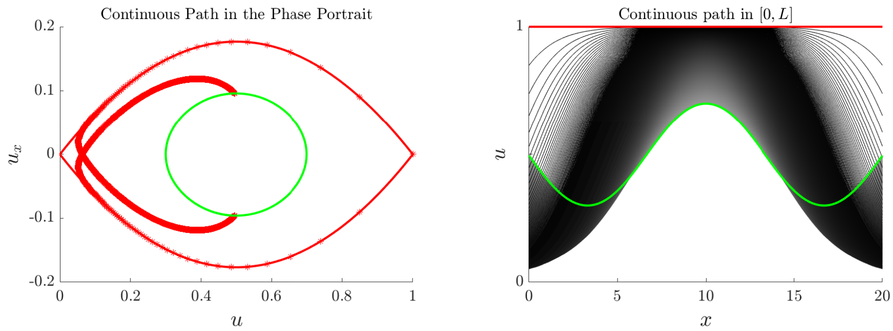

FIGURE 7.13. $L = 20$, $\theta = 0.5$. Phase portrait representation of the path of even steady-states of system (5.9) connecting the initial condition with the constant steady-state 1. Left: In black, the path of steady-states of (7.1) connecting to the stationary solution 1. In red, the values of the corresponding controls. Right: the stationary path plotted in the space domain, the green curve being the initial steady-state.

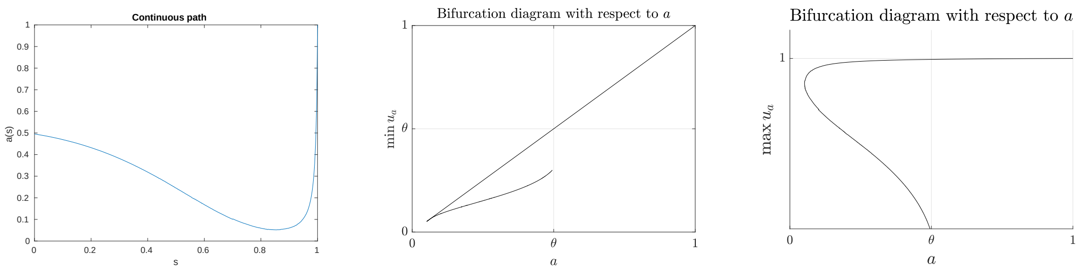

FIGURE 7.14. Graph associated with the path shown in Figure 7.13. Left: Boundary values of the connected path of steady-states as a function of the parameter $s \in [0, 1]$. Center: The minimum value of u_a, with respect to a, for the connected path of steady-states. Right: The maximum value of u_a, with respect to a.

8. Summary of the control results

We have analyzed and described the nature of the steady-state solutions with non-homogeneous boundary values, paying special attention to whether they lay without the limits established by the point-wise state-constraints. We have also discussed their control consequences. Moreover, we have seen how to construct paths of steady-states allowing to ensure controllability to steady-states, even when they are unstable.

8.1. Bistable Nonlinearities.

In Figures 8.1 we represent a summary of the main results plotting both the paths constituted by sections of the traveling wave profiles and the steady-states taking constant values over the boundary. In the case of $F(1)=0$, traveling waves are stationary. Therefore, as discussed previously, its restrictions on domains of length L give a connected path of steady-states between 0 and 1 naturally. However, note that this path does not pass through the constant stationary solution θ.

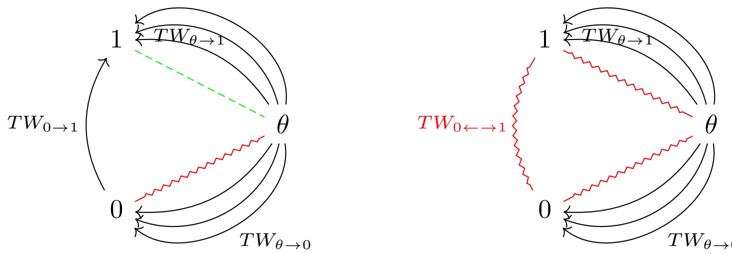

Figure 8.1. Left: Connectivity map for $F(1)>0$. In red, an admissible continuous path of steady-states connecting 0 and θ, existing for all $L>0$. In green, an admissible and continuous path of steady-states connecting θ and 1, whose existence is guaranteed for $L<L^*$. In black, traveling waves for the Cauchy problem. The traveling wave from 0 to 1 is unique, while there are infinitely many traveling waves from θ to 1 or to 0, since the nonlinearity, when restricted to $[\theta,1]$ or in $[0,\theta]$ is monostable and Theorem 3.10 applies. Right: Connectivity map for $F(1)=0$. In red, admissible continuous paths of steady-states connecting 0 and θ and θ to 1, respectively, existing for any $L>0$. The traveling wave from 0 to 1 is unique and stationary, giving a continuous path of admissible steady-states connecting 0 and 1. In black, infinitely many non-stationary traveling waves connecting θ to 1 and to 0, respectively.

We now present the main control results for the bistable equations, which is an extended version of Theorem 2 from [98].

Consider the following reaction-diffusion equation:

$$\begin{cases} \partial_t u - \partial_{xx} u = f(u) & (x,t)\in(0,L)\times(0,T),\\ u(0)=a_1(t), \quad u(L)=a_2(t) & t\in(0,T),\\ u(x,0)=u_0(x)\in[0,1], \end{cases} \tag{8.1}$$

where f is a C^2 bistable function satisfying $f(0)=f(1)=f(\theta)=0$ for a certain $\theta\in(0,1)$. Let F be defined as $F(t)=\int_0^t f(s)ds$ and $a_1,a_2\in L^\infty((0,T);[0,1])$.

Condition 8.1. Let w be any stationary solution of (8.1) satisfying:

$$\frac{1}{2}w_x(0)^2 + F(w(0)) \le F\left(\max_{x\in[0,L]} \underline{u}_L\right) \tag{8.2a}$$

$$\max_{x\in[0,L]} w(x) < \max_{x\in[0,L]} \underline{u}_L, \tag{8.2b}$$

where $\underline{u}_L$ is the minimum solution with respect to the infinity norm to the problem:

$$\begin{cases} -\partial_{xx}\underline{u}_L = f(\underline{u}_L) & x\in(0,L),\\ \underline{u}_L(0)=\underline{u}_L(L)=0, \\ 0<\underline{u}_L<1 & x\in(0,L), \end{cases} \tag{8.3}$$

In case the solution of (8.3) does not exist, the condition is not necessary.

Note that condition (8.2b), in particular, requires that the maximum of the steady-state to be below the maximum value of the minimum nontrivial solution $\underline{u}_L$. This fact, together with the energy requirement (8.2a), it assures that w is a supersolution with null boundary control. Since there is no barrier below w, the solution with null control will approach the null state as time evolves. Then one may apply the techniques exposed in the previous sections to reach the constant steady-state target θ. It is worth noting that, even if we state the main results with constant steady-states as targets, one can use the same methods to control to any element of the constructed paths of steady-states even if they are not constant. In particular, if the system can be controlled to the constant steady-state θ in large time, one can also control it to any target (possibly requiring a larger time) that is path-connected in an admissible way to θ (as long as the path is away of the prescribed bounds, see (4.12) in Theorem 4.2 and Subsection 7.2).

Theorem 8.2. *Assume f is bistable.*
Case 1: $\boldsymbol{F(1) > 0}$.

- *The solution of the system* (8.1) *can be driven asymptotically to* 0 *using the controls* $a_1, a_2 = 0$
  - *for any initial data u_0 iff $L < L^*$ (where L^* is defined in 5.4).*
  - *for any $L > 0$ if u_0 is below some steady-state $w(x)$ satisfying Condition 8.1.*
- *The solution of system* (8.1) *can be driven asymptotically to* 1 *using the controls* $a_1, a_2 = 1$ *for any initial data u_0 and for any L.*
- *There exists $T^*_{u_0,L} > 0$ such that for every $T \geq T^*_{u_0,L}$ the solution of system* (8.1) *can be controlled to θ iff u_0 asymptotically goes to* 0 *when $a \equiv 0$.*
- *Furthermore one has the following estimate for L^*:*

$$\frac{\pi}{\sqrt{\max_{s\in[0,1]} f'(s)}} \leq L^* \leq 2\sqrt{2}\sqrt{\frac{F(1)-F(\theta)}{F(1)^2}}.$$

Case 2: $\boldsymbol{F(1) = 0}$.

- *the solution of the system* (8.1) *can be driven asymptotically to* 0 *(resp.* 1*) using the controls $a_1, a_2 = 0$ (resp. $a_1, a_2 = 1$) for any $L > 0$ and any admissible u_0.*
- *there exists $T^*_{u_0,L} > 0$ such that for every $T \geq T^*_{u_0,L}$ the solution of the system* (8.1) *can be controlled to θ for every u_0 and every $L > 0$ and any admissible u_0.*

Proof. **Case 1:** $\boldsymbol{F(1) > 0}$.

- The parabolic solution with the steady-state fulfilling (8.1) as initial condition with control $a = 0$, asymptotically goes to 0 (due to Remark 8.3 and the fact that the steady states fulfilling condition (8.1) are path-connected with the steady-state θ). Therefore, by comparison, all initial conditions below a steady-state fulfilling condition (8.1) also converge asymptotically to 0 with control $a = 0$.

  When barrier steady states do not exist, any admissible initial condition tends asymptotically to 0 as $t \to +\infty$ with control $a = 0$.

  The estimates on L^* are consequence of Theorems 5.13 and 5.14.
- The result concerning the convergence to the constant steady-state 1 is a consequence of the uniqueness of the steady-state 1 in the admissible set.
- As explained in Section 7, once the dynamics with null control is guaranteed to converge to the steady state 0, the remaining task is to assure that the trajectory can be controlled to an element of the path of steady-states.

  Let u_0 be an initial datum such that the parabolic solution converges to 0 in C^0 with the boundary condition $a = 0$. Then, there exists a time t^* for which the solution will be below θ, i.e.

$$0 \leq u(x,t) < \theta, \quad \forall t \geq t^*, \quad \forall x \in (0,L).$$

  Note that the set $L^\infty((0,L),[0,\theta])$ is an invariant region for the dynamics (8.1) for every boundary condition $a_1, a_2 \in L^\infty((0,+\infty);[0,\theta])$, thanks to the comparison principle ($u = 0$ is a subsolution and $u = \theta$ a supersolution).

  By Theorem 3.1 ([78]), for any constant boundary condition, the state converges in C^0 to a steady-state. If the boundary condition is chosen to be in $[0,\theta]$, the state being within the invariant region $L^\infty((0,L),[0,\theta])$, the solution will converge to a steady state in $L^\infty((0,L),[0,\theta])$.

  Observe that an admissible steady-state fulfilling

$$0 < u < \theta, \qquad \partial_x u(L/2) = 0,$$

is in the invariant region Γ. Moreover, this steady-state is path connected to θ in an admissible way. Therefore, it is enough to set a constant control $a_1 = a_2 \in [0, \theta]$ to assure convergence to a steady-state u_1, which is symmetric with respect $x = L/2$ and, as a consequence, a steady-state belonging to Γ.

Once the solution, after a long enough time horizon, is close enough to the steady-state u_1, by local controllability it can be exactly controlled to u_1. Since u_1 is an element of the admissible path connecting to the target steady-state θ, the proof can be completed applying the staircase method.

Case 2: $F(1) = 0$.

- In this case, there do not exist non trivial admissible steady states taking both boundaries equal to 0 (or 1). This implies that, by setting the control $a_1 = a_2 = 0$ (resp. 1), any initial condition asymptotically goes to 0 (resp. 1).
- As discussed in section 7, in this case, the invariant region in the phase plane is enclosed by the two stationary travelling waves and the argument follows as in the case where $F(1) > 0$.

□

Remark 8.3. As noted in [106], 0 and θ have the same ω-limit when the control is set to 0 ($a = 0$). Moreover, one can only build admissible steady-state paths between steady-states whose ω limit, with $a = 0$ as boundary condition, are not in comparison. The proof of this fact follows by contradiction. Assume that there is an admissible path of steady-states between two admissible steady-states whose ω-limits for the $a = 0$ control are in comparison. Since, by hypothesis, a path exists, there will exist a positive control function that, in finite time, will bring the first steady-state to the second one. However, there will also exist another control function bringing the second steady state to the first one in finite time by means of a positive control. This enters in contradiction with the comparison principle.

Remark 8.4. Moreover, if the initial data is symmetric with respect $x = L/2$, the path can be constructed in a symmetric way and the results of Theorem 8.2 hold with symmetric controls $a_1(t) = a_2(t) = a(t)$.

Remark 8.5. Several remarks are in order

- The fact that the parabolic equation can be controlled along symmetric paths, with equal controls on both extremes $x = 0$ and $x = L$, is equivalent to controlling the dynamics in the half-interval $(0, L/2)$, with Dirichlet control at $x = 0$ and homogeneous Neumann boundary conditions at $x = L/2$.
- The construction of paths of steady-states in several space dimensions was addressed in [106], with the control being active in the whole boundary. The analysis of the multi-dimensional case with controls acting only on a subset of the boundary is an open problem (see Section 11).
- The continuous path of steady-states connecting 0 and 1 exists but it does not fulfill the constraints $0 \leq u \leq 1$. Thus, it is not of use when trying to control the dynamics within the constraints (see Figure 8.2).

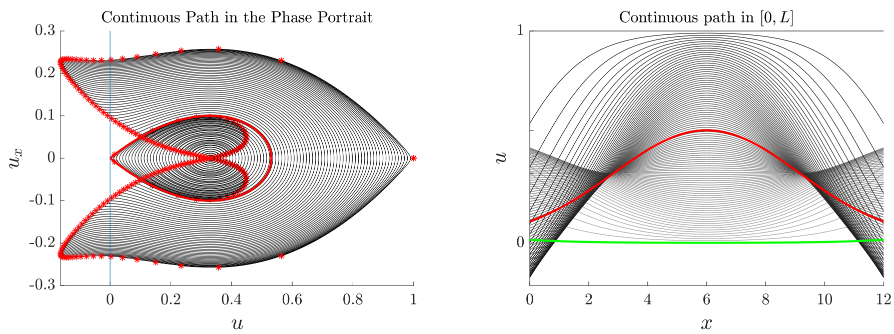

FIGURE 8.2. Non admissible continuous path from 0 to 1

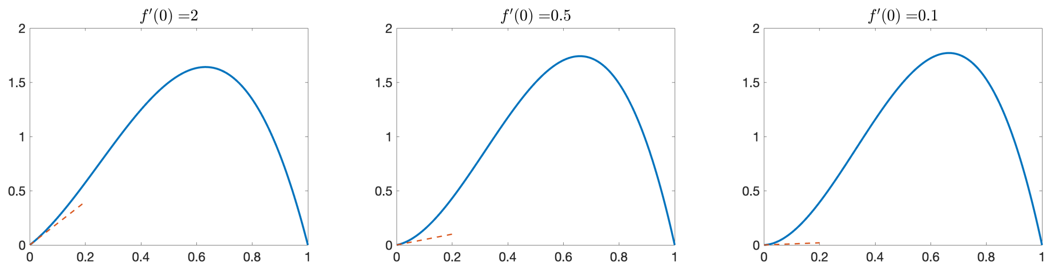

FIGURE 8.3. $f_d(s) = s\left(d + (12-3d)s + (3d-12)s^2\right)$ for different values of d. The integral between $(0,1)$ of the three functions is the same, i.e. $\int_0^1 f_d(s)ds = 1$. The red dashed line indicates the slope at the origin.

8.2. **Monostable Nonlinearities.** In this section we summarize the main results for monostable nonlinearities.

Theorem 8.6. *Assume that f is monostable.*

System (8.1) *can be asymptotically driven to:*

- *The constant steady-state* 1 *for any initial condition and any $L > 0$.*
- *The constant steady-state* 0 *for any initial condition if $L^2 < \lambda^*$ where*

$$\frac{\lambda_1(0,1)}{\max_{s\in[0,1]} f'(s)} \le \lambda^* \le \overline{\lambda} < +\infty,$$

$\lambda_1(0,1)$ *being the first eigenvalue of the Dirichlet Laplacian in* $(0,1)$.

Proof. As discussed in Section 5.4, for every $L > 0$ there exists a unique admissible steady-state with boundary conditions equal to 1, namely the constant steady-state $\equiv 1$. This implies that for every initial condition, setting the boundary controls equal to 1, the solution will asymptotically approach 1 as $t \to +\infty$.

In Section 5.4 upper and lower bounds for the existence of nontrivial solutions have been presented. When L is small enough, there does not exist a nontrivial admissible steady-state with 0 boundary conditions and this implies that for any admissible initial condition the solution will asymptotically go to 0 as $t \to +\infty$.

□

Following the computations in the proof of Theorem 5.14 for the bistable nonlinearity, it can be shown that $\overline{\lambda} = 8/F(1)$ for the monostable nonlinearity in dimension 1.

Remark 8.7. Note that, for $\lambda > \lambda^*$, the constant steady-state 0 may be stable. As an example, consider the nonlinearity

$$f_d(s) = s\left(d + (12-3d)s + (3d-12)s^2\right),$$

satisfying $F(1) = \int_0^1 f_d(s)ds = 1$, independently of d and $f_d'(0) = d$.

The steady-state 0 is linearly stable if:

$$\pi^2 - \lambda f_d'(0) > 0.$$

For every $L^2 > 8$, a non-trivial solution exists. Therefore, for every $L^2 > 8$ we can choose d small enough so that the steady state 0 is linearly stable while a barrier exists.

9. Numerical Simulations

This section is devoted to present numerical implementations of optimal control strategies and illustrate the results presented in the previous sections. We will perform four experiments, each one regarding one of the topics discussed in these lecture notes. All the experiments are developed in an optimization setting. More precisely, we will minimize various functionals J, depending on the controls objetive and we will also consider different classes of admissible controls $\mathcal{A}$. All the experiments will take the abstract form of

$$\min_{a\in\mathcal{A}} J(u_a; u_0) \tag{9.1}$$

where u_a solves:

$$\begin{cases} \partial_t u - \partial_{xx} u = u(1-u)(u-1/3) & (x,t) \in (0,L) \times (0,T), \\ u(0,t) = a_1(t), \quad u(L,t) = a_2(t), & t \in (0,T), \\ u(x,0) = u_0(x). \end{cases}$$

The optimization experiments are implemented with IpOpt ([121]) and are as follows:

(1) **Experiment 1: Control towards** θ. We analyze control strategy for $L_\theta < L < L^*$ considering:

$$J = \|u(T;a) - \theta\|^2, \quad \mathcal{A} = \Big\{a_1, a_2 \in L^\infty([0,T];[0,1])\Big\}.$$

In Figure 9.1 we show how the control takes values below $\theta = 1/3$ during a substantial time interval. This is natural since for $L > L_\theta$ there is a nontrivial steady-state solution above θ that the dynamics is forced to cross. Furthermore, it can be observed that the final state is close to θ as the theory predicts.

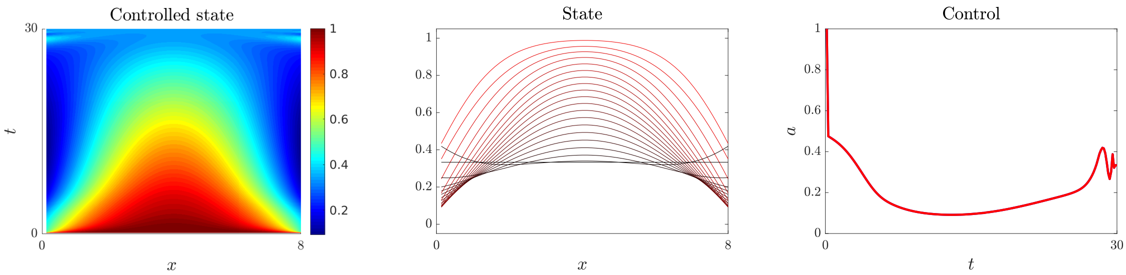

FIGURE 9.1. Experiment 1: Control towards θ. Controlled state for $L = 8$, $T = 30$ and initial datum $u_0(x) = 1$.

(2) **Experiment 2: Control in the presence of a barrier**. The lack of controllability towards 0 for $L > L^*$ with initial datum $u_0 = 1$, due to the emergence of the barrier steady-state, is illustrated in Figure 9.2. In this case, the functional setting considered is the following one:

$$J = \|u(T;a)\|^2, \quad \mathcal{A} = \Big\{a_1, a_2 \in L^\infty([0,T];[0,1])\Big\}.$$

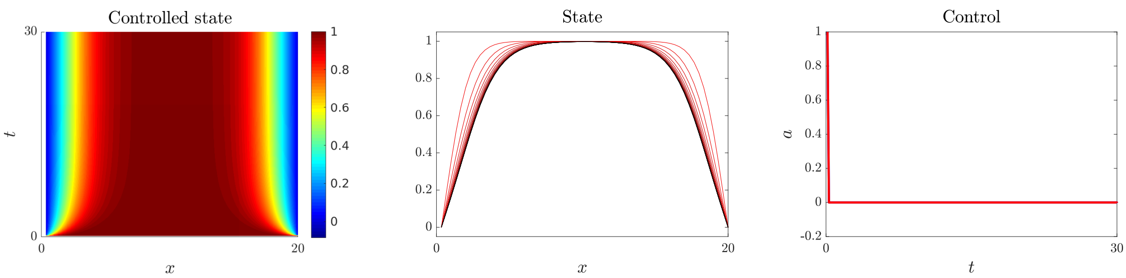

FIGURE 9.2. Experiment 2: Control in the presence of a barrier. Controlled state for $L = 20$, $T = 30$ and initial datum $u_0 \equiv 1$.

However, one can reach θ when starting from $u_0 = 0$. In Figure 9.3, the functional:

$$J = \|u(T;a) - \theta\|^2, \quad \mathcal{A} = \Big\{a_1, a_2 \in L^\infty([0,T];[0,1])\Big\},$$

is minimized.

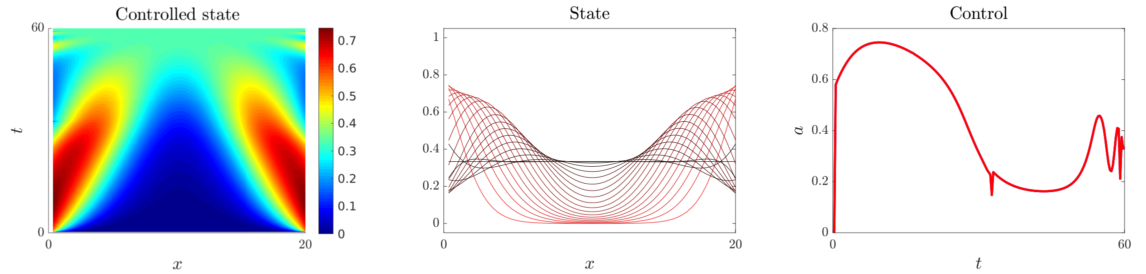

FIGURE 9.3. Experiment 2: Control in the presence of a barrier. Controlled state for $L = 20$, $T = 60$ and initial datum $u_0 \equiv 0$.

(3) **Experiment3. Positivity of the minimal control time.** First, we observe the lack of controllability in a short time horizon (Figure 9.4) for

$$J = \|u(T; a) - \theta\|^2, \quad \mathcal{A} = \Big\{a_1, a_2 \in L^\infty([0, T]; [0, 1])\Big\}.$$

In Figure 9.5 the functional minimized is the time horizon

$$J = T, \quad \mathcal{A} = \Big\{a_1, a_2 \in L^\infty([0, T]; [0, 1]) : \text{ the trajectory fulfills } \theta - \epsilon \leq u_a(T; u_0) \leq \theta + \epsilon\Big\}.$$

Numerically this can be achieved by first generating an initial guess minimizing the L^2 norm of the difference of the final datum with the target on a given time horizon. In a second step, fixing the number of time steps in the discretization scheme to be large enough, the minimization of the time-horizon for control is achieved by minimizing Δt, the discretization time step in the numerical approximation scheme. The control shown in Figure 9.5 shows that the minimal time control is of bang-bang type.

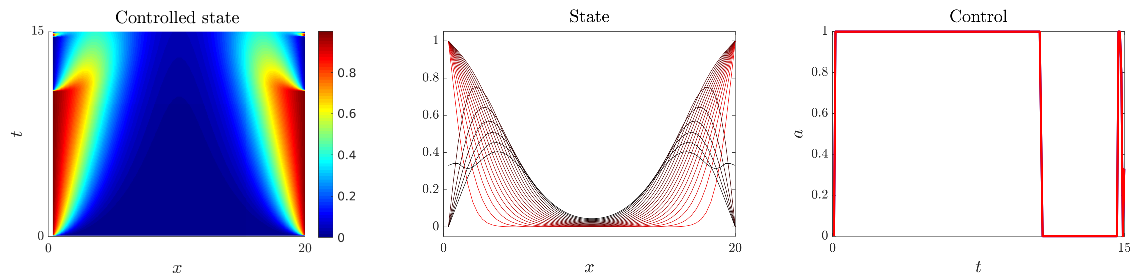

FIGURE 9.4. Experiment 3. Positivity of the minimal control time. Controlled state for $L = 20$, $T = 15$ and initial datum $u_0 \equiv 0$.

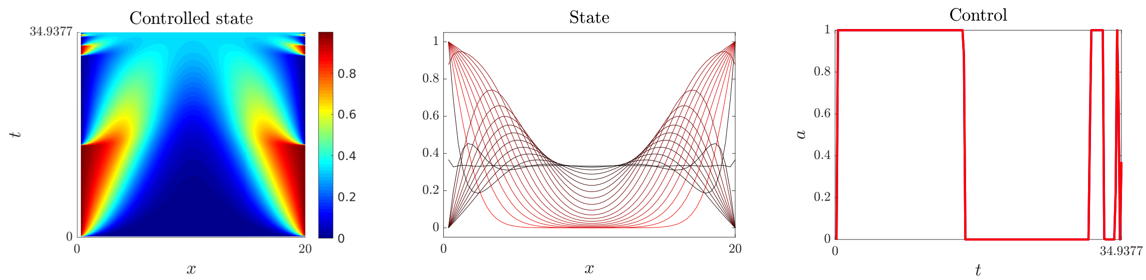

FIGURE 9.5. Experiment 3. Positivity of the minimal control time. Controlled state in minimal time for $L = 20$ and initial datum $u_0 \equiv 1$.

(4) **Experiment 4. Quasistatic control strategy**. In the following simulation we show a control that approximately follows a path of steady-states connecting the steady-state 0 with the steady state θ. For doing so, we set T large, and we minimize

$$J = \int_0^T \|a_t\|^2 dt, \quad \mathcal{A} = \Big\{ a_1, a_2 \in L^\infty([0,T];[0,1]) : \ \theta - \epsilon \leq u_a(T;u_0) \leq \theta + \epsilon \Big\}.$$

In Figure 9.7, we observe snapshots of the parabolic controlled state which, in the phase plane, are close to solutions of the elliptic equation.

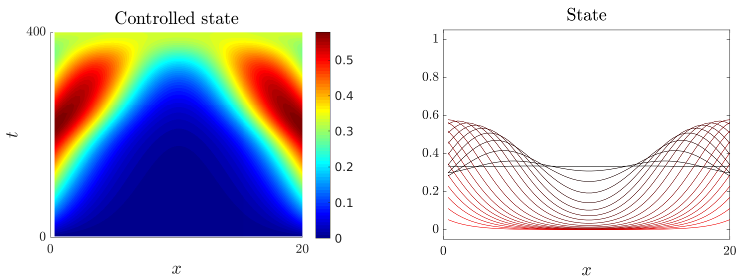

FIGURE 9.6. Experiment 4. Quasistatic control strategy. Controlled state for $L = 20$, $T = 400$ and initial datum $u_0 \equiv 0$.

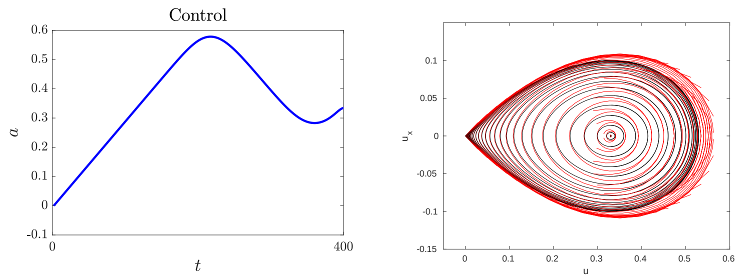

FIGURE 9.7. Experiment 4. Quasistatic control strategy. Left: Optimal constrained control associated with Figure 9.6. Right: Phase-plane plot of the steady-states (in black) and snapshots of the parabolic state (in red).

10. EXTENSIONS AND RELATED PROBLEMS

10.1. Combining Allee control and boundary control.

10.1.1. *Preliminaries.* The goal of this subsection is to present the control strategy in [114] schematically. In [114] the modeling of the control of a population of mosquitoes when sterile mosquitoes are released is presented. This way of acting on the system allows to regulate the nonlinearity and, more precisely, the Allee parameter θ:

$$\begin{cases} \partial_t v - \partial_{xx} v = v(1-v)(v - \theta(t)) & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ 0 \leq \theta(t) \leq 1, \\ 0 \leq v(x,0) \leq 1. \end{cases}$$

The goal in that article was to control the system approximately to a traveling wave solution in large time. For this to be done the initial datum is assumed to have two different asymptotes

$$\lim_{x \to \pm\infty} v(x,0) = l_\pm.$$

The strategy of the proof relies on the following arguments:

- Fix $\theta \in (\min(l_-, l_+), \max(l_-, l_+))$ for a long time interval and let the system converge exponentially to a traveling wave profile.

  Once the traveling wave is approximated a second step is needed to regulate (by translation) the position of the traveling wave profile.
- Using the fact that, for the cubic nonlinearity, the profile of the traveling wave is the same for any θ, in a second step one can modify the value of the control θ to move the center of the traveling wave. Note that if $\theta < 1/2$ (resp. $\theta > 1/2$), the traveling wave moves in a direction so that the value 1 (resp. 0) invades.

10.1.2. *Combined Strategy.* Combining both control strategies, i. e. the Allee threshold and the boundary control, more targets can be achieved. In particular, the dynamics can be driven beyond the barriers that might occur for a specific value of θ, by changing the value of the Allee threshold. Furthermore, one can easily obtain the controllability in large time, to any steady-state inside Γ, for any θ, and to sections of traveling waves. More precisely, let $L > 0$ and consider any target in the following set:

$$v_T \in \bigcup_{\theta \in [0,1/2]} \Gamma_\theta \cup \left\{ v \in L^\infty((0,L);[0,1]) : \ \exists c \in \mathbb{R} \text{ s.t. } v(x) = TW(x+c)\big|_{[0,L]} \right\}$$

where, by the subscript θ in Γ_θ, we indicate the dependence with respect to θ of the invariant region Γ and by TW we denote the traveling wave profile.

Consider the following control problem:

$$\begin{cases} \partial_t v - \partial_{xx} v = v(1-v)(v-\theta(t)) & (x,t) \in (0,L) \times (0,T), \\ v(0,t) = a_1(t) \quad v(L,t) = a_2(t) & t \in (0,T), \\ 0 \le v(x,0) \le 1 \quad v(T) = v_T. \end{cases}$$

One may develop the following strategy:

Take $\theta = 1/2$ and keep $a_1 = a_2 = 0$ for a long time interval. Since the null steady-state is the unique steady-state for $\theta = 1/2$ with null boundary conditions, by Theorem 3.1 ([78]), the solution tends to the constant steady-state 0 as $t \to +\infty$ for any initial datum in $L^\infty((0,L);[0,1])$.

- $v_T \in \bigcup_{\theta \in [0,1/2]} \Gamma_\theta$
  - Once the solution is close to the steady-state 0, one changes the value of the Allee control θ to be a constant in the desired value $\theta \in (0, 1/2)$.
  - Set $a_1 = a_2 = \epsilon$ and apply the strategy described in the previous sections, with the staircase method, allowing to control the system to any steady state inside Γ_θ.
- If $v_T \in \left\{ v \in L^\infty((0,L);[0,1]) : \ \exists c \in \mathbb{R} \text{ s.t. } v(x) = TW(x+c)\big|_{[0,L]} \right\}$

  Leaving the controls at $a_1 = a_2 = \epsilon \in (0,1)$, the solution will converge to a symmetric stationary solution. All the symmetric stationary solutions are inside the invariant region $\Gamma_{1/2}$.

  For $\theta = 1/2$, the traveling waves are stationary and they are seen in the phase portrait connecting $(0,0)$ and $(1,0)$. As mentioned in Subsection 7.3, the traveling waves are at the border of $\Gamma_{1/2}$.

  One can apply local controllability to control to a steady-state inside $\Gamma_{1/2}$ and then trace a path to the target arc of the traveling wave while being inside $\Gamma_{1/2}$.

10.2. **The multi-dimensional case.** The results and techniques explained before can be extended in several dimensions, [106]. Let us consider a bounded domain $\Omega \subset \mathbb{R}^d$ with C^2 boundary and Lebesgue measure $|\Omega| = 1$, $\lambda > 0$, and the following dynamics:

$$\begin{cases} \partial_t u - \Delta u = \lambda f(u) & (x,t) \in \Omega \times (0,T), \\ u(x,t) = a(x,t) & (x,t) \in \partial\Omega \times (0,T), \\ 0 \le u(x,t) \le 1 & (x,t) \in \Omega \times [0,T]. \end{cases} \tag{10.1}$$

Note that λ plays the role of a scaling parameter so to that the assumption $|\Omega| = 1$ does not reduce the generality of our analysis.

In this context, the bounds derived in Section 5 also apply by slightly adapting the proofs.

The proof of Theorem 5.14 is adapted by finding the largest ball inside the domain to find a subsolution.

The proof of Theorem 5.13 applies as well, and the final result will depend on the first eigenvalue that, in turn, depends on the geometry of the domain. Indeed, the first Dirichlet eigenvalue depends on the shape of the domain, not only on its size. This plays a relevant role since there are domains with large measure and large large first eigenvalue. The existence of steady-state barriers is a general feature of these problems that is illustrated in Figure 10.1.

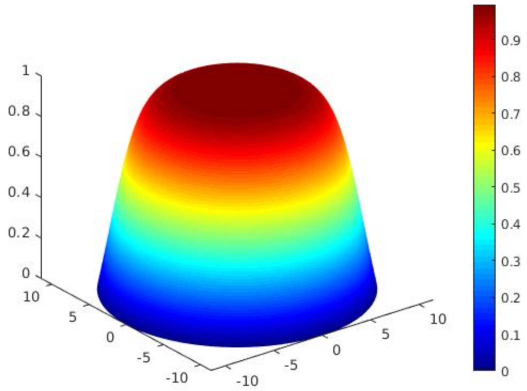

FIGURE 10.1. Numerical simulation of the semilinear heat equation with nonlinearity $f(s) = s(1-s)(1-\theta)$ with boundary value $a(x,t) = 0$ leading to a barrier steady-state.

Note that, by extending the one-dimensional traveling wave to be constant in all other $d-1$ remaining dimensions, one obtains a traveling wave profile for the multi-dimensional case. Thus, when $F(1) > 0$, the proof of the convergence to the constant steady-state 1 (with boundary controls $\equiv 1$) follows by comparison with the traveling wave solutions, as in the one-dimensional case, Proposition 5.25.

The construction of the continuous path of steady-states can be performed extending the domain Ω to a larger ball containing it, see Figure 10.2.

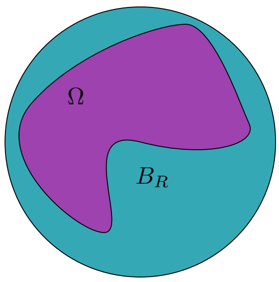

FIGURE 10.2. Ball containing the original domain Ω.

One can then consider the radially symmetric solutions of

$$-\Delta u = \lambda f(u) \qquad x \in B_r \subset \mathbb{R}^d, \tag{10.2}$$

as an ODE problem:

$$\begin{cases} u_{rr}(r) + \dfrac{d-1}{r} u_r(r) = -f(u(r)), & r \in (0, R_{B,\lambda}), \\ u(0) = a, \; u_r(0) = 0, \end{cases} \tag{10.3}$$

where $R_{B,\lambda}$ is the radius of the ball, after rescaling the parameter λ to $\lambda = 1$.

The key points of the qualitative behavior analysis of these radially symmetric solutions are the following:

(1) Radial solutions of the problem (10.3) dissipate: Assume that $F(1) \geq 0$ and consider the energy

$$E(u,v) = \frac{1}{2}v^2 + F(u)$$

where $F(u) = \int_0^u f(s)ds$. Define the region:

$$M := \{(u,v) \in \mathbb{R}^2 \quad \text{such that} \quad E(u,v) \leq 0\}.$$

Note that the region defined by

$$\Gamma := \{(u,v) \in [0,\theta_1] \times \mathbb{R} \quad \text{such that} \quad |v| \le \sqrt{-2F(u)}\}$$

satisfies $\Gamma \subset M$. In fact, this region is the same than in the one dimensional case.

Now one considers an initial datum of the form $(u_0, 0) \in \Gamma$, then the solution of (10.3) with initial datum $(u_0, 0)$ satisfies:

$$\frac{d}{dr}E(u,v) = vv_r + f(u)v = -\frac{d-1}{r}v^2 < 0.$$

As a consequence $(u,v) \in \Gamma$ for all $r > 0$ and the path is admissible and globally defined.

In Figure 10.3 we represent the construction of the path.

Then, the restriction of the path defined in B_r to the original domain Ω, leads the desired path.

(2) Another important feature of the analysis of the multi-dimensional problem is the control to the path of steady-states we just built. For this, one realizes that the minimum solution to

$$\begin{cases} -\Delta u = \lambda f(u) & x \in \Omega, \\ 0 < u < 1 & x \in \Omega, \\ u = a(x) & x \in \partial\Omega, \end{cases}$$

with a being the corresponding value related to the restriction of an element of the path in the ball into Ω, is radial with respect to some ball, and therefore it belongs to a path of steady-states [106].

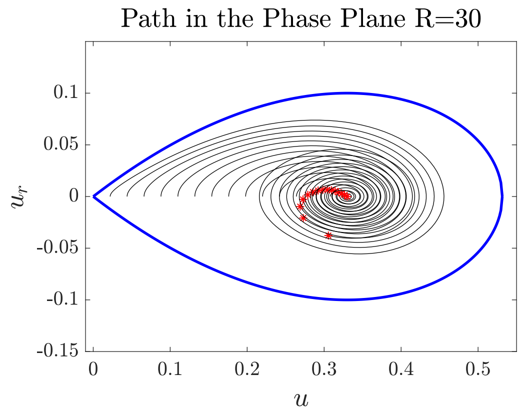

FIGURE 10.3. $\theta = 1/3$, $R = 30$ and $d = 2$. In blue the phase space description of the invariant region. In black the radial trajectories forming the continuous path of steady-states. The red stars indicate the values taken over the boundary.

10.3. **Spatially Heterogeneous case.** In this section, we discuss the more general model analyzed in [84]:

$$\begin{cases} \partial_t u - \Delta u + \left\langle \frac{2\nabla N}{N}, \nabla u \right\rangle = f(u) & x \in \Omega \times (0,T), \\ u(x,t) = a(x,t) & x \in \partial\Omega \times (0,T), \\ 0 \le u(x,t) \le 1, \end{cases} \tag{10.4}$$

where $N : \mathbb{R}^d \to (0, +\infty)$ is a C^∞ function.

10.3.1. *Modeling.* In Section 2 the whole population N was assumed to be driven by an homogeneous heat equation. But N can also be the solution of a non-homogeneous semilinear heat equation of the form:

$$\begin{cases} \partial_t N - \Delta N = N(\kappa(x) - N) & (x,t) \in \Omega \times (0,T), \\ \frac{\partial N}{\partial \nu} = 0 & (x,t) \in \partial\Omega \times (0,T), \\ N(x,0) = N_0(x) \ge 0, \end{cases} \tag{10.5}$$

where κ is a C^∞ function. It is known that, for $\kappa > 0$, there is only one positive steady-state of (10.5), [74]. Since the set of positive steady states is a singleton and the steady state 0 is unstable, due to the gradient structure of the semilinear heat equation under consideration, the solution of (10.5) converges to the solution of (see [16])

$$\begin{cases} -\Delta N = N(\kappa(x) - N) & x \in \Omega, \\ \dfrac{\partial N}{\partial \nu} = 0 & x \in \partial\Omega. \end{cases} \tag{10.6}$$

Taking N from (10.6) and employing the same approximation done in Section 2, one gets the heterogeneous bistable equation (10.4).

The effect of the mother population N on the behavior of the system depends very much on the shape of N.

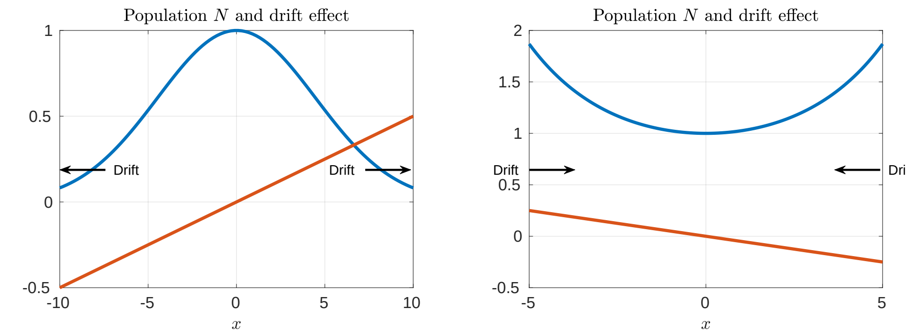

FIGURE 10.4. In blue the curve $N(x)$, in orange the quotient $-N_x(x)/N(x)$ responsible of the drift effect. Left $N(x) = e^{-\frac{x^2}{\sigma}}$, right $N(x) = e^{\frac{x^2}{\sigma}}$.

Figure 10.4 shows the effect of the drift produced by N. In the right hand side of Figure 10.4, the drift pushes from the boundary towards the interior. For this reason, one intuitively expects that the controllability will be easier to be achieved in that setting, since the effect of the boundary control will be enhanced. On the other hand, in the left hand side of Figure 10.4, one can see that the effect of the boundary control is diminished by the drift. This will lead to the existence of new nontrivial steady-state solutions that will act as barriers, as it is shown in the subsequent subsections.

These new barriers block the controllability from 0 to θ. In Figure 10.5 we illustrate the minimal controllability time, that may blow up for some specific profiles N.

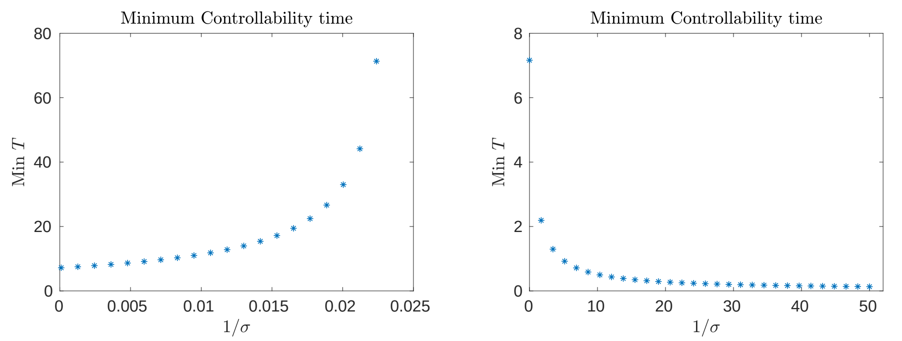

FIGURE 10.5. Minimal controllability time from 0 to $\theta = 1/3$ as a function $1/\sigma$. Left: minimal controllability time for the Gaussian $N(x) = e^{-\frac{x^2}{\sigma}}$. Right: $N(x) = e^{\frac{x^2}{\sigma}}$.

10.3.2. *Small drifts.* In [84] the controllability for small drifts is analyzed by means of a perturbation method of the homogeneous path corresponding to $N \equiv 0$.

Consider a finite number of states on a path of steady-states for the homogeneous equation (with $N \equiv 0$). We then consider the model with a small drift term:

$$\begin{cases} \partial_t u - \Delta u + \epsilon \langle \nabla p(x), \nabla u \rangle = f(u) & x \in \Omega \times (0,T), \\ u(x,t) = a(x,t) & x \in \partial\Omega \times (0,T), \\ 0 \le u(x,t) \le 1 & (x,t) \in \Omega \times [0,T]. \end{cases}$$

The implicit function theorem can be applied to all the elements selected from the homogeneous path to obtain a sequence of new steady-states for (10.3.2), in a way that they remain close enough to the original steady-states, and so that the staircase method can be applied (see Figure 10.6), see [84, Theorem 2] for details. Note that, by this argument, we do not know if a continuous path of steady-states for the perturbed problem exists. But the only requirement to apply the staircase method is to have a number successive steady-states that are close enough each other, see Figure 10.6 for an intuitive representation.

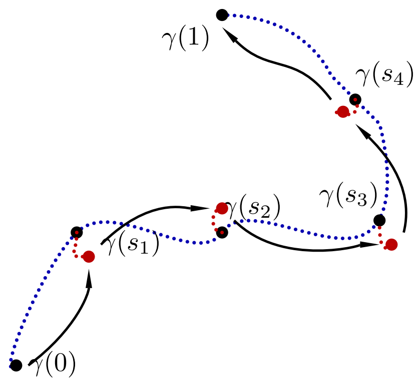

FIGURE 10.6. The blue dotted line represents the continuous path of steady-states for the homogeneous equation. In red, the perturbed steady-states, linked to the unperturbed steady-states (black) that belong to a continuous path for the homogeneous equation.

This perturbative argument is a variant/improvement of the staircase method [93].

The perturbation method only applies for small ϵ. In fact, Figure 10.5 shows that we cannot expect this method to work for large ϵ. In the subsection below we shall see the apparence of such obstructions.

10.3.3. *New barriers for large drifts.* In the case of radially symmetric solutions of the multi-D problem, the energy

$$E(u,v) = \frac{1}{2}v^2 + F(u)$$

satisfies

$$\frac{d}{dr}E = -\frac{N'}{N}v^2 - \frac{d-1}{r}v^2$$

If

$$\frac{d}{dr}N(r) \ge -\frac{d-1}{r}N(r),$$

then the energy decreases and the set:

$$E(u,v) \le 0$$

is positively invariant (see [84, Theorem 3]).

One can then conclude that if this differential inequality is fulfilled, there is an invariant region in the phase portrait to which the stationary states 0 and θ belong to.

However, in the opposite case, new barriers can appear even in the one-dimensional case. We sketch the proof of the existence of the upper barrier as in Figure 10.7.

Let us consider the a Gaussian drift $N(x) = e^{-x^2/\sigma}$. From the modeling perspective, this amounts to say that there is a big concentration of individuals around $x = 0$. Consider, in addition, the following boundary value problem:

$$\begin{cases} -\partial_{xx}u + \frac{4xu_x}{\sigma} = f(u) & x \in (-L,L), \\ u(-L) = u(L) = a \in \{0,1\}. \end{cases} \tag{10.7}$$

Theorem 10.1 (Theorem 4, [84])**.** *Let $F(1) > 0$. For every $\sigma > 0$ with $N(x) = e^{-x^2/\sigma}$, there exists $L_{\sigma,1} > 0$ such that if $L > L_{\sigma,1}$ a solution of* (10.7) *satisfying the state constraints $0 \leq u \leq 1$ with $a = 1$ exists. Moreover, there exists $L_\sigma > 0$ such that for every $L > L_\sigma$ there exist a nontrivial solution with $a = 0$.*

To prove the existence of the barrier function to reach 0, one can proceed with the same variational arguments above. However, for proving the existence of the upper barrier, the same methods do not apply, since the stationary solution 1 is a global minimum of the energy functional.

For this reason, in [84], Theorem 10.1 is proved using phase plane techniques (the shooting method). It consists on finding an initial condition for the system:

$$\begin{cases} \frac{d}{dx}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -f(u) \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{4}{\sigma}xv \end{pmatrix}, \\ \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \end{cases} \tag{10.8}$$

for which at a certain L, the trajectory of (10.8) reaches 1.

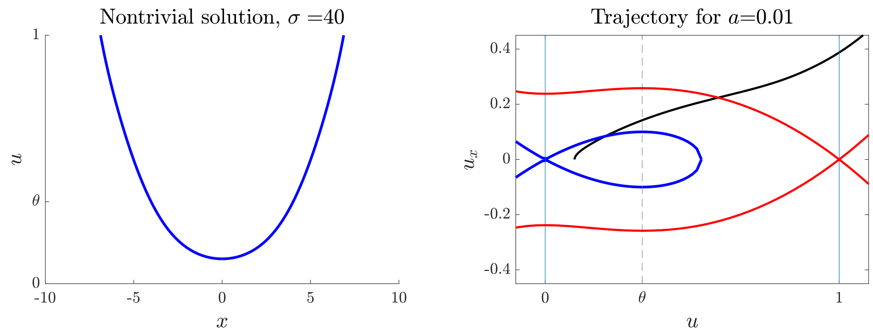

FIGURE 10.7. Left: Upper barrier, solution of (10.7) for $\sigma = 40$. Right: Sketch of the phase-plane analysis for the trajectory leading to a solution of (10.7)

Therefore, for large domains, for the Gaussian profile of N, one cannot guarantee the controllability to any of the constant steady-states.

11. Perspectives

To better describe the perspectives emerging out of the work described in these lecture notes, we first emphasize the limitations of the methods we have presented. One of the most illustrative limitations arises when non-autonomous dynamics are considered.

1) Nonautonomous dynamics. Let us consider the system

$$\begin{cases} \partial_t u - \mu(t)\Delta u = f(u) & (x,t) \in \Omega \times (0,T) \\ u(x,t) = a(x,t) & (x,t) \in \partial\Omega \times (0,T) \\ 0 \leq u(x,0) \leq 1 & x \in \Omega \end{cases} \tag{11.1}$$

where $a \in L^\infty((0,T) \times \partial\Omega; [0,1])$, $\mu \in C^1((0,T); \mathbb{R}^+)$ and $f : \mathbb{R} \to \mathbb{R}$ is bistable. Note that, since $\mu(t)$ is time dependent, the only possible steady-states (states w such that $\partial_t w = 0$ for all t) are the constant steady-states

$$w \equiv 0, \quad w \equiv \theta, \quad w \equiv 1$$

since they cancel both the bistable nonlinearity and the parabolic operator.

Therefore, a path of steady states connecting any pair of steady-states cannot exist. Moreover, observe that when μ is a decreasing function, i.e. $\mu' < 0$, then the steady-state $w \equiv \theta$ becomes, loosely speaking, *more unstable* as time advances.

Linearizing the system around $w \equiv \theta$ and the control $a \equiv \theta$, we obtain:

$$\begin{cases} \partial_t v - \mu(t)\Delta v = f'(\theta)v & (x,t) \in \Omega \times (0,T) \\ v(x,t) = 0 & (x,t) \in \partial\Omega \times (0,T) \\ v(x,t=0) = v_0. \end{cases} \tag{11.2}$$

Solving (11.2) one obtains

$$v(x,t) = \sum_{n=1}^{\infty} c_n(0) \exp\left\{ f'(\theta)t - \lambda_n \int_0^t \mu(s)ds \right\} e_n(x)$$

$c_n(0) = \langle v_0, e_n \rangle_{L^2(\Omega)}$, where $e_n \in H_0^1(\Omega)$ stands for n-th eigenfunction of the Dirichlet Laplacian and λ_n its eigenvalue $-\Delta e_n = \lambda_n e_n$ in Ω.

Note that if, for instance, $\int_0^\infty \mu(s)ds$ is finite, then since $f'(\theta) > 0$, the solution will grow exponentially after a certain critical time t^*. Conversely, if $\frac{1}{t}\int_0^t \mu(s)ds \to +\infty$ as $t \to \infty$, then every eigenmode will become stable in large times.

There are two main questions to address:

- **Can the time dependence of μ create an obstruction to the controllability?**
 Let us consider $F(1) = \int_0^1 f(s)ds \geq 0$. Is there an initial datum $u_0 \in L^\infty(\Omega; [0,1])$ such that the solution u of the problem

$$\begin{cases} \partial_t u - \mu(t)\Delta u = f(u) & (x,t) \in \Omega \times \mathbb{R}^+ \\ u = 1 & (x,t) \in \partial\Omega \times \mathbb{R}^+ \\ u(\cdot, t=0) = u_0 \end{cases} \tag{11.3}$$

 never approaches $w \equiv 1$ as $t \to +\infty$? Equivalently, for every $t \in \mathbb{R}^+$, there exists an open set $\omega \subset \Omega$ such that the solution of (11.3) satisfies:

$$u(x,t) < \theta \text{ if } x \in \omega?$$

 Note that, since we are working with a bistable nonlinearity, if the state satisfies $\theta < u < 1$, then, with control $a = 1$, it will asymptotically go to 1 as time goes to infinity. If the property above holds, then, by the comparison principle, we would have a fundamental obstruction to the controllability to $w \equiv \theta$ of non-autonomous nature, since, for the autonomous case $\mu(t) = \mu$, we can always approach the steady state $w \equiv 1$ asymptotically as $t \to +\infty$ by just choosing the boundary control to be equal to 1.
 One can gain further intuition about the fact that this situation is likely to occur in the following example. Consider μ such that there exists t^* for which $\mu(t^*) = 0$, then the equation becomes uncontrollable due to the lack of diffusion.
- **Universal control criteria?** We lack of systematic methods to guarantee controllability under state constraints. Existing methods are based either on dissipation plus local controllability or on the staircase method that relies on paths of steady-states.
 But, in the simple dynamics presented above, there cannot exist any steady-state paths and, furthermore, there is no dissipativity towards $w \equiv \theta$. Thus, more general and precise criteria for controllability are needed.

In Figure 11.1 we show two simulations corresponding to different decay rates of $\mu(\cdot)$. The simulation suggests that if $\mu' << -1$ the controllability might be lost, while it might still be achieved when $|\mu'|$ is small enough.

The dynamics of (11.1) serves as a simple example to illustrate that new phenomena can arise for non-autonomous dynamics. These type of dynamics are of paramount importance in mathematical biology. One of the simplest examples are reaction-diffusion equations in growing domains [21]

$$\begin{cases} \partial_t u - \mu \partial_{xx} u = f(u) & (x,t) \in (0, l(t)) \times (0,T) \\ u(x,t) = a(x,t) & (x,t) \in \{0, l(t)\} \times (0,T) \\ 0 \leq u(x,0) \leq 1 \end{cases} \tag{11.4}$$

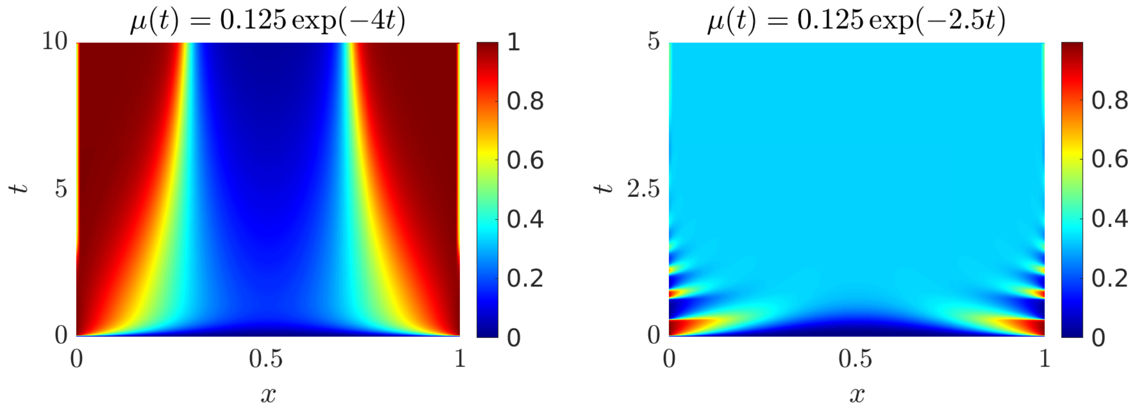

FIGURE 11.1. Space-time representation of the optimal control to $w \equiv \theta$ with initial data $w \equiv 0$. The spatial domain is the unit interval. The nonlinearity is $f(s) = s(s-\theta)(1-s)$ with $\theta = 0.33$ (therefore $\int_0^1 f(s) > 0$) and the controls have been limited to take values in $[0, 1]$. The optimal control problem consists on minimizing the L^2 distance of the final state to $w \equiv \theta$. At the left, with diffusivity $\mu(t) = 0.125\exp(-4t)$, we observe a lack of controllability from the initial state $w \equiv 0$ to the target $w \equiv \theta$. At the right, with diffusivity $\mu(t) = 0.125\exp(-2.5t)$, we observe controllability to the steaty-state $w \equiv \theta$.

where $a \in L^\infty((0,T) \times \partial\Omega; [0,1])$ and $l(t) \in C^1((0,T); \mathbb{R}^+)$. By scaling this model can be rewritten in the form:

$$\begin{cases} \partial_t u - \mu\Delta u = f(u,t) & (x,t) \in \Omega \times (0,T) \\ u(x,t) = a(x,t) & (x,t) \in \partial\Omega \times (0,T) \\ 0 \le u(x,0) \le 1 \end{cases} \tag{11.5}$$

where $a \in L^\infty((0,T) \times \partial\Omega; [0,1])$.

The treatment of the controllability of free boundary problems arising in many physical and biological phenomena is also worth mentioning. In these problems, the domain itself is part of the unknowns of the system and the evolution of the boundary is coupled with the state, as, for instance, in

$$\begin{cases} \partial_t u - \mu\partial_{xx} u = f(u) & (x,t) \in (0,l(t)) \times (0,T) \\ u(0,t) = a(t) & t \in (0,T) \\ l'(t) = g(l(t), u(1,t), \partial_x u(1,t)) & t \in (0,T) \\ 0 \le u(x,0) \le 1. \end{cases} \tag{11.6}$$

In such setting, even the controllability without state constraints is a challenging question [44].

In these notes we have mainly considered steady-states as targets. One could also consider, for instance, periodic (in time) trajectories as targets as well. The staircase method seems to be insufficient for tackling this problem. The need of developing new methods for handling more general target trajectories is relevant both for autonomous and non-autonomous dynamics.

Here we mainly focused on scalar equations. These problems are even more challenging in the context of reaction-diffusion systems with state constraints.

2) The construction of paths of steady-states. We lack of systematic methods to build paths of steady-states and this requires, in each case, a fine understanding of the properties of the system under consideration and a combination of ad-hoc arguments.

Building those paths for general equations of the form

$$\begin{cases} \partial_t u - \operatorname{div}(A(x)\nabla u) + \langle b(x), \nabla u\rangle = f(u,x) & (x,t) \in \Omega \times (0,T), \\ u = a(x,t) & (x,t) \in \partial\Omega \times (0,T), \\ 0 \le u(x,0) \le 1 & x \in \Omega, \end{cases}$$

is a highly non trivial task.

These problems are relevant since heterogeneities are essential for modeling purposes, since, most often, environments are diverse and/or space-dependent. For instance, the growth of the population can be larger in some locations due to a higher capacity, or the diffusion might be smaller because of the topopraphy of the territory where population diffuses. In [84], a particular type of heterogeneity is considered.

In the multidimensional setting the problem of controlling the system with controls acting only on part of the boundary is also worth considering. In [84, 106], the control is assumed to act on the whole boundary. But one could consider more general problems of the following form, with controls localized on a subset of the boundary:

$$\begin{cases} \partial_t u - \Delta u = f(u) & (x,t) \in \Omega \times (0,T), \\ u(x,t) = a(x,t) & (x,t) \in \eta \times (0,T), \\ \dfrac{\partial}{\partial \nu} u(x,t) = 0 & (x,t) \in \partial\Omega \backslash \eta \times (0,T), \\ 0 \leq u(x,t) \leq 1 & (x,t) \in \Omega \times [0,T], \end{cases}$$

with $\eta \subset \partial\Omega$.

In this setting it is unclear what the structure of the set of admissible state-states is and how the paths of steady-states can be constructed. The following general question arises:

Given two admissible steady-states, can we guarantee the existence (or not) of a path of admissible steady-states linking the two steady-states, without explicitly constructing a path?

A partial negative answer is given in [106], where, using the comparison principle, the authors establish a necessary condition for the existence of an admissible path between two steady-states.

Understanding the nature of the set of steady-states and its connectivity is a major problem, of independent mathematical interest, with many potential applications, aside from control.

3) More general diffusion terms and systems. In these lecture notes, diffusion has been modeled by the Laplace operator. However, there are several other relevant linear and nonlinear diffusion operators whose analysis is even more challenging. This is the case, for instance, when considering the porous medium equation [19, 43, 87, 119]:

$$\begin{cases} \partial_t u - \partial_{xx}(u^m) = f(u) & (x,t) \in (0,L) \times (0,T), \\ u(0,t) = a_1(t), \quad u(L,t) = a_2(t) & t \in (0,T), \\ 0 \leq u(x,t) \leq 1 & (x,t) \in (0,L) \times [0,T]. \end{cases}$$

or other nonlinear problems [3, 4].

The analysis of fractional diffusion, [2, 12, 47], and non-local reaction terms, [52], would require also of important further developments.

Reaction-diffusion systems deserve a special mention. For instance, in the context of evolutionary game theory, [53–55], one deals with more than two interaction strategies. Earlier in the paper we motivated the problem with a two strategy game that was reduced to a single evolution equation. However, if we deal with more strategies, the reduction to a scalar equation is not possible and one is led to consider systems of the form:

$$\begin{cases} \partial_t u_1 - \mu_1 \Delta u_1 = f_1(u_1, u_2, u_3) & (x,t) \in \Omega \times (0,T), \\ \partial_t u_2 - \mu_2 \Delta u_2 = f_2(u_1, u_2, u_3) & (x,t) \in \Omega \times (0,T), \\ \partial_t u_3 - \mu_3 \Delta u_3 = f_3(u_1, u_2, u_3) & (x,t) \in \Omega \times (0,T), \\ u_1 = a(x,t) \in [0,1] & (x,t) \in \partial\Omega \times (0,T), \\ \dfrac{\partial}{\partial \nu} u_j = 0 & (x,t) \in \partial\Omega \times (0,T) \quad j = 2,3, \\ 0 \leq u_i(x,0) \leq 1 & i = 1,2,3. \end{cases}$$

The constrained controllability of linear parabolic systems has been implemented in [69]. Note, however, that a phase plane analysis will be more intricate as the complexity of the nonlinear ODE system increases significantly with the dimension.

In practical situations it may make sense to consider a continuum of strategies. For instance, in linguistics, most of the individuals are not perfectly bilingual, and their mastery of the minority language can range within a continuum spectrum. Interactions among individuals can increase or decrease in this trait. We refer to [7] for the modelling of these scenarios.

4) Optimal constrained controls

We have seen that the length of the domain is a crucial parameter to determine the controllability of the equation. This fact can also be linked to the problem of *optimal placement of controls*. For large domains, boundary control is not sufficiently effective. However, the situation might be different if we consider interior control.

For instance, in the one-dimensional case, if we set homogeneous Neumann boundary conditions and a pointwise control in the middle of the interval, by symmetry, we also observe the existence of fundamental obstructions as the domain grows. But, if we allow ourselves to choose the region in which the control is to be placed, one can clearly split the control region into small pieces distributed over the domain so that barrier functions cannot exist.

This fact can be easily understood for one-dimensional homogeneous reaction-diffusion equations. However, the optimal placement of controllers in the multi-dimensional context is a challenging topic. For related literature on the placement of sensors and actuators we refer to [99–101].

The existence of a minimal controllability time and the nature of minimal-time controls is another interesting topic. Simulations suggest that a bang-bang control might be possible for the minimal time control. One possible way to address these questions would be to make use of the Pontryagin maximum principle (see [73] for similar questions).

It is also important to underline that the staircase method gives a way to control the system, in which the trajectory remains inside a tubular neighborhood of the path of steady-states. Knowing that, generally speaking, there is not a unique way to reach a specific configuration, the issue of analyzing the efficiency of the different control methodologies arises. This problem can be addressed from the perspective of different optimality criteria: the L^2-norms of control/state to go from 0 to θ, for instance, the minimal time, etc.

5) Other models and hyperbolic problems. The comparison principle played a strong role in the analysis we conducted for parabolic models. However, the controllability issues are also relevant for hyperbolic models, for instance, for the semilinear wave equation:

$$\begin{cases} \partial_{tt}u - \partial_{xx}u = f(u) & (x,t) \in (0,L) \times (0,T), \\ u(x,t) = a(x,t) & (x,t) \in \{0,L\} \times (0,T), \\ 0 \leq u(x,t) \leq 1 & (x,t) \in (0,L) \times [0,T). \end{cases}$$

The equation above has the same steady-states than the semilinear heat equation discussed in this text. This means that the paths of steady-states and nontrivial solutions are the same. However, for the wave equation, we do not have a maximum principle, and this means that barriers might be avoidable, see [94].

On the other hand, the damped semilinear wave equation (telegraph equation) might present, under certain conditions, a maximum principle (see [42]). The same problems arise for other relevant systems as well, for instance, in thermoelasticity, [125].

Generally speaking, control theory is intimately related to modeling issues. The natural constraints of the process and model under consideration will typically give rise to new mathematical challenges from a control perspective.

Diffusion models are appropriate for simple living species or chemicals. When dealing with intelligent animals, the understanding of their motion in the environment is essential. In this situation finite-dimensional models can also be appropriate, and the role of the network in which individuals move plays a crucial role in the dynamics [90, 113].

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