Optimal design of sensors via geometric criteria

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Abstract

We consider a convex set Ω and look for the optimal convex sensor $\omega \subset \Omega$ of a given measure that minimizes the maximal distance to the points of Ω. This problem can be written as follows

$$\inf\{d^H(\omega, \Omega) \mid |\omega| = c \text{ and } \omega \subset \Omega\},$$

where $c \in (0, |\Omega|)$, d^H being the Hausdorff distance.

We show that the parametrization via the support functions allows us to formulate the geometric optimal shape design problem as an analytic one. By proving a judicious equivalence result, the shape optimization problem is approximated by a simpler minimization of a quadratic function under linear constraints. We then present some numerical results and qualitative properties of the optimal sensors and exhibit an unexpected symmetry breaking phenomenon.

Contents

1 Introduction

The optimal shape and placement of sensors frequently arises in industrial applications such as urban planning and temperature and pressure control in gas networks. Roughly, a sensor is optimally designed and placed if it assures the maximum observation of the phenomenon under consideration. Naturally, it is often designed in a goal oriented manner, constrained by a suitable PDE, accounting for the physics of the process. For more examples and details, we refer to the following non exhaustive list of works [19, 18, 17, 10]. Recently, with the emergence of data driven methods, several authors considered approaches based on Machine Learning in order to accelerate the computational methods, we refer for example to [20, 22, 24, 26].

Here, we address the problem in a purely geometric setting, without involving the specific PDE model. We consider a simple and natural geometric criterion of performance, based on distance functions. But, as we shall see, tackling it will require to employ geometric analysis methods.

More precisely, we address the issue of designing an optimal sensor inside a given set in such a way to minimize the maximal distance from the sensor to all the points of the largest domain. This type of questions naturally arises in optimal resources distribution problems as one wants to minimize the maximal distance between the resources and the species present in the considered environment. Also in urban planning, it makes sense to look for the optimal design and placement of some facility (for example a park or an artificial lake) inside a city while taking into account a certain equity criterion that consists in minimizing the maximal distance from any point in the city to the facility. These problems can then be formulated in a shape optimization framework. Indeed, given a set $\Omega \subset \mathbb{R}^2$, and a mass fraction $c \in (0, |\Omega|)$, the problem can be mathematically formulated as follows:

$$\inf\{\sup_{x\in\Omega} d(x,\omega) \mid |\omega|= c \text{ and } \omega \subset \Omega\},$$

where $d(x,\omega) := \inf_{y\in\omega}\|x-y\|$ is the minimal distance from x to ω. In fact, the problem can be written in terms of the classical Hausdorff distance d^H (see Section 2.2) as when $\omega \subset \Omega$, one has

$$\sup_{x\in\Omega} d(x,\omega) = d^H(\omega,\Omega).$$

We are then interested in considering the following problem

$$\inf\{d^H(\omega,\Omega) \mid |\omega|= c \text{ and } \omega \subset \Omega\}, \tag{1}$$

where $c \in (0, |\Omega|)$.

By using a homogenization strategy, which consists in uniformly distributing the mass of the sensor over Ω (see Figure 1), we see that problem (1) does not admit a solution as the infimum vanishes and is asymptotically attained by a sequence of disconnected sets with an increasing number of connected components. It is then necessary to impose additional constraints on ω in order to obtain the existence of optimal solutions. In the present paper, we focus on the convexity constraint and assume that both the set Ω and the sensor ω are planar convex bodies. Then, given a convex bounded domain $\Omega \in \mathbb{R}^2$, we are interested in the numerical and theoretical study of the following problem:

$$\inf\{d^H(\omega,\Omega) \mid \omega \text{ is convex, } |\omega|= c \text{ and } \omega \subset \Omega\}, \tag{2}$$

where $c \in (0, |\Omega|)$.

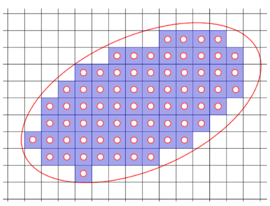


Figure 1: The homogenization strategy.

A first important result of the present paper is the following:

Theorem 1 *The function* $f : c \in [0, |\Omega|] \longmapsto \inf\{d^H(\omega, \Omega) \mid \omega$ *is convex,* $|\omega| = c$ *and* $\omega \subset \Omega\}$ *is continuous and strictly decreasing. Moreover, for every* $c \in [0, |\Omega|]$*, problem* (2) *admits solutions and is equivalent to the following shape optimization problems:*

(I) $\min\{d^H(\omega, \Omega) \mid \omega$ *is convex,* $|\omega| \leq c$ *and* $\omega \subset \Omega\}$.

(II) $\min\{|\omega| \mid \omega$ *is convex,* $d^H(\omega, \Omega) = f(c)$ *and* $\omega \subset \Omega\}$.

(III) $\min\{|\omega| \mid \omega$ *is convex,* $d^H(\omega, \Omega) \leq f(c)$ *and* $\omega \subset \Omega\}$,

in the sense that any solution of one of the problems also solves the other ones.

In addition to its importance from a theoretical point of view (as we shall see in Section 4), the equivalence result above allows to drastically simplify the numerical resolution of problem (2): indeed, as it is explained in Section 5.1 the equivalent problem (III) can be reformulated via the support functions h and h_Ω of the sets ω and Ω in the following analytical form:

$$\begin{cases} \inf\limits_{h \in H^1_{\text{per}}(0,2\pi)} \frac{1}{2}\int_0^{2\pi} (h'^2 - h^2)d\theta, \\ h'' + h \geq 0 \quad \text{(in the sense of distributions)}, \\ h_\Omega - f(c) \leq h \leq h_\Omega. \end{cases}$$

where $H^1_{\text{per}}(0, 2\pi)$ is the set of H^1 functions that are 2π-periodic and $c \in [0, |\Omega|]$. This analytical problem is then approximated by a finite dimensional problem involving the truncated Fourier series of the support functions h as in [2, 3], which yields to a simple minimization problem of a quadratic function under linear constraints. For more details on the support function parametrization, we refer to Section 2.1 and for the complete description of the numerical scheme used in the paper, we refer to Section 5.

One could expect that solutions of (2) will inherit the symmetries of the set Ω. We show that this is not always the case and highlight a symmetry breaking phenomenon appearing when Ω is a square, see Figure 2. Our result can be stated as follows:

Theorem 2 *Let* $\Omega = [0, 1] \times [0, 1]$ *be the unit square. There exists a threshold* $c_0 \in (0, 1)$ *such that:*

- *If* $c \in [c_0, 1]$*, then the solution of* (2) *is given by the square of area* c *and same axes of symmetry as* Ω.
- *If* $c \in [0, c_0)$*, then the solution of* (2) *is given by a suitable rectangle.*

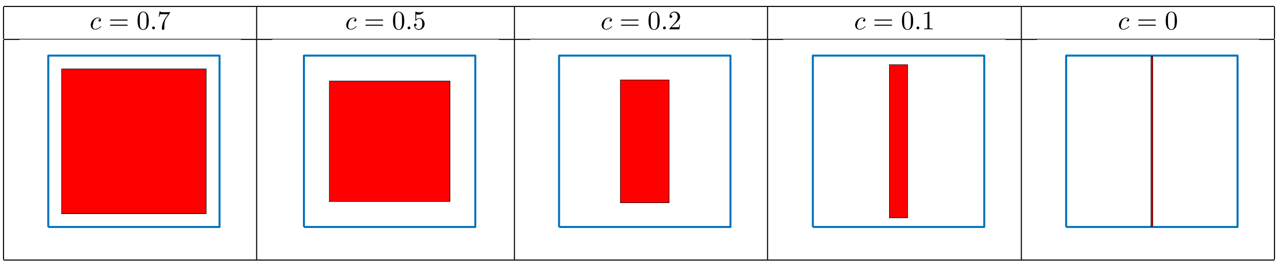


Figure 2: Optimal shapes when Ω is a square, for $c \in \{0.7, 0.5, 0.2, 0.1, 0\}$.

The paper is organized as follows: in Section 2, we present the notations used and recall some classical results on the support function which is a classical parametrization of convex sets that allows to formulate the considered geometric problems as purely analytic ones. In section 3, we present the proof of Theorem 1. Section 4 is devoted to the proof of Theorem 2 and some qualitative properties of intrinsic interest: namely, we prove that when the set Ω is a polygon, the optimal sensor is also a polygon. At last, in Section 5, we present a numerical framework for solving the problem and show that thanks to the equivalence result of Theorem 1, problem (2) can be numerically addressed by a simple minimization of a quadratic function under some linear constrains.

2 Notations and useful results

2.1 Definition of the support function and classical results

If Ω is convex (not necessarily containing the origin), its support function is defined as follows:

$$h_\Omega : x \in \mathbb{R}^2 \longmapsto \sup\{\langle x, y\rangle \mid y \in \Omega\}.$$

Since the functions h_Ω satisfy the scaling property $h_\Omega(tx) = th_\Omega(x)$ for $t > 0$, it can be characterized by its values on the unit sphere $\mathbb{S}^1$ or equivalently on the interval $[0, 2\pi]$. We then adopt the following definition:

Definition 3 *The support function of a planar bounded convex set Ω is defined on $[0, 2\pi]$ as follows:*

$$h_\Omega : [0, 2\pi) \longmapsto \sup\left\{\left\langle \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}, y\right\rangle \mid y \in \Omega\right\}.$$

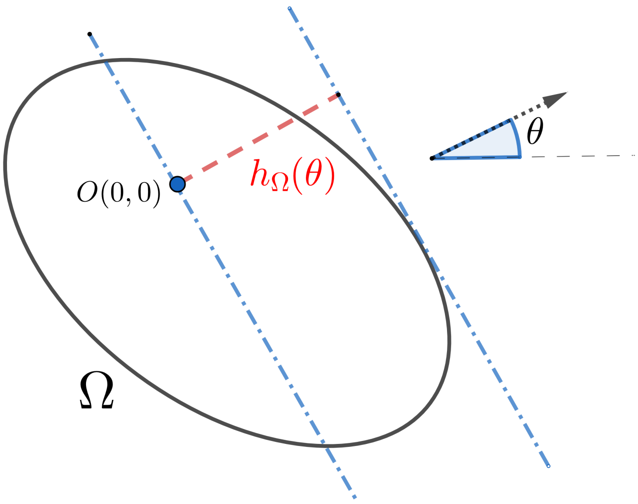


Figure 3: The support function of the convex Ω.

The support function has some interesting properties:

- It allows to provide a simple criterion of the convexity of Ω. Indeed, Ω is convex if and only if $h''_\Omega + h_\Omega \geq 0$ in the sense of distributions, see for example [21].
- It is linear for the Minkowski sum and dilatation. Indeed, if Ω_1 and Ω_2 are two convex bodies and $\alpha, \beta > 0$, we have
$$h_{\alpha\Omega_1+\beta\Omega_2} = \alpha h_{\Omega_1} + \beta h_{\Omega_2},$$
see [21, Section 1.7.1].
- It allows to parametrize inclusion in a simple way. Indeed, if Ω_1 and Ω_2 are two convex sets, we have
$$\Omega_1 \subset \Omega_2 \Longleftrightarrow h_{\Omega_1} \leq h_{\Omega_2}.$$
- It also provides elegant formulas for some geometrical quantities. For example the perimeter and the area of a convex body Ω are given by
$$P(\Omega) = \int_0^{2\pi} h_\Omega(\theta)d\theta \quad \text{and} \quad |\Omega| = \frac{1}{2}\int_0^{2\pi} h_\Omega(\theta)(h''_\Omega(\theta) + h_\Omega(\theta))d\theta = \frac{1}{2}\int_0^{2\pi} ({h'_\Omega}^2 - h_\Omega^2)d\theta,$$
and the Hausdorff distance between two convex bodies Ω_1 and Ω_2 is given by
$$d^H(\Omega_1, \Omega_2) = \max_{\theta\in[0,2\pi]} |h_{\Omega_1}(\theta) - h_{\Omega_2}(\theta)|,$$
see [21, Lemma 1.8.14].

2.2 Notations

- $\mathcal{K}_c$ corresponds to the class of planar, closed, bounded and convex subsets of Ω, where $c \in [0, |\Omega|]$.
- If X and Y are two subsets of $\mathbb{R}^n$, the Hausdorff distance between the sets X and Y is defined as follows

$$d^H(X, Y) = \max(\sup_{x \in X} d(x, Y), \sup_{y \in Y} d(y, X)),$$

  where $d(a, B) := \inf_{b \in B} \|a - b\|$ quantifies the distance from the point a to the set B. Note that when $\omega \subset \Omega$, as it is the case in the problems considered in the present paper, the Hausdorff distance is given by

$$d^H(\omega, \Omega) := \sup_{x \in \Omega} d(x, \omega).$$

- If Ω is a convex set, then h_Ω corresponds to its support function as defined in Section 2.1.
- Given a convex set Ω, we denote by Ω_{-t} its inner parallel set at distance $t \geq 0$, which is defined by

$$\Omega_{-t} := \{x \mid d(x, \partial\Omega) \geq t\}.$$

- $H^1_{\text{per}}(0, 2\pi)$ is the set of H^1 functions that are 2π-periodic.

3 Proof of Theorem 1

For the convenience of the reader, we decomposed the proof in 3 parts: first, we prove the existence of solutions of problem (2). Then, we prove the monotonicity and continuity of the function $f : c \in [0, |\Omega|] \longmapsto \min\{d^H(\omega, \Omega) \mid \omega \in \mathcal{K}_c\}$. At last, we present the proof of the equivalence between the four shape optimization problems stated in Theorem 1.

3.1 Existence of minimizers

Proposition 4 *Problem* (2) *admits solutions.*

Proof.
First, we note that the functional $\omega \longmapsto d^H(\omega, \Omega)$ is 1-Lipschitz (thus continuous) with respect to the Hausdorff distance. Indeed, for every convex sets ω_1 and ω_2, we have

$$|d^H(\omega_1, \Omega) - d^H(\omega_2, \Omega)| = |\|h_\Omega - h_{\omega_1}\|_\infty - \|h_\Omega - h_{\omega_2}\|_\infty| \leq \|h_{\omega_1} - h_{\omega_2}\|_\infty = d^H(\omega_1, \omega_2).$$

Let (ω_n) be a minimizing sequence for problem (2), i.e., such that $\omega_n \in \mathcal{K}_c$ and

$$\lim_{n \to +\infty} d^H(\omega_n, \Omega) = \inf\{d^H(\omega, \Omega) \mid \omega \in \mathcal{K}_c\}.$$

Since all the convex sets ω_n are included in the bounded set Ω, we have by Blaschke's selection Theorem (see [21, Th 1.8.7]) that there exists a convex set $\omega^* \subset \Omega$ such that (ω_n) converges up to a subsequence (that we also denote by (ω_n)) to ω^* with respect to the Hausdorff distance. By the continuity of the volume functional with respect to the Hausdorff distance, we have

$$|\omega^*| = \lim_{n \to +\infty} |\omega_n| = c,$$

which means that $\omega^* \in \mathcal{K}_c$. Moreover, by the continuity of $\omega \longmapsto d^H(\omega, \Omega)$ with respect to the Hausdorff distance, we have that

$$\lim_{n \to +\infty} d^H(\omega_n, \Omega) = d^H(\omega^*, \Omega).$$

This shows that ω^* is a solution of problem (2).

3.2 Monotonicity and continuity

Proposition 5 *The function* $f : c \in [0, |\Omega|] \longmapsto \min\{d^H(\omega, \Omega) \mid \omega \in \mathcal{K}_c\}$ *is continuous and strictly decreasing.*

Proof. ***Continuity:***
Let $c_0 \in (0, |\Omega|)$. By Proposition 4, for every $c \in [0, |\Omega|]$, there exists ω_c solution of the problem

$$\min\{d^H(\omega, \Omega) \mid \omega \in \mathcal{K}_c\}.$$

- We first show an **inferior limit inequality**. Let $(c_n)_{n\geq 1}$ a sequence converging to c_0 such that

$$\liminf_{c\to c_0} d^H(\omega_c, \Omega) = \lim_{n\to+\infty} d^H(\omega_{c_n}, \Omega).$$

  Since all the convex sets ω_{c_n} are included in the bounded set Ω, we have, by Blaschke selection theorem and the continuity of the functional $\omega \longmapsto d(\omega, \Omega)$ and the volume, the existence of a set $\omega^* \in \mathcal{K}_{c_0}$ that is a limit of a subsequence still denoted by (ω_{c_n}) with respect to the Hausdorff distance. We then have

$$f(c_0) \leq d^H(\omega^*, \Omega) = \lim_{n\to+\infty} d^H(\omega_{c_n}, \Omega) = \liminf_{c\to c_0} d^H(\omega_c, \Omega) = \liminf_{c\to c_0} f(c).$$

- It remains to prove **a superior limit inequality**. Let $(c_n)_{n\geq 1}$ a sequence converging to c_0 such that

$$\limsup_{c\to c_0} f(c) = \lim_{n\to+\infty} f(c_n).$$

  Let us now consider the following family of convex sets

$$Q_c := \begin{cases} (\omega_{c_0})_{-\tau_c} & \text{if } c \leq c_0, \\ (1-t_c)\omega_{c_0} + t_c\Omega & \text{if } c > c_0, \end{cases}$$

  where τ_c is chosen in $\mathbb{R}^+$ in such a way that

$$|(\omega_{c_0})_{-\tau_c}| = c$$

  and t_c is chosen in $[0, 1]$ in such a way that

$$|(1-t_c)\omega_{c_0} + t_c\Omega| = c.$$

  The map $c \in [0, |\Omega|] \longmapsto Q_c$ is continuous with respect to the Hausdorff distance and $Q_{c_0} = \omega_{c_0}$.
  Using the definition of f, we have

$$\forall n \in \mathbb{N}^*, \qquad f(c_n) \leq d^H(Q_{c_n}, \Omega).$$

  Passing to the limit, we get

$$\limsup_{c\to c_0} f(c) = \lim_{n\to+\infty} f(c_n) \leq \lim_{n\to+\infty} d^H(Q_{c_n}, \Omega) = d^H(\omega_{c_0}, \Omega) = f(c_0).$$

As a consequence, we finally get $\lim_{c\to c_0} f(c) = f(c_0)$, which proves the continuity of f.

Monotonicity:
Let $0 \leq x < y \leq |\Omega|$. We consider $\omega \in \mathcal{K}_x$ and such that $f(x) = d^H(\omega, \Omega)$. We have

$$f(y) \leq d^H((1-t_y)\omega + t_y\Omega, \Omega) = \|h_{(1-t_y)\omega+t_y\Omega} - h_\Omega\|_\infty = (1-t_y)\|h_\omega - h_\Omega\|_\infty = (1-t_y)f(x) < f(x),$$

where $t_y \in (0, 1]$ is chosen such that $|(1-t_y)\omega + t_y\Omega| = y$.

3.3 The equivalence between the problems

We then obtain the following important proposition that provides the equivalence between four different shape optimization problems.

Proposition 6 *Let $c \in [0, |\Omega|]$. The following shape optimization problems are equivalent*

(I) $\min\{d^H(\omega, \Omega) \mid \omega$ *is convex,* $|\omega| = c$ *and* $\omega \subset \Omega\}$.

(II) $\min\{d^H(\omega, \Omega) \mid \omega$ *is convex,* $|\omega| \leq c$ *and* $\omega \subset \Omega\}$.

(III) $\min\{|\omega| \mid \omega$ *is convex,* $d^H(\omega, \Omega) = f(c)$ *and* $\omega \subset \Omega\}$.

(IV) $\min\{|\omega| \mid \omega$ *is convex,* $d^H(\omega, \Omega) \leq f(c)$ *and* $\omega \subset \Omega\}$,

in the sense that any solution to one of the problems also solves the other ones.

Proof. Let us prove the equivalence between the four problems.

- We first show that any solution of (I) solves (II): let ω_c be a solution to (I). Then for every convex $\omega \subset \Omega$ such that $|\omega| \leq c$, one has
$$d^H(\omega, \Omega) \geq f(|\omega|) \geq f(c) = d^H(\omega_c, \Omega),$$
where we used the monotonicity of f given by Theorem 5: therefore ω_c solves (II).

- Reciprocally, let now ω^c be a solution of (II): we want to show that ω^c must be of volume c. We notice that
$$f(c) \geq d^H(\omega^c, \Omega) \geq f(|\omega^c|) \geq f(c),$$
where the first inequality follows as the problem (II) allows more candidates than in the definition of f, and the last inequality uses again the monotonicity of f. Therefore $f(c) = f(|\omega^c|)$, and since f is continuous and strictly decreasing, we obtain $|\omega^c| = c$, which implies that ω^c solves (I).

We proved the equivalence between problems (I) and (II); the equivalence between problems (III) and (IV) is shown by similar arguments. It remains to prove the equivalence between (I) and (III).

- Let ω_c be a solution of (I), which means that $\omega_c \in \mathcal{K}_c$ and $d^H(\omega_c, \Omega) = f(c)$. Then for every convex $\omega \subset \Omega$ such that $d^H(\omega, \Omega) = f(c)$, we have
$$f(c) = d^H(\omega, \Omega) \geq f(|\omega|),$$
thus, since f is decreasing, we get $c = |\omega_c| \leq |\omega|$, which means ω_c solves (III).

- Let now ω'_c be a solution of (III). We have
$$f(c) = d^H(\omega'_c, \Omega) \geq f(|\omega'_c|),$$
thus, by monotonicity of f we get $c \geq |\omega'_c|$. On the other hand, since ω'_c solves (III) and that there exists ω_c solution to (I), we have $|\omega'_c| \geq c$, which finally gives $|\omega'_c| = c$ and shows that ω'_c solves (I).

4 Proof of Theorem 2 and some qualitative results

4.1 Saturation of the Hausdorff distance

Proposition 7 *Let ω be a solution of problem* (2)*. Then, there exist (at least) two different couples of points* $(x_1, y_1), (x_2, y_2) \in \partial\omega \times \partial\Omega$ *such that*
$$\|x_1 - y_1\| = \|x_2 - y_2\| = d^H(\omega, \Omega).$$

Proof. Let us argue by contradiction. We assume that exist only one couple $(x_1, y_1) \in \partial\omega \times \partial\Omega$ such that
$$\|x_1 - y_1\| = d^H(\omega, \Omega).$$

Let $x \in \partial\omega$ different from x_1. By cutting an infinitesimal portion of the the convex ω (see Figure 4), we obtain a set ω_ε such that $d^H(\omega, \Omega) = d^H(\omega_\varepsilon, \Omega)$ (because we assumed that the Hausdorff distance is attained at only one couple of points) and $|\omega| > |\omega_\varepsilon|$, for sufficiently small values of ε. Thus, ω is not a solution of the third problem of Proposition 6, which is absurd since ω is assumed to be a solution of problem (2) (which is proven to be equivalent to the later one in Proposition 6).

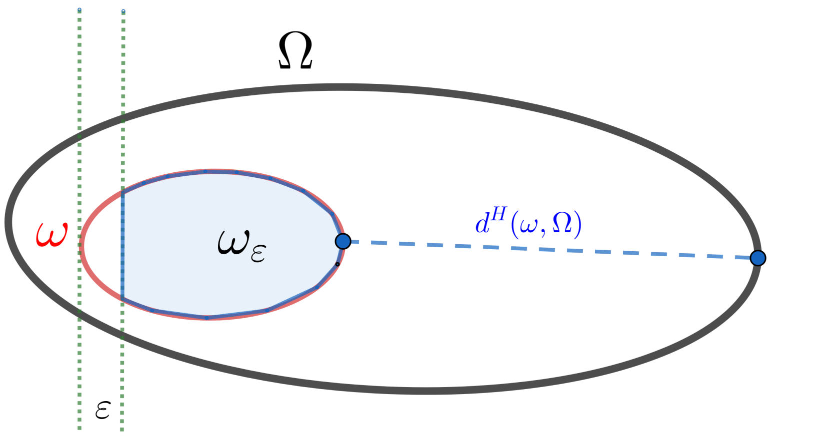


Figure 4: The sets ω (in red) and ω_ε (in blue).

4.2 Polygonal domains

Proposition 8 *If the set Ω is a polygon, then any solution of problem* (2) *is also a polygon.*

Proof. Let us denote by $v_1, \dots, v_N$, with $N \geq 3$, the vertices of the polygon Ω and consider ω a solution of problem (3).

The distance function $x \longmapsto \min_{y\in\omega}\|x-y\|$ is convex, thus, it is well known that its maximal value on the convex polygon Ω is attained at some vertices that we denote by $(v'_k)_{k\in[\![1,K]\!]}$, where $K \leq N$. Note that since ω a solution of problem (3), we have $K \geq 2$ by Proposition 7. Moreover, for every $k \in [\![1, K]\!]$ there exists a unique $u_k \in \partial\omega$ such that $\|v'_k - u_k\| = d^H(\omega, \Omega)$, which is the projection of the vertex v'_k onto the convex sensor ω. Let us consider two successive projection points u_1 and u_2 and assume without loss of generality that their coordinates are given by $(0, 0)$ and $(x_0, 0)$, with $x > 0$, see Figure 5.

We consider the altitude $h \geq 0$ defined as follows

$$h := \sup\{s \mid \exists x \in [0, x_0],\ \text{such that}\ \ (x, s) \in \omega\}.$$

Let us argue by contradiction and assume that $h > 0$. For $\varepsilon > 0$, we consider $\omega_\varepsilon := \omega \cap \{y \leq h - \varepsilon\}$, see Figure 5. For sufficiently small values of $\varepsilon > 0$, we have

$$d^H(\omega_\varepsilon, \Omega) = d^H(\omega, \Omega) \quad \text{and} \quad |\omega_\varepsilon| < |\omega|,$$

which means that ω is not a solution of the problem

$$\min\{|\omega| \mid d^H(\omega, \Omega) = f(c) \text{ and } \omega \subset \Omega\},$$

that is equivalent to problem (2) by Proposition 6. This provides a contradiction since ω is assumed to be a solution of problem (2). We then have that $h = 0$, which means that the segment of extremities u_1 and u_2 is included in the boundary of the optimal set ω. By repeating the same argument with the successive couple of points u_k and u_{k+1} (with the convention $u_{k+1} = u_1$), we prove that the boundary of the optimal set ω is exactly given by the union of the segments of extremities u_k and u_{k+1} which means that ω is a polygon (of K sides).

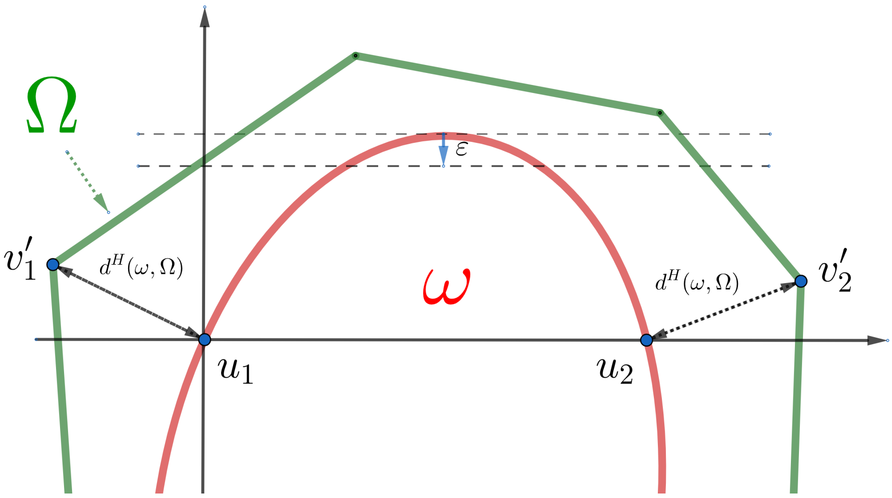


Figure 5: The polygon Ω and the sensor ω.

4.3 Application to the square: symmetry breaking

In this section, we combine the results of Propositions 6 and 8 to solve problem (3) when Ω is a square. This leads to observe the non uniqueness of the optimal shape and a symmetry breaking phenomenon. The phenomenon might seem surprising as one could expect that the optimal sensor will inherit all the symmetries of Ω.

Let $\Omega = [0,1] \times [0,1]$, we are interested in solving problem (2) stated as follows

$$\min\{d^H(\omega,\Omega) \mid \omega \subset \Omega \text{ is convex and } |\omega| = c\}, \tag{3}$$

with $c \in [0, |\Omega|]$.

Before presenting the proof, we present the solutions for different values of c:

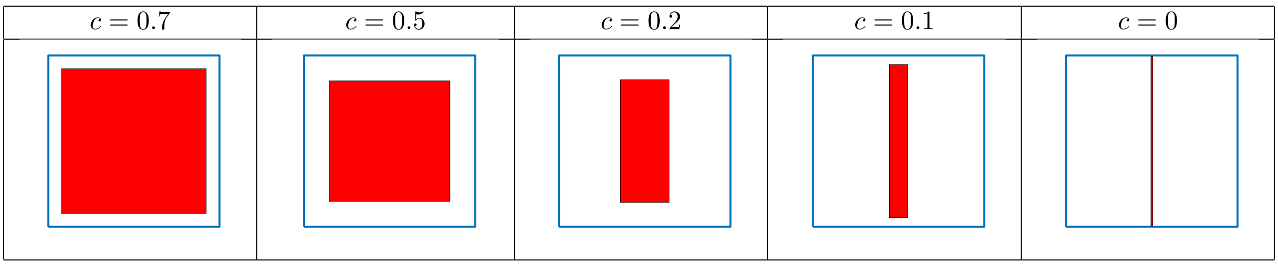


Figure 6: Optimal shapes when Ω is a square, for $c \in \{0.7, 0.5, 0.2, 0.1, 0\}$.

Remark 9 *As one observes in Figure 6, for values of c close to $|\Omega| = 1$, the optimal sensor is a square and thus has the same symmetries of Ω, but for small values of c, the optimal sensor is no longer the square but a certain rectangle. One should then note that the optimal sensor is not necessarily unique (as one can consider rotating the rectangle with an angle $\pi/2$) and it does not necessarily inherit all the symmetries of the shape Ω (as it is not symmetrical with respect to the diagonals of Ω).*

Let us now present the details of the proof. By Propositions 6 and 8, problem (3) is equivalent to the problem

$$\min\{|\omega| \mid \omega \subset \Omega \text{ is a convex quadrilateral and } d^H(\omega,\Omega) = \delta\}, \tag{4}$$

with $\delta \in [0, \frac{1}{2}]$. In the following proposition, we completely solve problem (4).

Proposition 10 *Let $\Omega = [0,1] \times [0,1]$ be the unit square and $\delta \in [0, \frac{1}{2})$. The solution of problem (4) is given by*

- *the square of vertices*

$$\begin{cases} M_1(\delta\frac{\sqrt{2}}{2},\delta\frac{\sqrt{2}}{2}), \\ M_2(1-\delta\frac{\sqrt{2}}{2},\delta\frac{\sqrt{2}}{2}), \\ M_3(1-\delta\frac{\sqrt{2}}{2},1-\delta\frac{\sqrt{2}}{2}), \\ M_4(\delta\frac{\sqrt{2}}{2},1-\delta\frac{\sqrt{2}}{2}), \end{cases}$$

if $\delta \leq \frac{1}{2\sqrt{2}}$,

- *and by one of the two rectangles of vertices*

$$\begin{cases} M_1(\delta\cos\theta_\delta,\delta\sin\theta_\delta), \\ M_2(1-\delta\cos\theta_\delta,\delta\sin\theta_\delta), \\ M_3(1-\delta\cos\theta_\delta,1-\delta\sin\theta_\delta), \\ M_4(\delta\cos\theta_\delta,1-\delta\sin\theta_\delta), \end{cases}$$

with $\theta_\delta \in \{\arcsin\left(\frac{1}{2\sqrt{2}\delta}\right)-\frac{\pi}{4},\frac{3\pi}{4}-\arcsin\left(\frac{1}{2\sqrt{2}\delta}\right)\}$, *if* $\delta \in [\frac{1}{2\sqrt{2}},\frac{1}{2}]$.

Proof. We denote by $A_1(0,0)$, $A_2(1,0)$, $A_3(1,1)$ and $A_4(0,1)$ the vertices of the square Ω and by B_1, B_2, B_3 and B_4 the balls of radius δ and centers respectively A_1, A_2, A_3 and A_4, see Figure 7.

Let ω be a solution of problem (4) (it is then also a solution of problem (3) by Proposition 6). By the result of Proposition 8, since Ω is a square (in particular a polygon), the optimal shape ω is also a polygon with at most four vertices. Since $d^H(\omega,\Omega)=\delta$, the polygon ω has four different vertices. Each one of them is contained in a set $B_k\cap\Omega$, with $k\in[\![1,4]\!]$. In fact, since the optimal set ω minimises the area for a given Hausdorff distance, we deduce that all its vertices are located on the arcs of circles $\partial B_k\cap\Omega$ given by the intersection of the boundaries of the balls B_k and the square Ω. Indeed, if it were not the case, one could easily construct a convex polygon strictly included in ω (thus, with strictly less volume) such that its Hausdorff distance to the square Ω is equal to δ, see Figure 7

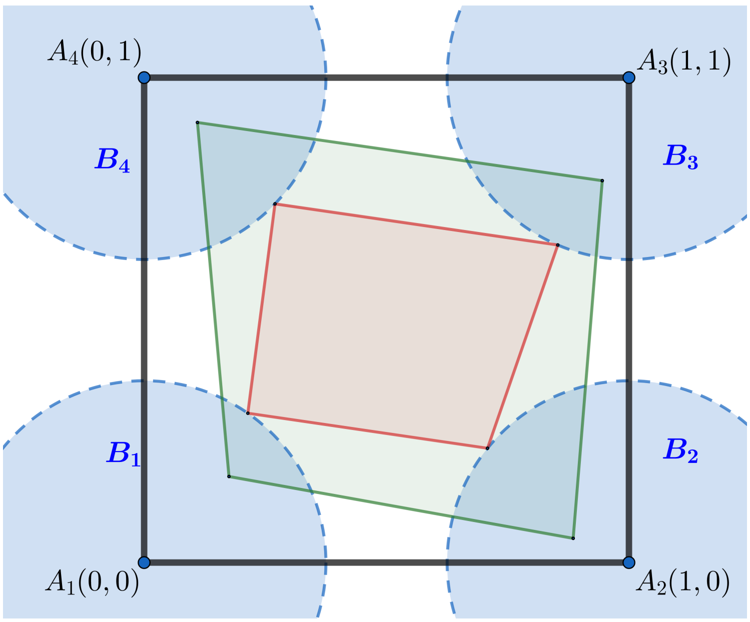


Figure 7: The polygon in red has a smallest area than the one in green and its Hausdorff distance to the square Ω is equal to δ.

Now that we know that each vertex of the optimal sensor ω is located on a (different) arc of circle $\partial B_k \cap \Omega$, with $k \in [\![1, 4]\!]$, let us denote them by

$$\begin{cases} M_1(\delta \cos\theta_1, \delta \sin\theta_1), \\ M_2(1 - \delta \cos\theta_2, \delta \sin\theta_2), \\ M_3(1 - \delta \cos\theta_3, 1 - \delta \sin\theta_3), \\ M_4(\delta \cos\theta_4, 1 - \delta \sin\theta_4), \end{cases}$$

where $\theta_1, \theta_2, \theta_3, \theta_4 \in [0, \frac{\pi}{2}]$, see Figure 8.

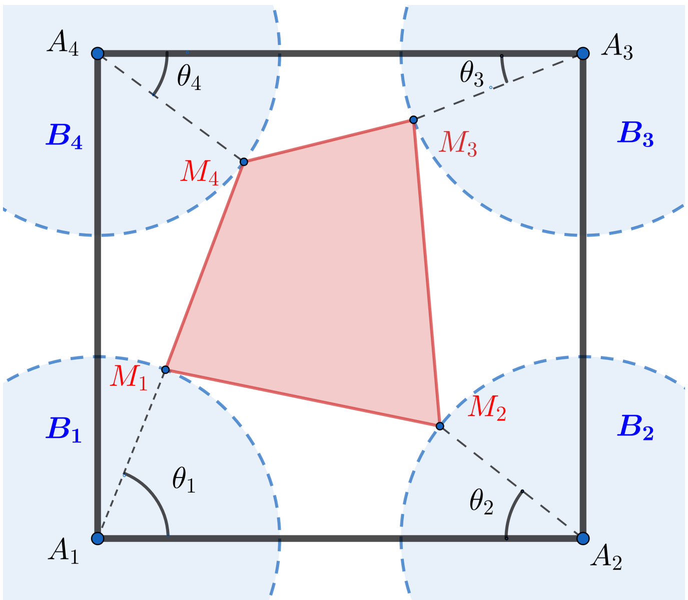


Figure 8: Parametrization via the angles θ_1, θ_2, θ_3 and θ_4.

The area of the polygon ω can be expressed via the coordinates of its vertices as follows:

$$|\omega| = \frac{1}{2}\sum_{k=1}^{4}(x_k y_{k+1} - x_{k+1} y_k),$$

where (x_k, y_k) correspond to the coordinates of the points M_k, with the convention $(x_5, y_5) := (x_1, y_1)$.
We obtain by straightforward computations:

$$|\omega| = 1 - \frac{1}{2}\delta\sum_{k=1}^{4}\cos\theta_k - \frac{1}{2}\delta\sum_{k=1}^{4}\sin\theta_k + \frac{1}{2}\delta^2\sum_{k=1}^{4}(\cos\theta_k \sin\theta_{k+1} + \cos\theta_{k+1}\sin\theta_k),$$

with the convention $\theta_5 = \theta_1$.
We then perform a judicious factorization to obtain the following formula

$$|\omega| = \frac{1}{2}((1 - \delta\cos\theta_1 - \delta\cos\theta_3)(1 - \delta\sin\theta_2 - \delta\sin\theta_4) + (1 - \delta\cos\theta_2 - \delta\cos\theta_4)(1 - \delta\sin\theta_1 - \delta\sin\theta_3)).$$

We then use the inequality $a+b \geq 2\sqrt{ab}$, where the equality holds if and only of $a=b$, and obtain

$$\begin{cases} 1-\delta\cos\theta_1-\delta\cos\theta_3 = (\frac{1}{2}-\delta\cos\theta_1)+(\frac{1}{2}-\delta\cos\theta_3) \geq 2\sqrt{(\frac{1}{2}-\delta\cos\theta_1)(\frac{1}{2}-\delta\cos\theta_3)}, \\ 1-\delta\sin\theta_2-\delta\sin\theta_4 = (\frac{1}{2}-\delta\sin\theta_2)+(\frac{1}{2}-\delta\sin\theta_4) \geq 2\sqrt{(\frac{1}{2}-\delta\sin\theta_2)(\frac{1}{2}-\delta\sin\theta_4)}, \\ 1-\delta\cos\theta_2-\delta\cos\theta_4 = (\frac{1}{2}-\delta\cos\theta_2)+(\frac{1}{2}-\delta\cos\theta_4) \geq 2\sqrt{(\frac{1}{2}-\delta\cos\theta_2)(\frac{1}{2}-\delta\cos\theta_4)}, \\ 1-\delta\sin\theta_1-\delta\sin\theta_3 = (\frac{1}{2}-\delta\sin\theta_1)+(\frac{1}{2}-\delta\sin\theta_3) \geq 2\sqrt{(\frac{1}{2}-\delta\sin\theta_1)(\frac{1}{2}-\delta\sin\theta_3)}, \end{cases}$$

with equality if and only if

$$\theta_1=\theta_3 \quad \text{and} \quad \theta_2=\theta_4. \tag{5}$$

We then write

$$\begin{aligned} |\omega| \geq & \sqrt{(\frac{1}{2}-\delta\cos\theta_1)(\frac{1}{2}-\delta\cos\theta_3)}\sqrt{(\frac{1}{2}-\delta\sin\theta_2)(\frac{1}{2}-\delta\sin\theta_4)} \\ & + \sqrt{(\frac{1}{2}-\delta\cos\theta_2)(\frac{1}{2}-\delta\cos\theta_4)}\sqrt{(\frac{1}{2}-\delta\sin\theta_1)(\frac{1}{2}-\delta\sin\theta_3)} \end{aligned}$$

and use again the inequality $a+b \geq 2\sqrt{ab}$ to obtain

$$\begin{aligned} |\omega| \geq & 2\left((\frac{1}{2}-\delta\cos\theta_1)(\frac{1}{2}-\delta\cos\theta_3)(\frac{1}{2}-\delta\sin\theta_2)(\frac{1}{2}-\delta\sin\theta_4)\right)^{\frac{1}{4}} \\ & \cdot \left((\frac{1}{2}-\delta\cos\theta_2)(\frac{1}{2}-\delta\cos\theta_4)(\frac{1}{2}-\delta\sin\theta_1)(\frac{1}{2}-\delta\sin\theta_3)\right)^{\frac{1}{4}}, \end{aligned} \tag{6}$$

where the equality holds if and only if one has

$$(\frac{1}{2}-\delta\cos\theta_1)(\frac{1}{2}-\delta\cos\theta_3)(\frac{1}{2}-\delta\sin\theta_2)(\frac{1}{2}-\delta\sin\theta_4) = (\frac{1}{2}-\delta\cos\theta_2)(\frac{1}{2}-\delta\cos\theta_4)(\frac{1}{2}-\delta\sin\theta_1)(\frac{1}{2}-\delta\sin\theta_3). \tag{7}$$

By combining the equality conditions (5) and (7), we show that the inequality (6) is an equality if and only if $\theta_1=\theta_3$, $\theta_2=\theta_4$ and

$$(\frac{1}{2}-\delta\cos\theta_1)(\frac{1}{2}-\delta\sin\theta_2) = (\frac{1}{2}-\delta\sin\theta_1)(\frac{1}{2}-\delta\cos\theta_2),$$

which is equivalent to

$$\frac{\frac{1}{2}-\delta\cos\theta_1}{\frac{1}{2}-\delta\sin\theta_1} = \frac{\frac{1}{2}-\delta\cos\theta_2}{\frac{1}{2}-\delta\sin\theta_2},$$

which holds if and only if $\theta_1=\theta_2$, because the function $\theta \longmapsto \frac{\frac{1}{2}-\delta\cos\theta}{\frac{1}{2}-\delta\sin\theta}$ is a bijection from $[0,\frac{\pi}{2}]$ to $[1-2\delta, \frac{1}{1-2\delta}]$.

We then conclude that the equality in (6) holds if and only if $\theta_1=\theta_2=\theta_3=\theta_4$, which means that the optimal sensor is a rectangle that corresponds to the value of θ_δ that minimizes the function

$$f_\delta : \theta \in [0,\frac{\pi}{2}] \longmapsto \left(\frac{1}{2}-\delta\cos\theta\right)\left(\frac{1}{2}-\delta\sin\theta\right).$$

Since we have $f_\delta(\frac{\pi}{2}-\theta)=f_\delta(\theta)$ for every $\theta\in[0,\frac{\pi}{4}]$, we deduce by symmetry that it is sufficient to study the function f_δ on the interval $[0,\frac{\pi}{4}]$; we have

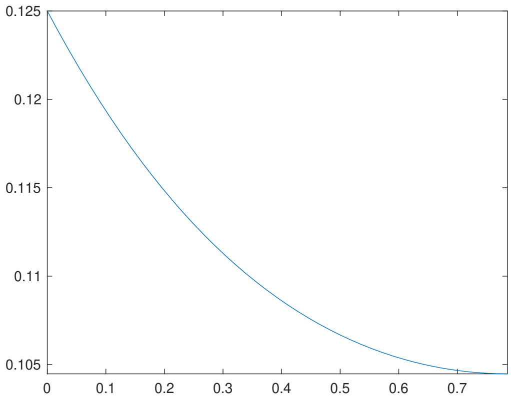
$$\forall\theta\in[0,\frac{\pi}{4}], \quad f'_\delta(\theta)=\delta(\cos\theta-\sin\theta)\left(-\frac{1}{2}+\delta\cos\theta+\delta\sin\theta\right).$$

The function $g_\delta : \theta \longmapsto -\frac{1}{2} + \delta\cos\theta + \delta\sin\theta$ is continuous and strictly increasing on $[0, \frac{\pi}{4}]$. Thus,

$$g_\delta([0, \frac{\pi}{4}]) = [g_\delta(0), g_\delta(\frac{\pi}{4})] = [-\frac{1}{2} + \delta, -\frac{1}{2} + \delta\sqrt{2}].$$

Then, the sign of g_δ on $[0, \frac{\pi}{4}]$ (and thus the variation of f_δ, see Figure 9) depends on the value of $\delta \in [0, \frac{1}{2})$. Indeed:

- If $\delta \leq \frac{1}{2\sqrt{2}}$ (i.e., $g_\delta(\frac{\pi}{4}) \leq 0$), then $g' < 0$ on $(0, \frac{\pi}{4})$, which means that f_δ is strictly decreasing on $[0, \frac{\pi}{4}]$ and thus attains its minimal value at $\theta_\delta = \frac{\pi}{4}$.
- If $\delta > \frac{1}{2\sqrt{2}}$ (i.e., $g_\delta(\frac{\pi}{4}) > 0$), then straightforward computations show that the function f_δ is strictly decreasing on $[0, \theta_\delta]$ and increasing on $[\theta_\delta, \frac{\pi}{4}]$, with $\theta_\delta = \arcsin\left(\frac{1}{2\sqrt{2}\delta}\right) - \frac{\pi}{4}$. Thus, f_δ attains its minimal value at θ_δ.



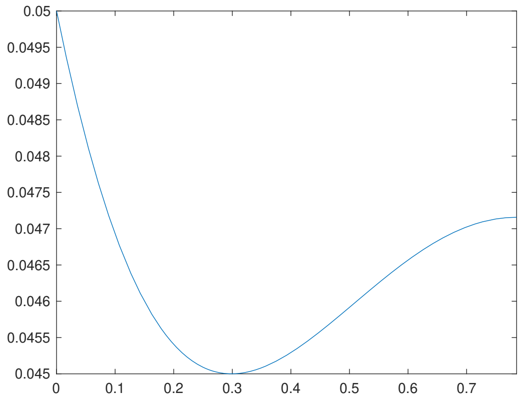


Figure 9: The graphs of the function f_δ for the cases: $\delta = 0.25$ in the left and $\delta = 0.4$ in the right.

5 Numerical simulations

In this section, we present the numerical scheme adopted to solve the problems in consideration in the present paper. In particular, we focus on the following (equivalent) problems:

$$\min\{d^H(\omega, \Omega) \mid \omega \text{ is convex, } |\omega| = c \text{ and } \omega \subset \Omega\}, \tag{8}$$

and

$$\min\{|\omega| \mid \omega \text{ is convex, } d^H(\omega, \Omega) \leq d \text{ and } \omega \subset \Omega\}, \tag{9}$$

where $c, d \geq 0$.

As we shall see, even-though the problems are equivalent (see Theorem 6), problem (9) is much easier to solve numerically as it is approximated by a simple problem of minimizing a quadratic function under linear constraints.

5.1 Parametrization of the functionals

It is recalled in Section 2.1 that if both Ω and ω are convex, we have the following formulae for the Hausdorff distance between ω and Ω

$$d^H(\omega, \Omega) = \|h_\Omega - h_\omega\|_\infty := \max_{\theta\in[0,2\pi]} |h_\Omega(\theta) - h_\omega(\theta)|$$

and the area of ω

$$|\omega| = \frac{1}{2}\int_0^{2\pi} h_\omega(h''_\omega + h_\omega)d\theta = \frac{1}{2}\int_0^{2\pi}(h_\omega^2 - {h'_\omega}^2)d\theta,$$

where h_Ω and h_ω respectively correspond to the support functions of the convex sets Ω and ω.

On the other hand, the inclusion constraint $\omega \subset \Omega$ can be expressed by $h_\omega \leq h_\Omega$ on $[0, 2\pi]$ and the convexity of the sensor ω can also be analytically expressed as follows

$$h''_\omega + h_\omega \geq 0,$$

in the sense of distributions. We refer to [21] for more details and results on convexity.

Therefore, the use of the support functions allows to respectively transform the purely geometrical problems (8) and (9) into the following analytical ones:

$$\begin{cases} \inf\limits_{h \in H^1_{\text{per}}(0,2\pi)} \|h_\Omega - h\|_\infty, \\ h \leq h_\Omega, \\ h'' + h \geq 0, \\ \frac{1}{2}\int_0^{2\pi} h(h'' + h)d\theta = c. \end{cases} \tag{10}$$

and

$$\begin{cases} \inf\limits_{h \in H^1_{\text{per}}(0,2\pi)} \frac{1}{2}\int_0^{2\pi} h(h'' + h)d\theta, \\ h \leq h_\Omega, \\ h'' + h \geq 0, \\ \|h_\Omega - h\|_\infty \leq d. \end{cases} \tag{11}$$

where $H^1_{\text{per}}(0, 2\pi)$ is the set of H^1 functions that are 2π-periodic.

Since

$$\begin{cases} h \leq h_\Omega, \\ \|h_\Omega - h\|_\infty \leq d \end{cases} \Longleftrightarrow h_\Omega - d \leq h \leq h_\Omega,$$

problem (11) can be reformulated as follows

$$\begin{cases} \inf\limits_{h \in H^1_{\text{per}}(0,2\pi)} \frac{1}{2}\int_0^{2\pi} h(h'' + h)d\theta, \\ h'' + h \geq 0, \\ h_\Omega - d \leq h \leq h_\Omega. \end{cases} \tag{12}$$

To perform numerical approximation of optimal shape, we have to retrieve a finite dimensional setting. We then follow the same ideas in [2, 3] and parametrize the sets via Fourier coefficients of their support functions truncated at a certain order $N \geq 1$. Thus, we look for solutions in the set

$$\mathcal{H}_N := \left\{\theta \longmapsto a_0 + \sum_{k=1}^{N}(a_k\cos(k\theta) + b_k\sin(k\theta)) \mid a_0, \dots, a_N, b_1, \dots, b_N \in \mathbb{R}\right\}.$$

This approach is justified by the following approximation proposition:

Proposition 11 *([21, Section 3.4])*
Let $\Omega \in \mathcal{K}^2$ and $\varepsilon > 0$. Then there exists N_ε and Ω_ε with support function $h_{\Omega_\varepsilon} \in \mathcal{H}_{N_\varepsilon}$ such that $d^H(\Omega, \Omega_\varepsilon) < \varepsilon$.

We refer to [2, 4] for other and applications to different problems and some theoretical convergence results.

Let us now consider the regular subdivision $(\theta_k)_{k \in [\![1,M]\!]}$ of $[0, 2\pi]$, where $\theta_k = 2k\pi/M$ and $M \in \mathbb{N}^*$. The inclusion constraints $h_\Omega(\theta) - d \leq h(\theta) \leq h_\Omega(\theta)$ and the convexity constraint $h''(\theta) + h(\theta) \geq 0$ are approximated by the following $3M$ linear constraints on the Fourier coefficients:

$$\forall k \in [\![1, M]\!], \qquad \begin{cases} h_\Omega(\theta_k) - d \leq a_0 + \sum\limits_{j=1}^{N} (a_j \cos(j\theta_k) + b_j \sin(j\theta_k)) \leq h_\Omega(\theta_k), \\ a_0 + \sum\limits_{j=1}^{N} ((1-j^2)\cos(j\theta_k) a_j + (1-j^2)\sin(j\theta_k) b_j) \geq 0. \end{cases}$$

At last, the area of the convex set corresponding to the truncated support function of ω at the order N is given by the following quadratic formula:

$$|\omega| = \pi a_0^2 + \frac{\pi}{2} \sum_{j=1}^{N} (1-j^2)(a_j^2 + b_j^2).$$

Thus, the infinitely dimensional problems (10) and (12) are approximated by the following finitely dimensional ones:

$$\begin{cases} \inf\limits_{(a_0, a_1, \ldots, a_N, b_1, \ldots, b_N) \in \mathbb{R}^{2N+1}} \max\limits_{\theta \in [0, 2\pi]} h_\Omega(\theta) - a_0 - \sum\limits_{j=1}^{N} (a_j \cos(j\theta) + b_j \sin(j\theta)), \\ \forall k \in [\![1, M]\!], \quad a_0 + \sum\limits_{j=1}^{N} (a_j \cos(j\theta_k) + b_j \sin(j\theta_k)) \leq h_\Omega(\theta_k), \\ \forall k \in [\![1, M]\!], \quad a_0 + \sum\limits_{j=1}^{N} ((1-j^2)\cos(j\theta_k) a_j + (1-j^2)\sin(j\theta_k) b_j) \geq 0, \\ \pi a_0^2 + \frac{\pi}{2} \sum\limits_{j=1}^{N} (1-j^2)(a_j^2 + b_j^2) = c. \end{cases} \tag{13}$$

and

$$\begin{cases} \inf\limits_{(a_0, a_1, \ldots, a_N, b_1, \ldots, b_N) \in \mathbb{R}^{2N+1}} \pi a_0^2 + \frac{\pi}{2} \sum\limits_{j=1}^{N} (1-j^2)(a_j^2 + b_j^2), \\ \forall k \in [\![1, M]\!], \quad h_\Omega(\theta_k) - d \leq a_0 + \sum\limits_{j=1}^{N} (a_j \cos(j\theta_k) + b_j \sin(j\theta_k)) \leq h_\Omega(\theta_k), \\ \forall k \in [\![1, M]\!], \quad a_0 + \sum\limits_{j=1}^{N} ((1-j^2)\cos(j\theta_k) a_j + (1-j^2)\sin(j\theta_k) b_j) \geq 0. \end{cases} \tag{14}$$

Remark 12 *We conclude that the shape optimization problems considered in the present paper are approximated by problem* (14)*, which simply consists in minimizing a quadratic function under linear constraints.*

5.2 Computation of the gradients

A very important step in shape optimization is the computation of the gradients. In our case, the convexity and inclusion constraints are linear and the area constraint is quadratic. Thus, its gradients are obtained by direct computations. Nevertheless, the computation of the gradient of the objective function in Problem (13) is not straightforward as it is defined as a supremum. This is why we use a Danskin's differentiation scheme [8] to compute the derivative.

Proposition 13 *Let us consider*

$$g : (\theta, a_0, \ldots, b_N) \longmapsto h_\Omega(\theta) - a_0 - \sum_{k=1}^{N} (a_k \cos(k\theta) + b_k \sin(k\theta))$$

and

$$j : (a_0, \ldots, b_N) \longmapsto \max_{\theta \in [0, 2\pi]} g(\theta, a_0, \ldots, b_N).$$

The function j admits directional derivatives in every direction and we have

$$\frac{\partial j}{\partial a_0} = -1,$$

and for every $k \in [\![1, N]\!]$,

$$\begin{cases} \frac{\partial j}{\partial a_k} = \max_{\theta^* \in \Theta} -\cos(k\theta^*), \\ \frac{\partial j}{\partial b_k} = \max_{\theta^* \in \Theta} -\sin(k\theta^*), \end{cases}$$

where

$$\Theta := \{\theta^* \in [0, 2\pi] \mid G(\theta^*, a_0, \dots, b_N) = \max_{\theta \in [0,2\pi]} G(\theta, a_0, \dots, b_N)\}.$$

Proof. Since the same scheme is followed for every coordinate, we limit our selves to present the proof for the fist coordinate a_0. In order to simplify the notations, we will write for every $x \in \mathbb{R}$, $j(x) = j(a_0, \dots, a_{k-1}, x, a_{k+1} \dots, b_N)$ and $G(\theta, x) = G(\theta, a_0, \dots, a_{k-1}, x, a_{k+1} \dots, b_N)$.

For every $t \geq 0$, we denote by $\theta_t \in [0, 2\pi]$ a point such that

$$G(\theta_t, a_k + t) = j(a_k + t) = \max_{\theta \in [0,2\pi]} G(\theta, a_k + t).$$

We have

$$j(a_k + t) - j(a_k) = G(\theta_t, a_k + t) - G(\theta_0, a_k) \geq G(\theta_0, a_k + t) - G(\theta_0, a_k) = -t\cos(k\theta_0).$$

Thus,

$$\forall \theta_0 \in \Theta, \quad \liminf_{t \to 0^+} \frac{j(a_k + t) - j(a_k)}{t} \geq -\cos(k\theta_0),$$

which means that

$$\liminf_{t \to 0^+} \frac{j(a_k + t) - j(a_k)}{t} \geq \max_{\theta_0 \in \Theta} -\cos(k\theta_0). \tag{15}$$

Let us now consider a sequence (t_n) of positive numbers decreasing to 0, such that

$$\lim_{n \to +\infty} \frac{j(a_k + t_n) - j(a_k)}{t_n} = \limsup_{t \to 0^+} \frac{j(a_k + t) - j(a_k)}{t}.$$

We have, for every $n \geq 0$,

$$j(a_k + t_n) - j(a_k) = G(\theta_{t_n}, a_k + t_n) - G(\theta_0, a_k) \leq G(\theta_{t_n}, a_k + t_n) - G(\theta_{t_n}, a_k) = -t_n \cos(k\theta_{t_n}).$$

Thus,

$$\limsup_{t \to 0^+} \frac{j(a_k + t) - j(a_k)}{t} = \lim_{n \to +\infty} \frac{j(a_k + t_n) - j(a_k)}{t_n} \leq \limsup_{n \to +\infty} -\cos(k\theta_n) = -\cos(k\theta_\infty),$$

where θ_∞ is an accumulation point of the sequence (θ_n). It is not difficult to check that $\theta_\infty \in \Theta$. Thus, we have

$$\limsup_{t \to 0^+} \frac{j(a_k + t) - j(a_k)}{t} \leq \max_{\theta_0 \in \Theta} -\cos(k\theta_0). \tag{16}$$

By the inequalities (15) and (16) we deduce the announced formula for the derivative.

5.3 Numerical results

Now that we have parameterized the problem and computed the gradients, we are in position to perform shape optimization. We use the 'fmincon' Matlab routine. In the following figures we present the results obtained for different shapes and different mass fractions $c_0 := \alpha_0|\Omega|$, where $\alpha_0 \in \{0.01, 0.1, 0.4, 0.7\}$.

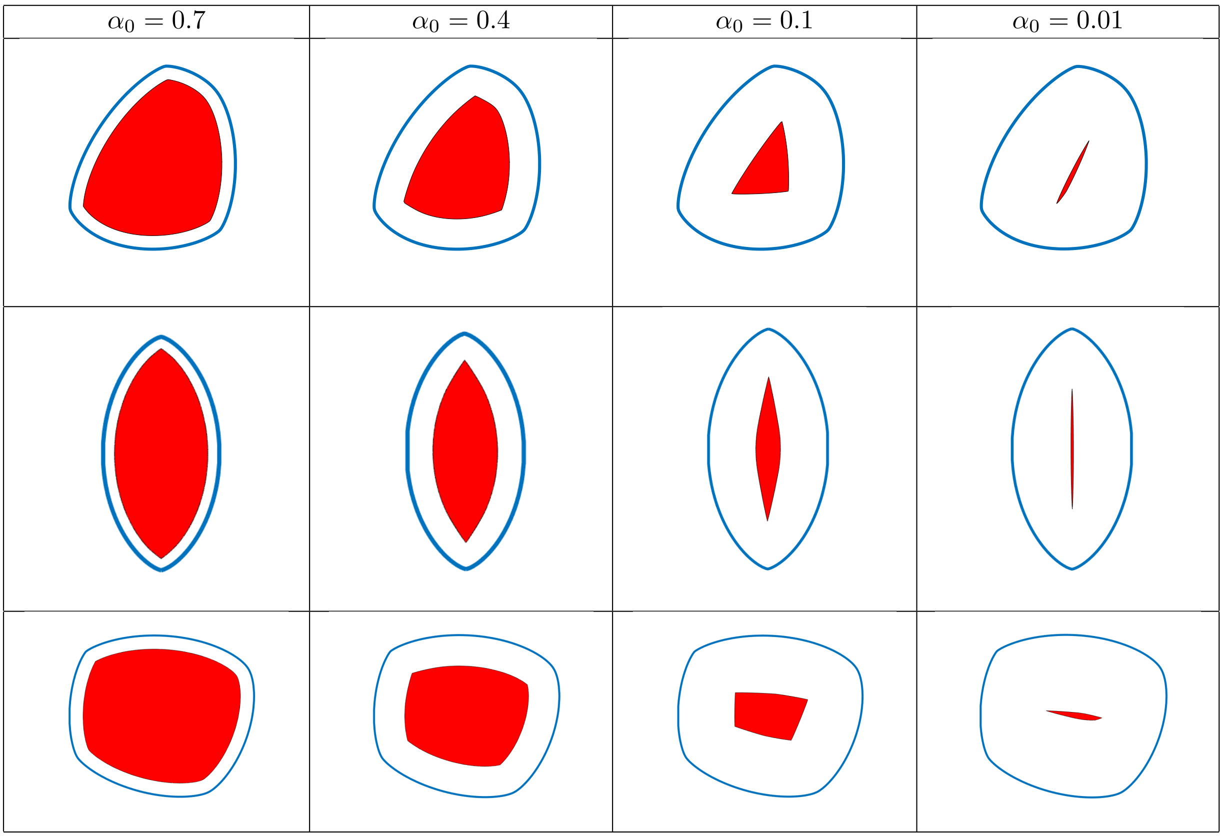


Figure 10: Obtained optimal shapes for $\alpha_0 \in \{0.01, 0.1, 0.4, 0.7\}$ and different choices of Ω.

5.4 Optimal spherical sensors and relation with Chebychev centers

In this section, we show that the ideas developed in the last sections can be efficiently used to numerically solve the problem of optimal placement of a spherical sensor inside the convex set Ω. We show also that this problem is related to the task of finding the Chebychev center of the set, i.e., the center of the minimal-radius ball enclosing the entire set Ω.

We are then considering the following optimal placement problem

$$\min\{d^H(B,\Omega) \mid B \text{ is a ball included in } \Omega \text{ and of radius } R\}, \tag{17}$$

with $R \in [0, r(\Omega)]$, where $r(\Omega)$ is the inradius of Ω that is the radius of the biggest ball contained in Ω.

Since the support function of a ball B of center (x,y) and radius R is simply given by $h_B : \theta \longmapsto R + x\cos\theta + y\sin\theta$, problem (17) can be formulated in terms of support functions as follows:

$$\min_{(x,y)}\{\|h_\Omega - R + x\cos\theta + y\sin\theta\|_\infty \mid \forall\theta \in [0,2\pi], \quad R + x\cos\theta + y\sin\theta \leq h_\Omega(\theta)\}. \tag{18}$$

Here also, as in Section 5.1, the inclusion constraint $B \subset \Omega$ (i.e., $h_B \leq h_\Omega$) can be approximated by a finite number of linear inequalities

$$R + x\cos\theta_k + y\sin\theta_k \leq h_\Omega(\theta_k),$$

where $\theta_k := 2k\pi/M$, with $k \in [\![1, M]\!]$ and M chosen equal to 500. Thus, we retrieve a problem of minimizing the non linear function $(x,y) \longmapsto \|h_\Omega - R + x\cos\theta + y\sin\theta\|_\infty$ (whose gradient is computed by using the result of Proposition 13) with a finite number of linear constraints. In the following figures, we present some numerical results:

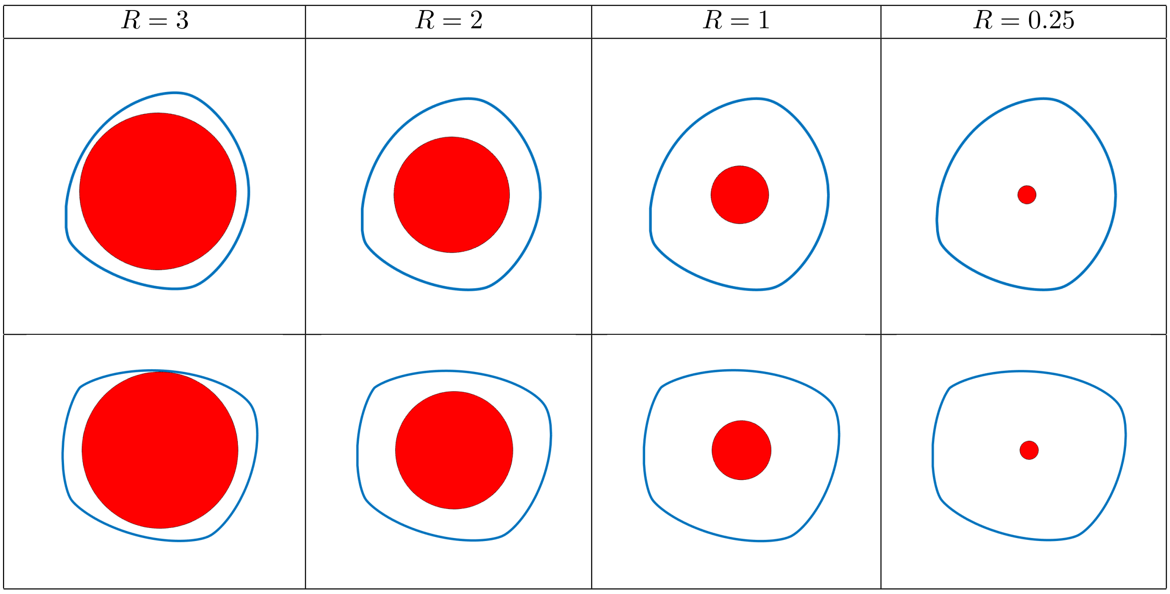


Figure 11: Optimal placement of spherical sensors with different radii $R \in \{0.25, 1, 2, 3\}$.

At last, we note that solving problem (17) with $R = 0$ is equivalent to finding the Chebychev center of Ω that is the center of the minimal-radius ball enclosing the entire set Ω, see Figure 12. This center has been considered by several authors in different settings especially in functional analysis, we refer for example to [1, 12, 15, 16].

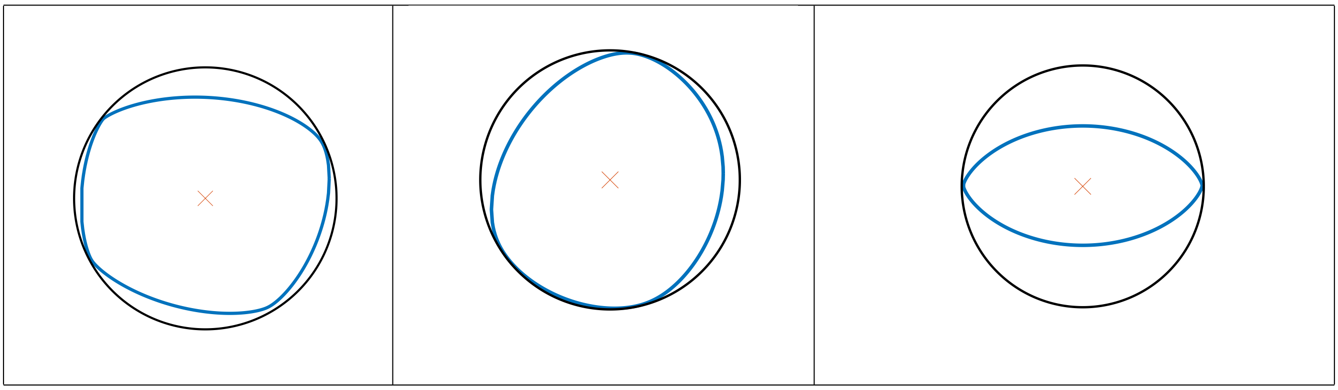


Figure 12: Chebychev centers and circumcircles of different convex sets.

6 Conclusion and perspectives

The problems studied in the present paper involve the distance function, that is quite difficult to deal with from a numerical perspective especially when performing numerical shape optimization. We would like to mention that it can be interesting to use a suitable approximation of the distance function based on some PDE results in same spirit of K. Crane et al. in [7], where the authors introduce a new approach to computing distance based on a heat flow result of Varadhan [25], which says that the geodesic distance $\phi(x, y)$ between any pair of points x and y on a Riemannian manifold can be recovered via a simple pointwise transformation of the heat kernel:

$$\phi(x, y) = \lim_{t \longrightarrow 0} \sqrt{-4t \log k_{t,x}(y)},$$

where $k_{t,x}(y)$ is called the heat kernel, which measures the heat transferred from a source x to a destination y after time t. We refer to [7] and [25] for more details and to [23] for an extension to graphs. In the same spirit, one could use a suitable approximation of the distance function in terms of the solution of an elliptic PDE, inspired by the following classical result:

Theorem 14 *([25, Th. 2.3])*
Let Ω be an open subset of $\mathbb{R}^n$ and $\varepsilon > 0$, we consider the problem

$$\left\{ \begin{array}{rl} w_\varepsilon - \varepsilon \Delta w_\varepsilon = 0 & \text{in } \Omega \\ w_\varepsilon = 1 & \text{on } \partial\Omega \end{array} \right. \tag{19}$$

We have

$$\lim_{\varepsilon \to 0} -\sqrt{\varepsilon} \ln w_\varepsilon(x) = d(x, \partial\Omega) := \inf_{y \in \partial\Omega} \|x - y\|,$$

uniformly over compact subsets of Ω.

In the following figures, we plot the approximation of the distance function to the boundary obtained via the result of Theorem 14.

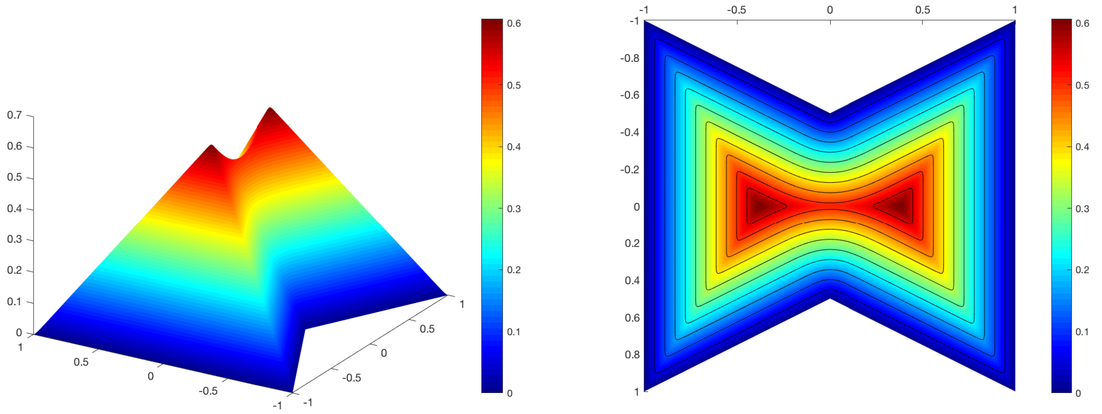


Figure 13: Approximation of the distance function to the boundary via Varadhan's result of Theorem 14, where Matlab's toolbox `'PDEtool'` to solve problem (19), with $\varepsilon = 10^{-4}$.

We note that there are other results of approximation of the distance function via PDEs, see [9] and references therein. We recall for example the following result of Bernd Kawohl:

Theorem 15 *([11, Th. 1]) We consider the problem*

$$\left\{ \begin{array}{rl} -\Delta_p u_p = 1 & \text{in } \Omega \\ u_p = 0 & \text{on } \partial\Omega \end{array} \right.$$

where Δ_p corresponds to the p-Laplace operator, defined as follows

$$\Delta_p v = div(|\nabla v|^{p-2} \nabla v).$$

We have

$$\lim_{p \to +\infty} u_p(x) = d(x, \partial\Omega) \text{ uniformly in } \Omega.$$

One advantage of such approximation methods is that they allow to introduce relevant PDE based problems that may be easier to consider from a numerical point of view than the initial problems involving the distance function and that are of intrinsic interest. Let us conclude by presenting some examples of such problems:

- **Case of multiple sensors on sets and networks.** A natural problem related to the topic of the paper is to optimally place N sensors (S_k) inside a set or a network Ω in such a way that any point in Ω is "easily" reachable from one of the sensors. This problem can be mathematically formulated as follows:

$$\min\{d^H(\Omega, \cup_{k=1}^N S_k) = \max_{y\in\Omega} d(y, \cup_{k=1}^N S_k) \ \mid \forall k \in [\![1,N]\!],\ S_k \subset \Omega\}. \tag{20}$$

where $d(y, \cup_{k=1}^N S_k)$ is the minimal (geodesic if Ω is a network) distance from the point y to the union of the sensors. If we consider (v_ε) a family of functions approximating the distance function $y \longmapsto d(y, \cup_{k=1}^N S_k)$ when ε goes to 0 (such as the ones defined in Theorems 14 and 15), we may consider approximating problem (20) by the following one

$$\min\{\max_{y\in\Omega} v_\varepsilon \ \mid \forall k \in [\![1,N]\!],\ S_k \subset \Omega\}. \tag{21}$$

The advantage of such approximated problems is that they involve elliptic equations that are much easier to deal with from both theoretical and numerical points of views.

Once problem (21) is solved, the next natural step would be to justify that the obtained solutions converge to solutions of the initial problem (20). This is classically done by proving Γ-convergence results, see for example [14, Section 6].

- **About the average distance problem.** Given a set $\Omega \subset \mathbb{R}^n$ and a subset $\Sigma \subset \Omega$, the average distance to Σ is defined as follows:

$$\mathcal{J}_p(\Sigma) := \int_\Omega d(x,\Sigma)^p dx,$$

where p is a positive parameter. The main focus here is to study the shapes Σ that minimize the average distance and investigate their properties such as symmetries and regularity. This problem has been introduced in [5, 6] and was studied by many authors in the last years. For a presentation of the problem, we refer to [13] and to the references therein for related results. Even if these problems are easy to formulate, they are quite difficult to tackle both theoretically and numerically. It is then interesting use the approximation results of the distance function to approximate the functional $\mathcal{J}_p$ by some functional $\mathcal{J}_{p,\varepsilon}(\Sigma) := \int_\Omega v_\varepsilon^p dx$, where (v_ε) is a family of functions uniformly converging to $d(\cdot,\Sigma)$ on Ω when ε goes to 0. We are then led to consider shape optimization problems of functionals involving solutions of simple elliptic PDEs. Several results for such functionals are easier to obtain such as Hadamard formulas for the shape derivatives which are of crucial importance for numerical simulations.

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