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Hamiltonization of the nonholonomic system given by a homogeneous ball rolling on a convex surface of revolution

Orientador: Profa. Paula Balseiro

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HAMILTONIZATION OF THE NONHOLONOMIC SYSTEM GIVEN BY A HOMOGENEOUS BALL ROLLING ON A CONVEX SURFACE OF REVOLUTION

Tese apresentada por **Luis Pánfilo Yapu Quispe** ao Curso de Doutorado em Matemática - Universidade Federal Fluminense, como requisito parcial para a obtenção do Grau de Doutor. Linha de Pesquisa: mecânica geométrica.

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A mi familia.

A la memoria de mi abuela.

Abstract

In this thesis, we study geometrical aspects of nonholonomic systems in connection with the hamiltonization problem. In particular, we study the classical example of an homogeneous ball rolling without sliding in the interior side of a convex surface of revolution from a geometric perspective and we observe that it is *hamiltonizable* after a reduction process by a Lie group.

Indeed, we compute the nonholonomic bracket, which is an almost Poisson bracket describing the dynamics, and we observe that the associated reduced bracket, defined on the reduced space, is not Poisson. Afterwards, following [5, 6, 8], we find a gauge transformation that preserves the dynamics and as a consequence we compute a new reduced bracket that is Poisson and describes the reduced dynamics. Moreover, we observe that this new bracket is rank-two on a dense set of the reduced phase space and has a symplectic foliation determined by the level sets of two Casimirs induced by first integrals of the system that are *horizontal gauge momenta* [41]. From the theoretical point of view, we extendend some results given in [6] so that the current example fits in that framework.

Resumo

Nesta tese estudamos aspectos geométricos de sistemas não-holônomos en conexão com o problema de hamiltonização. Em particular, estudamos o exemplo clássico de uma bola homogênea rolando sem deslizar no lado interior de uma superfície de revolução convexa de um ponto de vista geométrico e observamos que é *hamiltonizável* após uma redução pela ação de um grupo de Lie.

Começamos calculando o colchete não-holônomo, que é um colchete almost Poisson que descreve a dinâmica, e observamos que o colchete reduzido associado, definido no espaço reduzido, não é Poisson. Logo, seguindo [5, 6, 8], encontramos uma *transformação de gauge* que preserva a dinâmica e como consequência calculamos um novo colchete reduzido que é Poisson e descreve a dinâmica reduzida. Alem disso, observamos que esse novo colchete tem posto dois num subconjunto denso do espaço reduzido onde tem uma folhação simplética determinada pelos conjuntos de nível de dois Casimires induzidos por integrais primeiras do sistema que são *horizontal gauge momenta* [41]. Desde o ponto de vista teorico, generalizamos resultados dados em [6] para que o exemplo possa ser estudado usando essas ferramentas.

Contents

Introduction

Mechanical systems are described by positions and velocities (in *Lagrangian mechanics*) or positions and momenta (in *Hamiltonian mechanics*). Sometimes not all configurations of position and velocities/momenta are allowed and the system is then called *constrained*. Nonholonomic systems are defined by constraints in the velocities which cannot be reduced to equations in the positions (i.e. the constraints are not integrable). Those systems describe typically objects rolling over surfaces with non-sliding constraints.

Standard references that develop the mathematical formalism of mechanics are the books of Arnold [3], Abraham & Marsden [1], Marsden & Ratiu [71], Cushman & Bates [35] and D. Holm [53, 54]. Books treating nonholonomic mechanics and related systems are Neimark & Fufaev [76], Bloch et al. [15], Cortés [32] and Cushman, Duistermaat & Sniatycki [36].

The problem of a homogeneous ball constrained to roll without sliding inside a convex surface of revolution with vertical axis under the action of gravity has been treated since Routh [79] and more recently caught the attention of reserchers studing conserved quantities, integrability and qualitative aspects of the dynamics, [24, 40, 52, 77, 88]. See an illustration of the mechanical system in Fig. 1.

In this thesis we perform the hamiltonization of this example using tools of differential geometry and the existence of first integrals which are related to the symmetries of the system [5, 6, 8, 47]. More precisely, we show that the reduced system is described by a Poisson structure induced by the reduction by symmetries of an almost Poisson bracket which is obtained by a *gauge transformation* of the *nonholonomic*

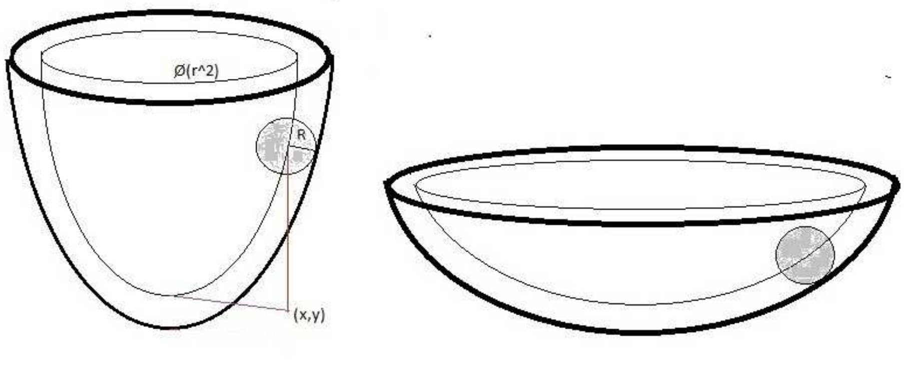


Figure 1: Ball inside a convex surface of revolution.

bracket following the ideas of [8] and extending some of the theory in [6] so that the current example fits in that framework.

Nonholonomic systems

From a more geometric point of view it is known that the equations of motion for a mechanical system with nonholonomic constraints cannot be formulated as a classical Hamiltonian system with respect to a symplectic or Poisson structure. Instead, they are written using an almost Poisson bracket called *nonholonomic bracket* which fails to satisfy the Jacobi identity [55, 69, 86]. More precisely, let Q denote the configuration manifold of the mechanical system. The nonholonomic constraints are described by a *nonintegrable* distribution D on Q and from the Lagrangian $L : TQ \to \mathbb{R}$ one constructs a triple $(\mathcal{M}, \{\cdot,\cdot\}_{nh}, H_{\mathcal{M}})$, where $\mathcal{M} \subset T^*\mathcal{M}$ is the *constraint manifold*, $\{\cdot,\cdot\}_{nh}$ is the *nonholonomic bracket* on $\mathcal{M}$ and $H_{\mathcal{M}}$ denotes the Hamiltonian restricted to $\mathcal{M}$. The dynamics of the nonholonomic system is given by the integral curves of the vector field X_{nh} called *nonholonomic vector field* defined with respect to the almost Poisson bracket $\{\cdot,\cdot\}_{nh}$,

$$X_{nh} = \{\cdot, H_{\mathcal{M}}\}_{nh}. \tag{I.1}$$

It is worth mentioning here that the bracket $\{\cdot,\cdot\}_{nh}$ is never Poisson because of the nonintegrability of the distribution of permitted velocities D.

When the nonholonomic system admits *symmetries* given by the action of a Lie group, it is possible to perform a reduction obtaining a reduced system (with fewer degrees of freedom). More precisely, let G denote the Lie group symmetry of the nonholonomic system which means that the G-action preserves the triple $(\mathcal{M}, \{\cdot,\cdot\}_{nh}, H_{\mathcal{M}})$. By performing the reduction by the symmetry group we get the reduced triple $(\mathcal{M}/G, \{\cdot,\cdot\}_{red}, H_{red})$, where $\{\cdot,\cdot\}_{red}$ is, in principle, an almost Poisson bracket. The *reduced dynamics* is given by the integral curves of the vector field X_{red} given by

$$X_{red} = \{\cdot, H_{red}\}_{red}. \tag{I.2}$$

There are several differences between hamiltonian dynamics and nonholonomic dynamics. Nonholonomic systems are not variational, in the sense that equations of motion are derived from Lagrange d'Alembert Principle and not from Hamilton's Principle. Conservation of energy also occurs in nonholonomic systems though, often, these systems do not have associated momentum conservation laws as Hamiltonian systems. In other words, the presence of symmetries does not necessarily induce first integrals (there is no Noether Theorem). Examples of this fact are the rolling disk, the rattleback and the snakeboard [16]. On the other hand, for Hamiltonian systems we can assert the preservation of volume on the phase space but, again this is not the case for nonholonomic systems [14]. This fact leads to interesting asymptotic stability in some cases, despite energy conservation.

Finally, we see that if we are interested in the integrability of a nonholonomic systems (in the sense of the presence of a foliation by tori of the manifold), we cannot use Arnold-Liouville Theorem since the bracket describing the dynamics is not Poisson[3, 35]. In this case, there are other theorems studying integrability whose formulations are independent of the geometric structure describing the dynamics, see e.g. [4, 18].

The hamiltonization problem and its geometric approach

As it was observed in [24, 58, 77, 40, 46, 6, 47] in a number of examples, the reduced equations of motion allow a Hamiltonian formulation (sometimes after a time reparametrization), recalling that the equations of motion for a nonholonomic system are never hamiltonian before reduction. When this phenomenon occurs, we say that the system is *hamiltonizable*. The question of whether a nonholonomic system admits a hamiltonian formulation after a reduction process is called the *hamiltonization problem*, see [8, 6, 21, 22, 23, 47, 43, 57, 58, 87, 65, 87].

Recall that, from a geometric standpoint, the dynamics of a nonholonomic system is described by the *nonholonomic bracket* as in (I.1) which is an *almost* Poisson bracket. However, after a reduction process, the reduced bracket may become a Poisson bracket. Therefore, the reduced dynamics will have a hamiltonian formulation in terms of a Poisson bracket. Hence, from a geometric perspective, we say that the nonholonomic system is hamiltonizable if there exists a Poisson bracket describing the reduced dynamics (in particular, if $\{\cdot,\cdot\}_{red}$ is Poisson).

$$
\begin{tikzcd}
\boxed{\begin{array}{c}(\mathcal{M}, \{\cdot,\cdot\}_{nh}) \\ X_{nh} = \{\cdot, H_{\mathcal{M}}\}_{nh}\end{array}} \arrow[r, squiggly, "\substack{\text{geometric}\\ \text{structure}}"] \arrow[d, "\substack{\text{reduction}\\ \text{by symmetries}}"'] & \text{almost Poisson manifold} \arrow[d, "\substack{\text{reduction}\\ \text{by symmetries}}"] \\
\boxed{\begin{array}{c}(\mathcal{M}/G, \{\cdot,\cdot\}_{red}) \\ X_{red} = \{\cdot, H_{red}\}_{red}\end{array}} \arrow[r, squiggly] & \begin{array}{c}\text{almost Poisson manifold?} \\ \text{Poisson manifold?}\end{array}
\end{tikzcd}
$$

As we already mentioned, hamiltonian systems have strong properties that, in general, are not satisfied by nonholonomic systems. If after a reduction process, the reduced nonholonomic system is hamiltonian then this reduced system will have all the known properties of hamiltonian systems. This process is central to study different aspects of nonholonomic systems such as integrability, Hamilton-Jacobi theory, and even numerical methods (e.g., variational integrators) [40, 19, 20, 17, 75, 32, 33, 67, 49, 50].

In this thesis we follow the techniques of [5, 6, 8, 46, 47] using *gauge transformations by a 2-form*, introduced by Severa and Weinstein in [80], in order to hamiltonize our nonholonomic system. The idea is that we use a 2-form B to generate another almost Poisson bracket $\{\cdot,\cdot\}_B$ on $\mathcal{M}$ by means of a gauge transformation. Then, if we impose the *dynamical* condition that $\mathbf{i}_{X_{nh}}B = 0$ (see [46, 8]) the new bracket $\{\cdot,\cdot\}_B$ will also describe the dynamics, i.e.,

$$X_{nh} = \{\cdot, H_{\mathcal{M}}\}_B,$$

and we say that B defines a *dynamical gauge transformation.*

In the presence of symmetries, if $\{\cdot,\cdot\}_B$ is G-invariant, it can be reduced to the quotient space $\mathcal{M}/G$ and then, we obtain a new reduced bracket $\{\cdot,\cdot\}^B_{red}$ that describes the reduced dynamics

$$X_{red} = \{\cdot, H_{red}\}^B_{red}.$$

It was observed in [8, 46] that the reduced brackets $\{\cdot,\cdot\}_{red}$ and $\{\cdot,\cdot\}^B_{red}$ may have different properties, for example one can be Poisson while the other not. The geometrical approach to hamiltonization proposed in [46] and then developed in [5, 6, 8, 47] is to find a B so that the reduced bracket $\{\cdot,\cdot\}^B_{red}$ is Poisson (or conformally Poisson, or twisted Poisson). If such a B is found, then the reduced nonholonomic system is described by a Poisson bracket and we say also that the system is *hamiltonizable.* We illustrate the gauge transformation by a 2-form B with the following diagram:

$$\begin{tikzcd}
\boxed{\begin{array}{c} (\mathcal{M}, \{\cdot,\cdot\}_{nh}) \\ \text{Not Poisson} \end{array}} \arrow[r, squiggly, "\text{gauge transformation}"] \arrow[d, "\text{reduction by symmetries}"'] & \boxed{\begin{array}{c} (\mathcal{M}, \{\cdot,\cdot\}_{B}) \\ \text{Not Poisson} \end{array}} \arrow[d, "\text{reduction by symmetries}"] \\
\boxed{\begin{array}{c} (\mathcal{M}/G, \{\cdot,\cdot\}_{red}) \\ X_{red} = \{\cdot, H_{red}\}_{red} \\ \text{Poisson? / almost Poisson?} \end{array}} & \boxed{\begin{array}{c} (\mathcal{M}/G, \{\cdot,\cdot\}^B_{red}) \\ X_{red} = \{\cdot, H_{red}\}^B_{red} \\ \text{Poisson? / almost Poisson?} \end{array}}
\end{tikzcd} \tag{I.3}$$

In this thesis, we study the hamiltonization problem of the set of examples described by a homogeneous ball rolling without sliding on a convex surface of revolution using the approach of the diagram (I.3). In particular, we study first integrals $J^{(1)}$ and $J^{(2)}$ induced by the presence of symmetries (called *horizontal gauge momenta*, [41]) and we observe that the corresponding hamiltonian vector fields $\pi^\sharp_{nh}(dJ^{(1)})$ and $\pi^\sharp_{nh}(dJ^{(2)})$ are not vertical with respect to the orbit projection $\rho : \mathcal{M} \to \mathcal{M}/G$. We compute a 2-form B so that $\pi^\sharp_B(dJ^{(1)})$ and $\pi^\sharp_B(dJ^{(2)})$ are, in fact, vertical. As a consequence of the G-invariance of the first integrals $J^{(1)}$ and $J^{(2)}$ they become Casimirs of the reduced bracket $\{\cdot,\cdot\}^B_{red}$. We use these Casimirs to prove that the reduced bracket $\{\cdot,\cdot\}^B_{red}$ is Poisson and moreover, it describes the reduced dynamics. This approach to find a 2-form B was also studied in [6] and [47], however the present work was done independently. In particular, we extended some of the results in [6] so that this example fits in the theory.

Previous work on hamiltonization

One very interesting example of hamiltonization concerns the Chaplygin ball or Chaplygin sphere. Even though the formulation and integration of the equations of motion dates to Chaplygin [29], the hamiltonian property of the reduced equations was only discovered in 2001 by Borisov and Mamaev [21]. The geometric understanding of this example has been performed by Luis García-Naranjo [46] and his subsequent work with P. Balseiro [8] has explicitly introduced and used the method of gauge transformation to show the hamiltonization of the Chaplygin ball. More recently the hamiltonization of a family of examples consisting of convex solids of revolution rolling on a plane was proved in [6] and [47]. In particular, in [47] the authors presented a coordinate method to find a gauge transformation so that the first integrals becomes Casimirs on the reduced bracket.

The hamiltonization of a mechanical system is also an interesting feature because it is related to integrability. In fact, this may have been the original motivation for Chaplygin to consider the problem of hamiltonization and to prove his *Chaplygin reducing multiplier Theorem* [30]. Recently, Jovanovic [58] proved that the n-dimensional version of the Chaplygin sphere problem is also hamiltonizable and integrable at the zero level of the $SO(n-1)$-moment map, and the hamiltonization was a crucial step in his proof of integrability. Moreover, Fedorov and Jovanovic [43] also used hamiltonization to prove the integrability of the multidimensional Veselova system.

More related to our mechanical example, Borisov, Mamaev and Kilin discovered that a number of classical nonholonomic systems with symmetries give rise to reduced systems which are hamiltonian with respect to a Poisson structure after a time reparametrization [24, 25]. In these references, rank-two Poisson structures are proven to exist for a variety of examples in which a heavy body rolls on a surface. This includes the so-called Routh sphere (a ball rolling on a plane but where its geometric center does not coincide with its center of mass), the disk rolling on a plane and the sphere rolling inside a convex surface of revolution. A detailed study of the form of these Poisson tensors has been developed in [77], where it was also observed that time reparametrization is not necessary for rank-two bivectors. In all these cases, the reduced systems share common features: their phase space is four-dimensional, they possess three independent integrals of motion and their dynamics is periodic (at least in some open dense subset of the reduced phase space). One of the integrals of motion is the Hamiltonian of the reduced system and the other two are Casimirs of the reduced Poisson structure. For the sphere rolling inside a convex surface of revolution, the authors in [40] showed that the existence of the reduced Poisson structure is linked to a strong property: the fact that the reduced periodic orbits are the fibers of a locally trivial fibration. The periodicity property was already known and has been studied by Hermans [52], who also showed that the reconstructed dynamics is quasi-periodic on tori of dimension at most three.

Main results in this thesis

We now present the main results obtained in this thesis. In our geometric framework, a nonholonomic system is then described by a triple $(\mathcal{M}, \{\cdot,\cdot\}_{nh}, H_{\mathcal{M}})$ and the reduction by the proper G-symmetry defines a triple $(\mathcal{M}/G, \{\cdot,\cdot\}_{red}, H_{red})$. In order to verify the failure of the Jacobi identity for the brackets $\{\cdot,\cdot\}_{nh}$ and $\{\cdot,\cdot\}_{red}$, we use the *Jacobiator* formulas proved in [5]. These formulas depend on the choice of a G-invariant vertical complement of the constraints (2.3.10), that we denoted by W. In [6] was proved that, for a proper action, a vertical complement W (of constant rank) always exists and in this thesis we observe that it is also possible to choose W so that it is G-invariant, see Prop. 2.3.2.

Afterwards, in Chapter 3, we study dynamical gauge transformations by a 2-form B and first integrals of a nonholonomic system induced by the presence of symmetries (horizontal gauge momenta). In particular, we characterize the properties that B has to satisfy in order to transform the horizontal gauge momenta of the system into Casimirs of the reduced bracket $\{\cdot,\cdot\}_{red}^B$, Prop. 3.3.2. We study also the system of equations defining the 2-form B so that, if the nonholonomic system has k G-invariant horizontal gauge momenta (functionally independent), then the k functions become Casimirs of the reduced bracket $\{\cdot,\cdot\}_{red}^B$.

Next, we prove that if the amount of Casimirs coincides with the rank of $S := D \cap V$, where V is the vertical space associated to the G-action, then the characteristic distribution of $\{\cdot,\cdot\}_{red}^B$ is involutive, Prop. 3.4.1. This fact was already observed in [6] for the particular case of *vertical symmetries* (Remark 2.3.1) and here we extend it for the general case.

Finally, away from singularities we observe that if $\text{rank}(D) = \text{rank}(S) + 1$ and the nonholonomic system admits l (independent) horizontal gauge momenta $\{J_1, ..., J_l\}$ (where $l = \text{rank}(S)$) then there is a unique gauge transformation by a 2-form B which transforms these horizontal gauge momenta into Casimirs of the reduced bracket $\{\cdot,\cdot\}_{red}^B$. Consequently, the reduced bracket $\{\cdot,\cdot\}_{red}^B$ is a rank 2 Poisson bracket and we give an explicit simple formulation in coordinates, Lemma 3.5.4. Under one extra (technical) condition, the mentioned gauge transformation is dynamical, Prop. 3.5.5.

Chapter 4 is devoted to the hamiltonization problem of the mechanical system describing a homogeneous ball rolling without sliding on a convex surface of revolution. First we study the geometrical framework underlying the example and we write the *nonholonomic bracket* $\{\cdot,\cdot\}_{nh}$ on the constraint manifold $\mathcal{M}$. We also perform the reduction by a Lie group G that acts properly on Q obtaining the reduced bracket $\{\cdot,\cdot\}_{red}$ on the *differential space* $\mathcal{M}/G$. In order to verify that $\{\cdot,\cdot\}_{red}$ is not Poisson we compute the formulas characterizing the failure of the Jacobi identity for this specific example, Prop. 4.2.13.

Following [41, 40] we study the existence of two first integrals of this example which are, in fact, horizontal gauge momenta associated to the G-symmetry. Then, in Section 4.3.2, we set the system of equations that the 2-form B has to satisfy in order to be *dynamical* and to transform the horizontal gauge momenta into Casimirs of the reduced bracket $\{\cdot,\cdot\}_{red}^B$. We compute the appropriate dynamical gauge transformation B and we write the corresponding brackets $\{\cdot,\cdot\}_B$ and $\{\cdot,\cdot\}_{red}^B$. Finally we check that $\{\cdot,\cdot\}_{red}^B$ is Poisson using two different approaches. On one hand, we check the Jacobi identity using the formulas computed in [6] adapted to the example. On the other hand, away from the singularities, we check that it is Poisson using Prop. 3.4.1. Hence, we conclude that the example of a homogeneous ball rolling without slipping on a convex surface of revolution is *hamiltonizable* after a *gauge transformation* and a reduction by symmetries as diagram I.3 shows. We ended this section by recalling some integrability and qualitative aspects of the dynamics of this nonholonomic system.

This dissertation is organized as follows. In Chapter 1. we give some preliminaries on nonholonomic mechanical system and present some examples. Chapter 2. gives a presentation of reduction by symmetries, mainly for proper actions. Chapter 3. treats the problem of finding a dynamical gauge transformation from the existence of horizontal gauge momenta. Chapter 4 studies the nonholonomic example and states the main results of the thesis. We show that the first integrals $J^{(1)}$ and $J^{(2)}$ of our system are G-invariant horizontal gauge momenta, and compute the 2-form B giving the gauge transformation of the nonholonomic bracket whose reduction by the symmetry group is Poisson.

Chapter 1

Geometric formalism of nonholonomic systems

Nonholonomic systems are mechanical systems with constraints in the velocities which cannot be derived from constraints in the positions [15, 32, 36, 76]. Typical examples of such mechanical systems are solids rolling over a surface without slipping since in that case the angular velocity due to rolling is tied to the linear velocity of the center of mass of the solid.

One of the initial tasks is to present the geometric framework of nonholonomic mechanics and illustrate the theory with some standard examples. We suppose that all the differentiable objects treated in this dissertation are smooth, i.e C^∞. The standard notation $\mathfrak{X}(Q)$ indicates the set of smooth vector fields on the manifold Q. Finally, given a bundle $E \to Q$, we use the usual notation $\Gamma(E)$ or $\Gamma(Q, E)$ to mean the set of all smooth sections of the bundle.

1.1 Holonomic vs. nonholonomic constraints

Let Q be a n-dimensional differentiable manifold representing the *space of configurations* of a mechanical system. The local charts in this manifold define *generalized coordinates*, i.e. locally Q have coordinates (q^i), $i = 1, \cdots, n$. The work of Lagrange in the second half of the $18th$ century allowed the generalization de Newton's Second Law to the generalized coordinates and the derivation of the equations of motion from physically motivated variational principles.

Let us recall some basic facts from classical lagrangian mechanics. We start from a *Lagrange function* (or symply *Lagrangian*) $L : TQ \to \mathbb{R}$, $(q^i, v^i) \mapsto L(q^i, v^i)$. In mechanical examples we take as Lagrangian the difference between the kinetic energy and the potential energy V, thus $L(q^i, v^i) = \frac{1}{2}\kappa_{ij}(q)v^iv^j - V(q)$, which is called a *Lagrangian of mechanical type* and κ is a Riemmanian metric called the *kinetic energy metric*.

Applying the Hamilton's principle (sometimes called principle of least action, see discussion in Section 1.2 of [15]), we are led to the Euler-Lagrange equations of motion:

$$\frac{d}{dt}\frac{\partial L}{\partial v^i}(q(t),\dot{q}(t)) - \frac{\partial L}{\partial q^i}(q(t),\dot{q}(t)) = 0, \quad i = 1,\cdots,n. \tag{1.1.1}$$

On the other hand, Hamiltonian mechanics is defined in the cotangent bundle T^*Q. The *Hamiltonian* function $H : T^*Q \to \mathbb{R}$, $(q^i, p_i) \mapsto H(q^i, p_i)$ induces the *Hamilton's equations* given by

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad i = 1,\cdots,n, \tag{1.1.2}$$

and the function H (independent of the time) is a conserved quantity of the flow of Hamilton's equation. In fact, the flow of (1.1.2) has other properties such as the existence of an invariant measure in T^*Q and thus it cannot have attracting or repelling fixed points. These properties are based on the symplectic structure associated canonically to T^*Q, as we recall in Section 1.3.

The relationship between the Lagrangian and Hamiltonian mechanics is given by means of the *Legendre transformation.*

Definition 1.1.1. Let $L : TQ \to \mathbb{R}$ be a given Lagrangian. We define the *fiber derivative* $\mathbb{F}L : TQ \to T^*Q$ by

$$\langle \mathbb{F}L(v), w\rangle = \frac{d}{dt}|_{t=0} L(x, v + tw),$$

for any $w \in \Gamma(TQ)$, where x is the base point of the vector v. In coordinates (q^i, v^i) of TQ and (q^i, p_i) of T^*Q we have

$$\mathbb{F}L(q^i, v^i) = \left(x^i, p_i = \frac{\partial L}{\partial v^i}\right).$$

The change of variables induced by $\mathbb{F}L$ is called the *Legendre transformation*, also denoted *Leg*.

Under the assumption of a mechanical-type Lagrangian, the Legendre transformation $Leg = \kappa^\flat : TQ \to T^*Q$ is a global diffeomorphism and such Lagrangians are called *hyperregular*, where $\kappa^\flat(X)(Y) = \kappa(X,Y)$ for $X, Y \in \Gamma(TQ)$. It that case, given a hyperregular Lagrangian one defines the corresponding Hamiltonian H by $H := E \circ (\mathbb{F}L)^{-1}$, where $E : TQ \to \mathbb{R}$ is called the *energy* associated to L and is defined in coordinates (q^i, v^i) by

$$E = v^i \frac{\partial L}{\partial v^i} - L.$$

The next example illustrated a simple mechanical systems with constraints in the positions.

Example 1.1.1. *Consider a pendulum oscillating under the influence of gravity. The simplest model consists of a mass $m = 1$ attached at one end of a massless rod of length $l = 1$, the other end being fixed to some point considered as the origin. Then,* *the mass has coordinates* $(x, y, z) \in \mathbb{R}^3$. *In the absence of Coriolis' force, equilibrium of forces and other basic considerations permit to conclude that the motion is actually constrained to a plane, which we can take to be the* xy*-plane with coordinates* $(x, y, 0)$. *The constraint fixed by the length of the rod indicates that the coordinates* x *and* y *lie on the submanifold* $\{(x, y, 0) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$. *Then it becomes clear that the system is better described letting the configuration space to be the 1-dimensional manifold* S^1, *with only one generalized coordinate, the angle* ϕ.

The Lagrangian is given by

$$L(\phi, \dot{\phi}) = \frac{1}{2}\dot{\phi}^2 - g\cos(\phi),$$

where g *is the constant of gravity. From Euler-Lagrange equations (1.1.1) one derives easily the dimensionless* ($g = 1$) *equations of motion:*

$$\ddot{\phi} = -\sin(\phi).$$

A good qualitative comprehension of the structure of the orbits of this nonlinear differential equation can be achieved considering the phase space, $S^1 \times \mathbb{R}$, *that is a cylinder with coordinates* (ϕ, p), *where the equations of motion are:*

$$\begin{aligned} \dot{\phi} &= p \\ \dot{p} &= -\sin(\phi). \end{aligned}$$

This example and the orbits on the phase space are illustrated in Chapter 2 of [31].

In the following we are interested in constrained mechanical systems subject to constraints in the velocities. More precisely we study mechanical system with *linear constraints* in the velocities. In mechanical examples these constraints are originated by instance when one object rolls over another. In that case the velocity of the center of mass of the object is tied with the angular velocity. In fact there is a linear relation between both velocities and therefore, for each point q of the configuration space Q, only a linear subspace D_q of the tangent space T_qQ defines the *permitted velocities*. More precisely we have the following

Definition 1.1.2. A *distribution* D on the manifold Q is a collection of linear subspaces of TQ such that for every $q \in Q$ we have $D_q \subset T_qQ$.

A distribution D on Q is called *regular* if the linear subspaces D_q, have the same dimension for any $q \in Q$ and that dimension is called the *rank* of the distribution, otherwise it is called a *singular* or *generalized* distribution. The regular case is actually equivalent to define a subbundle of the tangent bundle TQ. Many aspects of the differential geometry studied in this dissertation are related to almost Poisson and Poisson manifolds where generalized distributions are natural objects. Thus we recall the notion of smoothness in this context.

The (generalized) distribution D on Q is *smooth* at a point $q \in Q$ if any tangent vector $X|_q \in D_q$ can be extended to some neighbourhood U of q to a vector field X

such that $X|_p \in D_p$ for any $p \in U$. The distribution is smooth if it is smooth at any point.

The subbundle $D \subset TQ$ is *involutive* if it is closed with respect to the Lie bracket, i.e. $X, Y \in D$ implies $[X, Y] \in D$. Moreover the subbundle D is called *integrable* if for any $q \in Q$, there exists a (local) submanifold $N \subset Q$ called *(local) integral manifold* such that $T_q N = D_q$. The classic formulation of *Frobenius theorem* claims that a regular smooth distribution (or subbundle) is integrable if and only if it is involutive. We discuss integrability in more detail in Section 1.2.

Constraints on the positions are called *holonomic* and constraints in the velocities which cannot be reduced to holonomic constraints are said *nonholonomic*. More precisely we give the following

Definition 1.1.3. A *mechanical system with constraints in velocities* in the Lagrangian formalism consists in:

1. A smooth manifold Q, called space of configurations.
2. A smooth function $L : TQ \to \mathbb{R}$, called the Lagrangian function.
3. A smooth (regular) distribution $D \subset TQ$ defining the permitted velocities (constraints in velocities).

Definition 1.1.4. A mechanical system with constraints in velocities is holonomic if the distribution D is integrable, otherwise the system is called nonholonomic.

Nonholonomic mechanical systems are qualitatively different from nonholonomic systems in the geometric and dynamical point of view. We summarize some features of both systems in the following table:

Feature	*Holonomic system*	*Nonholonomic system*
Energy	Preserved by the dynamics	Preserved by the dynamics
Volume in phase space	Preserved by the dynamics	Not preserved by the dynamics
Geometry	Integrable structures Symplectic Poisson	Non-integrable structures Almost symplectic Almost Poisson
First integrals	Noether's theorem	Not all symmetries induce first integrals and some first integrals do not come from symmetries.

Example 1.1.2. *(The nonholonomic particle) This is a very simple an ideal model introduced by Rosenberg [78] and used usually in the nonholonomic mechanics literature. It consists of a punctual particle moving in $Q = \mathbb{R}^3$, without potential energy*

and having a linear constraint in the velocities given by

$$\dot{z} = y\dot{x}. \tag{1.1.3}$$

The associated distribution D describing the permitted velocities verifying (1.1.3) *is given by*

$$D = span\{Y_x := \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \frac{\partial}{\partial y}\}. \tag{1.1.4}$$

The computation $[Y_x, \frac{\partial}{\partial y}] = -\frac{\partial}{\partial z}$ shows that the distribution is not involutive. Since D is regular of rank 2 and smooth (generated by vector fields) we conclude by Frobenius theorem that it is not integrable.

Example 1.1.3. *(The vertical rolling disk) The physical system is a disk rolling without slipping on a horizontal plane and consider for simplicity that the plane containing the disk is orthogonal to the horizontal plane, that is the disk is not allowed to 'fall'. We can imagine a coin rolling vertically on a horizontal table. We can imagine a coin rolling vertically without sliding on a horizontal table. The configuration space is $Q = \mathbb{R}^2 \times S^1 \times S^1$, with coordinates (x, y, θ, ϕ), where (x, y) indicates the contact position of the disk and the xy-plane, θ is the rotation angle of the disk, and ϕ indicates the 'heading angle', i.e. the angle between the xz-plane and the plane containing the disk. If R is the radius of the disk then the condition of non-slipping is represented by the (linear) constraints:*

$$\dot{x} - R(\cos\phi)\dot{\theta} = 0, \quad \dot{y} - R(\sin\phi)\dot{\theta} = 0,$$

Then the constraint distribution is given by

$$D = span\{Y_\theta := R\cos(\phi)\frac{\partial}{\partial x} + R\sin(\phi)\frac{\partial}{\partial y} + \frac{\partial}{\partial \theta}, Y_\phi := \frac{\partial}{\partial \phi}\}.$$

Computing the Lie bracket we get $[Y_\theta, Y_\phi] = R(\sin\phi\frac{\partial}{\partial x} - \cos\phi\frac{\partial}{\partial x})$ we conclude that the (rank 2) smooth distribution D is not involutive, then it is not integrable and the constraints are nonholonomic.

1.2 Integrability of distributions and foliations

In general, the constrained distribution D of a nonholonomic system is described by the kernel of certain number of differential 1-forms. Before studying general results, let us consider some examples where the constraints are defined by the kernel of 1-forms and integrability/non-integrability can be directly stated. For a unique constraint, Example 1.2.3 bellow gives a necessary and sufficient condition for the integrability. The general case for regular distributions is treated using the theorem of Frobenius (see Thm. 1.2.3) and for generalized distributions, integrability conditions are given by the theorem o Stefan-Sussmann (see Thm. 1.2.5.

Before stating the theorems, let us present some examples where integrability is illustrated from elementary results of Calculus and differential geometry.

Example 1.2.1. *Consider the 1-form $\omega \in \Omega^1(\mathbb{R}^2 - \{0\})$ given by $\omega = 2xdx + 2ydy$. It is clear that ω is an exact form, i.e. $\omega = df$, where $f(x, y) = x^2 + y^2$ so that the integral curves are given by $f(x, y) = c$, where c is a constant, and form a foliation whose leaves are circles (dimension 1). Which is important here is the fact that the initial constraints have been reduced to a constraint in the positions $f(x, y) = c$, i.e. the constraints are holonomic. Note that for $c > 0$, the configuration space is better described by the manifold S^1 with a local coordinate θ representing the angle.*

Example 1.2.2. *(From [44, p. 92-93]) Take $\omega = yzdx+xzdy+dz$. Now it is not clear if the form is exact as in the last example. In fact, it is not exact, but it possesses and 'integrant factor' it can be shown that $\omega = e^{-xy}d(ze^{xy})$ with 'integral surfaces' (leaves) given by $ze^{xy} = c$, with c a constant. Thus the system is integrable, and suitable coordinates can be found in the leaves where the dynamics is constrained. On the contrary, the more easier-looking form $\omega = dz - ydx - dy$, defines a non integrable distribution as can be verified easily using the following example.*

Example 1.2.3. *More generally, the kernel of a 1-form ω defines a smooth codimension-one distribution. Then, the formula of the exterior derivative gives*

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]), \tag{1.2.5}$$

which implies that if ω is a closed 1-form then its kernel is involutive. In fact, formula (1.2.5) implies more: denoting $D = Ker\ \omega$, then $d\omega|_D = 0$ if and only if D is a (regular) involutive distribution, then integrable.

Example 1.2.4. *(The nonholonomic particle) Continuing with Example 1.1.2, from the linear constraint in the velocities given by $\dot{z} = y\dot{x}$, we obtain easily the constraint 1-form given by*

$$\epsilon^1 = dz - ydx.$$

Using the basis of the constraint distribution D in (1.1.4), we compute easily

$$d\epsilon^1|_D = dx \wedge dy|_D \neq 0,$$

so, by Example 1.2.3, the constraint distribution is nonintegrable and the system is nonholonomic.

There exists a criterion to decide if several 1-forms ω^a, $a = 1, \cdots, k$, are (pointwise) independent, that consists in verifying that $\omega^1 \wedge \cdots \wedge \omega^k \neq 0$. In this case the issue of integrability is more complicated and we need an appropriate formulation of the Frobenius Theorem.

The presentation here is based on [2], others useful references are [74] and [44]. Recall that the classic formulation of Frobenius theorem claims that a regular smooth distribution (or subbundle) is integrable if and only if it is involutive.

The (global) theorem of Frobenius is related to the concept of *foliation*. In fact the integral manifolds can be smoothly collected such that they are disjoint and the union of all them is Q. More precisely we have the following

Definition 1.2.1. A (regular) *foliation* $\mathcal{F}$ of dimension k of the n-dimensional manifold Q is a decomposition $\mathcal{F} = \{\mathcal{F}_\alpha, \alpha \in A\}$ of the manifold Q in arc-connected disjoint sets called leaves and verifying the following property: for any $q \in Q$ there exists a local chart $\phi = (x^1, \cdots, x^k, y^1, \cdots, y^{n-k}) : U \to \mathbb{R}^k \times \mathbb{R}^{n-k}$ such that the connected components of $\mathcal{F}_\alpha \cap U$ have the form

$$\{q \in U : y^1(q) = const., \cdots, y^{n-k}(q) = const.\}.$$

In fact each leaf of the foliation is a k-dimensional immersed submanifold of Q. In general the leaves are not embedded as any leaf can accumulate over itself. We refer the reader to the references mentioned in the beginning of this section for the proofs of those facts. We that that a distribution D *is tangent to a foliation* $\mathcal{F}$, and denoted $D = T\mathcal{F}$ if for any $q \in Q$, we have $D_q = T_q\mathcal{F}_\alpha$, where $\mathcal{F}_\alpha$ is the unique maximal integral manifold passing through q. A *maximal integral manifold* is a connected integral manifold which contains every connected integral manifold with which it has a point in common. Consequently, the global version of Frobenius Theorem is stated as:

Theorem 1.2.2. *(Global Frobenius theorem) For every smooth involutive (regular) distribution D on the smooth manifold Q, the collection of all maximal integral manifolds forms a foliation $\mathcal{F}$ of Q, and $D = T\mathcal{F}$.*

We are interested in a dual formulation of the Frobenius theorem in terms of differential forms.

Theorem 1.2.3. *(Frobenius Theorem, [2]) Let M be an n-manifold and $E \subset TM$ be a subbundle with k-dimensional fiber, and $I(E)$ the associated ideal. The following are equivalent:*

1. *E is integrable.*
2. *E is involutive.*
3. *For every point of M there exists an open set U and $\omega^1, ..., \omega^{n-k}$ in $\Omega^1(U)$ generating $I(E)$ such that:*
$$d\omega^i = \sum_{j=1}^{n-k} \omega^{ij} \wedge \omega^j \tag{1.2.6}$$
for some ω^{ij} in $\Omega^1(U)$.
4. *As in the last item, but ω^i verifying*
$$d\omega^i \wedge \omega^1 \wedge \cdots \wedge \omega^{n-k} = 0. \tag{1.2.7}$$

We illustrate the theorem with some examples.

Example 1.2.5. *Consider the following 1-form on $\mathbb{R}^2$ in standard (x, y) coordinates:*

$$\omega = P(x, y)dx + Q(x, y)dy. \tag{1.2.8}$$

We seek a solution to $\omega = 0$. *Using the Frobenius theorem, since* $d\omega \wedge \omega = 0 \in \Omega^3(\mathbb{R}^2)$, *integral manifolds exist and are unique. In* $\mathbb{R}^3$, $d\omega \wedge \omega = 0$ *is a genuine condition. Take the 1-form:*

$$\omega = P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz, \tag{1.2.9}$$

so that

$$d\omega \wedge \omega = \left[P(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}) + Q(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}) + R(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\right] dxdydz. \tag{1.2.10}$$

Therefore the distribution defined by $Ker\omega$ *is integrable if and only if the term in square brackets is zero.*

Example 1.2.6. *(Pfaffian systems) Let consider 1-forms* $\epsilon_i \in \Omega^1(U)$, $i = 0, \cdots, n-k$, *where* $U \subset \mathbb{R}^n$ *is an open set. A solution to the system:* $\epsilon^i = 0$, $i = 0, \cdots, n-k$, *is a* k*-dimensional submanifold* N *of* U *such that all* ϵ_i *annihilates* TN. *If the forms* ϵ^i *are independent, their common kernel defines a subbundle* $E = \{v \in T(U) : \epsilon^i(v) = 0, i = 1, .., n-k\}$. *By Frobenius theorem this subbundle is integrable if*

$$d\epsilon^i \wedge \epsilon^1 \wedge \cdots \wedge \epsilon^{n-k} = 0, \quad i = 1, \cdots, n-k.$$

Example 1.2.7. *(The vertical rolling disk) We continue with the example started in Example 1.1.3. The configuration space is* $Q = \mathbb{R}^2 \times S^1 \times S^1$, *with coordinates* (x, y, θ, ϕ). *If* R *is the radius of the disk then the condition of non-slipping is represented by the (linear) constraints:*

$$\dot{x} - R(\cos\phi)\dot{\theta} = 0, \quad \dot{y} - R(\sin\phi)\dot{\theta} = 0,$$

with associated constraint 1*-forms*

$$\epsilon^1 = dx - R\cos(\phi)d\theta, \quad \epsilon^1 = dy - R\sin(\phi)d\theta.$$

We have $d\epsilon^i \wedge \epsilon^1 \wedge \epsilon^2 \neq 0$, $i = 1, 2$, *so the constraint distribution is nonintegrable and the system is nonholonomic.*

Example 1.2.8. *(From [2]) In* $\mathbb{R}^n$ *with coordinates* $(q^i) = (r^\alpha, s^a) \in \mathbb{R}^{(n-m)} \times \mathbb{R}^m$, *consider the following equation:*

$$\omega^a(q) = ds^a - A^a_\alpha(r,s)dr^\alpha, \quad a = 1, \cdots, m. \tag{1.2.11}$$

By Frobenius theorem, the subbundle $E = \{v \in T(U) : \varepsilon^i(v) = 0, i = 1, .., n-k\}$ *is integrable if there exist 1-forms* ω^{ab}*such that*

$$d\omega^a = \sum_{b=1}^{n-m} \omega^{ab} \wedge \omega^b. \tag{1.2.12}$$

A straightforward computation shows that:

$$d\omega^a = K^a_{\alpha\beta} dr^\alpha \wedge dr^\beta + \frac{\partial A^a_\alpha}{\partial s^b} dr^\alpha \wedge \omega^b, \tag{1.2.13}$$

where:

$$K^a_{\alpha\beta} = \frac{\partial A^a_\alpha}{\partial r^\beta} - \frac{\partial A^a_\beta}{\partial r^\alpha} + A^b_\alpha \frac{\partial A^a_\beta}{\partial s^b} - A^b_\beta \frac{\partial A^a_\alpha}{\partial s^b}. \tag{1.2.14}$$

Since $\{dr^\alpha, \omega^a\}$ form a basis of $T^*\mathbb{R}^n$, we see that $K^a_{\alpha\beta} = 0$ iff

$$d\omega^a = \sum_{b=1}^{n-m} \omega^{ab} \wedge \omega^b \tag{1.2.15}$$

for some 1-forms ω^{ab}. We conclude that the system is integrable if and only if $K^a_{\alpha\beta} = 0$.

Generalized foliations and Stefan-Sussmann Theorem

Generalized or singular foliations are the analogous of foliations for generalized distribution. Intuitively the dimension of the leaves can vary.

Definition 1.2.4. A *generalized foliation* $\mathcal{F}$ of the n-dimensional manifold Q is a decomposition $\mathcal{F} = \{\mathcal{F}_\alpha, \alpha \in A\}$ of the manifold Q in smooth immersed arc-connected submanifolds called leaves and verifying the following property: for any $q \in Q$, denote by $\mathcal{F}_q$ the leaf containing q and d its dimension. Then there exists a local chart $\phi = (x^1, \cdots, x^d, y^1, \cdots, y^{n-d}) : U \to \mathbb{R}^k \times \mathbb{R}^{n-k}$ such that the connected components of $\mathcal{F}_q \cap U$ have the form

$$\{q \in U : y^1(q) = 0, \cdots, y^{n-d}(q) = 0\}.$$

and the sets with local coordinates

$$\{q \in U : y^1(q) = const., \cdots, y^{n-d}(q) = const.\},$$

are contained in some leaf $\mathcal{F}_\alpha$ of $\mathcal{F}$.

A foliation $\mathcal{F}$ has an associated *tangent distribution* $D^{\mathcal{F}}$ such that, at each $q \in Q$, $D^{\mathcal{F}}_q$ is the tangent space to the leaf $\mathcal{F}_q$ containing q. Then, the tangent distribution of a smooth generalized foliation is a smooth generalized distribution. A smooth generalized foliation D on Q is an *integrable distribution* if every point $q \in Q$ is contained in a maximal integral manifold of D.

Let $\mathfrak{F}$ be a family of smooth vector fields on Q. The distribution $D^{\mathfrak{F}}$ *generated by* the family $\mathfrak{F}$ is the generalized distribution given, at each $q \in Q$, by $D^{\mathfrak{F}}_q = span\{X_q \in T_qQ : X \in D^{\mathfrak{F}}\}$.

A distribution D is called *invariant* with respect to the family of vector fields $\mathfrak{F}$ if for any $X \in \mathfrak{F}$, the distribution D is invariant under the flow of X, i.e. if ϕ^t_X indicated the flow of X at time t then $(\phi^t_X)_* D_q = D_{\phi^t_X(q)}$ whenever $\phi^t_X(q)$ is well defined.

The following result generalizes Frobenius Theorem for generalized distribution.

Theorem 1.2.5. *(Stefan-Sussman Theorem, [83, 84, 37]) Let D be a smooth generalized distribution on Q. Then the following conditions are equivalent:*

1. *D is integrable.*

2. *D is generated by a family $\mathfrak{F}$ of smooth vector fields, and it is invariant with respect to $\mathfrak{F}$.*

3. *D is the tangent distribution $D^{\mathcal{F}}$ of a smooth generalized foliation $\mathcal{F}$.*

A smooth distribution D on Q is called *locally finitelly generated* if for any $q \in Q$ there is neighbourhood U of q such that there exist a finite number of smooth vector fields $X_1, \cdot, X_n$ in U which are tangent to D, such that any smooth vector field Y in U tangent to D can be written $Y = \sum f_i X_i$ with $f_i \in C^\infty(U)$. In that case, by a result of Hermann [51], any locally finitely generated smooth involutive distribution on Q is integrable.

Example 1.2.9. *The distribution D on $\mathbb{R}^2$ generated by the vector field $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial x}$ has rank 0 at the origin $(0,0)$ and rank 1 otherwise. It is integrable, with foliation by maximal integral manifolds given by the radial rays arising from the origin, and the origin itself (as a leaf).*

We will see many relevant examples of generalized distributions and foliations in the next Section (Poisson hamiltonain system) and in Chapter 2 when considering action of groups.

1.3 Hamiltonian formalism for classical mechanics

Symplectic hamiltonian systems

The basic mathematical objects in the study of the Hamiltonian formalism of classical mechanics are symplectic manifolds.

Definition 1.3.1. An *almost symplectic* manifold is a pair, (M, Ω), where M is a differentiable manifold and Ω is a nondegenerate two-form on M. An almost symplectic manifold is *symplectic* if the Ω is closed.

The nondegeneracy of the 2-form Ω implies the existence of the 'musical' isomorphisms, which are inverse one of another,

$$\begin{aligned} \Omega^\flat : \mathcal{X}(M) &\rightarrow \Omega^1(M), \\ \Omega^\sharp : \Omega^1(M) &\rightarrow \mathcal{X}(M), \end{aligned}$$

defined by $\Omega^\flat(X) = \mathbf{i}_X\Omega$ and $\Omega^\sharp = (\Omega^\flat)^{-1}$. Given a function $f \in C^\infty(M)$, we define the corresponding *Hamiltonian vector field* by

$$X_f = \Omega^\sharp(df), \tag{1.3.16}$$

or equivalently

$$\mathbf{i}_{X_f}\Omega = df. \tag{1.3.17}$$

Vector fields X on M which leave invariant the symplectic form, i.e. $\pounds_X\Omega = 0$, are called *symplectic*. By Cartan's formula Hamiltonian vector field are symplectic.

The cotangent bundle T^*Q of a smooth manifold Q carries a canonical symplectic structure, i.e. (T^*Q, Ω_Q). In fact, as a consequence of Darboux's theorem, every symplectic manifold is locally isomorphic to a cotangent bundle. Let us recall the canonical symplectic structure of T^*Q.

Let Q be a smooth manifold. Consider the cotangent bundle $M := T^*Q$, with projection map $\tau : M \to Q$ and tangent map $T\tau : TM \to TQ$. The cotangent bundle admits a canonical global 1-form called *tautological form* defined, for $p \in T^*_{\tau(p)}Q$, by

$$\Theta_Q(X_p) := \langle T\tau(X), p\rangle.$$

The coordinates (q^i) of Q induces coordinates (q^i, p_i) on M and the local expression of Θ_Q is $\Theta_Q(q^i, p_i) = \sum_i p_i dq^i$. The canonical symplectic form on T^*Q is defined by $\Omega_Q = -d\Theta_Q$. Indeed, this form is clearly closed, with local expression $\sum_i dq^i \wedge dp_i$. The n-th power of Ω_Q is $\pm dq^1 \wedge \cdots \wedge dq^n \wedge dp^1 \wedge \cdots \wedge dp^n \neq 0$, which implies that Ω_Q is nondegenerate and consequently symplectic.

Recall that the nondegeneracy of Ω_Q is equivalent to the fact that $\Omega^\flat_Q|_q : T_qQ \to T^*_qM$ is an isomorphism for any $q \in Q$, which implies that 1-forms and vector fields are in one-to-one correspondence.

Definition 1.3.2. Let H be a smooth function on T^*Q called Hamiltonian function. The unique vector field $X_H \in \mathcal{X}(M)$ verifying

$$\mathbf{i}_{X_H}\Omega_Q = dH \tag{1.3.18}$$

is called the *Hamiltonian vector field associated to H*. The vector field X_H describes the dynamics of the mechanical hamiltonian system and (1.3.18) are the (free) Hamilton's equations written in intrinsic form.

In the local coordinates (q^i, p_i) of T^*Q, it is straightforward to find the form of the Hamiltonian vector field:

$$X_H = \sum_j \left(\frac{\partial H}{\partial p_j}\frac{\partial}{\partial q^j} - \frac{\partial H}{\partial q^j}\frac{\partial}{\partial p_j} \right). \tag{1.3.19}$$

The integral curve of this vector field is a curve written in local coordinates as $(q(t), p(t))$ verifying, at least near a point $(q(0), p(0))$, the equation $(\dot q(t), \dot p(t)) = \dot q(t)\frac{\partial}{\partial q^j} + \dot p(t)\frac{\partial}{\partial p_j} = X_H(q(t), p(t))$ which, by (1.3.19), is equivalent to the classic form of the (free) Hamilton's equations (1.1.2).

Poisson hamiltonian systems

Symplectic manifolds are examples of more general structures defining *Poisson manifolds.* Let us recall the basic definitions and properties of Poisson structures. Our references for this section are [85] and [37].

Definition 1.3.3. A Poisson bracket on a smooth manifold M is an $\mathbb{R}$-bilinear skew-symmetric operation

$$C^\infty(M) \times C^\infty(M) \to C^\infty(M), \quad (f, g) \mapsto \{f, g\}$$

which verifies *Leibniz identity*

$$\{f, gh\} = \{f, g\}h + g\{f, h\},$$

and the *Jacobi identity*

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0,$$

for any $f, g, h \in C^\infty(M)$. The pair $(M, \{\cdot, \cdot\})$ is called a *Poisson manifold*. A bracket verifying all the conditions but not necessarily the Jacobi identity is called an *almost Poisson bracket*.

Every symplectic manifold (M, Ω) is naturally equipped with a bracket of functions given by

$$\begin{aligned} C^\infty(M) \times C^\infty(M) &\to C^\infty(M) \\ (f, g) &\mapsto \{f, g\} := \Omega(X_f, X_g). \end{aligned}$$

The Jacobi identity is equivalent to the fact that the 2-form Ω is closed and then this bracket is a Poisson bracket. Thus every symplectic manifold is a Poisson manifold.

The Poisson bracket $\{\cdot, \cdot\}$ defines a bivector field $\pi \in \Gamma(\bigwedge^2 TM)$ on M by the relation

$$\pi(df, dg) = \{f, g\}, \quad \text{for } f, g \in C^\infty(M).$$

We denote by $\pi^\sharp : T^*M \to TM$ the map given, at each $\alpha, \beta \in T^*Q$, by $\beta(\pi^\sharp(\alpha)) = \pi(\alpha, \beta)$.

Hamiltonian vector fields can in general be defined for Poisson manifolds. By the Leibniz property, given a function $f \in C^\infty(M)$, the map $g \mapsto \{f, g\}$ is a *derivation* and then defines a vector field X_f on M called the *Hamiltonian vector field of f*, such that

$$X_f(g) = -\{f, g\},$$

for any $g \in C^\infty(M)$. It is straightforward to see that this definition coincides with the definition given of symplectic manifolds in (1.3.16).

Hamiltonian vector fields are closed with respect to Lie bracket. More precisely the Jacobi identity implies that the map $f \mapsto X_f$ is a anti-homomorphism of Lie algebras from $C^\infty(M)$ with the Poisson bracket to the Lie algebra of vector fields on M under the usual Lie bracket. In other words the following formula holds

$$[X_f, X_g] = -X_{\{f,g\}}. \tag{1.3.20}$$

The *characteristic distribution* of the Poisson manifold $(M, \{\cdot, \cdot\})$ is the (generalized) distribution C, given at each point $q \in M$, by

$$C_q = \{X_f(q) : f \in C^\infty(M)\}.$$

The dimension of C_q is called the *rank* of the Poisson structure at q, it is denoted $rank\, \pi|_q$ and it can be shown that it is lower-semicontinuous function of q, i.e. for any $q \in M$ there is some neighbourhood U containing q such that $rank\, \pi|_p \geq rank\, \pi|_q$

for any $p \in U$. When $rank\, \pi|_q$ is equal to $dim(M)$ we say that the Poisson structure is *nondegenerate* at q. If the rank does not depend on the point q, then the bracket defines a *regular Poisson structure*.

The hamiltonian vector fields preserve the characteristic distribution and then $\mathcal{C}$ is integrable by the Stefan-Sussmann theorem and corresponds to a foliation $\mathcal{F}$. Denote by $\mathcal{F}_q$ the leaf passing by $q \in M$. For $\tilde{f}, \tilde{g} \in C^\infty(\mathcal{F}_q)$, define the following bracket on $\mathcal{F}_q$

$$\{\tilde{f}, \tilde{g}\}(q) = \{f, g\}(q) = X_f(g)(q), \tag{1.3.21}$$

where $f, g \in C^\infty(M)$ are any extension of $\tilde{f}, \tilde{g}$, respectively. It can be shown that (1.3.21) does not depend of the extensions f, g and that it defines a nondegenerate Poisson bracket on $\mathcal{F}_q$. Therefore the leaf $\mathcal{F}_q$ is a symplectic manifold and $\mathcal{F}$ is called the *symplectic foliation* of the Poisson manifold $(M, \{\cdot, \cdot\})$.

The local structure of Poisson manifolds is well described using Weinstein's splitting theorem which asserts that in a neighbourhood of a point a Poisson manifold of rank $2k$ is a product of a $2k$-dimensional symplectic manifold (the 'local' symplectic leaf) and a Poisson manifold with bivector of rank 0 at the point considered, see e.g. [37, 85].

Given a Hamiltonian $H : M \to \mathbb{R}$, the *dynamics* or *equations of motion* is given the hamiltonian vector field X_H on M, i.e.

$$X_H = \{\cdot, H\},$$

or equivalently, given any function (*observable*) $f \in C^\infty(M)$, the evolution of f under the dynamics verifies

$$\dot{f} = \{f, H\},$$

where $\dot{f} := X_H(f)$ is usually called *orbital derivative* of f. Consequently, a *first integral* (or *constant of motion* or *preserved quantity*) is some function $f \in C^\infty(M)$ such that $\dot{f} = 0$. Observe that, by definition, X_H is tangent to the symplectic leaves, then the dynamics evolves on the leaves under the symplectic structure and the Hamiltonian restricted on the leaves of the symplectic foliation of M.

Two functions $f, g \in C^\infty(M)$ such that $\{f, g\} = 0$ are said in *involution*, then the collection of first integrals is exactly the collection of functions which are in involution with the Hamiltonian H. As easy consequences, by the skew-symmetry of the Poisson bracket, the Hamiltonian H is a first integral of X_H, and by Jacobi identity, if f and g are first integrals, then $\{f, g\}$ is also a first integral (not necessarily independent of f and g).

A functions $f \in C^\infty(M)$ which commutes with any other function is called a *Casimir* of the Poisson structure. In particular Casimirs are first integrals for any Hamiltonian. In other words, Casimirs are constant on the symplectic leaves, or equivalently, symplectic leaves are in the level sets of Casimirs.

Using the notations and notion defined we now present the hamiltonian formalism of nonholonomic systems.

1.4 Nonholonomic hamiltonian systems

Consider a mechanical system on the n-dimensional configuration manifold Q with a Lagrangian $L : TQ \to \mathbb{R}$ of mechanical type, i.e., L is of the form $L = \frac{1}{2}\kappa - \tau_{TQ}^* V$, where κ is the kinetic energy metric, $V : Q \to \mathbb{R}$ the potential energy and $\tau_{TQ} : TQ \to Q$ the canonical projection.

Let us assume that the system admits k constraints in the velocities –that in the case of the rolling ball are the non-sliding conditions (4.1.1)– which are geometrically represented by a constant rank nonintegrable distribution D on Q (of constant rank $n-k$), that is, at the configuration point $q \in Q$, the permitted velocities belong to a subspace D_q of T_qQ.

Under the assumption of a mechanical-type Lagrangian, the Legendre transform $Leg = \kappa^\flat : TQ \to T^*Q$ is a global diffeomorphism, where $\kappa^\flat(X)(Y) = \kappa(X,Y)$ for $X, Y \in \mathfrak{X}(Q)$. Hence, the distribution D on Q induces a submanifold $\mathcal{M}$ of T^*Q,

$$\mathcal{M} := Leg(D) \subset T^*Q, \tag{1.4.22}$$

called the *constraint manifold*. Since $Leg : TQ \to T^*Q$ is linear on the fibers, $\mathcal{M} \to Q$ is also a subbundle of $\tau_{T^*Q} : T^*Q \to Q$ of rank $n-k$. Note that, as a manifold, $\mathcal{M}$ is of dimension $2n-k$. Let us denote by $\iota_{\mathcal{M}} : \mathcal{M} \hookrightarrow T^*Q$ the inclusion and $\tau_{\mathcal{M}} : \mathcal{M} \to Q$ the canonical projection.

The distribution D also induces a nonintegrable (constant rank) distribution $\mathcal{C}$ on $\mathcal{M}$ given, at a point $m \in \mathcal{M}$, by

$$\mathcal{C}_m = \{v_m \in T_m\mathcal{M} \mid T\tau_{\mathcal{M}}(v_m) \in D_{\tau(m)}\}. \tag{1.4.23}$$

We will denote by $\Omega_{\mathcal{M}}$ the pull back to $\mathcal{M}$ of the canonical 2-form Ω_Q in T^*Q and by $\Omega_{\mathcal{C}}$ the point-wise restriction of $\Omega_{\mathcal{M}}$ to $\mathcal{C}$, i.e.

$$\Omega_{\mathcal{M}} := \iota_{\mathcal{M}}^*\Omega_Q \quad \text{and} \quad \Omega_{\mathcal{C}} := \Omega_{\mathcal{M}}|_{\mathcal{C}}. \tag{1.4.24}$$

Since the 2-section $\Omega_{\mathcal{C}}$ is nondegenerate (see [11]), it is possible to define the *nonholonomic bracket* $\{\cdot,\cdot\}_{nh}$ on $\mathcal{M}$ [55, 69, 86] such that, for any $f, g \in C^\infty(\mathcal{M})$,

$$\{f,g\}_{nh} = -X_f(g) \quad \text{where} \quad X_f \in \Gamma(\mathcal{C}) \quad \text{such that} \quad \mathbf{i}_{X_f}\Omega_{\mathcal{C}} = df|_{\mathcal{C}}, \tag{1.4.25}$$

which is an almost Poisson bracket, i.e. it is bilinear, skew-symmetric, verifies Leibniz property but does not necessarily satisfies the Jacobi identity. Moreover, the distribution on $\mathcal{M}$ generated by the hamiltonian vector fields X_f –called *characteristic distribution*– is the nonintegrable distribution $\mathcal{C}$ of constant rank $2(n-k)$.

The nonholonomic bracket defines a bivector field π_{nh} on $\mathcal{M}$ by the relation,

$$\pi_{nh}(df, dg) = \{f,g\}_{nh}, \qquad \text{for } f, g \in C^\infty(\mathcal{M}). \tag{1.4.26}$$

We denote by $\pi_{nh}^\sharp : T^*\mathcal{M} \to T\mathcal{M}$ the map given, at each $\alpha, \beta \in T^*\mathcal{M}$, by $\beta(\pi_{nh}^\sharp(\alpha)) = \pi_{nh}(\alpha,\beta)$.

The bivector field $\pi_{nh} \in \Gamma(\bigwedge^2 T\mathcal{M})$ defined in (1.4.26) is, of course, not Poisson. This implies that that map $f \mapsto X_f$ is not an antihomomorphism of Lie algebras as in the Poisson case (see (1.3.20)). In fact, using the *Schouten bracket*, (see e.g. [85]),

$$[\cdot,\cdot] : \Gamma(\bigwedge^{p} T\mathcal{M}) \times \Gamma(\bigwedge^{q} T\mathcal{M}) \to \Gamma(\bigwedge^{p+q-1} T\mathcal{M}),$$

the failure of the Jacobi identity is given by the *Jacobiator*, defined as

$$Jac(f,g,h) := \frac{1}{2}[\pi,\pi](df,dg,dh). \tag{1.4.27}$$

Using the properties of the Schouten bracket we also have

$$\frac{1}{2}[\pi,\pi](df,dg,dh) = cyclic[\{f,\{g,h\}\}], \tag{1.4.28}$$

where *cyclic* indicates the sum of cyclic permutations of its parameters. Hence, for almost Poisson the following holds instead of the antihomomorphism in (1.3.20).

Proposition 1.4.1 ([26])**.** *Let π be any bivector field with associated almost Poisson bracket $\{\cdot,\cdot\}$. Then, for any $f,g \in C^\infty(M)$, the following formula holds*

$$-X_{\{f,g\}} = [X_f, X_g] - \frac{1}{2}\mathbf{i}_{df\wedge dg}[\pi,\pi],$$

or equivalently, for any $h \in C^\infty(M)$,

$$-X_{\{f,g\}}(h) = [X_f, X_g](h) - \frac{1}{2}[\pi,\pi](df,dg,dh),$$

Proof. By definition:

$$-X_{\{f,g\}}(h) = -\{h,\{f,g\}\},$$

and

$$\begin{aligned}[X_f, X_g](h) &= X_f(X_g(h)) - X_g(X_f(h)) = X_f(\{h,g\}) - X_g(\{h,f\}) \\ &= \{f,\{g,h\}\} + \{g,\{h,f\}\}.\end{aligned}$$

Collecting the latter expressions we get $Jac(f,g,h) = [X_f, X_g](h) + X_{\{f,g\}}(h)$. □

Example 1.4.1. *(Twisted Poisson structures) Twisted Poisson structures can be considered as intermediate structures between Poisson and (non-Poisson) almost Poisson structures. In fact the Jacobi identity is not verified but the Jacobiator is controlled by a closed 3-form ϕ. That is, for $f,g,h \in C^\infty(M)$, the following holds*

$$cyclic[\{f,\{g,h\}\}] = \phi(X_f, X_g, X_h). \tag{1.4.29}$$

The main property of ϕ-twisted Poisson structures is that its characteristic distribution is integrable and that the leaf-wise bracket induces a nondegenerate 2-form $\Omega_{\mathcal{F}}$ which is not closed but $d\Omega_{\mathcal{F}} = \phi$.

If π is the bivector field on M defined by a nondegenerate 2-form Ω, then pi is ϕ-twisted with $\phi = d\Omega$. On the other hand, any bivector having a regular integrable characteristic distribution is ϕ-Twisted Poisson but ϕ is not uniquely determined, see [8] (or Appendix 1.). Note that a ϕ-twisted Poisson manifold having a foliation of dimension at most 2 is Poisson (trivially $d\Omega_{\mathcal{F}} = 0$).

The dynamics of the nonholonomic system is given by the (intrinsic) constrained Hamilton's equations,

$$\mathbf{i}_{X_{nh}}\iota^*_{\mathcal{M}}\Omega_Q = \iota^*_{\mathcal{M}}(dH + \lambda_a \tau^*_{\mathcal{M}}\epsilon^a), \tag{1.4.30}$$

where the scalar functions λ^a are referred to as Lagrange multipliers. Restricting (1.4.30) to the distribution $\mathcal{C}$ we get

$$\mathbf{i}_{X_{nh}}\Omega_{\mathcal{C}} = dH_{\mathcal{M}}|_{\mathcal{C}}, \tag{1.4.31}$$

which, by nondegeneracy of $\Omega_{\mathcal{C}}$, defines uniquely the *nonholonomic vector field* X_{nh} on $\mathcal{M}$. The function $H_{\mathcal{M}} \in C^\infty(\mathcal{M})$ is the restriction to $\mathcal{M}$ of the Hamiltonian function $H : T^*Q \to \mathbb{R}$ induced by the Lagrangian $L : TQ \to \mathbb{R}$, [11]. Equivalently, the vector field X_{nh} is defined by $X_{nh} = -\pi^\sharp_{nh}(dH_{\mathcal{M}})$. Hence, we say that a nonholonomic system is *described by* the triple $(\mathcal{M}, \pi_{nh}, H_{\mathcal{M}})$, or that the nonholonomic dynamics is *described by* the bivector field π_{nh}.

Recall that the constraint distribution $D \subset TQ$ can be written as $D = \{(q, \dot{q}) : \epsilon^a(q)(\dot{q}) = 0, a = 1, \cdots, k\}$. If we consider canonical coordinates (q^i, p_i) on T^*Q, equation (1.4.30) becomes

$$\dot{q^i} = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i} + \lambda_a \varepsilon^a_i$$

and the constraints are give by:

$$\epsilon^a_i(q)\frac{\partial H}{\partial p^i} = 0, \quad a = 1, \cdots, k. \tag{1.4.32}$$

1.5 Examples

We present some examples to illustrate the geometric framework of hamiltonian nonholonomic systems explained in this chapter. We will return to the examples in the following chapters in order to illustrate the new theoretical framework and results.

1.5.1 The nonholonomic particle

We continue with the ideal example introduced in Ex. 1.1.2 and Ex. 1.2.4. Recall that the configuration manifold is $Q = \mathbb{R}^3$, the constraint 1-form is given by $\epsilon^1 = dz - ydx$ and the constraint distribution D is given in (1.1.4) and is nonintegrable. The Lagrangian of the system is $L : (q, v) \mapsto q \cdot q$, where the dot means the standard dot product in $\mathbb{R}^3$.

The constraint distribution D is the kernel of the 1-form ϵ^1. Writing an arbitrary tangent vector in $\mathbb{R}^3$ as $v = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}$, the condition $v \in Ker(\epsilon^1)$ implies $c = ay$, so that $v = a(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}) + b(\frac{\partial}{\partial y})$. Hence we get

$$D = span\{Y_x := \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \frac{\partial}{\partial y}\}.$$

Completing to a base of TQ we have

$$TQ = span\{Y_x, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\},$$

with dual basis

$$T^*Q = span\{dx, dy, \epsilon^1\}, \tag{1.5.33}$$

and associated coordinates (p_x, p_y, p_z). The constraint manifold in T^*Q is given by $\mathcal{M} = \{\kappa^\flat(X) : X \in D\}$. The metric κ here is proportional to the standard Euclidean metric, hence, writing $\mathcal{M} = \{a\kappa^\flat(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}) + b\kappa^\flat(\frac{\partial}{\partial y})\}$, it is not difficult to find

$$\mathcal{M} = span\{(1+y^2)dx + y\epsilon^1, dy\}.$$

In the coordinates of T^*Q, $\mathcal{M}$ is described by

$$\mathcal{M} = \{(x, y, z, p_x, p_y, p_z) : p_z = \frac{y}{1+y^2}p_x\}.$$

Then,

$$T\mathcal{M} = span\{Y_x, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial p_x}, \frac{\partial}{\partial p_y}\},$$

and, from the definition of $\mathcal{C}$ in (1.4.23),

$$\mathcal{C} = span\{Y_x, \frac{\partial}{\partial y}, \frac{\partial}{\partial p_x}, \frac{\partial}{\partial p_y}\}.$$

The Hamiltonian restricted to $\mathcal{M}$ is given by

$$H_{\mathcal{M}} = \frac{1}{2}(\frac{p_x^2}{1+y^2} + p_y^2).$$

Now we compute the nonholonomic bivector field. The Liouville 1-form restricted to $\mathcal{M}$ is $\Theta_{\mathcal{M}} = p_x dx + p_y dy + \frac{y}{1+y^2}p_x\epsilon^1$. Then $\Omega_{\mathcal{M}} = dx \wedge dp_x + dy \wedge dp_y - d(\frac{yp_x}{1+y^2}) \wedge \epsilon^1 - \frac{y}{1+y^2}p_x d\epsilon^1$ and

$$\Omega_{\mathcal{C}} = dx \wedge dp_x + dy \wedge dp_y - \frac{y}{1+y^2}p_x dx \wedge dy.$$

The nonholonomic bivector field is given by

$$\pi_{nh} = Y_x \wedge \frac{\partial}{\partial p_x} + \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial p_y} + \frac{y}{1+y^2}p_x \frac{\partial}{\partial p_x} \wedge \frac{\partial}{\partial p_y},$$

and the nonholonomic dynamics $X_{nh} = -\pi_{nh}^{\#}(dH_{\mathcal{M}})$ is given by

$$X_{nh} = \frac{p_x}{1+y^2}Y_x + p_y\frac{\partial}{\partial y} + \frac{y}{1+y^2}p_x p_y\frac{\partial}{\partial p_x}.$$

1.5.2 Vertical rolling disk

We continue with the example presented in Ex. 1.1.3. The configuration space is $Q = \mathbb{R}^2 \times S^1 \times S^1$, with coordinates (x, y, θ, ϕ). The Lagrangian L is given by the kinetic energy,

$$L(x, y, \theta, \phi, \dot{x}, \dot{y}, \dot{\theta}, \dot{\phi}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\phi}^2, \tag{1.5.34}$$

where m is the mass of the disk, R the radius of the disk, I is the moment of inertia of the disk about its axis and J is the moment of inertia about an axis in the plane of the disk, both axes passing through the center of the disk. The associated constraint 1-forms are

$$\epsilon^1 = dx - R\cos(\phi)d\theta, \quad \epsilon^2 = dy - R\sin(\phi)d\theta.$$

We have $d\epsilon^i \wedge \epsilon^1 \wedge \epsilon^2 \neq 0$, $i = 1, 2$, so the constraint distribution is nonintegrable and the system is nonholonomic.

Writing an arbitrary tangent vector in TQ as $v = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial \theta} + d\frac{\partial}{\partial \phi}$, the condition $v \in Ker(\epsilon^1) \cap Ker(\epsilon^2)$ is equivalent to:

$$a - R\cos(\phi)c = 0, \quad b - R\sin(\phi)c = 0,$$

so that the constraint distribution is given by

$$D = span\{Y_\theta := R\cos(\phi)\frac{\partial}{\partial x} + R\sin(\phi)\frac{\partial}{\partial y} + \frac{\partial}{\partial \theta}, Y_\phi := \frac{\partial}{\partial \phi}\}.$$

Details for the computation of the constraint manifold $\mathcal{M}$ are explained in the next example (snakeboard). Here we just present the final result:

$$\mathcal{M} = \{(x, y, \theta, \phi, \tilde{p}_x, \tilde{p}_y, \tilde{p}_\theta, \tilde{p}_\phi) : \tilde{p}_x = \frac{Rm\cos(\phi)}{E}\tilde{p}_\theta, \quad \tilde{p}_y = \frac{Rm\sin(\phi)}{E}\tilde{p}_\theta\},$$

where $E = I + R^2m$, where we use the (adapted) coordinates $(\tilde{p_\theta}, \tilde{p_\phi}, \tilde{p_x}, \tilde{p_y})$ associated to the dual basis in T^*Q given by $\{d\theta, d\phi, \epsilon^1, \epsilon^2\}$.

The Hamiltonian restricted in our coordinates is given by

$$H_{\mathcal{M}} = \frac{1}{2E}\tilde{p}_\theta^2 + \frac{1}{2I}\tilde{p}_\phi^2, \tag{1.5.35}$$

the 2-section $\Omega_{\mathcal{C}}$ is given by

$$\Omega_{\mathcal{C}} = d\theta \wedge d\tilde{p_\theta} + d\phi \wedge d\tilde{p_\phi}, \tag{1.5.36}$$

and the nonholonomic bivector field π_{nh} by

$$\pi_{nh} = Y_\theta \wedge \frac{\partial}{\partial \tilde{p_\theta}} + Y_\phi \wedge \frac{\partial}{\partial \tilde{p_\phi}}. \tag{1.5.37}$$

Finally the nonholonomic vector field $X_{nh} = -\pi_{nh}^{\#}(dH_{\mathcal{M}})$ is given by

$$X_{nh} = \frac{\tilde{p_\theta}}{E}Y_\theta + \frac{\tilde{p_\phi}}{I}Y_\phi. \tag{1.5.38}$$

1.5.3 Snakeboard

It is a variation of a standard skate permitting the axis of the wheels to rotate by the effect of a human rider creating a torque allowing the board to spin about a vertical axis. A better description of the geometry of the system, including a figure, is given in [15, p. 271]. The configuration manifold is $Q = SE(2) \times S^1 \times S^1$, and the Lagrangian is the kinetic energy of the system given by

$$L(q, \dot{q}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + r^2\dot{\theta}^2) + \frac{1}{2}J\dot{\psi}^2 + J\dot{\psi}\dot{\theta} + J_1\dot{\phi}^2.$$

The non-sliding constraints are given by the kernels of the 1-forms:

$$\begin{aligned}
\epsilon^1 &= dx + r\cot(\phi)\cos(\theta)d\theta, \\
\epsilon^2 &= dy + r\cot(\phi)\sin(\theta)d\theta.
\end{aligned}$$

Writing a general vector in TQ as $v = \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{\theta}\frac{\partial}{\partial \theta} + \dot{\phi}\frac{\partial}{\partial \phi} + \dot{\psi}\frac{\partial}{\partial \psi}$, then to verify the constraints it must annihilate the constraint 1-forms, i.e.,

$$\begin{aligned}
\epsilon^1(v) &= \dot{x} + rcot(\phi)cos(\theta)\dot{\theta} = 0, \\
\epsilon^2(v) &= \dot{y} + rcot(\phi)sin(\theta)\dot{\theta} = 0.
\end{aligned}$$

Using these relations between the coefficients of v, we obtain the constraint distribution,

$$D = span\{Y_\theta := \frac{\partial}{\partial\theta} - rcot(\phi)cos(\theta)\frac{\partial}{\partial x} - rcot(\phi)sin(\theta)\frac{\partial}{\partial y}, Y_\phi := \frac{\partial}{\partial \phi}, Y_\psi := \frac{\partial}{\partial \psi}\}. \tag{1.5.39}$$

We complete D to form bases for TQ and T^*Q,

$$\begin{aligned}
TQ &= \{Y_\theta, Y_\phi, Y_\psi, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\} \\
T^*Q &= \{d\theta, d\phi, d\psi, \epsilon^1, \epsilon^2\},
\end{aligned}$$

with coordinates $(\tilde{p_\theta}, \tilde{p_\phi}, \tilde{p_\psi}, \tilde{p_x}, \tilde{p_y})$ on T^*Q.

The constraint manifold $\mathcal{M}$ is computed from $\mathcal{M} = \{\kappa^\flat(X) : X \in D\}$. From the Lagrangian, the metric tensor is given by

$$\kappa = \frac{m}{2}dx\otimes dx + \frac{m}{2}dy\otimes dy + \frac{mr^2}{2}d\theta\otimes d\theta + J_1 d\phi\otimes d\phi + \frac{J}{2}d\psi\otimes d\psi + \frac{J}{2}(d\theta\otimes d\psi + d\psi\otimes d\theta).$$

Using the expression for the metric we find

$$\begin{aligned}
\kappa^\flat(\frac{\partial}{\partial\theta}) &= \frac{mr^2}{2}d\theta + \frac{J}{2}d\psi, \\
\kappa^\flat(\frac{\partial}{\partial\phi}) &= J_1 d\phi, \\
\kappa^\flat(\frac{\partial}{\partial\psi}) &= \frac{J}{2}(d\theta + d\psi), \\
\kappa^\flat(\frac{\partial}{\partial x}) &= \frac{m}{2}\varepsilon^1 - \frac{rm}{2}\cot(\theta)\cos(\theta)d\theta, \\
\kappa^\flat(\frac{\partial}{\partial y}) &= \frac{m}{2}\varepsilon^2 - \frac{rm}{2}\cot(\theta)\sin(\theta)d\theta.
\end{aligned} \tag{1.5.40}$$

Then

$$\mathcal{M} = span\{\kappa^\flat(Y_\theta), \kappa^\flat(Y_\phi), \kappa^\flat(Y_\psi)\}, \tag{1.5.41}$$

where $\kappa^\flat(Y_\phi)$ and $\kappa^\flat(Y_\psi)$ are read directly from (1.5.40) and

$$\kappa^\flat(Y_\theta) = -\frac{rm}{2}\cot(\phi)\cos(\theta)\varepsilon^1 - \frac{rm}{2}\cot(\phi)\sin(\theta)\varepsilon^2 + \frac{r^2m}{2}d\theta + \frac{r^2m}{2}\cot^2(\phi)d\theta + \frac{J}{2}d\psi.$$

Writing a general element of $\mathcal{M}$ in the form $\alpha = \tilde{p_\theta}d\theta + \tilde{p_\phi}d\phi + \tilde{p_\psi}d\psi + \tilde{p_x}\varepsilon^1 + \tilde{p_y}\varepsilon^2$, we find

$$\begin{aligned}
\tilde{p_\theta} &= a\frac{r^2m}{2}(1+\cot^2(\phi)) + c\frac{J}{2},\\
\tilde{p_\phi} &= bJ_1,\\
\tilde{p_\psi} &= \frac{J}{2}(a+c),\\
\tilde{p_x} &= -a\frac{rm}{2}\cot(\phi)\cos(\theta),\\
\tilde{p_y} &= -a\frac{rm}{2}\cot(\phi)\sin(\theta),
\end{aligned}$$

where a, b, c are the coefficients of α with respect to (1.5.41). Then, $\mathcal{M}$ is given by

$$\begin{aligned}
\mathcal{M} = \{(q^i, \tilde{p_\theta}, \tilde{p_\phi}, \tilde{p_\psi}, \tilde{p_x}, \tilde{p_y}) : &\tilde{p_x} = -\frac{mr\cos(\theta)\sin^2(\phi)\cot(\phi)}{mr^2 - J\sin^2(\phi)}(\tilde{p_\theta} - \tilde{p_\psi}),\\
&\tilde{p_y} = -\frac{mr\sin(\theta)\sin^2(\phi)\cot(\phi)}{mr^2 - J\sin^2(\phi)}(\tilde{p_\theta} - \tilde{p_\psi})\}.
\end{aligned}$$

The restricted Hamiltonian $H_\mathcal{M}$ is computed in our adapted variables and we get

$$H_\mathcal{M} = \frac{\sin^2(\phi)}{2(mr^2 - J\sin^2(\phi))}(\tilde{p_\theta} - \tilde{p_\psi})^2 + \frac{1}{J}\tilde{p_\psi}^2 + \frac{1}{4J_1}\tilde{p_\phi}^2.$$

The 2-section $\Omega_\mathcal{C}$ is given by

$$\Omega_\mathcal{C} = d\theta\wedge d\tilde{p_\theta} + d\phi\wedge d\tilde{p_\phi} + d\psi\wedge d\tilde{p_\psi} + \frac{mr^2\cot(\phi)}{mr^2 - J\sin^2(\phi)}(\tilde{p_\theta} - \tilde{p_\psi})d\theta\wedge d\phi,$$

which allows the computation of the bivector field π_{nh}.

1.5.4 The rigid body and the Chaplygin ball

To understand the configuration space we fix an orthonormal basis (*inertial frame*) in space $(\boldsymbol{e}^1, \boldsymbol{e}^2, \boldsymbol{e}^3)$, and attach other orthonormal basis $(\boldsymbol{E}^1, \boldsymbol{E}^2, \boldsymbol{E}^3)$ to the body at the center of mass (*the body frame*). If the body rotate about its center of mass, the two frames are related by and orthonormal matrix $g(t) \in SO(3)$:

$$\boldsymbol{e}^i(t) = g(t)\boldsymbol{E}^i, \quad i = 1, 2, 3.$$

The center of mass can be in any point of the space, so that the configuration space of the system is the direct product of Lie groups, $Q = SO(3) \times \mathbb{R}^3$. A point $q \in Q$ will be denoted $q = (g, (x, y, z))$.

We identify, as usual, the Lie algebra of $SO(3)$, denoted $\mathfrak{so}(3)$, with $\mathbb{R}^3$ using the 'hat-map':

$$v = \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} \mapsto \hat{v} = \begin{pmatrix} 0 & -v^3 & v^2 \\ v^3 & 0 & -v^1 \\ -v^2 & v^1 & 0 \end{pmatrix},$$

which is a Lie algebra isomorphism taking the vector product $\times$ to be the commutator in $\mathfrak{so}(3)$.

An important example of a rigid body with nonholonomic constraints is the Chaplygin sphere or Chaplygin ball [29], consisting in an inhomogeneous sphere whose geometric center coincides with its center of mass and which is allowed to roll without sliding over a horizontal plane.

If $\boldsymbol{x} \in \mathbb{R}^3$ describes the coordinates of the center of mass and $\boldsymbol{\omega}$ the angular velocity of the body with respect to the space frame, then the rolling constraint can be written as a linear equation relating $\dot{\boldsymbol{x}}$ and $\boldsymbol{\omega}$. These constraints are written

$$\dot{\boldsymbol{x}} = rA\boldsymbol{\omega},$$

where r is the radius of the sphere, and A is matrix whose rank specifies a special kind of rolling. For instance the rolling over a plane uses the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.5.42}$$

Denote by $\boldsymbol{\Omega}$ the angular velocity in body coordinates, then we have the relations

$$\hat{\boldsymbol{\omega}} = \dot{g}(t)g^{-1}(t), \quad \hat{\boldsymbol{\Omega}} = g^{-1}(t)\dot{g}(t),$$

where $(g(t), x(t)) \in Q$ describes the motion of the body. Hence $\boldsymbol{\omega} = g\boldsymbol{\Omega}$, and the constraints can also be written as $\dot{\boldsymbol{x}} = rAg\boldsymbol{\Omega}$.

Consider the left and right Maurer-Cartan forms on $SO(3)$, $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$, respectively. These are $\mathfrak{so}(3)$-valued 1-forms verifying $\boldsymbol{\rho} = g\boldsymbol{\lambda}$ and such that for a tangent vector $v_g \in T_gSO(3)$, identifying $\mathfrak{so}(3)$ with $\mathbb{R}^3$, the following holds for $g \in SO(3)$,

$$\boldsymbol{\omega} = \boldsymbol{\rho}(g)(v_g), \quad \boldsymbol{\Omega} = \boldsymbol{\lambda}(g)(v_g).$$

Using these 1-forms, the constraints can be defined by a $\mathbb{R}^3$-valued 1-form $\boldsymbol{\epsilon} = (\epsilon^1, \epsilon^2, \epsilon^3)$ as:

$$\boldsymbol{\epsilon} = dx - rA\boldsymbol{\rho} = dx - rAg\boldsymbol{\lambda}.$$

Denote $\boldsymbol{X}^R$ the moving frame of $SO(3)$ dual to $\boldsymbol{\rho}$, and $\boldsymbol{X}^L$ the moving frame dual to $\boldsymbol{\lambda}$. Then, the constraint distribution D is given by

$$D = span\{\boldsymbol{X}^R + rA\frac{\partial}{\partial \boldsymbol{x}}\} = span\{\boldsymbol{X}^L + rAg\frac{\partial}{\partial \boldsymbol{x}}\}. \tag{1.5.43}$$

If $\mathbb{I}$ denotes the *inertia tensor* (a symmetric definite positive 3×3 matrix), we assume additionally that the body frame is aligned with the principal axes of inertia (the eigenvectors of $\mathbb{I}$) of the body. In that basis $\mathbb{I}$ is represented by a diagonal matrix with positive entries I_1, I_2, I_3, the principal moments of inertia (the eigenvalues of $\mathbb{I}$).

The Lagrangian of the rigid body is the total kinetic energy, i.e.,

$$L(g, \boldsymbol{x}, \dot{g}, \dot{\boldsymbol{x}}) = \frac{1}{2}\boldsymbol{\Omega}^T \mathbb{I}\boldsymbol{\Omega} + \frac{1}{2}m\boldsymbol{x}^T\boldsymbol{x},$$

where we write $\boldsymbol{x}$ and $\boldsymbol{\Omega}$ as column vectors, and ^{T} indicate the matrix transpose. Defining the linear momentum by $\boldsymbol{p} = m\dot{\boldsymbol{x}}$, Lagrange-d'Alembert equations give the system:

$$\begin{aligned} \dot{\boldsymbol{p}} &= \boldsymbol{\mu}, \\ \mathbb{I}\dot{\boldsymbol{\Omega}} &= \mathbb{I}\boldsymbol{\Omega} \times \boldsymbol{\Omega} - rg^{-1}A^T\boldsymbol{\mu}, \end{aligned}$$

where $\boldsymbol{\mu}$ are Lagrange multipliers. Using the constraint and the fact that $\dot{g}\boldsymbol{\Omega} = 0$, we find $\boldsymbol{\mu} = mrAg\dot{\boldsymbol{\Omega}}$, which permits to decouple de system as

$$\mathbb{I}\dot{\boldsymbol{\Omega}} = \mathbb{I}\boldsymbol{\Omega} \times \boldsymbol{\Omega} - mr^2 g^{-1}A^T Ag\dot{\boldsymbol{\Omega}}.$$

On the cotangent bundle T^*Q we consider the frame $\{\boldsymbol{\lambda}, d\boldsymbol{x}\}$ with coordinates $(\boldsymbol{M}, \boldsymbol{p})$, then a co-vector $\alpha \in T^*Q$ can be written as

$$\alpha = \boldsymbol{M} \cdot \boldsymbol{\lambda} + \boldsymbol{p} \cdot d\boldsymbol{x}. \tag{1.5.44}$$

The Legendre transformation $\mathbb{F}L : TQ \to T^*Q$ gives the new momentum coordinates,

$$\begin{aligned} \boldsymbol{M} &= \mathbb{I}\boldsymbol{\Omega}, \\ \boldsymbol{p} &= m\dot{\boldsymbol{x}}. \end{aligned}$$

In order to deal with the constraints, we work with the frame $\{\boldsymbol{\lambda}, \boldsymbol{\varepsilon}\}$ of T^*Q, with coordinates $(\tilde{\boldsymbol{M}}, \tilde{\boldsymbol{p}})$. Comparing with (1.5.44) we obtain

$$\begin{aligned} \tilde{\boldsymbol{M}} &= \boldsymbol{M} + rg^T A^T \boldsymbol{p}, \\ \tilde{\boldsymbol{p}} &= \boldsymbol{p}. \end{aligned} \tag{1.5.45}$$

Now we compute the constraint manifold $\mathcal{M} = Leg(D) \subset T^*\mathcal{M}$. We get

$$\mathcal{M} = \{(g, \boldsymbol{x}, \tilde{\boldsymbol{M}}, \tilde{\boldsymbol{p}}) : \tilde{\boldsymbol{p}} = mrAg\boldsymbol{\Omega}\}, \tag{1.5.46}$$

where

$$\tilde{\boldsymbol{M}} = \mathbb{I}\boldsymbol{\Omega} + mr^2 g^T A^T Ag\boldsymbol{\Omega}. \tag{1.5.47}$$

The Hamiltonian in the coordinates $(\boldsymbol{M}, \boldsymbol{p})$ is given by

$$H = \frac{1}{2}\boldsymbol{M} \cdot \mathbb{I}^{-1}\boldsymbol{M} + \frac{1}{2m}\boldsymbol{p} \cdot \boldsymbol{p}.$$

Passing to the adapted variables $(\tilde{\boldsymbol{M}}, \tilde{\boldsymbol{p}})$ and using the relation in (1.5.46), we get the restriction of the Hamiltonian to $\mathcal{M}$,

$$H_{\mathcal{M}} = \frac{1}{2}\tilde{\boldsymbol{M}} \cdot \boldsymbol{\Omega},$$

where $\boldsymbol{\Omega}$ can be written in the variables on $\mathcal{M}$ using (1.5.47).

The nonintegrable distribution $\mathcal{C}$, defined in (1.4.23), is given by

$$\mathcal{C} = \{\boldsymbol{X}^L + rg^T A^T \frac{\partial}{\partial \boldsymbol{x}}, \frac{\partial}{\partial \tilde{\boldsymbol{M}}}\}.$$

The canonical two-form Q on T^*Q is given by

$$\begin{aligned}\Omega_Q &= -d(\tilde{\boldsymbol{M}} \cdot \boldsymbol{\lambda} + \tilde{\boldsymbol{p}} \cdot \boldsymbol{\epsilon}) \\ &= \boldsymbol{\lambda} \cdot d\tilde{\boldsymbol{M}} - \tilde{\boldsymbol{M}} \cdot d\boldsymbol{\lambda} - \tilde{\boldsymbol{p}} \cdot d\boldsymbol{\epsilon} + \boldsymbol{\epsilon} \cdot d\tilde{\boldsymbol{p}},\end{aligned}$$

and a small computation using Maurer-Cartan equations gives $d\boldsymbol{\epsilon} = rAgd\boldsymbol{\lambda}$. Therefore,

$$\Omega_Q = \boldsymbol{\lambda} \cdot d\tilde{\boldsymbol{M}} - (\tilde{\boldsymbol{M}} + mr^2 g^T A^T \tilde{\boldsymbol{p}}) \cdot d\boldsymbol{\lambda} + \boldsymbol{\epsilon} \cdot d\tilde{\boldsymbol{p}}.$$

Recalling that $\iota_{\mathcal{M}} : \mathcal{M} \hookrightarrow T^*Q$ denotes the inclusion and using that $\tilde{\boldsymbol{p}} = mrAg\boldsymbol{\Omega}$ holds along $\mathcal{M}$, we have

$$\Omega_{\mathcal{M}} = \iota^*(\Omega_Q) = \boldsymbol{\lambda} \cdot d\tilde{\boldsymbol{M}} - (\tilde{\boldsymbol{M}} + mr^2 g^T A^T Ag\boldsymbol{\Omega}) \cdot d\boldsymbol{\lambda} + \iota_{\mathcal{M}}^*(\boldsymbol{\epsilon} \cdot d\tilde{\boldsymbol{p}}).$$

Finally, considering that $\boldsymbol{\epsilon}$ vanishes over $\mathcal{C}$, we have

$$\Omega_{\mathcal{C}} = \Omega_{\mathcal{M}}|_{\mathcal{C}} = \boldsymbol{\lambda} \cdot d\tilde{\boldsymbol{M}} - (\tilde{\boldsymbol{M}} + mr^2 g^T A^T Ag\boldsymbol{\Omega}) \cdot d\boldsymbol{\lambda}. \tag{1.5.48}$$

We compute the nonholonomic bivector field π_{nh} using (1.4.25) and (1.4.26),

$$\pi_{nh} = \boldsymbol{X}^L \wedge \frac{\partial}{\partial \tilde{\boldsymbol{M}}} + rAg\frac{\partial}{\partial \tilde{\boldsymbol{x}}} \wedge \frac{\partial}{\partial \tilde{\boldsymbol{M}}} + (\tilde{\boldsymbol{M}} + mr^2 g^T A^T Ag\boldsymbol{\Omega})\frac{\partial}{\partial \tilde{\boldsymbol{M}}} \wedge \frac{\partial}{\partial \tilde{\boldsymbol{M}}}. \tag{1.5.49}$$

Finally, computing the nonholonomic vector field $X_{nh} = -\pi_{nh}^\sharp(dH_{\mathcal{M}})$, we obtain

$$X_{nh} = \boldsymbol{\Omega} \cdot \boldsymbol{X}^L + rAg\boldsymbol{\Omega} \cdot \frac{\partial}{\partial \boldsymbol{x}} + (\tilde{\boldsymbol{M}} \times \boldsymbol{\Omega}) \cdot \frac{\partial}{\partial \tilde{\boldsymbol{M}}}. \tag{1.5.50}$$

1.5.5 Homogeneous ball in a cylinder

The systems consist of a homogeneous ball of radius r and scalar moment of inertia $\mathbb{I} = I \cdot id$ rolling without sliding inside a circular cylinder of radius R and subject to the force of gravity, see [5, 10, 70].

The configuration space is $Q = R \times S^1 \times SO(3)$ with coordinates (z, θ, g) where (z, θ) indicate the vertical position and the angle of the center of mass in cylindrical coordinates, and g as in the last example indicates the orientation of the ball with respect to some fixed frame.

Following [5, 70], from the right invariant frame $\boldsymbol{X}^R = (X_1^R, X_2^R, X_3^R)$, we construct a new (moving) frame (X_n, X_θ, X_z) obtained by a rotation of $\boldsymbol{X}^R$ of angle θ with respect to the z-axis, and with associated coordinates of the angular velocity given by $\boldsymbol{\omega} = (\omega_n, \omega_\theta, \omega_z)$, and dual basis denoted by $(\beta_n, \beta_\theta, \beta_z)$.

The velocity constraints are given by

$$\dot{z} = r\omega_n, \quad \dot{\theta} = -\frac{r}{R-r}\omega_z, \tag{1.5.51}$$

with constraint 1-forms

$$\epsilon^\theta = \beta_\theta - \frac{dz}{r}, \quad \epsilon^z = \beta_z + \frac{R-r}{r}d\theta, \tag{1.5.52}$$

which define de constraint distribution D given by

$$D = span\{Y_z := \frac{\partial}{\partial z} + \frac{1}{r}X_\theta, Y_\theta := \frac{\partial}{\partial \theta} - \frac{R-r}{r}X_z, X_n\}. \tag{1.5.53}$$

We observe that, by instance, $[Y_\theta, X_n] = \frac{R-r}{r}\frac{\partial}{\partial\theta} \notin \Gamma(D)$, then the regular distribution D is not integrable.

The Lagrangian $L : TQ \to \mathbb{R}$ is given by

$$L(z, \theta, g, \dot{z}, \dot{\theta}, \boldsymbol{\omega}) = \frac{m}{2}((R-r)^2\dot{\theta}^2 + \dot{z}^2) + \frac{I}{2}\boldsymbol{\omega}\cdot\boldsymbol{\omega} + ma_g z, \tag{1.5.54}$$

where a_g denotes the acceleration of gravity.

We choose the following basis of TQ

$$TQ = span\{Y_z, Y_\theta, X_n, X_\theta, X_z\},$$

and dual basis of T^*Q,

$$T^*Q = span\{dz, d\theta, \beta_n, \epsilon^\theta, \epsilon^z\},$$

with associated coordinates $(\tilde{p_z}, \tilde{p_\theta}, \tilde{M}_n, \tilde{M}_\theta, \tilde{M}_z)$.

Using the kinetic energy metric one computes the constraint manifold $\mathcal{M} = Leg(D)$,

$$\mathcal{M} = \{(z, \theta, g, \tilde{p_z}, \tilde{p_\theta}, \tilde{M}_n, \tilde{M}_\theta, \tilde{M}_z) : \tilde{M}_\theta = \frac{Ir}{E}\tilde{p_z}, \tilde{M}_z = -\frac{Ir}{E(R-r)}\tilde{p_\theta}\},$$

where $E = I + mr^2$. On the other hand, using

$$d\beta_n|_\mathcal{C} = -\frac{R}{r^2}dz \wedge d\theta, \quad d\beta_\theta|_\mathcal{C} = \frac{R}{r}\beta_n \wedge d\theta, \quad d\beta_z|_\mathcal{C} = \frac{1}{r}\beta_n \wedge dz,$$

the 2-section $\Omega_\mathcal{C}$ is given by

$$\Omega_\mathcal{C} = -d\tilde{p_z} \wedge dz - d\tilde{p_\theta} \wedge d\theta - d\tilde{M}_n \wedge d\beta_n - \frac{I}{E}\frac{R}{r}\tilde{p_z}\beta_n \wedge d\theta + \frac{I}{E(R-r)}\tilde{p_\theta}\beta_n \wedge dz,$$

and the nonholonomic bivector field is

$$\begin{aligned}\pi_{nh} =& Y_z \wedge \frac{\partial}{\partial \tilde{p_z}} + Y_\theta \wedge \frac{\partial}{\partial \tilde{p_\theta}} + X_n \wedge \frac{\partial}{\partial \tilde{M_n}} \\ &- \frac{R}{r^2}\tilde{M_n}\frac{\partial}{\partial \tilde{p_z}} \wedge \frac{\partial}{\partial \tilde{p_\theta}} + \frac{I}{E(R-r)}\tilde{p_\theta}\frac{\partial}{\partial \tilde{p_z}} \wedge \frac{\partial}{\partial \tilde{M_n}} - \frac{IR}{E}\tilde{p_z}\frac{\partial}{\partial \tilde{p_\theta}} \wedge \frac{\partial}{\partial \tilde{M_n}}.\end{aligned}$$

Computing the Hamiltonian, writing it in the adapted coordinates and restricting to $\mathcal{M}$ we get

$$H_{\mathcal{M}} = \frac{r^2}{2E}\left(\tilde{p_z}^2 + \frac{\tilde{p_\theta}^2}{(R-r)^2}\right) + \frac{\tilde{M_n}^2}{2I} - ma_g z.$$

Then, the nonholonomic vector field is

$$X_{nh} = \frac{r^2}{(R-r)^2E}Y_\theta + \frac{r^2}{E}\tilde{p_z}Y_z + \frac{\tilde{M_n}}{I}X_n - \frac{r\tilde{p_\theta}\tilde{M_n}}{E(R-r)^2}\frac{\partial}{\partial \tilde{p_z}} + \frac{Ir^3\tilde{p_\theta}\tilde{p_z}}{E^2(R-r)^2}\frac{\partial}{\partial \tilde{M_n}} - a_g m\frac{\partial}{\partial \tilde{p_z}}. \tag{1.5.55}$$

We can compare with the formulas obtained in [5] using the relations

$$\begin{aligned}\tilde{p_z} = \frac{E}{mr^2}p_z, \quad \tilde{p_\theta} = \frac{E}{mr^2}p_\theta, \\ \tilde{M_n} = M_n, \quad \tilde{M_\theta} = M_\theta, \quad \tilde{M_z} = M_z.\end{aligned}$$

1.5.6 Body of revolution

Following [6, 36] we consider a strictly convex body of revolution rolling without sliding over a horizontal plane which we take to be described by $\{z = 0\}$. We denote by m the mass of the body and by $\mathbb{I} = (\mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_3)$ the moment of inertia with respect to an orthonormal frame $(\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3})$ attached to the body where $\boldsymbol{e_3}$ is oriented along the axis of symmetry of the body. By the rotational symmetry we have $\mathbb{I}_1 = \mathbb{I}_2$. Moreover we indicate by $\mathbf{a} \in \mathbb{R}^3$ the coordinates of the center of mass and by g the orthogonal matrix indicating the orientation of the body. Then, the configuration manifold of the free mechanical system is given by $\mathbb{R}^3 \times SO(3)$ with coordinates $(\mathbf{a}, g)$ and the Lagrangian $L : TQ \to \mathbb{R}$ (of mechanical type) is given by

$$L(\mathbf{a}, g, \dot{\mathbf{a}}, \boldsymbol{\Omega}) = \frac{1}{2}m\langle \dot{\mathbf{a}}, \dot{\mathbf{a}}\rangle + \frac{1}{2}\langle \mathbb{I}\boldsymbol{\Omega}, \boldsymbol{\Omega}\rangle - ma_g\langle \dot{\mathbf{a}}, \boldsymbol{e_3}\rangle, \tag{1.5.56}$$

where $\boldsymbol{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$ is the angular velocity in the body frame and $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^3$.

We start considering the constraints. First let us describe the holonomic constraint and write the configuration manifold Q. We call $\vec{a}$ the vector in the space frame joining the center of mass to the contact point. Thus, the holonomic constraint imposing the fact that the body is on the plane is written

$$a_3 = -\langle \vec{a}, \boldsymbol{e_3}\rangle.$$

Since here we work in the body frame, it is convenient to call $\boldsymbol{s} = g^{-1}\vec{a}$ and $\boldsymbol{\gamma} = g^{-1}\boldsymbol{e_3}$, the corresponding coordinates of $\vec{a}$ and $\boldsymbol{e_3}$ in the body frame. More explicitly

$$\boldsymbol{s}(\boldsymbol{\gamma}) = (\varrho(\gamma_3)\gamma_1, \varrho(\gamma_3)\gamma_2, \zeta(\gamma_3)),$$

where $\varrho : (-1,1) \to \mathbb{R}$ and $\zeta : (-1,1) \to \mathbb{R}$ are smooth functions depending on the parametrization of the surface of revolution given in [36, Sec. 6.7.1] that can be extended smoothly to a neighbourhood of ± 1.

Consequently the configuration manifold is given by

$$Q = \{(\mathbf{a}, g) \in \mathbb{R}^3 \times SO(3) : a_3 = -\langle \boldsymbol{s}, \boldsymbol{\gamma} \rangle\}. \tag{1.5.57}$$

We define a local basis of TQ given by $\{X_1^L, X_2^L, X_3^L, \frac{\partial}{\partial a_1}, \frac{\partial}{\partial a_1}\}$ with coordinates $(\boldsymbol{\Omega}, \dot{a_1}, \dot{a_2})$, where we denote by (X_1^L, X_2^L, X_3^L) the left-invariant vector fields, with dual basis $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ given by the left Maurer-Cartan 1-forms. Then, the nonholonomic constraint of rolling without slipping is given in the body coordinates by

$$g^{-1}\dot{\boldsymbol{a}} = -\boldsymbol{\Omega} \times \boldsymbol{s},$$

or equivalently

$$\dot{\boldsymbol{a}} = -g(\boldsymbol{\Omega} \times \boldsymbol{s}).$$

Taking the first two coordinates of the last relation, and denoting $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ (this is coherent with our previous use of $\boldsymbol{\gamma}$) the rows of the matrix g we get

$$\dot{a_1} = -\langle \boldsymbol{\alpha}, \boldsymbol{\Omega} \times \boldsymbol{s} \rangle, \quad \dot{a_1} = -\langle \boldsymbol{\beta}, \boldsymbol{\Omega} \times \boldsymbol{s} \rangle,$$

and therefore the constraint 1-forms are written:

$$\epsilon^1 = da_1 + \langle \boldsymbol{\alpha}, \boldsymbol{\lambda} \times \boldsymbol{s} \rangle, \quad \epsilon^2 = da_2 + \langle \boldsymbol{\beta}, \boldsymbol{\lambda} \times \boldsymbol{s} \rangle. \tag{1.5.58}$$

We get the constraint distribution D computing $Ker\{\epsilon^1, \epsilon^2\}$, that is

$$D = span\{Y_1, Y_2, Y_3\}, \tag{1.5.59}$$

where $Y_i := X_i^L + (\boldsymbol{\alpha} \times \boldsymbol{s})_i \frac{\partial}{\partial a_1} + (\boldsymbol{\beta} \times \boldsymbol{s})_i \frac{\partial}{\partial a_2} + (\boldsymbol{\gamma} \times \boldsymbol{s})_i \frac{\partial}{\partial a_3}$.

We denote $\boldsymbol{Y} = (Y_1, Y_2, Y_3)$ and complete D to form a basis of TQ with the vectors $\frac{\partial}{\partial a_1}$ and $\frac{\partial}{\partial a_2}$, thus

$$TQ = span\{Y_1, Y_2, Y_3, \frac{\partial}{\partial a_1}, \frac{\partial}{\partial a_2}\} = \{\boldsymbol{Y}, \frac{\partial}{\partial a_1}, \frac{\partial}{\partial a_2}\}.$$

Using the dual basis in T^*Q given by $\{\boldsymbol{\lambda}, \epsilon^1, \epsilon^2\}$ with coordinates $(\tilde{\boldsymbol{M}}, \tilde{p_1}, \tilde{p_2})$, and using the kinetic energy metric given in (1.5.56) we can compute the constraint manifold $\mathcal{M} = Leg(D)$. We get

$$\mathcal{M} = \{(a_1, a_2, g, \tilde{\boldsymbol{M}}, \tilde{p_1}, \tilde{p_2}) : \tilde{p_1} = m\langle \boldsymbol{\alpha}, \boldsymbol{s} \times \boldsymbol{\Omega} \rangle, \quad \tilde{p_2} = m\langle \boldsymbol{\beta}, \boldsymbol{s} \times \boldsymbol{\Omega} \rangle\}, \tag{1.5.60}$$

where $\tilde{\boldsymbol{M}} = \mathbb{I}\boldsymbol{\Omega} + m\boldsymbol{s} \times (\boldsymbol{\Omega} \times \boldsymbol{s})$. Then, the coordinates of $\mathcal{M}$ are $(a_1, a_2, g, \tilde{\boldsymbol{M}})$ and the bundle $\tau_{\mathcal{M}} : \mathcal{M} \to Q$ is given by $(a_1, a_2, g, \tilde{\boldsymbol{M}}) \mapsto (a_1, a_2, g)$.

The Liouville 1-form written in our basis of T^*Q is $\Theta_Q = \tilde{\boldsymbol{M}} \cdot \boldsymbol{\lambda} + \tilde{p_1}\epsilon^1 + \tilde{p_2}\epsilon^2$. The computation of the 2-section $\Omega_{\mathcal{C}}$ has been performed in [6, Prop. 3.3]. We do not repeat the details here since they are better explained considering symmetries and other constructions which are recalled and explianed in Chapter 2. The final form of the nonholonomic bivector π_{nh} is

$$\pi_{nh} = \boldsymbol{Y} \wedge \frac{\partial}{\partial \tilde{\boldsymbol{M}}} + (\tilde{\boldsymbol{M}} + \boldsymbol{K}) \frac{\partial}{\partial \tilde{\boldsymbol{M}}} \times \wedge \frac{\partial}{\partial \tilde{\boldsymbol{M}}}, \tag{1.5.61}$$

where $\frac{\partial}{\partial \tilde{\boldsymbol{M}}} = (\frac{\partial}{\partial \tilde{M}_1}, \frac{\partial}{\partial \tilde{M}_2}, \frac{\partial}{\partial \tilde{M}_3})$ and

$$\boldsymbol{K} = -m\varrho\langle \boldsymbol{\gamma}, \boldsymbol{s}\rangle \boldsymbol{\Omega} + \mathcal{Q}\boldsymbol{\gamma} + \mathcal{P}\boldsymbol{e_3}, \tag{1.5.62}$$

with

$$\begin{aligned} &\mathcal{Q} = -m(\varrho^2\langle \boldsymbol{\Omega}, \boldsymbol{\gamma}\rangle + \varrho' c_3), \qquad \mathcal{P} = m(L\varrho\langle \boldsymbol{\Omega}, \boldsymbol{\gamma}\rangle + L' c_3), \\ &c_3 = (\boldsymbol{\gamma} \times (\boldsymbol{\Omega} \times \boldsymbol{s}))_3, \qquad L = L(\gamma_3) = \varrho\gamma_3 - \zeta(\gamma_3). \end{aligned} \tag{1.5.63}$$

The Hamiltonian restricted to $\mathcal{M}$ has been computed in [36, Sec. 6.4] (in the Lagrangian formalism) and is given by

$$H_{\mathcal{M}} = \frac{1}{2}\langle \tilde{\boldsymbol{M}}, \boldsymbol{\Omega}\rangle + m a_g \langle \boldsymbol{a}, \boldsymbol{e_3}\rangle. \tag{1.5.64}$$

The nonholonomic vector field has also been computed in [36] (see also [6]):

$$X_{nh} = \langle \boldsymbol{\Omega}, \boldsymbol{Y}\rangle + \langle \dot{\tilde{\boldsymbol{M}}}, \frac{\partial}{\partial \tilde{\boldsymbol{M}}}\rangle, \tag{1.5.65}$$

where $\dot{\tilde{\boldsymbol{M}}} = \tilde{\boldsymbol{M}} \times \boldsymbol{\Omega} + m(\boldsymbol{\Omega} \times \dot{\boldsymbol{s}}) \times \boldsymbol{s} - m a_g \boldsymbol{\gamma} \times \boldsymbol{s}$.

Chapter 2

Nonholonomic systems with symmetries

In this chapter we present the geometric framework we use to write the equations of motion before and after reduction by symmetries. We follow the previous works in [5, 8, 11, 16, 46, 55, 86].

2.1 G-actions in nonholonomic mechanics

The objective of this section is to recall the reduction of nonholonomic systems with a proper G-action. First we recall some standard facts about smooth group actions and fix some notations, see e.g. [1, 38, 66].

2.1.1 Free and proper G-actions

Let G be a Lie group and M an n-dimensional smooth manifold. A *smooth action* (or simply *action*) of G on M is a smooth map $\psi : G \times M \to M$ such that, if we denote $\psi(g, m)$ by $g \cdot m$, then,

$$(i)\ \ e \cdot m = m \qquad \text{and} \qquad (ii)\ \ (gh) \cdot m = g \cdot (h \cdot m),$$

for any $g, h \in G$, $m \in M$, and where $e \in G$ denotes the unit of the Lie group. For any $g \in G$ the *action diffeomorphism* $\psi_g : M \to M$, is given by $m \mapsto g \cdot m$. Equivalently one can define the action as a group homomorphism $G \to Diff(M)$ given by $g \mapsto \psi_g$ where the product in $Diff(M)$ is the composition.

We say that the action is *free* if when $g \cdot m = m$ implies $g = e$, i.e. if $g \neq e$ then the map $\psi_g : M \to M$ has no fixed points. The action is called *proper* if the map given by $G \times M \to M \times M$, such that $(g, x) \mapsto (x, g \cdot x)$ is a proper map (the preimage of a compact set is compact).

The *orbit* of a point $m \in M$ is the set denoted $G \cdot m$ or $Orb(m)$ given by $\{g \cdot m : g \in G\}$. The orbits of a smooth action are immersed submanifolds of M and if the action is proper (in particular if G is compact) then the orbits are embedded closed submanifolds of M. If, in addition, the action is free then the orbits are diffeomorphic to G.

The Lie algebra $\mathfrak{g}$ of the Lie group G, defines and *infinitesimal action* on M, given by *fundamental vector fields* or *infinitesimal generators*. Given a element $\xi \in \mathfrak{g}$, the associated infinitesimal generator of the action on M, denoted ξ_M, is the vector field defined by

$$\xi_M(m) := \frac{d}{dt}|_{t=0}\ exp(t\xi) \cdot m, \tag{2.1.1}$$

where $exp : \mathfrak{g} \to G$ is the exponental map (see [38]). Then, the tangent space to the orbit passing through $m \in M$, $Orb(m)$, is generated the infinitesimal generators at m, i.e.

$$T_m Orb(m) = \{\xi_M(m) \ : \ \xi \in \mathfrak{g}\}. \tag{2.1.2}$$

If the action is free then $\xi_M(m) \neq 0$, for any $\xi \in \mathfrak{g}$, and then the orbits $Orb(m)$ are $rank(\mathfrak{g})$-dimensional submanifolds of M. In fact, Thm. 2.1.1 bellow shows that the orbits are the fibers of a G-principal bundle.

The *isotropy group* or *stabilizer* G_m of a point $m \in M$ is the subgroup: $G_m = \{g \in G | g \cdot m = m\}$. Since G_m is a closed subgroup then it is a Lie subgroup. By definition, an action is free if all the isotropy groups are trivial. Moreover isotropy groups for proper actions are compact.

The quotient space M/G is a topological space with the quotient topology and the quotient map $\rho : M \to M/G$ is usually called *orbit projection* and is continuous and open. If the action is proper, the fact that the orbits are closed imply that M/G is Hausdorff.

Theorem 2.1.1. *If the Lie group G acts freely and properly over a manifold M, then the quotient space M/G is a smooth manifold and the quotient map $\rho : M \to M/G$ is a G-principal bundle.*

In particular the projection map $\rho : M \to M/G$ is a submersion. Thus a smooth function $f : M/G \to \mathbb{R}$ on the quotient manifold M/G is equivalent to a smooth function $F : M \to \mathbb{R}$ constant on the orbits (or G-*invariant*). We denote $C^\infty(M)^G$ the collection of all G-invariant functions on M and then we have the equivalence $C^\infty(M/G) \simeq C^\infty(M)^G$.

We present here two examples of smooth actions which illustrate symmetries in mechanical systems.

Example 2.1.1. *(The nonholonomic particle) Continuing with the example developed in Section 1.5.1, we recall that the configuration space is $Q = R^3$. We will see in Section 2.4.1 that the system has symmetry group $G = \mathbb{R}^2$. The action of $(a, b) \in G$ on $(x, y, z) \in Q$ is given by*

$$(a, b) \cdot (x, y, z) = (x + a, y, z + b),$$

and it is straightforward to see that it is free and proper. The orbit projection $\rho : Q \to Q/G \simeq \mathbb{R}^1$ is given in coordinates by $(x, y, z) \mapsto (y)$.

Example 2.1.2. *(A ball on a plane under gravity) The configuration space is* $\mathbb{R}^2 \times SO(3)$ *with coordinates* (x, y, g) *and since the position of the center of mass as well as the orientation in the plane are not relevant (homogeneity and isotropy of space, in the physics nomenclature), we consider a symmetry group* $G = SE(2) = \mathbb{R}^2 \times SO(2) \simeq \mathbb{R}^2 \times S^1$ *acting as*

$$((a, b), \varphi) \cdot (x, y, g) = (h_\varphi(x, y) + (a, b), \hat{h}_\varphi g),$$

where h_φ *is the* 2×2 *orthogonal matrix representing counter-clockwise rotation of angle* φ*, and* $\hat{h}_\varphi$ *is the* 3×3 *orthogonal matrix representing a rotation of angle* φ *with respect to the* z*-axis. The action is free and proper and the quotient space* Q/G*, is diffeomorphic to* $SO(3)/S^1 \simeq S^2$*. In this case the map* $\rho : Q \to S^2$ *is given in coordinates by* $(x, y, g) \mapsto \boldsymbol{\gamma}$*, where* $\boldsymbol{\gamma}$ *is the third row of the orthogonal matrix* g*.*

2.1.2 Proper G-actions

If the action $\psi : G \times Q \to Q$ is not free then the quotient space Q/G is not necessarily a smooth manifold but it can be understood as a *differential space*, see for instance [13, 35, 36, 82].

Definition 2.1.2. A differential space M is a topological space endowed with the ring $C^\infty(M)$ of continuous functions that satisfy the following conditions:

1. The family $\{f^{-1}(I) | f \in C^\infty(M)$ and I an open interval of $\mathbb{R}\}$ is a subbasis of the topology of M.

2. If $f_1, \cdots, f_n \in C^\infty(M)$ and $F \in C^\infty(\mathbb{R}^n)$, then $F(f_1, \cdots, f_n) \in C^\infty(M)$.

3. If $f : M \to \mathbb{R}$ is a function such that for each $m \in M$ there is a neighbourhood U of p and a function $f_p \in C^\infty(M)$ satisfying $f_p|_U = f|_U$, then $f \in C^\infty(M)$.

When necessary we use the pair $(M, C^\infty(M))$ to denote the differential space. Let $(M, C^\infty(M))$ and $(N, C^\infty(N))$ be two differential spaces, a continuous map $\varphi : M \to N$ is said a *smooth map* from $(M, C^\infty(M))$ to $(N, C^\infty(N))$ if $\varphi^*(C^\infty(N)) \subset C^\infty(M)$. If φ is a homeomorphism and φ^{-1} is also smooth then φ is called a *diffeomorphism* of the differential spaces $(M, C^\infty(M))$ and $(N, C^\infty(N))$.

Example 2.1.3. *A smooth manifold* M *with its ring of smooth functions* $C^\infty(M)$ *is a differential space.*

Example 2.1.4. *If* G *is a Lie group acting properly on a smooth manifold* Q*, then the quotient space* Q/G *is a Hausdorff topological space. Moreover* Q/G *is also a differential space by declaring the ring of smooth functions on* Q/G *as being the ring of smooth* G*-invariant functions on* Q *denoted by* $C^\infty(Q)^G$*. That is, if we denote by* $\rho : Q \to Q/G$ *the orbit projection, then a smooth function* $\bar{f} \in C^\infty(Q/G)$ *is identified with the corresponding* G*-invariant function* f *on* Q *s.t.* $\rho^* \bar{f} = f$*. It can be shown that* $(Q/G, C^\infty(Q/G))$ *is a differential space, that* $\rho : Q \to Q/G$ *is smooth map and that the quotient topology coincides with the differential space topology of* Q/G*, see e.g [35, Sec. VII.3.2].*

If the action is not free at $m \in M$, then there exits $\xi \in \mathfrak{g}$ such that its infinitesimal generator vanishes at m, i.e. $\xi_M(m) = 0$. Denote by $\mathfrak{g}_m$ the Lie subalgebra given by $\mathfrak{g}_m = \{\xi \in \mathfrak{g} : \xi_M(m) = 0\}$, then it is equal to the Lie algebra $Lie(G_m)$ of the isotropy group G_m even if the action is non-proper. In this case, as we mentioned in Section 2.1.1, the orbit $Orb(m)$ is a embedded closed submanifold of M, but not necessarily of dimension $rank(\mathfrak{g})$. In fact, its tangent space is still given by (2.1.2), but has dimension $dim(\mathfrak{g}/\mathfrak{g}_m)$. See the book [38] for details and other result for proper actions on manifolds.

2.1.3 Reduction of nonholonomic systems

Consider a Lie group G acting properly on the configurations manifold Q such that the tangent lift of the action to TQ leaves invariant the Lagrangian L and the distribution D. As a consequence the corresponding cotangent lift of the action to T^*Q leaves invariant the constraint manifold $\mathcal{M} \subset T^*Q$, the bivector field π_{nh} on $\mathcal{M}$ and the restricted Hamiltonian $H_{\mathcal{M}}$.

Definition 2.1.3. A G-action on the constraint manifold $\mathcal{M}$ is called a *G-symmetry* of the nonholonomic system $(\mathcal{M}, \pi_{nh}, H_{\mathcal{M}})$ if it preserves the bivector field π_{nh} and the restricted Hamiltonian $H_{\mathcal{M}}$.

Consequently, a G-symmetry of the nonholonomic system $(\mathcal{M}, \pi_{nh}, H_{\mathcal{M}})$ induces the reduced dynamics on the quotient space $\mathcal{M}/G$. We have seen in Example 2.1.4 that the orbit space $\mathcal{M}/G$ is a Hausdorff differential space such that $C^\infty(\mathcal{M}/G) \simeq C^\infty(\mathcal{M})^G$. Then, the G-invariant nonholonomic bracket $\{\cdot,\cdot\}_{nh}$ on $\mathcal{M}$ and the orbit projection $\rho : \mathcal{M} \to \mathcal{M}/G$ induce a reduced almost Poisson bracket on the differential space $\mathcal{M}/G$ given by,

$$\{\bar{f}, \bar{g}\}_{red} \circ \rho = \{f, g\}_{nh} \ , \forall \bar{f}, \bar{g} \in C^\infty(\mathcal{M}/G), \tag{2.1.3}$$

where $f = \rho^*\bar{f}$ and $g = \rho^*\bar{g}$ belong to $C^\infty(\mathcal{M})^G$.

Since the Hamiltonian $H_{\mathcal{M}}$ is G-invariant, the reduced bracket $\{\cdot,\cdot\}_{red}$ describes the reduced dynamics: $X_{red} = \{\cdot, H_{red}\}_{red} \in \mathfrak{X}(\mathcal{M}/G)$, where $H_{red} : \mathcal{M}/G \to \mathbb{R}$ is the reduced Hamiltonian give by $\rho^* H_{red} = H_{\mathcal{M}}$ (see [6, 12]). Observe that X_{red} is *a priori* only a derivation on the differential space $\mathcal{M}/G$. We will see in the next Section that, for a proper G-action, the differential space $\mathcal{M}/G$ has also the structure of a subcartesian space and a stratified space, which allow one to understand the meaning of vector fields in $\mathfrak{X}(\mathcal{M}/G)$.

Example 2.1.5. *(An axisymmetric ball on a plane under gravity) Recall the mechanical system in Example 2.1.2. Suppose that the ball has an axis of symmetry which induces a new (left) S^1-action and commuting with the $SE(2)$-action.*

Recall that the configuration space is $Q = \mathbb{R}^2 \times SO(3)$ with coordinates (x, y, g) and now $G = SE(2) \times S^1$ acts as

$$((a, b), \varphi, \alpha) \cdot (x, y, g) = (h_\varphi(x, y) + (a, b), \hat{h}_\varphi g \hat{h}_\alpha),$$

where, as before, h_φ is the 2×2 orthogonal matrix representing counter-clockwise rotation of angle φ, and $\hat{h}_\varphi$ (resp. $\hat{h}_\alpha$) is the 3×3 orthogonal matrix representing a rotation of angle φ (resp. α) with respect to the z-axis.

The action is proper but it is not free. Indeed, for any $g_z \in SO(3)$ representing a rotation around the vertical axis (then commuting with $\hat{h}_\alpha$), the element $((0,0), \varphi, -\varphi)$ fixes the point $(0, 0, g_z)$. In this case the quotient space Q/G, is diffeomorphic to $S^2/S^1 \simeq I$, where I denotes a closed interval, which is not a smooth manifold. In this case the map $\rho : Q \to Q/G$ given in local coordinates by $(x, y, g) \mapsto \gamma_3$, where γ_3 is the z-component of the vector $\boldsymbol{\gamma}$ (using the usual notation for the rows of the matrix g, see Example 2.1.2).

Example 2.1.6. *The last example is an instance on a solid of revolution rolling on a plane, developed in [6, 36, 47] and presented in Section 1.5.6. As a consequence, the G-symmetry for that example does not come from a free action. We will illustrate in Section 2.4.6 the use of invariant functions to show the results about the reduced Poisson structure on $\mathcal{M}/G$. Moreover, the example treated in thesis and developed in Chapter 4 also admits a proper non-free action.*

2.2 The quotient $\mathcal{M}/G$ seen as a subcartesian and stratified space

We now briefly explain the structure of the quotient space $\mathcal{M}/G$ for proper actions, more specifically we discuss the existence of vector fields and the orbit type stratification. For more details see the books [35, 36, 38, 82] and the paper [81].

For a proper G-action, the space $\mathcal{M}/G$ has more properties in addition of being a differential space. First, it is a *subcartesian space*, meaning that it is a Hausdorff differential space, locally diffeomorphic (as a differential space) to a open subset of the Cartesian space $\mathbb{R}^n$. On the other hand $\mathcal{M}/G$ is a *stratified space*[1] given by the *orbit type stratification* associated to the G-action.

A *derivation* X on a subcartesian space C is called a *vector field* if the unique maximal integral curve passing by a given point is a local one-parameter group of local diffeomorphisms. Given a family $\mathfrak{F}$ of vector fields in a subcartesian space C, the *orbit* of $\mathfrak{F}$ passing through $p \in C$ is the subset of C formed by points which can be attained from p following, during finite times, the integral curves of a finite number of vector fields of the family $\mathfrak{F}$. For such families $\mathfrak{F}$, Sniatycki [81] has shown an analogue of Stefan-Sussmann orbit theorem for subcartesian spaces. In particular, the orbits of the family $\mathfrak{X}(C)$ of all vector fields in a subcartesian space C are smooth manifolds immersed in C and gives rise to a generalized foliation of C where the leaves are the orbits of $\mathfrak{X}(C)$. The orbits of a (sub-)family of vector fields are also manifolds contained in the orbits of all vector fields $\mathfrak{X}(C)$ but are not necessarily maximal.

[1] Recall that a *stratification* of a paracompact Hausdorff space S is a partition of S on a locally finite number of locally closed subspaces called *strata* such that: (i) each stratum M is a smooth manifold with the indiced topology, and (ii) if M, N are strata and $M \cap \bar{N} \neq \emptyset$, then either $M = N$ or $M \subset \bar{N} - N$, where $\bar{N}$ denotes the topological closure of N, [81].

Let us recall the *orbit type stratification* associated to a proper action. Consider in general a proper action of a Lie group G on a manifold M. The isotropy group G_m of a point $m \in M$ is then compact. Let H be a compact subgroup of G. The subset of M of orbit type H is given by

$$M_{(H)} = \{m \in M | G_m = gHg^{-1} \text{ for some } g \in G\}.$$

It can be shown that each connected component of $M_{(H)}$ is a submanifold of M. The family of connected components of $M_{(H)}$ as H varies over compact subgroups of G gives rise to a stratification of M called *orbit type stratification of M*. Moreover the projection of the orbit type stratification of M by the orbit map ρ gives a (minimal) stratification in M/G called the *orbit type stratification of* M/G where the strata are the orbit projections $M_{(H)}/G$ of the strata of the orbit type stratification of M.

Returning to the quotient $\mathcal{M}/G$ by a proper action. It is proved in [35, Sec. VII.3.3] (see also [36, Sec. 2.5]) that $\mathcal{M}/G$ with its differential space structure given by G-invariant functions is a subcartesian space. Moreover, we get a generalized foliation of $\mathcal{M}/G$ by the orbits of all vector fields $\mathfrak{X}(\mathcal{M}/G)$, which coincide with the strata of the orbit type stratification of $\mathcal{M}/G$, see [81, Sec. 7].

We summarize now the results of Chapter 8 of [82] for almost Poisson manifolds. Recall that a proper G-symmetry induces, by (2.1.3), an almost Poisson bracket $\{\cdot,\cdot\}_{red}$ on the stratified differential space $\mathcal{M}/G$. Then, each stratum N of the orbit type stratification of $\mathcal{M}/G$ is an almost Poisson manifold with bracket $\{\cdot,\cdot\}_{red}|_N$. For a smooth function $\bar{f}$ on $\mathcal{M}/G$, the derivation of $\mathcal{M}/G$ given by

$$X_{\bar{f}}(\bar{g}) = -\{\bar{f}, \bar{g}\}_{red}$$

defines a vector field on $\mathcal{M}/G$ that it is also called the *Hamiltonian vector field of* $\bar{f}$. The orbits of the family of all Hamiltonian vector fields in $\mathcal{M}/G$ are smooth manifolds immersed in the strata of $\mathcal{M}/G$ and are not necessarily integral manifolds. Let us call O, with $O \subset N \subset \mathcal{M}/G$, one of such orbits, then any Hamiltonian vector field on $\mathcal{M}/G$ induces a smooth vector field in the manifold O by restriction.

In the case of a nonholonomic system $(\mathcal{M}, \pi_{nh}, H_{\mathcal{M}})$ with a proper G-symmetry, the space $\mathcal{M}/G$ is an almost Poisson differential space with bracket $\{\cdot,\cdot\}_{red}$ and it is stratified by orbit type, each stratum is an almost Poisson manifold and X_{red} is a vector field in $\mathcal{M}/G$ inducing a smooth vector field on each stratum which preserves the orbit type stratification of $\mathcal{M}/G$.

We are interested in the integrability properties of the reduced bracket $\{\cdot,\cdot\}_{red}$ in $\mathcal{M}/G$, that is, the failure of the Jacobi identity. In order to compute this failure, we will use the formulas proved in [5] which are based in certain splittings of the tangent bundle TQ explained in detail in the next Section.

2.3 The failure of the Jacobi identity of nonholonomic brackets

In this section we present formulas for the Jacobiator of nonholonomic bracket $\{\cdot,\cdot\}_{nh}$ and the reduced bracket $\{\cdot,\cdot\}_{red}$ for nonholonomic systems with symmetries. These formulas were proved in [5] for nonholonomic systems with free and proper G-symmetries and are based on the construction of the 3-form $dJ \wedge K_{\mathcal{W}}$, see Section 2.3.2. The formulas for the Jacobiator were generalized to proper G-actions in [6] where one key assumption was the so-called *vertical symmetry condition*, see Rmk. 2.3.1. The main result in the following section is that, if the action not necessarily free and G is compact, then the vertical symmetry condition is no longer necessary to compute the Jacobiator, see Eqs. (2.3.20) and (2.3.21)

2.3.1 Splittings adapted to the constraints

Following our discussion of Section 2.1.3, let us consider a nonholonomic system $(\mathcal{M}, \pi_{nh}, H_{\mathcal{M}})$ with a proper G-symmetry. Let us denote by V the (generalized) smooth distribution on Q, called *vertical distribution*, given, at each $q \in Q$, by the tangent space to the orbit by G passing through the point q, i.e. $V_q := T_qOrb(q)$. In this thesis we will suppose that the *dimension assumption* [16] holds, i.e.

$$T_qQ = D_q + V_q, \tag{2.3.4}$$

for $q \in Q$. Since the action is not necessarily free, the rank of the vertical distribution $V \subset TQ$ may vary.

Now let us define the (generalized) smooth distribution S in Q given, at each $q \in Q$, by

$$S_q := D_q \cap V_q. \tag{2.3.5}$$

It was shown in [6, Prop. 2.2] that the dimension assumption implies the existence of a constant rank smooth distribution W on Q such that for all $q \in Q$,

$$T_qQ = D_q \oplus W_q \quad \text{and} \quad W_q \subset V_q, \tag{2.3.6}$$

which is equivalent to the following splitting of the vertical distribution,

$$V_q = S_q \oplus W_q. \tag{2.3.7}$$

Let $\mathfrak{g}$ denote the m-dimensional Lie algebra of G. Following [16], for each $q \in Q$, let us define the vector subspace $\mathfrak{g}_S|_q$ of $\mathfrak{g}$ by

$$\eta \in \mathfrak{g}_S|_q \Leftrightarrow \eta_Q(q) \in S_q, \tag{2.3.8}$$

where η_Q denotes the infinitesimal generator of the action of G on Q associated to the element $\eta \in \mathfrak{g}$. Moreover, it was shown in [6] that the dimension assumption implies that $\mathfrak{g}_S \to Q$ is, in fact, a vector subbundle of the trivial bundle $\mathfrak{g} \times Q \to Q$. Then $\eta \in \Gamma(\mathfrak{g}_S)$ if $\eta_Q(q) \in S_q$, where $\eta_Q(q) := (\eta|_q)_Q(q)$. The bundle $\mathfrak{g}_S \to Q$ admits a (non-unique) bundle complement $\mathfrak{g}_W \to Q$ such that, for any $q \in Q$,

$$(\mathfrak{g} \times Q)|_q = \mathfrak{g}_S|_q \oplus \mathfrak{g}_W|_q. \tag{2.3.9}$$

Then, the distribution $W \subset V$ in (2.3.7) is defined by

$$W_q = span\{\xi_Q(q) : \xi \in \mathfrak{g}_W|_q\}, \tag{2.3.10}$$

and it is smooth with constant rank. The bundle W is called a *vertical complement of the constraints* if, in addition to (2.3.6), it is G-invariant.

Remark 2.3.1. Following [5, 6], let us recall that the bundle W in (2.3.10) satisfy the *vertical symmetry condition* is the bundle $\mathfrak{g}_W \to Q$ is given by a Ad-invariant subspace $\mathfrak{w} \in \mathfrak{g}$, i.e. $\mathfrak{g}_W|_q \simeq \mathfrak{w}$ for any $q \in Q$. In fact, the vertical symmetry condition is equivalent to the existence of a normal Lie subgroup $G_W \subset G$, such that W is the vertical space associated to the corresponding G_W-action, i.e. $T_q Orb_{G_W}(q) \simeq W_q$, for any $q \in Q$, where $Orb_{G_W}(q)$ is the orbit of q by the action of G_W. ◇

Proposition 2.3.2. *Let G be a compact (and connected) Lie group acting (properly) on Q. If the dimension assumption is satisfied, then there exists a G-invariant complement $W \subset V$ verifying (2.3.6) (or equivalently there exists an Ad-invariant subbundle $\mathfrak{g}_W \to Q$ of $\mathfrak{g} \times Q \to Q$ verifying (2.3.9)).*

Proof. Since the group G is compact, the Lie algebra $\mathfrak{g}$ admits an Ad-invariant scalar product ([60, Proposition 4.24]), which induces an Ad-invariant bundle metric in the trivial bundle $\mathfrak{g} \times Q \to Q$.

For any $g \in G$, let us consider the bundle map $Ad_g : \mathfrak{g} \times Q \to \mathfrak{g} \times Q$, over the action diffeomorphism $\psi_g : Q \to Q$, given at each $\eta \in \Gamma(\mathfrak{g} \times Q)$ by $Ad_g(\eta)|_{\psi_g(q)} := Ad_g(\eta|_q)$. By the dimension assumption $g_S \to Q$ is a subbundle of the trivial bundle $\mathfrak{g} \times Q \to Q$ [6], and since the distribution S is G-invariant then $\mathfrak{g}_S$ is Ad-invariant and thus Ad_g restricts to a bundle map $Ad_g : \mathfrak{g}_S \to \mathfrak{g}_S$, see [32, Lemma. 4.4.8]. By Ad-invariance of the bundle metric on $\mathfrak{g} \times Q \to Q$, if $\mathfrak{g}_W$ in (2.3.9) is chosen to be the orthogonal complement of the bundle $\mathfrak{g}_S$ with respect to this metric, then $\mathfrak{g}_W \to Q$ is an Ad-invariant subbundle of $\mathfrak{g} \times Q \to Q$. Defining the distribution W as in (2.3.10), we obtain that it is G-invariant.

□

Remark 2.3.3. In the last proof we used the fact that if S is G-invariant then $\mathfrak{g}_S$ is Ad-invariant and thus Ad_g restricts to a bundle map $Ad_g : \mathfrak{g}_S \to \mathfrak{g}_S$. Indeed since S is G-invariant, the Ad-invariance of $\mathfrak{g}_S$ follows from the formula (see [32, Lemma. 4.4.8]),

$$T_q\psi_g(\xi_Q(q)) = (Ad_g(\xi))_Q(\psi_g(q)),$$

where $T_q\psi_g$ denotes the derivative of ψ_g at the point $q \in Q$ and $\xi \in \Gamma(\mathfrak{g} \times Q)$. When there is no risk of confusion we will use the short notation for the G-action: $g \cdot q := \psi_g(q)$. ◇

Remark 2.3.4. For completeness we recall the averaging procedure used to show that the bundle $\mathfrak{g} \times Q \to Q$ admits an Ad-invariant bundle metric. First, the trivial bundle $\mathfrak{g} \times Q \to Q$ as a vector bundle admits a bundle metric, i.e. a family of scalar products on the fibers varying smoothly with the base point, see e.g. [56, Sec. 2.1]. Let us denote by $\langle\cdot,\cdot\rangle_q$ the bundle metric of $\mathfrak{g} \times Q \to Q$ on the fibre over $q \in Q$. Since the Lie group G is compact, it admits a left (and right) Haar measure denoted by dg. We claim that the following bilinear form $\langle\langle\cdot,\cdot\rangle\rangle$ on $\mathfrak{g} \times Q \to Q$,

$$\langle\langle \eta|_q, \xi|_q \rangle\rangle_q := \int_G \langle Ad_g(\xi|_q), Ad_g(\eta|_q)\rangle_{g\cdot q} dg,$$

is an Ad-invariant metric on $\mathfrak{g} \times Q \to Q$. It is straightforward to show that it is in fact a scalar product, then it remains to see that $\langle\langle\cdot,\cdot\rangle\rangle$ is Ad_h-invariant for any $h \in G$.

Indeed,

$$\begin{aligned}\langle\langle Ad_h(\eta|_q), Ad_h(\xi|_q)\rangle\rangle_{h\cdot q} &= \int_G \langle Ad_g(Ad_h(\xi|_q)), Ad_g(Ad_h(\eta|_q))\rangle_{g\cdot(h\cdot q)} dg \\ &= \int_G \langle Ad_{gh}(\xi|_q)), Ad_{gh}(\eta|_q))\rangle_{(gh)\cdot q} dg.\end{aligned} \tag{2.3.11}$$

Since for any function $f : G \to \mathbb{R}$, we have that $\int_G f dg = \int_G (f \circ R_h) dg$, where $R_h : G \to G$ is the right multiplication, $R_h(g) = gh$, then (2.3.11) is equivalent to $\int_G \langle Ad_g(\xi|_q), Ad_g(\eta|_q)\rangle_{g\cdot q} dg$ and thus we obtain the Ad-invariance. $\diamond$

Remark 2.3.5. The Killing form K of a compact Lie algebra is an Ad-invariant, negative semidefinite bilinear form. If G is semisimple K is non-degenerate, thus for compact semisimple Lie algebras, $-K$ works well as Ad-invariant scalar product. However our example has symmetry group $S^1 \times SO(3)$ which is not semisimple because the factor S^1 is abelian. $\diamond$

Remark 2.3.6. There is a general theory for Lie algebras admitting Ad-invariant scalar products. The following statements are equivalent for a connected Lie group G with Lie algebra $\mathfrak{g}$: G admits a bi-invariant metric, G is isomorphic to the Cartesian product of a compact group and a vector space, the Lie algebra $\mathfrak{g}$ admits an Ad-invariant scalar product, see for instance [45, Chap. 18]. This would allow to apply the last proposition for others examples such as the solid of revolution rolling on a plane, see [6, 47], where the symmetry group is not compact but has the mentioned Cartesian product structure. $\diamond$

Following [5], the splitting (2.3.6) on TQ induces a splitting on $\mathcal{M}$. More precisely, the G-action on $\mathcal{M}$ defines a (generalized) vertical distribution $\mathcal{V}$ on $\mathcal{M}$ by $\mathcal{V}_m := T_m Orb(m)$, for $m \in \mathcal{M}$. Then, the constant rank distribution $\mathcal{W}$ given by

$$\mathcal{W}_m = span\{\xi_\mathcal{M}(m) : \xi \in \mathfrak{g}_W|_q,\ q = \tau_\mathcal{M}(m)\}, \tag{2.3.12}$$

is a *vertical complement of the constraints* $\mathcal{C}$ since

$$T\mathcal{M} = \mathcal{C} \oplus \mathcal{W} \quad \text{and} \quad \mathcal{W} \subset \mathcal{V}. \tag{2.3.13}$$

As in Prop. 2.3.2, the distribution $\mathcal{W}$ is G-invariant as long as $\mathfrak{g}_W \to Q$ is Ad-invariant. We thus define the (generalized) distribution $\mathcal{S}$ on $\mathcal{M}$ given, at each point $m \in \mathcal{M}$, by

$$\mathcal{S}_m := \mathcal{C}_m \cap \mathcal{V}_m, \tag{2.3.14}$$

and we see that

$$\mathcal{V}_m = \mathcal{S}_m \oplus \mathcal{W}_m. \tag{2.3.15}$$

2.3.2 The Jacobiator and the 3-form $dJ \wedge K_\mathcal{W}$

Now, we recall the formula that casts the failure of the Jacobi identity of the nonholonomic bracket $\{\cdot,\cdot\}_{nh}$ and the reduced one $\{\cdot,\cdot\}_{red}$, see [5, 7]. First, let us consider the $\mathfrak{g}$-valued 1-form A_W on Q given, at each $q \in Q$ and $v_q \in T_qQ$, by

$$A_W(v_q) = \xi \in \mathfrak{g} \quad \text{if and only if} \quad P_W(v_q) = \xi_Q(q), \tag{2.3.16}$$

where $P_W : TQ \to W$ is the projection to the second factor in the splitting (2.3.6). Analogously we denote by $P_D : TQ \to D$ the projection to the first factor in the splitting (2.3.6) of TQ. Then we define the $\mathfrak{g}$-valued 2-form on Q given by

$$K_W(X,Y) = dA_W(P_D(X), P_D(Y)), \quad X, Y \in \mathfrak{X}(Q). \tag{2.3.17}$$

The $\mathcal{W}$-*curvature* $K_{\mathcal{W}}$ is the $\mathfrak{g}$-valued 2-form on $\mathcal{M}$ induced by (2.3.17) given by

$$K_{\mathcal{W}}(\tilde{X}, \tilde{Y}) = K_W(T\tau_{\mathcal{M}}(\tilde{X}), T\tau_{\mathcal{M}}(\tilde{Y})), \quad \tilde{X}, \tilde{Y} \in \mathfrak{X}(\mathcal{M}). \tag{2.3.18}$$

Second, let $J : \mathcal{M} \to \mathfrak{g}^*$ be the restriction to $\mathcal{M}$ of the canonical moment map on T^*Q, i.e. $\langle J(m), \eta\rangle = \mathbf{i}_{\eta_{\mathcal{M}}(m)}\Theta_{\mathcal{M}}(m)$, for $\eta \in \mathfrak{g}$ and $\Theta_{\mathcal{M}}$ the Liouville 1-form restricted to $\mathcal{M}$. Finally, we have the 3-form on $\mathcal{M}$,

$$dJ \wedge K_{\mathcal{W}}(\tilde{X}, \tilde{Y}, \tilde{Z}) = \mathit{cyclic}[\langle dJ(\tilde{X}), K_{\mathcal{W}}(\tilde{Y}, \tilde{Z})\rangle], \quad \tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{X}(\mathcal{M}), \tag{2.3.19}$$

where the pairing $\langle\cdot,\cdot\rangle$ is the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$ (recall that $dJ(\tilde{X}) \in \Gamma(\mathcal{M} \times \mathfrak{g}^*)$ and $K_{\mathcal{W}}(\tilde{X}, \tilde{Y}) \in \Gamma(\mathcal{M} \times \mathfrak{g})$) and *cyclic* indicates the sum of cyclic permutations of the variables (see [5] and [7] for more details).

The 3-form $dJ \wedge K_{\mathcal{W}}$ is G-invariant, see [5]. This is a consequence of the fact that the moment map $J : \mathcal{M} \to \mathfrak{g}^*$ is Ad^*-equivariant, and, if W is G-invariant, the $\mathcal{W}$-curvature $K_{\mathcal{W}}$ is Ad-equivariant, see [5, Prop. 4.4].

The 3-form (2.3.19) appears in [5] to describe intrinsic formulas for the Jacobiator of π_{nh} and $\{\cdot,\cdot\}_{red}$ in the presence of symmetries. More precisely, let $(\mathcal{M}, \{\cdot,\cdot\}_{nh})$ be the G-invariant almost Poisson manifold associated to a nonholonomic system with symmetries satisfying the dimension assumption (2.3.4). If $\mathcal{W}$ is a G-invariant vertical complement of $\mathcal{C}$, then:

1. The nonholonomic bracket $\{\cdot,\cdot\}_{nh}$ on $\mathcal{M}$ satisfies, for all $f, g, h \in C^\infty(\mathcal{M})$,

$$\begin{aligned} \mathit{cyclic}[\{f, \{g,h\}_{nh}\}_{nh}] =& (dJ \wedge K_{\mathcal{W}})(\pi^\#_{nh}(df), \pi^\#_{nh}(dg), \pi^\#_{nh}(dh)) \\ &- \psi_{\pi_{nh}}(df, dg, dh), \end{aligned} \tag{2.3.20}$$

   where $\psi_{\pi_{nh}}$ is the trivector given by $\psi_{\pi_{nh}}(\alpha, \beta, \gamma) = \mathit{cyclic}[\gamma(K_{\mathcal{W}}(\pi^\#_{nh}(\alpha), \pi^\#_{nh}(\beta))_{\mathcal{M}})]$, for α, β, γ 1-forms in $\mathcal{M}$, and which vanishes if and only if the distribution $\mathcal{C}$ is involutive.

2. For $\bar{f}, \bar{g}, \bar{h} \in C^\infty(\mathcal{M}/G)$,

$$\mathit{cyclic}[\{\bar{f}, \{\bar{g}, \bar{h}\}_{red}\}_{red} \circ \rho] = (dJ \wedge K_{\mathcal{W}})(\pi^\#_{nh}(d\rho^*\bar{f}), \pi^\#_{nh}(d\rho^*\bar{g}), \pi^\#_{nh}(d\rho^*\bar{h})), \tag{2.3.21}$$

   where $\{\cdot,\cdot\}_{red}$ is the reduced bracket on $\mathcal{M}/G$ defined in (2.1.3). We observe that (2.3.21) also works for the case where the quotient $\mathcal{M}/G$ is a differential space.

From formula (2.3.20) one concludes that $\{\cdot,\cdot\}_{nh}$ is never Poisson because the trivector $\psi_{\pi_{nh}}$ is never zero if $\mathcal{C}$ is non-integrable (as is our case). Indeed, by definition the $\mathcal{W}$-curvature $K_{\mathcal{W}}$ is zero if and only if the regular distribution $\mathcal{C}$ is involutive, see [5, Sec. 4.1]. In fact, the latter argument shows that $\{\cdot,\cdot\}_{nh}$ is not even twisted Poisson, see Example 1.4.1, since the $\mathcal{C}$ is the characteristic distribution of $\{\cdot,\cdot\}_{nh}$.

On the other hand, using formula (2.3.21) one observe that the reduced bracket $\{\cdot,\cdot\}_{red}$ is Poisson if the RHS of (2.3.21) is zero. Moreover, if the action is free and proper, Cor. 4.10 of [5] shows that if 3-form $dJ \wedge K_{\mathcal{W}}$ is closed and basic with respect to $\rho : \mathcal{M} \to \mathcal{M}/G$, then $\{\cdot,\cdot\}_{red}$ is ϕ-twisted Poisson, where $\phi \in \Omega^3(\mathcal{M}/G)$ is given by $\rho^*\phi = dJ \wedge K_{\mathcal{W}}$. In fact, it is enough to check that the 3-form $dJ \wedge K_{\mathcal{W}}$ is semi-basic with respect to $\rho : \mathcal{M} \to \mathcal{M}/G$, since in that case the G-invariance of $dJ \wedge K_{\mathcal{W}}$ implies that it is basic.

We close the section illustrating the reduction process where the reduced bracket $\{\cdot,\cdot\}_{red}$ could in principle have better integrability properties that the nonholonomic bracket $\{\cdot,\cdot\}_{nh}$:

$$\begin{gathered}(\mathcal{M}, \{,\}_{nh}) \\ \big\downarrow \\ (\mathcal{M}/G, \{,\}_{red})\end{gathered}$$

Poisson/twisted Poisson?

2.3.3 Formulas in local basis

Let us fix some notations in order to write formulas of the $\mathcal{W}$-curvature $K_{\mathcal{W}}$ and the 3-form $dJ \wedge K_{\mathcal{W}}$ in coordinates. Take a basis of sections $\{\eta_i, \xi_a\}$ of $\mathfrak{g} \times Q \to Q$ where $\eta_i \in \Gamma(\mathfrak{g}_S)$ and $\xi_a \in \Gamma(\mathfrak{g}_W)$, for $i = 1, \cdots, l$; $a = l+1, \cdots, l+k$ and $dim(\mathfrak{g}) = m = l+k$. In the following, we will use indices i, j for objects related to S, indices a, b for objects related to W, and uppercase indices I, J, K when considering objects both in S and in W.

In order to apply formula (2.3.21) in the example treated in the next section, we compute the 3-form $dJ \wedge K_{\mathcal{W}}$ in coordinates adapted to the splitting (2.3.9). The basis of sections $\{\xi_a\}$, $a = l+1, \cdots, m$, of the bundle $\mathfrak{g}_W \to Q$, induces a basis $\{Z_a = (\xi_a)_Q\}$ of W by (2.3.10). Choosing a basis $\{X_i\}$, $i = 1, \cdots, l$, of the distribution D, we get a basis of sections $\{X_i, Z_a\}$ of TQ, with dual basis $\{X^i, \epsilon^a\}$. Let us denote by $\tilde{\epsilon}^a$, $a = l+1, \cdots, m$ the pull-back to $\mathcal{M}$ of the constraint 1-forms, i.e. $\tilde{\epsilon}^a = \tau_{\mathcal{M}}^*\epsilon^a$, where as usual $\tau_{\mathcal{M}} : \mathcal{M} \to Q$ is the canonical projection.

Using these notations we first compute in coordinates the $\mathcal{W}$-curvature $K_{\mathcal{W}}$ defined in (2.3.18) which is semi-basic with respect to the bundle $\mathcal{M} \to Q$ [5].

Lemma 2.3.7. *In the basis of sections $\{\xi_a\}$, $a = l+1, \cdots, m$, of the bundle $\mathfrak{g}_W \to Q$, the $\mathcal{W}$-curvature $K_{\mathcal{W}}$ is given by*

$$K_{\mathcal{W}}|_{\mathcal{C}} = d\tilde{\epsilon}^a|_{\mathcal{C}} \otimes \xi_a \quad \text{and} \quad K_{\mathcal{W}}|_{\mathcal{W}} = 0. \tag{2.3.22}$$

Proof. Let $\{\chi_I\}$ be a basis of $\mathfrak{g}$ (i.e. χ_I, $I = 1, \cdots, m$, are constant sections of $\mathfrak{g} \times Q \to Q$), then we can write the sections $\xi_a \in \mathfrak{g}_W$ using Einstein convention as

$$\xi_a = h_a^I \chi_I, \quad a = l+1, \cdots, m$$

for functions $h_a^I \in C^\infty(Q)$.

The projection $P_W : TQ \to W$, associated to the splitting $TQ = D \oplus W$, is given by $P_W = \epsilon^a \otimes Z_a$, where $Z_a = (\xi_a)_Q \in \Gamma(W)$ by (2.3.10). The $\mathfrak{g}$-valued 1-form A_W given in (2.3.16) is written as $A_W = \epsilon^a \otimes \xi_a$, and therefore $A_W = h_a^I \epsilon^a \otimes \chi_I$. Differentiating A_W we obtain:

$$\begin{aligned} dA_W &= d(\epsilon^a \otimes \xi_a) \\ &= d\epsilon^a \otimes \xi_a + dh_a^I \wedge \epsilon^a \otimes \chi_I. \end{aligned}$$

Therefore the $\mathcal{W}$-curvature $K_{\mathcal{W}}$ given in (2.3.18) is computed as follows, for $\tilde{X}, \tilde{Y} \in \mathfrak{X}(\mathcal{M})$,

$$\begin{aligned} K_{\mathcal{W}}(\tilde{X}, \tilde{Y}) &= K_W(T\tau_{\mathcal{M}}\tilde{X}, T\tau_{\mathcal{M}}\tilde{Y}) = dA_W(P_D T\tau_{\mathcal{M}}\tilde{X}, P_D T\tau_{\mathcal{M}}\tilde{Y}) \\ &= d\epsilon^a(P_D T\tau_{\mathcal{M}}\tilde{X}, P_D T\tau_{\mathcal{M}}\tilde{Y}) \otimes \xi_a, \end{aligned} \tag{2.3.23}$$

where in the last equality we used that ϵ^a annihilates the distribution D. By definition of the distribution $\mathcal{C}$ in $\mathcal{M}$ (1.4.23), we see that the following diagram commutes,

$$\begin{array}{ccc} T\mathcal{M} & \xrightarrow{P_{\mathcal{C}}} & \mathcal{C} \\ {\scriptstyle T\tau_{\mathcal{M}}}\big\downarrow & & \big\downarrow{\scriptstyle T\tau_{\mathcal{M}}} \\ TQ & \xrightarrow{P_D} & D, \end{array}$$

where $P_{\mathcal{C}} : T\mathcal{M} \to \mathcal{C}$ denotes the projection associated to splitting $T\mathcal{M} = \mathcal{C} \oplus \mathcal{W}$ in (2.3.13). Then, since $\tilde{\epsilon}^a = \tau_{\mathcal{M}}^* \epsilon^a$, we see that the last term in (2.3.23) is equal to $d\tilde{\epsilon}^a(P_{\mathcal{C}}\tilde{X}, P_{\mathcal{C}}\tilde{Y}) \otimes \xi_a$ and hence we obtain (2.3.22). $\square$

Before presenting the coordinate expression of $dJ \wedge K_{\mathcal{W}}|_{\mathcal{C}}$, let us fix some notations concerning change of basis of sections of the bundle $\mathfrak{g} \times Q \to Q$. As in the proof of the last Lemma consider a basis of $\mathfrak{g}$ denoted by $\{\chi_1, \cdots, \chi_m\}$, with dual basis $\{\chi^1, \cdots, \chi^m\}$. Also consider the basis of sections of $\mathfrak{g} \times Q \cong \mathfrak{g}_S \oplus \mathfrak{g}_W \to Q$ given by $\{\eta_i, \xi_a\}$, with dual basis $\{\eta^i, \xi^a\}$, $i = 1, \cdots, l$, $a = l+1, \cdots, m$. We can relate the corresponding basis by

$$\eta_i = h_i^K \chi_K, \quad \xi_a = h_a^K \chi_K, \tag{2.3.24}$$

for functions h_i^K, $h_a^K \in C^\infty(Q)$, and form the $m \times m$ square matrix $\mathbf{h}$ with entries h_J^K, where the l first rows are given by h_i^K and the next rows (for $J = l+1, \cdots, m$) are given by h_a^K. We denote by $\bar{\mathbf{h}}$ the inverse transpose of $\mathbf{h}$. With this notation the dual basis $\{\eta^i, \xi^a\}$ and $\{\chi^K\}$ are related by

$$\eta^i = \bar{h}_K^i \chi^K, \quad \xi^a = \bar{h}_K^a \chi^K. \tag{2.3.25}$$

Proposition 2.3.8. *Consider a basis $\{\eta_i, \xi_a\}$ of $\mathfrak{g} \times Q$ adapted to the splitting (2.3.9) and define the functions J_L on $\mathcal{M}$ by $J_i := \mathbf{i}_{(\eta_i)_{\mathcal{M}}}\Theta_{\mathcal{M}}$ and $J_a := \mathbf{i}_{(\xi_a)_{\mathcal{M}}}\Theta_{\mathcal{M}}$, $L = 1, \cdots, m$; $i = 1, \cdots, l$; $a = l+1, \cdots, m$. Then the 3-form $dJ \wedge K_{\mathcal{W}}$ restricted to $\mathcal{C}$ verifies*

$$dJ \wedge K_{\mathcal{W}}|_{\mathcal{C}} = \left(dJ_a \wedge d\tilde{\epsilon}^a + J_L \phi^L\right)|_{\mathcal{C}}, \tag{2.3.26}$$

where ϕ^L are 3-forms (basic with respect to $\tau_{\mathcal{M}} : \mathcal{M} \to Q$) defined by $\phi^L = h_a^K d\bar{h}_K^L \wedge d\tilde{\epsilon}^a$ and the functions h_a^K, $\bar{h}_K^L \in C^\infty(Q)$ are given in (2.3.24) and (2.3.25), respectively.

Proof. The moment map $J : \mathcal{M} \to \mathfrak{g}^*$ can be written as

$$J = \mathbf{i}_{(\eta_i)_\mathcal{M}} \Theta_\mathcal{M} \otimes \eta^i + \mathbf{i}_{(\xi_a)_\mathcal{M}} \Theta_\mathcal{M} \otimes \xi^a = J_a \otimes \xi^a + J_i \otimes \eta^i.$$

Therefore from (2.3.25),

$$\begin{aligned} dJ &= d(\bar{h}^a_K J_a) \otimes \chi^K + d(\bar{h}^i_K J_i) \otimes \chi^K \\ &= dJ_a \otimes \xi^a + dJ_i \otimes \eta^i + J_L d\bar{h}^L_K \otimes \chi^K. \end{aligned}$$

Using Lemma 2.3.7 we get,

$$dJ \wedge K_\mathcal{W}|_\mathcal{C} = \left(dJ_a \wedge d\tilde{\epsilon}^a + h^K_a J_L d\bar{h}^L_K \wedge d\tilde{\epsilon}^a \right) |_\mathcal{C},$$

and we observe that the last term has the form $J_L \phi^L$, a semi-basic 3-form with respect to the bundle $\tau_\mathcal{M} : \mathcal{M} \to Q$, and ϕ^L is the basic 3-form appearing in (2.3.26). $\square$

Remark 2.3.9. If the bundle W verifies the vertical symmetry condition (Rmk. 2.3.1), then the functions $barh^a_K$ in 2.3.24 are constant and hence in Prop. 2.3.8 we have $\phi^L = 0$. Consequently (2.3.26) becomes

$$dJ \wedge K_\mathcal{W}|_\mathcal{C} = (dJ_a \wedge d\tilde{\epsilon}^a)|_\mathcal{C},$$

, which implies that

$$dJ \wedge K_\mathcal{W}|_\mathcal{C} = d\langle J, K_\mathcal{W} \rangle,$$

where $\langle J, K_\mathcal{W} \rangle = J_a d\tilde{\epsilon}^a$. The 2-form $\langle J, K_\mathcal{W} \rangle$ is semi-basic with respect to the bundle $\tau_\mathcal{M} : \mathcal{M} \to Q$ and G-invariant, see [5], where it has been studied thoughtfully. $\diamond$

In Section 4.2.3 we apply the geometric framework and reduction by symmetries together with Proposition 2.3.8 to show that, for the example studied in this thesis, the reduced bracket $\{\cdot, \cdot\}_{red}$ in the quotient system $\mathcal{M}/G$ is not Poisson.

2.4 Examples

2.4.1 The nonholonomic particle

Continuing with the example of Section 1.5.1 and Ex. (2.1.1) , we recall that the system has a symmetry group $G = \mathbb{R}^2$ with Lie algebra $\mathfrak{g} \simeq \mathbb{R}^2$. The action is free and proper with the vertical distribution V given by $V = span\{\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\}$. Recalling the constraint distribution D computed in (1.1.4), we have that $S = D \cap V = \{Y_x\}$. We choose a vertical complement $W = span\{\frac{\partial}{\partial z}\}$ which verifies the vertical symmetry condition. Indeed, W is generated by the Lie subalgebra $span\{(0, 1)\} \subset \mathfrak{g}$.

The lifted action to $\mathcal{M}$ is given by

$$(a, b) \cdot (x, y, z, \tilde{p_x}, \tilde{p_y}) = (x + a, y, z + b, \tilde{p_x}, \tilde{p_y}), \tag{2.4.27}$$

then the vertical space is $\mathcal{V} = span\{\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\}$, with vertical complement $\mathcal{W} = span\{\frac{\partial}{\partial z}\}$, which verifies the vertical symmetry condition.

The bundle $\mathfrak{g}_S \to Q$ associated to S is generated by the section $\eta = (1, \frac{y}{1+y^2}) \in \mathfrak{g} \times Q$. On the other hand, as a consequence of the vertical symmetry condition we have that $dJ \wedge K_{\mathcal{W}} = d\langle J, K_{\mathcal{W}} \rangle$. Moreover

$$\langle J, K_{\mathcal{W}} \rangle = \frac{y\tilde{p_x}}{1+y^2} dx \wedge dy.$$

We can now compute the Jacobiator of the reduced bracket $\{\cdot,\cdot\}_{red}$ in the reduced space $\mathcal{M}/G \simeq \mathbb{R}^3$. We get

$$d\langle J, K_{\mathcal{W}} \rangle (\pi_{nh}(\rho^* dy), \pi_{nh}(\rho^* d\tilde{p_x}), \pi_{nh}(\rho^* d\tilde{p_y}) = 0,$$

and therefore the reduced bracket $\{\cdot,\cdot\}_{red}$ is Poisson.

2.4.2 The vertical rolling disk

We continue the study of the example described in Section 1.5.2. Recall that the configuration manifold is $Q = R^2 \times S^1 \times S^1$ and that the group $G = \mathbb{R}^2 \times S^1$ acts as a symmetry of the nonholonomic system. The action of an element (a, b, α) on a point of $\mathcal{M}$ is given by

$$(a, b, \alpha) \cdot (x, y, \theta, \phi, \tilde{p_\theta}, \tilde{p_\phi}) = (x + a, y + b, \theta + \alpha, \phi, \tilde{p_\theta}, \tilde{p_\phi}),$$

with orbit projection $\rho : \mathcal{M} \to \mathcal{M}/G$ given by

$$(x, y, \theta, \phi, \tilde{p_\theta}, \tilde{p_\phi}) \mapsto (\phi, \tilde{p_\theta}, \tilde{p_\phi}),$$

and vertical space $V = span\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta}\}$. We choose the vertical complement W as $W = span\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$, which verifies the vertical symmetry condition in Rmk. 2.3.1.

We compute the 3-form $dJ \wedge K_{\mathcal{W}}$ and we get zero. In fact, $\langle J, K_{\mathcal{W}} \rangle = 0$ and use that here $dJ \wedge K_{\mathcal{W}} = d\langle J, K_{\mathcal{W}} \rangle$. As a consequence, by the Jacobiator formula (2.3.21), the reduced bivector field is Poisson.

2.4.3 Snakeboard

We continue with the example started in Section 1.5.3. The symmetry group of the system is given by the action of the group $G = \mathbb{R}^2 \times S^1$ such that,

$$(a, b, \alpha) \cdot (x, y, \theta, \phi, \psi) = (x + a, y + b, \theta, \phi, \psi + \alpha),$$

which is clearly free and proper. Using the canonical basis of the Lie algebra $\mathfrak{g} \simeq \mathbb{R}^3$, we compute the vertical space:

$$V = span\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \psi}\}. \tag{2.4.28}$$

The constraint distribution D was computed in (1.5.39). We choose the vertical complement as $W = span\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ and we observe that it verifies the vertical symmetry condition in Rmk. 2.3.1.

The (free and proper) lifted action to $\mathcal{M}$ is given by

$$(a,b,\alpha)\cdot(x,y,\theta,\phi,\psi,\tilde{p_\theta},\tilde{p_\phi},\tilde{p_\psi}) = (x+a, y+b, \theta, \phi, \psi+\alpha, \tilde{p_\theta}, \tilde{p_\phi}, \tilde{p_\psi}),$$

and then the quotient $\mathcal{M}/G$ is a smooth manifold diffeomorphic to $\mathbb{T}^2\times R^3$ with orbit map $\rho:\mathcal{M}\to\mathcal{M}/G$ given by

$$\rho(x,y,\theta,\phi,\psi,\tilde{p_\theta},\tilde{p_\phi},\tilde{p_\psi}) = (\theta,\phi,\tilde{p_\theta},\tilde{p_\phi},\tilde{p_\psi}).$$

Using the canonical basis of the Lie algebra $\mathfrak{g}$, we compute the moment map $J:\mathcal{M}\to\mathfrak{g}^*$ and get $J(m) = (\iota^*_{\mathcal{M}}(\tilde{p_x}), \iota^*_{\mathcal{M}}(\tilde{p_y}), \tilde{p_\psi})$, with $\iota_{\mathcal{M}}:\mathcal{M}\to T^*Q$ the inclusion. The $\mathcal{W}$-curvature is given in our basis by $K_{\mathcal{W}} = (d\epsilon^1, d\epsilon^1, 0)$, and then

$$\langle J, K_{\mathcal{W}}\rangle = -\frac{mr^2\cot\phi}{mr^2 - J\sin^2\phi}(\tilde{p_\theta}-\tilde{p_\psi})d\theta\wedge d\phi, \tag{2.4.29}$$

which is *basic* with respect to the bundle $\mathcal{M}\to\mathcal{M}/G$. Since, as in the previous examples, $d\langle J, K_{\mathcal{W}}\rangle = dJ\wedge K_{\mathcal{W}}$, we conclude that the reduced bracket $\{\cdot,\cdot\}_{red}$ is twisted Poisson and then it has an integrable characteristic distribution (see the discussion after formula (2.3.21)).

2.4.4 Rigid body and Chaplygin ball

The nonholonomic system presented in Section 1.5.4 admits a symmetry group $G = \{(h,a)\in SO(3)\times\mathbb{R}^2 : h\boldsymbol{e_3} = \boldsymbol{e_3}\}\simeq SO(2)\times\mathbb{R}^2$ which acts on a point $(g,x)\in Q$ as

$$(h,a)\cdot(g,x) = (hg, hx+a). \tag{2.4.30}$$

The lifted action to TQ and $\mathcal{M}$ preserves and nonholonomic structure and the Hamiltonian $H_{\mathcal{M}}$. One see easily that the action is free and proper.

The vertical space V is computed from the infinitesimal generators computed from the (abelian) Lie algebra $\mathfrak{g}\simeq\mathbb{R}\times\mathbb{R}^2$. We get

$$\begin{aligned}(1,0,0)_Q =& \langle\boldsymbol{\gamma}, \boldsymbol{X}^L\rangle - x^2\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial x^2},\\ (0,1,0)_Q =& \frac{\partial}{\partial x^1}, \qquad (0,0,1)_Q = \frac{\partial}{\partial x^2},\end{aligned} \tag{2.4.31}$$

where recall that $\boldsymbol{\gamma}$ is the third row of the matrix $g\in SO(3)$. Then, the vertical space V is given by

$$V = span\{\langle\boldsymbol{\gamma}, \boldsymbol{X}^L\rangle - x^2\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\}, \tag{2.4.32}$$

and the system verifies the dimension assumption (2.3.4). Moreover, the distribution S is given by

$$S = span\{\langle\boldsymbol{\gamma}, \boldsymbol{X}^L + rAg\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}}\rangle\} = span\{\langle\boldsymbol{\gamma}, \boldsymbol{X}^L\rangle\}, \tag{2.4.33}$$

where in the last equality we used the form of the matrix A given in (1.5.42). The distribution S is generated using the section $\eta\in\mathfrak{g}_S$, such that $\eta = (1,-y,x)$ and

$\eta_Q = \langle \boldsymbol{\gamma}, \boldsymbol{X}^L \rangle$. The vertical complement to the distribution D given in (1.5.43) is chosen as $W = span\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\}$, which verifies the vertical symmetry condition (see Rmk. 2.3.1) and is generated by the Lie subalgebra $\mathfrak{w} = span\{(0,1,0),(0,0,1)\}$.

Then we get the splitting $TQ = D \oplus W$ generated by $\{\boldsymbol{Y}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\}$, with dual basis $\{\boldsymbol{\lambda}, \boldsymbol{\epsilon}\}$, where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ denotes the left Maurer-Cartan forms and $\boldsymbol{\epsilon} = (\epsilon^1, \epsilon^2)$ denotes the constraint 1-forms. The associated coordinates on T^*Q are $(\tilde{\boldsymbol{M}}, \tilde{\boldsymbol{p}})$.

The constraint manifold $\mathcal{M} \subset T^*Q$ has been computed in (1.5.46). The lifted G-action to $\mathcal{M}$ is given by

$$(h,a)\cdot(g,x,\tilde{\boldsymbol{M}}) = (hg, hx + a, \tilde{\boldsymbol{M}}), \tag{2.4.34}$$

where we observe that $\tilde{\boldsymbol{M}}$ is left invariant. This action is free and proper and thus its quotient space $\mathcal{M}/G$ is a smooth manifold diffeomorphic to $S^2 \times \mathbb{R}^3$. In fact the orbit projection $\rho : \mathcal{M} \to \mathcal{M}/G$ is given by $((g, x, \tilde{\boldsymbol{M}}) \mapsto (\boldsymbol{\gamma}, \tilde{\boldsymbol{M}})$.

The 2-form $\langle J, K_{\mathcal{W}} \rangle$ has been computed in [5, Sec. 7.3] and it is given by

$$\langle J, K_{\mathcal{W}} \rangle|_{\mathcal{C}} = r^2 m (g\boldsymbol{\Omega}^T A^T A g \boldsymbol{\lambda} \times \boldsymbol{\lambda}|_{\mathcal{C}}.$$

Using the expression of the matrix A given in (1.5.42), we have

$$\langle J, K_{\mathcal{W}} \rangle|_{\mathcal{C}} = r^2 m \langle \boldsymbol{\Omega}, \boldsymbol{\lambda} \times \boldsymbol{\lambda} \rangle|_{\mathcal{C}} - r^2 m \langle \boldsymbol{\gamma}, \boldsymbol{\Omega} \rangle \langle \boldsymbol{\gamma}, d\boldsymbol{\gamma} \times d\boldsymbol{\gamma})|_{\mathcal{C}}. \tag{2.4.35}$$

Using the Jacobiator formula for the reduced bracker $\{\cdot,\cdot\}_{red}$ we can verify that it is not Poisson. In fact, by the vertical symmetry condition $dJ \wedge K_{\mathcal{W}} = d\langle J, K_{\mathcal{W}} \rangle$ and moreover it is stated in [5, Sec. 7.3] that $\mathbf{i}_X d\langle J, K_{\mathcal{W}} \rangle \neq 0$, for $X \in \Gamma(\mathcal{S})$. Consequently the reduced bracket $\{\cdot,\cdot\}_{red}$ cannot be twisted Poisson and, since $\{\cdot,\cdot\}_{red}$ is regular [8], the characteristic distribution of $\{\cdot,\cdot\}_{red}$ is not integrable, as was proved in [46].

2.4.5 Homogeneous ball in a cylinder

We study the symmetries of the example presented in Section 1.5.5. That nonholonomic system admits a symmetry given by the group $G = S^1 \times SO(3)$ where the action of an element (φ, h) of G on a point (z, θ, g) of Q is given by

$$(\varphi, h)\cdot(z,\theta,g) = (z, \theta+\varphi, R_\varphi g h), \tag{2.4.36}$$

where R_φ denotes a 3×3 orthogonal matrix representing a rotation of angle φ with respect to the vertical axis. The action is easily seen to be free, and since G is compact, the action is also proper.

The lift of the action in (2.4.36) to $\mathcal{M} \subset T^*Q$ is given by

$$(\varphi, h)\cdot(z,\theta,g,\tilde{p_z},\tilde{p_\theta},M_n) = (z,\theta+\varphi,R_\varphi g h,\tilde{p_z},\tilde{p_\theta},M_n),$$

and the orbit projection $\rho : \mathcal{M} \to \mathcal{M}/G$ is given by $(z,\theta,g,\tilde{p_z},\tilde{p_\theta},M_n) \mapsto (z,\tilde{p_z},\tilde{p_\theta},M_n)$.

Considering the Lie algebra $\mathfrak{g} \simeq \mathbb{R}\times\mathfrak{so}(3)$ of G, with the canonical basis $\{(1,\mathbf{0}),(0,\boldsymbol{e_i})\}$ for $i = 1,2,3$, we get the infinitesimal generators:

$$(1,\mathbf{0})_Q = \frac{\partial}{\partial\theta} + X_3^R, \tag{2.4.37}$$

and

$$(0, \boldsymbol{e_i})_Q = \alpha_i X_1^R + \beta_i X_2^R + \gamma_i X_3^R, \quad i = 1, 2, 3, \tag{2.4.38}$$

where $\alpha_i, \beta_i, \gamma_i$ are the components of the vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ indicating, as usual, the rows of the matrix g. Then, the vertical space V is generated as follows,

$$V = span\{\frac{\partial}{\partial\theta} + X_3^R, X_1^R, X_2^R, X_3^R\} = span\{\frac{\partial}{\partial\theta} + X_z, X_n, X_\theta, X_z\}, \tag{2.4.39}$$

where in the last equality we used that (X_1^R, X_2^R, X_3^R) and (X_n, X_θ, X_z) are both bases of sections in $TSO(3)$. Recalling the basis of D in (1.5.53), we compute $S = D \cap V = \{Y_\theta, X_n\}$. The vertical complement W is chosen as $W = span\{X_\theta, X_z\}$ and we observe that it does not verify the vertical symmetry condition.

The G-invariant 3-form $dJ \wedge K_{\mathcal{W}}$ is given by

$$\frac{IR}{r^2 m} dp_z \wedge \beta_n \wedge d\theta - \frac{I}{r^2 m(R-r)} dp_\theta \wedge \beta_n \wedge dz, \tag{2.4.40}$$

which is not basic. Since $dJ \wedge K_{\mathcal{W}}(\pi_{nh}^\sharp(dM_n), \pi_{nh}^\sharp(dM_n), \pi_{nh}^\sharp(dM_n)) \neq 0$, the Jacobiator formula for the reduced bivector (2.3.21) shows that the reduced bivector field π_{nh}^{red} is not Poisson. In fact π_{nh}^{red} is not even Twisted Poisson because $dJ \wedge K_{\mathcal{W}}$ is not semi-basic, see [5, Cor. 4.7].

2.4.6 Body of revolution on a plane

We continue with the example introduced in Section 1.5.6. The configuration manifold Q is given in (1.5.57) and it is diffeomorphic to $\mathbb{R}^2 \times SO(3)$, the Lagrangian L is given in (1.5.56) and the constraint distribution D in (1.5.59). The mechanical system has the symmetry group $SE(2)$ acting on the left (space frame) as in the case of the Chaplygin ball (see Section 2.4.4) and because the body has a symmetry axis, there is a S^1-action by the right (body frame). The two actions commute (see [6, 36]) and therefore an element $(x, y, \phi, \theta) \in \mathbb{R}^2 \times SO(2) \times S^1$ acts on a point $(a_1, a_2, g) \in Q$ as

$$(x, y, \phi, \theta) \cdot (a_1, a_2, g) = (R_\phi(a_1, a_2) + (x, y), \hat{R_\phi} g \hat{R_{-\theta}}), \tag{2.4.41}$$

where R_ϕ is a 2×2 rotation matrix with angle ϕ and $\hat{R_\phi}$ and $\hat{R_{-\theta}}$ denote 3×3 orthogonal matrices representing a rotation about the z-axis of angles ϕ and $-\theta$, respectively. It is shown in [36] that this G-action is a symmetry of the nonholonomic system.

We have seen this action in Example 2.1.5 and we observed that it is proper but not free. The Lie algebra $\mathfrak{g}$ of G is isomorphic to $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$, and the infinitesimal generators for the S^1-action is given by

$$((0,0), 0, 1)_Q = -X_3^L, \tag{2.4.42}$$

where, as usual, we denote by $\boldsymbol{X}^L = (X_1^L, X_2^L, X_3^L)$ the left invariant vector fields for $SO(3)$. On the other hand the infinitesimal generators for the $SE(2)$-action are given by

$$\begin{aligned} ((1,0),0,0)_Q &= \frac{\partial}{\partial a_1}, \qquad ((0,1),0,0)_Q = \frac{\partial}{\partial a_1}, \\ ((0,0),1,0)_Q &= \langle \boldsymbol{\gamma}, \boldsymbol{X}^L \rangle - a_2 \frac{\partial}{\partial a_1} + a_1 \frac{\partial}{\partial a_2}. \end{aligned} \tag{2.4.43}$$

As was observed in Example 2.1.5, at the point $(0, 0, g_z) \in Q$, where g_z has a third row $(0, 0, \pm 1)$, the infinitesimal generators $(0, 0, 0, 1)_Q$ and $(0, 0, 1, 0)_Q$ coincide, and the action is not free. Then, the vertical space V generated by the infinitesimal generators (2.4.42) and (2.4.43) does not have constant rank and more precisely the rank decreases exactly at the points $(0, 0, g_z)$ which have nontrivial isotropy group. Nevertheless, the dimension assumption (2.3.4) is verified.

Using the basis of D given in (1.5.59) we compute $S = D \cap V = span\{Y_3, \langle \boldsymbol{\gamma}, \boldsymbol{Y} \rangle\}$, where $\boldsymbol{Y} = (Y_1, Y_2, Y_3)$, and observe that its rank is also nonconstant. The bundle $\mathfrak{g}_S \to Q$ is generated by 2 sections $\eta_1, \eta_2 \in \Gamma(\mathfrak{g}_S)$ (see Eq. (3.41) in [6]), which verifies $(\eta_1)_Q = Y_3$, and $(\eta_2)_Q = \langle \boldsymbol{\gamma}, \boldsymbol{Y} \rangle$.

On the other hand, we choose the vertical complement W as

$$W = span\{\frac{\partial}{\partial a_1}, \frac{\partial}{\partial a_2}\},$$

and we observe that it is G-invariant. Moreover, W verifies the vertical symmetry condition (see Rmk. 2.3.1) because it is generated by the subalgebra

$$\mathfrak{w} = span\{((1, 0), 0, 0), ((0, 1), 0, 0)\}.$$

Then, we have the splittings $TQ = D \oplus W$ generated by $\{\boldsymbol{Y}, \frac{\partial}{\partial a_1}, \frac{\partial}{\partial a_2}\}$ with dual basis $\{\boldsymbol{\lambda}, \epsilon^1, \epsilon^2\}$ and associated coordinates $(\tilde{\boldsymbol{M}}, \tilde{\boldsymbol{p}}) = (\tilde{M}_1, \tilde{M}_2, \tilde{M}_3, \tilde{p}_1, \tilde{p}_2)$ (recall that $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ denotes the left Maurer-Cartan 1-form).

The constraint manifold $\mathcal{M} \subset T^*Q$ has been computed in (1.5.60). Since G is a symmetry of the system it leaves invariant $\mathcal{M}$ and the lifted action of the element $(x, y, \phi, \theta) \in G$ on a point $((a_1, a_2), g, \tilde{\boldsymbol{M}}) \in \mathcal{M}$ is given by

$$(x, y, \phi, \theta) \cdot ((a_1, a_2), g, \tilde{\boldsymbol{M}}) = (R_\phi(a_1, a_2) + (x, y), \hat{R}_\phi g \hat{R}_\theta, \hat{R}_\theta \tilde{\boldsymbol{M}}),$$

where we observe that $\tilde{\boldsymbol{M}}$ is invariant by the left $E(2)$-action. The action on $\mathcal{M}$ is also proper but not free and we treat the quotient $\mathcal{M}/G$ as a stratified differential space. Reducing by stages, observe that $E(2)$ is a normal subgroup of G and the $E(2)$-action is free and proper. Then, the quotient $\mathcal{M}/SE(2)$ is a manifold which is diffeomorphic to $S^2 \times \mathbb{R}^3$ with coordinates $(\boldsymbol{\gamma}, \tilde{\boldsymbol{M}})$ (see Example 2.1.5). The S^1-action on $\mathcal{M}/SE(2)$ is not free and we describe the resulting differential space $\mathcal{M}/G$ using *invariant theory*, see [6, 36]. The ring of S^1-invariant polynomials in $S^2 \times \mathbb{R}^3$ is generated by

$$\begin{aligned}
\tau_1 =& \gamma_3, \quad \tau_2 = \gamma_1 M_2 - \gamma_2 M_1, \quad \tau_3 = \gamma_1 M_1 + \gamma_2 M_2, \\
\tau_4 =& M_3, \quad \tau_5 = M_1^2 + M_2^2,
\end{aligned}$$

and the quotient space $\mathcal{M}/G$ is represented by the following semi-algebraic set in $\mathbb{R}^5$ with coordinates $(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5)$:

$$\{(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5) \in \mathbb{R}^5 \ : \ |\tau_1| \leq 1, \tau_5 \geq 0, \tau_2^2 + \tau_3^2 = (1 - \tau_1^2)\tau_5\}.$$

To describe the stratification of $\mathcal{M}/G$, observe that the S^1-action on $\mathcal{M}/E(2) \simeq S^2 \times \mathbb{R}^3 \subset \mathbb{R}^3 \times \mathbb{R}^3$ is given by

$$\theta \cdot (\boldsymbol{\gamma}, \tilde{\boldsymbol{M}}) = (\hat{R}_\theta \boldsymbol{\gamma}. \hat{R}_\theta \tilde{\boldsymbol{M}}),$$

and then the points $(0,0,\pm 1,0,0,M_3)$ are fixed and have S^1 as isotropy group. Thus, the *singular stratum* of $\mathcal{M}/G$ corresponding to S^1-isotropy type are the points of the 1-dimensional manifold

$$\mathcal{M}_{sing} = \{(\pm 1, 0, 0, \tau_4, 0) \in \mathcal{M} \mid \tau_4 \in \mathbb{R}\}, \tag{2.4.44}$$

and corresponds to the configuration where the body of revolution is spinning over one the two poles which remains fixed on the plane. Since there are no other isotropy types, the regular stratum is the complementary 4-dimensional manifold, i.e. $\mathcal{M}_{reg} := \mathcal{M} - \mathcal{M}_{sing}$.

To perform the reduction recall that the nonholonomic bivector field π_{nh} given in (1.5.61) and its associated nonholonomic bracket $\{\cdot,\cdot\}_{nh}$ induce a reduced bracket $\{\cdot,\cdot\}_{red}$ on $\mathcal{M}/G$ from (2.1.3). Since our choice of W verifies the vertical symmetry condition the 3-form $dJ \wedge K_{\mathcal{W}}$ verifies $dJ \wedge K_{\mathcal{W}} = d\langle J, K_{\mathcal{W}}\rangle$ and the 2-form $\langle J, K_{\mathcal{W}}\rangle$ is G-invariant and semi-basic with respect to $\mathcal{M} \to Q$ (see [5]). It has been computed in [6, Lemma 3.2] and can be written as

$$\langle J, K_{\mathcal{W}}\rangle = \langle \boldsymbol{K}, d\boldsymbol{\lambda}\rangle,$$

where $\boldsymbol{K} = (K_1, K_2, K_3)$ is given (1.5.62). Using the Jacobiator formula (2.3.21) it is shown in [6, Prop. 3.3] that the reduced bracket $\{\cdot,\cdot\}_{red}$ is not Poisson.

Chapter 3

Gauge transformations and conserved quantities

Following [5, 8], we will generate a new almost Poisson bracket π_B on $\mathcal{M}$ preserving the dynamics so that its reduction by symmetries to the differential space $\mathcal{M}/G$ becomes Poisson. In order to do that we assume that the system has first integrals and study properties of the first integrals of the system that are *horizontal gauge momenta* [40] and their relation with *gauge transformations*, see also [6, 47]. Our work has been developed independently of [47], which treat generally the problem of finding gauge transformation to induce Casimirs from horizontal gauge momenta.

3.1 Gauge transformations

Consider a nonholonomic system $(\mathcal{M}, \pi_{nh}, H_{\mathcal{M}})$. In this section we recall the concept of *dynamical gauge transformation of π_{nh} by a 2-form* [8] (see also [80]), which is a "deformation" of π_{nh} that generates a new bivector π_B which also *describes the dynamics*: $\pi_B^\#(dH_{\mathcal{M}}) = -X_{nh}$.

First, following [8], observe that a regular almost Poisson manifold (M, π) is always defined by a distribution F on M (its characteristic distribution) and a nondegenerate 2-section Ω on F such that

$$\pi^\#(\alpha) = -X \Leftrightarrow \mathbf{i}_X\Omega|_F = \alpha|_F, \quad \text{for } \alpha \in T^*M.$$

Definition 3.1.1 ([80])**.** Consider a 2-form B on M such that the 2-section $\Omega + B|_F$ is nondegenerate. The *gauge transformation of the bivector field π by the 2-form B* induces a new bivector field π_B on M defined by

$$\pi_B^\#(\alpha) = -X \Leftrightarrow \mathbf{i}_X(\Omega + B)|_F = \alpha|_F, \qquad \text{for } \alpha \in T^*M. \tag{3.1.1}$$

In this case, the new bivector π_B is defined by the same distribution F on M and the 2-section $\Omega + B|_F$, and we say that π_{nh} and π_B are *gauge related*.

In particular observe that the almost Poisson manifold $(\mathcal{M}, \pi_{nh})$ presented in Section 1.4 is defined by the (non-integrable) distribution $\mathcal{C}$ in (1.4.23) and the nondegenerate 2-section $\Omega_{\mathcal{C}}$ on $\mathcal{C}$ defined in (1.4.24). Following (3.1.1), if we consider a 2-form

B on $\mathcal{M}$ such that $\Omega_{\mathcal{C}} + B|_{\mathcal{C}}$ is a nondegenerate 2-section, we get a new bivector π_B gauge related to π_{nh}. Moreover, since we are interested in bivector fields π_B describing the dynamics in the sense that $\pi_B^\sharp(dH_{\mathcal{M}}) = -X_{nh}$, we recall the following

Definition 3.1.2 ([8])**.** A 2-form B on $\mathcal{M}$ defines a *dynamical gauge transformation* if $\Omega_{\mathcal{C}} + B|_{\mathcal{C}}$ is a nondegenerate 2-section and $\mathbf{i}_{X_{nh}}B = 0$. The *dynamical gauge related* bivector field π_B is the bivector defined by

$$\pi_B^\sharp(\alpha) = -X \Leftrightarrow \mathbf{i}_X(\Omega_{\mathcal{M}} + B)|_{\mathcal{C}} = \alpha|_{\mathcal{C}}, \qquad \text{for } \alpha \in T^*\mathcal{M}, \tag{3.1.2}$$

and it also describes the nonholonomic dynamics.

Remark 3.1.3. (*i*) If we consider a *semi-basic* (or *horizontal*) 2-form B with respect to the bundle $\tau_{\mathcal{M}} : \mathcal{M} \to Q$ (i.e. $\mathbf{i}_X B = 0$ if $T\tau_{\mathcal{M}}(X) = 0$), then $\Omega_{\mathcal{C}} + B|_{\mathcal{C}}$ is nondegenerate, see [8]. The motivation behind this result is the standard construction given by addition of a 'magnetic type term' to the canonical symplectic manifold (T^*Q, Ω_Q). In fact, if B is a closed 2-form, *basic* w.r.t. the bundle $T^*Q \to Q$, then it induces the symplectic manifold $(T^*Q, \Omega_Q + B)$.

(*ii*) If $\Omega_{\mathcal{C}} + B|_{\mathcal{C}}$ is nondegenerate, then by definition the bivector fields π_{nh} and π_B have the same (non-integrable) characteristic distribution $\mathcal{C}$, then π_B cannot be Poisson.

(*iii*) From (3.1.2), we see that the bivector field π_B only depends on the restriction of B to the characteristic distribution $\mathcal{C}$. Therefore, following [5], once we choose a vertical complement $\mathcal{W}$ so that $T\mathcal{M} = \mathcal{C} \oplus \mathcal{W}$, we will also ask that

$$\mathbf{i}_X B \equiv 0, \quad X \in \Gamma(\mathcal{W}). \tag{3.1.3}$$

(*iv*) Gauge transformations can be defined in the more general context of Dirac structures [80] and were introduced in nonholonomic mechanics for almost Dirac structures in [8]. We give a summary of this more general construction in Appendix 1.

$\diamond$

One interesting feature when considering a dynamical gauge related bivector π_B of π_{nh} is that, under the presence of symmetries, we may consider the reduction of both bivectors and obtain, in the reduced space $\mathcal{M}/G$, two brackets $\{\cdot,\cdot\}_{red}^B$ and $\{\cdot,\cdot\}_{red}$, respectively, describing the reduced dynamics. It was already observed in previous examples [5, 6, 8, 47] that the integrability properties of $\{\cdot,\cdot\}_{red}^B$ can be very different from the original reduced bracket $\{\cdot,\cdot\}_{red}$. In particular they can have different characteristic distributions or even one can be Poisson while the other not. More precisely, consider a nonholonomic system $(\mathcal{M}, \pi_{nh}, H_{\mathcal{M}})$ with a proper G-symmetry satisfying the dimension assumption and let B be a 2-form defining a dynamical gauge transformation. If B is G-invariant, then the gauge related bivector π_B is G-invariant and it induces a reduced almost Poisson bracket $\{\cdot,\cdot\}_{red}^B$ on the differential space $\mathcal{M}/G$ given, for $\bar{f}, \bar{g} \in C^\infty(\mathcal{M}/G)$, by

$$\{\bar{f}, \bar{g}\}_{red}^B \circ \rho = \{\bar{f} \circ \rho, \bar{g} \circ \rho\}_B, \tag{3.1.4}$$

where as usual $\rho : \mathcal{M} \to \mathcal{M}/G$ is the orbit projection, and $\{\cdot,\cdot\}_B$ is the bracket associated to π_B. Moreover, since $\mathbf{i}_{X_{nh}} B = 0$, the reduced bracket $\{\cdot,\cdot\}_{red}^B$ describes the nonholonomic reduced dynamics

$$X_{red} = \{\cdot, H_{red}\}_{red}^B \in \mathfrak{X}(\mathcal{M}/G), \tag{3.1.5}$$

where $H_{red} : \mathcal{M}/G \to \mathbb{R}$ is the reduced Hamiltonian. We illustrated the gauge transformation by a 2-form B and reduction by the G-symmetry in Fig. (I.3) in the introduction chapter.

Following [5], in order to analyse the failure of the Jacobi identity of $\{\cdot,\cdot\}_{red}^B$ we use an analogous formula to (2.3.21) but now considering the gauge transformation. That is, let π_B be a bivector field on $\mathcal{M}$ gauge related to π_{nh} by a G-invariant 2-form B satisfying that $\mathbf{i}_Z B \equiv 0$, for $Z \in \Gamma(\mathcal{W})$. If $\mathcal{W}$ is a G-invariant vertical complement of $\mathcal{C}$, then:

1. The nonholonomic bracket $\{\cdot,\cdot\}_B$ on $\mathcal{M}$ satisfies, for all $f, g, h \in C^\infty(\mathcal{M})$,

$$\begin{aligned} cyclic[\{f,\{g,h\}_B\}_B] =& (dJ \wedge K_{\mathcal{W}} - dB)(\pi_B^\#(df), \pi_B^\#(dg), \pi_B^\#(dh)) \\ &- \psi_{\pi_B}(df, dg, dh), \end{aligned} \tag{3.1.6}$$

   where ψ_{π_B} is the trivector given by $\psi_{\pi_B}(\alpha,\beta,\gamma) = cyclic[\gamma(K_{\mathcal{W}}(\pi_B^\#(\alpha), \pi_B^\#(\beta))_{\mathcal{M}})]$, for α, β, γ 1-forms in $\mathcal{M}$.

2. For $\bar{f}, \bar{g}, \bar{h} \in C^\infty(\mathcal{M}/G)$,

$$cyclic[\{\bar{f},\{\bar{g},\bar{h}\}_{red}^B\}_{red}^B \circ \rho] = (dJ \wedge K_{\mathcal{W}} - dB)(\pi_B^\#(d\rho^*\bar{f}), \pi_B^\#(d\rho^*\bar{g}), \pi_B^\#(d\rho^*\bar{h})), \tag{3.1.7}$$

   where $\{\cdot,\cdot\}_{red}^B$ is the reduced bracket on $\mathcal{M}/G$ defined in (2.1.3). As in the case of (2.3.21), we observe that (3.1.7) also works for the case where the quotient $\mathcal{M}/G$ is a differential space.

By definition, the gauge transformation does not change the (non-integrable) characteristic distribution $\mathcal{C}$ of the nonholonomic bracket $\{\cdot,\cdot\}_{nh}$. Consequently (and also from formula (3.1.6)), the bracket $\{\cdot,\cdot\}_B$ is never Poisson. In fact, by the same argument, the bracket $\{\cdot,\cdot\}_B$ is never twisted Poisson (see Example 1.4.1).

From formula (3.1.7) one observe that the reduced bracket $\{\cdot,\cdot\}_{red}^B$ is Poisson if the RHS of (3.1.7) is zero. More generally, the reduced bracket $\{\cdot,\cdot\}_{red}^B$ is Poisson if

$$(dJ \wedge K_{\mathcal{W}} - dB)|_{U_B} = 0,$$

where U_B is the distribution on $\mathcal{M}$ given by

$$U_B = span\{\pi_B^\sharp(df) \ : \ f \in C^\infty(\mathcal{M})^G\}.$$

On the other hand, if the action is free and proper, Cor. 4.10 of [5] shows that if 3-form $dJ \wedge K_{\mathcal{W}}$ is closed and $dJ \wedge K_{\mathcal{W}} - dB$ is basic with respect to $\rho : \mathcal{M} \to \mathcal{M}/G$, then $\{\cdot,\cdot\}_{red}^B$ is ϕ-twisted Poisson, where $\phi \in \Omega^3(\mathcal{M}/G)$ is given by $\rho^*\phi = dJ \wedge K_{\mathcal{W}} - dB$.

Remark 3.1.4. Following [6], suppose that G acts freely and properly on $\mathcal{M}$ and that the bivectors fields π_{nh} and π_B are gauge related by a 2-form B *basic* with respect to the (principal) bundle $\rho : \mathcal{M} \to \mathcal{M}/G$, then the corresponding reduced brackets $\{\cdot,\cdot\}_{red}$ and $\{\cdot,\cdot\}_{red}^B$ are gauge related by the 2-form $\bar{B}$ on $\mathcal{M}/G$ such that $\rho^*\bar{B} = B$. Therefore, in that case the gauge transformation does not change the integrability properties of the reduced brackets. $\diamond$

Next, we present formulas in local coordinates and afterwards we see how to find an appropriate 2-form B in such a way that the reduced bracket $\{\cdot,\cdot\}_{red}^B$ has enough Casimirs so that its characteristic distribution is integrable. In particular, in the example studied in this thesis we show that $\{\cdot,\cdot\}_{red}^B$ is Poisson. In order to do that we study the first integrals of the nonholonomic system induced by the presence of symmetries, see also [6, 47].

3.2 Formulas in local basis

In this section we present formulas in local basis for a gauge transformation and see how it affects the expression of the nonholonomic bivector field π_{nh} when we write it in adapted bases to the constraints and the symmetry, i.e. adapted to the splitting $TQ = D \oplus W$ in (2.3.6).

If Q is n-dimensional and we have k constraints, take a basis of (local) sections of D and W such that $D = span\{X_i\}$, $i = 1, \cdots, n-k$, and $W = span\{Z_a\}$, $a = 1, \cdots, n$. Then, we have

$$TQ = span\{X_i, Z_a\}, \tag{3.2.8}$$

with dual basis $\{X^i, \epsilon^a\}$ s.t.

$$T^*Q = span\{X^i, \epsilon^a\}, \tag{3.2.9}$$

and associated coordinates $(\tilde{p_i}, \tilde{p_a})$. In the rest of this section we use Einstein notation, the indices of i, j, l, m vary in the range $1, \cdots, n-k$, and a in the range $1, \cdots, k$.

From the basis (3.2.9) we construct a basis of $T^*\mathcal{M}$ using the map $\tau_{\mathcal{M}} : \mathcal{M} \to Q$. We denote $\tilde{X^i} = \tau_{\mathcal{M}}^* X^i$, and $\tilde{\epsilon^a} = \tau_{\mathcal{M}}^*\epsilon^a$, and we form the basis of $T^*\mathcal{M}$ such that

$$T^*\mathcal{M} = span\{\tilde{X^i}, \tilde{\epsilon}^a, d\tilde{p_i}\}, \tag{3.2.10}$$

with corresponding dual basis generating $T\mathcal{M}$, that is,

$$T\mathcal{M} = span\{\tilde{X}_i, \tilde{Z}_a, \frac{\partial}{\partial \tilde{p_i}}\}.$$

Then the distribution $\mathcal{C}$ defined in (1.4.23) is generated by

$$\mathcal{C} = span\{\tilde{X}_i, \frac{\partial}{\partial \tilde{p_i}}\}.$$

The Liouville 1-form restricted to $\mathcal{M}$ is given by

$$\Theta_{\mathcal{M}} = \tilde{p_i}\tilde{X^i} + \iota_{\mathcal{M}}^*(\tilde{p_a})\tilde{\epsilon}^a,$$

where, as usual, we use the inclusion $\iota_{\mathcal{M}} : \mathcal{M} \to T^*Q$. Then the canonical 2-form restricted to $\mathcal{M}$ (see(1.4.24)) is

$$\Omega_{\mathcal{M}} = \tilde{X}^i \wedge d\tilde{p}_i - \tilde{p}_i d\tilde{X}^i + \tilde{\epsilon}^a \wedge \iota^*_{\mathcal{M}}(d\tilde{p}_a) - \iota^*_{\mathcal{M}}(\tilde{p}_a) d\tilde{\epsilon}^a,$$

and after point-wise restriction to $\mathcal{C}$ we get

$$\Omega_{\mathcal{C}} = (\tilde{X}^i \wedge d\tilde{p}_i - \tilde{p}_i d\tilde{X}^i - \iota^*_{\mathcal{M}}(\tilde{p}_a) d\tilde{\epsilon}^a)|_{\mathcal{C}}. \tag{3.2.11}$$

The 2-forms $d\tilde{X}^i$ and $d\tilde{\epsilon}^a$ can be written in the basis of the exterior algebra associated to (3.2.9). After restriccion to $\mathcal{C}$ they take the form

$$d\tilde{X}^i|_{\mathcal{C}} = f^i_{lm} \tilde{X}^l \wedge \tilde{X}^m, \quad d\tilde{\epsilon}^a|_{\mathcal{C}} = g^a_{lm} \tilde{X}^l \wedge \tilde{X}^m, \tag{3.2.12}$$

where $f^i_{lm} = d\tilde{X}^i(\tilde{X}_l, \tilde{X}_m)$ and $g^a_{lm} = d\tilde{\epsilon}^a(\tilde{X}_l, \tilde{X}_m)$. On the one hand, using the standard Eq. (1.2.5), we have

$$\begin{aligned} d\tilde{X}^i(\tilde{X}_l, \tilde{X}_m) &= \tilde{X}_l(\tilde{X}^i(\tilde{X}_m)) - \tilde{X}_m(\tilde{X}^i(\tilde{X}_l)) - \tilde{X}^i([\tilde{X}_l, \tilde{X}_m]) \\ &= -\tilde{X}^i(c^n_{lm} \tilde{X}_n + d^a_{lm} \tilde{Z}_a) = -c^i_{lm}, \end{aligned} \tag{3.2.13}$$

where we used properties of the dual basic and we denoted by c^i_{lm} and d^a_{lm} the structure functions of the Lie bracket associated to the basis (3.2.8) of TQ. More precisely, we write

$$[X_l, X_m] = c^n_{lm} X_n + d^a_{lm} Z_a. \tag{3.2.14}$$

On the other hand, we get

$$d\tilde{\epsilon}^a(\tilde{X}_l, \tilde{X}_m) = -\tilde{\epsilon}^a(c^n_{lm} \tilde{X}_n + d^b_{lm} \tilde{Z}_b) = -d^a_{lm}.$$

Then, Eq. (3.2.12) become

$$d\tilde{X}^i|_{\mathcal{C}} = -c^i_{lm} \tilde{X}^l \wedge \tilde{X}^m, \quad d\tilde{\epsilon}^a|_{\mathcal{C}} = -d^a_{lm} \tilde{X}^l \wedge \tilde{X}^m, \tag{3.2.15}$$

and, replacing in (3.2.11),

$$\Omega_{\mathcal{C}} = \tilde{X}^i \wedge d\tilde{p}_i + \tilde{p}_i \; c^i_{lm} \tilde{X}^l \wedge \tilde{X}^m + \iota^*_{\mathcal{M}}(\tilde{p}_a) d^a_{lm} \tilde{X}^l \wedge \tilde{X}^m,$$

From the definition of the nonholonomic bivector field π_{nh}, Eqs. (1.4.25)-(1.4.26), we compute

$$\begin{aligned} \pi^\sharp_{nh}(\tilde{X}^i) &= \frac{\partial}{\partial \tilde{p}_i}, \\ \pi^\sharp_{nh}(d\tilde{p}_j) &= -\tilde{X}_j - \tilde{p}_i c^i_{jm} \frac{\partial}{\partial \tilde{p}_m} - \iota^*_{\mathcal{M}}(\tilde{p}_a) d^a_{jm} \frac{\partial}{\partial \tilde{p}_m}, \\ \pi_{nh}(\tilde{\epsilon}^a) &= 0, \end{aligned}$$

or equivalently

$$\pi_{nh} = \tilde{X}_i \wedge \frac{\partial}{\partial \tilde{p}_i} - \tilde{p}_i c^i_{jm} \frac{\partial}{\partial \tilde{p}_j} \wedge \frac{\partial}{\partial \tilde{p}_m} - \iota^*_{\mathcal{M}}(\tilde{p}_a) d^a_{jm} \frac{\partial}{\partial \tilde{p}_j} \wedge \frac{\partial}{\partial \tilde{p}_m}. \tag{3.2.16}$$

Now we study the form of the gauge related bivector field π_B, for a 2-form B on $\mathcal{M}$ semi-basic with respect to $\mathcal{M} \to Q$ and such that $\mathbf{i}_X B = 0$, for $X \in \Gamma(\mathcal{W})$. In the basis (3.2.10), the 2-form B has the form

$$B = b_{ij} \tilde{X}^i \wedge \tilde{X}^j,$$

and then the 2-section $\Omega_{\mathcal{C}} + B$ defining the gauge transformed bivector field π_B is written

$$\Omega_{\mathcal{C}} + B = \tilde{X}^i \wedge d\tilde{p}_i + (\tilde{p}_i \ c^i_{lm} + \iota^*_{\mathcal{M}}(\tilde{p}_a) d^a_{lm} + b_{lm}) \tilde{X}^l \wedge \tilde{X}^m,$$

hence

$$\pi_B = \tilde{X}_i \wedge \frac{\partial}{\partial \tilde{p}_i} - \tilde{p}_i c^i_{jm} \frac{\partial}{\partial \tilde{p}_j} \wedge \frac{\partial}{\partial \tilde{p_m}} - \iota^*_{\mathcal{M}}(\tilde{p}_a) d^a_{jm} \frac{\partial}{\partial \tilde{p}_j} \wedge \frac{\partial}{\partial \tilde{p_m}} - b_{jm} \frac{\partial}{\partial \tilde{p}_j} \wedge \frac{\partial}{\partial \tilde{p_m}}. \quad (3.2.17)$$

Comparing (3.2.16) and (3.2.17), we see that the gauge transformation only changes the terms associated to $\frac{\partial}{\partial \tilde{p}_j} \wedge \frac{\partial}{\partial \tilde{p_m}}$. Thus, $\pi_B^\sharp(\tilde{X}^i) = \frac{\partial}{\partial \tilde{p}_i}$.

3.3 Horizontal gauge symmetries and the choice of a gauge transformation

Consider a nonholonomic system $(\mathcal{M}, \pi_{nh}, H_{\mathcal{M}})$ with a proper G-symmetry satisfying the dimension assumption (2.3.4). Recall from Section 2.3.1 that the dimension assumption induces a splitting of the trivial bundle $\mathfrak{g} \times Q \to Q$ as in (2.3.9):

$$(\mathfrak{g} \times Q)|_q = \mathfrak{g}_S|_q \oplus \mathfrak{g}_W|_q.$$

The *nonholonomic moment map* [16] is the map $J^{nh} : \mathcal{M} \to \mathfrak{g}_S^*$ defined, at each $m \in \mathcal{M}$ and $\eta \in \Gamma(\mathfrak{g}_S)$, by

$$J_\eta(m) = \langle J^{nh}, \eta \rangle (m) = \langle J^{nh}(m), \eta(\tau_{\mathcal{M}}(m)) \rangle := \mathbf{i}_{\eta_{\mathcal{M}}} \Theta_{\mathcal{M}}(m), \quad (3.3.18)$$

where $\Theta_{\mathcal{M}}$ is the restriction to $\mathcal{M}$ of the Liouville 1-form Θ_Q on T^*Q. Observe that the function J_η on $\mathcal{M}$ is linear on the fibers of the bundle $\tau_{\mathcal{M}} : \mathcal{M} \to Q$. However, contrarily to the standard moment map for Hamiltonian systems, J_η is not necessarily a first integral of the dynamics X_{nh}. In fact, using the G-invariance of the restricted Hamiltonian $H_{\mathcal{M}}$, we have (see [9])

$$X_{nh}(J_\eta) = (\pounds_{\eta_{\mathcal{M}}} \Theta_{\mathcal{M}})(X_{nh}),$$

where $\pounds$ is the Lie derivative. Indeed, using (3.3.18), and using Cartan's formula we have

$$X_{nh}(J_\eta) = dJ_\eta(X_{nh}) = (\pounds_{\eta_{\mathcal{M}}} \Theta_{\mathcal{M}})(X_{nh}) - (\mathbf{i}_{\eta_{\mathcal{M}}} d\Theta_{\mathcal{M}})(X_{nh}). \quad (3.3.19)$$

Writing $\eta \in \Gamma(\mathfrak{g}_S)$ as $\eta = \sum f_i \chi_i$, with $\chi_i \in \mathfrak{g}$ and $f_i \in C^\infty(Q)$, we have that the last term of the RHS of (3.3.19) is equal to

$$\Omega_{\mathcal{M}}(\eta_{\mathcal{M}}, X_{nh}) = -\Omega_{\mathcal{M}}(X_{nh}, \eta_{\mathcal{M}}) = -dH_{\mathcal{M}}(\eta_{\mathcal{M}}) = -\eta_{\mathcal{M}}(H_{\mathcal{M}}) = -f_i(\chi_i)_{\mathcal{M}}(H_{\mathcal{M}}) = 0,$$

where in the last equality we used that $H_{\mathcal{M}}$ is G-invariant.

Following [41] we call *horizontal gauge momentum* of X_{nh} a function of the type $J_\zeta = \langle J^{nh}, \zeta \rangle \in C^\infty(\mathcal{M})$ such that J_ζ is a first integral of X_{nh}, and the section $\zeta \in \Gamma(\mathfrak{g}_S)$ is called *horizontal gauge symmetry.* Moreover, it was proven in [41] (see also [9]) that ζ is a horizontal gauge symmetry if and only if the cotangent lift of the vector field ζ_Q leaves the Hamiltonian invariant, i.e. $\zeta_Q^{T^*Q}(H_{\mathcal{M}})|_{\mathcal{M}} = 0$.

Let us consider $\{\eta_i\}$, $i = 1, \cdots, l$, a basis of sections of the bundle $\mathfrak{g}_S \to Q$. The *components of the nonholonomic moment map* in this basis are the functions $J_i \in C^\infty(\mathcal{M})$, $i = 1, \cdots, l$, given by

$$J_i := \langle J^{nh}, \eta_i \rangle. \tag{3.3.20}$$

Therefore, any $\zeta \in \Gamma(\mathfrak{g}_S)$ can be written as $\zeta = \sum_i f_i \eta_i$, for functions $f_1, \cdots, f_l \in C^\infty(Q)$, and its associated function J_ζ given in (3.3.18) is of the form $J_\zeta = \sum_i f_i J_i$.

Next, we show how the knowledge of horizontal gauge momenta J_ζ can give conditions on the (dynamical) gauge transformation B such that $\pi_B^\#(dJ_\zeta)$ becomes a section of $\Gamma(\mathcal{V})$. More precisely, we look for a B so that $\pi_B^\#(dJ_\zeta) = -\zeta_{\mathcal{M}}$. The importance of the mentioned property of $\pi_B^\#(dJ_\zeta)$ is illustrated in Lemma 3.3.1. These ideas are generalizations of the results presented in [6] to the case when the vertical complement W does not satisfy the vertical symmetry condition (see Remark 2.3.1).

Lemma 3.3.1. *If f is a G-invariant function on $\mathcal{M}$ such that $\pi_B^\#(df) \in \Gamma(\mathcal{V})$, then the reduced function $\bar{f}$ is a Casimir of the reduced bracket $\{\cdot,\cdot\}_{red}^B$ where $\bar{f}$ is the function on $\mathcal{M}/G$ such that $f = \rho^*\bar{f}$.*

Proof. For all $\bar{g} \in C^\infty(\mathcal{M}/G)$, we have

$$\{\bar{g}, \bar{f}\}_{red}^B \circ \rho = \{\rho^*\bar{g}, f\}_B = \pi_B^\#(df)(\rho^*\bar{g}) = 0,$$

because $\pi_B^\#(df) \in \Gamma(\mathcal{V})$ and $\rho^*\bar{g}$ is G-invariant. $\square$

Proposition 3.3.2. *Let ζ be a section of the bundle $\mathfrak{g}_S \to Q$ and J_ζ the corresponding associated function as in (3.3.18). Then:*

(i) $\pi_{nh}^\#(dJ_\zeta - \Lambda) = -\zeta_{\mathcal{M}}$, *for Λ the 1-form given by*

$$\Lambda|_{\mathcal{C}} = -\mathbf{i}_{\zeta_{\mathcal{M}}}\Omega_{\mathcal{C}} + dJ_\zeta|_{\mathcal{C}} = \pounds_{\zeta_{\mathcal{M}}}\Theta_{\mathcal{M}}|_{\mathcal{C}}. \tag{3.3.21}$$

(ii) If $\mathbf{i}_{\zeta_{\mathcal{M}}}B = \Lambda$, *where Λ is given in (3.3.21), then*

$$\pi_B^\#(dJ_\zeta) = -\zeta_{\mathcal{M}}. \tag{3.3.22}$$

Proof. (i) By definition of the nonholonomic bivector we have that $\pi_{nh}^\#(dJ_\zeta - \Lambda) = -\zeta_{\mathcal{M}}$ if and only if $\mathbf{i}_{\zeta_{\mathcal{M}}}\Omega_{\mathcal{C}} = dJ_\zeta|_{\mathcal{C}} - \Lambda|_{\mathcal{C}}$. The second equality in (3.3.21) holds using (3.3.18) and Cartan's formula.

(ii) From item (i) we get that the 1-form Λ satisfies that $\mathbf{i}_{\zeta_{\mathcal{M}}}\Omega_{\mathcal{C}} = dJ_\zeta|_{\mathcal{C}} - \Lambda|_{\mathcal{C}}$. If $\mathbf{i}_{\zeta_{\mathcal{M}}}B = \Lambda$ then $\mathbf{i}_{\zeta_{\mathcal{M}}}(\Omega_{\mathcal{C}} + B)|_{\mathcal{C}} = dJ_\zeta|_{\mathcal{C}}$ and hence $\pi_B^\#(dJ_\zeta) = -\zeta_{\mathcal{M}}$. $\square$

Note that if J_ζ is G-invariant and B satisfies that $\mathbf{i}_{\zeta_{\mathcal{M}}}B = \Lambda$ then, by Proposition 3.3.2 (ii) and Lemma 3.3.1, $\bar{J}_\zeta$ becomes a Casimir of $\{\cdot,\cdot\}_{red}^B$, where we denote by $\bar{J}_\zeta$ the associated reduced function on $\mathcal{M}/G$ (i.e. $J_\zeta = \rho^*\bar{J}_\zeta$).

3.4 Sufficient condition for the integrability of the characteristic distribution of $\{\cdot,\cdot\}^B_{red}$

The following proposition has been proved in [6] in the case where the G-invariant vertical complement W in (2.3.6) satisfies the vertical symmetry condition (see Remark 2.3.1) and the G-action is free and proper. We prove here that the same conclusion is valid without vertical symmetry condition. Let us denote by $\mathcal{M}_{reg}$ the stratum of $\mathcal{M}$ where the G-action is free and $\bar{\mathcal{M}}_{reg} := \mathcal{M}_{reg}/G$ the quotient manifold.

Proposition 3.4.1. *Let π_B be a bivector gauge related to π_{nh} by a G-invariant 2-form B in $\mathcal{M}$. If there are G-invariant functions $J_1,\cdots,J_l \in C^\infty(\mathcal{M})$ such that over points in $\mathcal{M}_{reg}$, the set $\{\pi_B^\#(dJ_i)\}$, $i=1,\cdots,l$, generates $\mathcal{S}=\mathcal{C}\cap\mathcal{V}$, then the characteristic distribution on $\bar{\mathcal{M}}_{reg}$ of the reduced bracket $\{\cdot,\cdot\}^B_{red}$ is involutive.*

Proof. The proof follows the same idea as Proposition 2.19 in [6], but taking into consideration the Jacobiator formula (3.1.7). We are going to show that, given two elements $(\pi^B_{red})^\#(d\bar{f})$ and $(\pi^B_{red})^\#(d\bar{g})$ in the characteristic distribution of $\{\cdot,\cdot\}^B_{red}$, the Lie bracket $[(\pi^B_{red})^\#(d\bar{f}),(\pi^B_{red})^\#(d\bar{g})]$ remains in the characteristic distribution. Consequently, for each $f,g\in C^\infty(\mathcal{M}_{reg})^G$, let us define a 1-form Υ on $\mathcal{M}_{reg}$ such that $\Upsilon|_\mathcal{C}=\mathbf{i}_{\pi_B^\#(df)\wedge\pi_B^\#(dg)}(dJ\wedge K_\mathcal{W}-dB)$ and $\Upsilon|_\mathcal{W}=0$. Using that the $\pi_B^\#(dJ_i)$, $i=1,\cdots,l$, generate $\mathcal{S}$ and by the G-invariance of J_i we get that the reduced functions $\bar{J}_i$ are Casimirs of $\{\cdot,\cdot\}^B_{red}$ and hence, using (3.1.7) and the fact that the $\{\pi_B^\#(dJ_i)\}$ generates $\mathcal{S}$, we show that $\Upsilon|_\mathcal{S}=0$. Therefore $\Upsilon|_\mathcal{V}=0$, and since Υ is G-invariant we have that Υ is a basic 1-form, i.e. $\Upsilon=\rho^*(\bar{\Upsilon})$ for a 1-form $\bar{\Upsilon}$ in $\bar{\mathcal{M}}_{reg}$. If we denote by π^B_{red} the bivector field on $\bar{\mathcal{M}}_{reg}$ associated to the bracket $\{\cdot,\cdot\}^B_{red}$ we have that, for each $\bar{f},\bar{g},\bar{h}\in C^\infty(\bar{\mathcal{M}}_{reg})$, (see [6]),

$$[(\pi^B_{red})^\#(d\bar{f}),(\pi^B_{red})^\#(d\bar{g})](\bar{h}) = (\pi^B_{red})^\#(d\{\bar{f},\bar{g}\}^B_{red})(\bar{h}) + cyclic[\{\{\bar{f},\bar{g}\}^B_{red},\bar{h}\}^B_{red}].$$

Therefore from (3.1.7) we obtain that

$$[(\pi^B_{red})^\#(d\bar{f}),(\pi^B_{red})^\#(d\bar{g})] = (\pi^B_{red})^\#(d\{\bar{f},\bar{g}\}^B_{red}) - (\pi^B_{red})^\#(\bar{\Upsilon}).$$

Since the characteristic distribution of the bivector π^B_{red} is the image by $(\pi^B_{red})^\#$ of all 1-forms in $\bar{\mathcal{M}}_{reg}$, we have shown that it is involutive. □

We remark that if the reduced bracket has regular characteristic distribution then this distribution is integrable and the reduced bracket is twisted Poisson [59, 80]. In that case the almost-sympletic leaves are the common level sets of the l Casimirs $\bar{J}_i$, $i=1,\cdots,l$.

The last proposition implies that, if we find $l=rank(\mathfrak{g}_S)$ independent Casimirs $\bar{J}_i$ of $\{\cdot,\cdot\}^B_{red}$ such that the fields $\pi_B^\sharp(d\rho^*\bar{J}_i)$ degenerate $\mathcal{S}$ on the regular stratum $\mathcal{M}_{reg}$, then the bracket $\{\cdot,\cdot\}^B_{red}$ has involutive characteristic distribution. This explains the utility of Lemma 3.3.1 and Prop. 3.3.2 in order to find a 2-form B inducing a bracket $\{\cdot,\cdot\}^B_{red}$ with better integrability properties than $\{\cdot,\cdot\}_{red}$.

In the regular stratum $\bar{\mathcal{M}}_{reg}$ the ranks of $\mathcal{V}$ and $\mathcal{S}$ are constant. The dimension of $\bar{\mathcal{M}}_{reg}$ is $dim(\bar{\mathcal{M}}_{reg})=dim(\mathcal{M})-rank(\mathcal{V})=rank(\mathcal{C})-rank(\mathcal{S})=rank(\mathcal{C})-l$. Under the hypothesis of Proposition 3.4.1 we have l Casimirs of $\{\cdot,\cdot\}_{red}^B$ and, if the characteristic distribution of $\{\cdot,\cdot\}_{red}^B$ is regular, the almost symplectic leaves in $\bar{\mathcal{M}}_{reg}$ are of dimension $dim(\bar{\mathcal{M}}_{reg}) - l = rank(\mathcal{C}) - 2l$. In particular, when the almost symplectic leaves of the reduced bracket $\{\cdot,\cdot\}_{red}^B$ are 2-dimensional, then $\{\cdot,\cdot\}_{red}^B$ is Poisson by dimensionality properties and thus the nonholonomic system is hamiltonizable through a reduction procedure. We will see that this is the case for the mechanical system we are considering in this thesis and in the next chapter we will study in detail its hamiltonization.

3.5 System of equations determining the 2-form B

Let us consider a G-symmetry of the nonholonomic system $(\mathcal{M}, \pi_{nh}, H_{\mathcal{M}})$ given by a *free* and proper action and denote $l = \text{rank}(\mathfrak{g}_S)$. Suppose that we have $J_1, ..., J_l$ horizontal gauge momenta such that the associated horizontal gauge symmetries $\{\zeta_1, ..., \zeta_l\}$ form a set of generators of sections of $\mathfrak{g}_S$. Prop. 3.3.2 (i) implies that the vector field $\pi_{nh}^\sharp(dJ_i)$ is not necessarily equal to $Y_i := (\zeta_i)_{\mathcal{M}}$. In fact

$$\pi_{nh}^\sharp(dJ_i) = -Y_i + \pi_{nh}^\sharp(\Lambda_i) \qquad \text{for} \quad i = 1, \cdots, l, \tag{3.5.23}$$

where each Λ_i is a 1-form on $\mathcal{M}$ satisfying $\Lambda_i|_{\mathcal{C}} = -\mathbf{i}_{Y_i}\Omega_{\mathcal{C}} + dJ_i|_{\mathcal{C}} = \pounds_{Y_i}\Theta_{\mathcal{M}}|_{\mathcal{C}}$.

Lemma 3.5.1. *The 1-forms Λ_i, $i = 1, \cdots, l$, in (3.5.23) satisfy*

(i) $\Lambda_i(X_{nh}) = 0$*,*

(ii) $\Lambda_i(Y_j) = -\Lambda_j(Y_i)$*,*

(iii) Λ_i *are semi-basic with respect to the bundle* $\tau_{\mathcal{M}} : \mathcal{M} \to Q$*.*

Proof. (i) From (3.5.23) we get

$$\Lambda_i(X_{nh}) = -dH_{\mathcal{M}}(\pi_{nh}^\#(\Lambda_i)) = dH_{\mathcal{M}}(-\pi_{nh}^\#(dJ_i) - Y_i) = dJ_i(X_{nh}) - Y_i(H_{\mathcal{M}}) = 0,$$

where in the last equality we used that J_i are first-integrals of X_{nh} and $H_{\mathcal{M}}$ is G-invariant.

(ii) Since the horizontal gauge momenta J_i, $i = 1, \cdots, l$, are G-invariant, then $Y_i(J_j) = 0$, $i, j = 1, \cdots, l$. Thus

$$\Lambda_i(Y_j) = -\mathbf{i}_{Y_j}\mathbf{i}_{Y_i}\Omega_{\mathcal{C}} + dJ_i(Y_j) = \mathbf{i}_{Y_i}\mathbf{i}_{Y_j}\Omega_{\mathcal{C}} - dJ_j(Y_i) = -\Lambda_j(Y_i).$$

(iii) Let us take $X \in Ker(T\tau_{\mathcal{M}}) \cap \Gamma(\mathcal{C})$, then

$$\begin{aligned} \mathbf{i}_X\Lambda_i &= \mathbf{i}_X\mathbf{i}_{Y_i}d\Theta_{\mathcal{M}} + \mathbf{i}_X dJ_i \\ &= \mathbf{i}_X\pounds_{Y_i}\Theta_{\mathcal{M}} - \mathbf{i}_X d\mathbf{i}_{Y_i}\Theta_{\mathcal{M}} + \mathbf{i}_X dJ_i, \end{aligned} \tag{3.5.24}$$

where we used Cartan's formula. If we write $\zeta_i = g_{ij}\chi_j$, for functions $g_{ij} \in C^\infty(Q)$, and $\{\chi_j\}$ a basis of $\mathfrak{g}$, then we have

$$\pounds_{Y_i}\Theta_{\mathcal{M}} = \mathbf{i}_{(\chi_j)_{\mathcal{M}}}\Theta_{\mathcal{M}}dg_{ij} + g_{ij}\pounds_{(\chi_j)_{\mathcal{M}}}\Theta_{\mathcal{M}}. \tag{3.5.25}$$

Recall that since G is a symmetry of the nonholonomic system, then $\pounds_{(\chi_j)_\mathcal{M}}\Theta_\mathcal{M} = 0$. Thus, using that $J_i = \mathbf{i}_{Y_i}\Theta_\mathcal{M}$, the expression (3.5.24) becomes

$$\mathbf{i}_X\Lambda_i = \mathbf{i}_{(\chi_j)_\mathcal{M}}\Theta_\mathcal{M}(\mathbf{i}_X dg_{ij}) = 0,$$

since the functions g_{ij} are basic with respect to the bundle $\mathcal{M} \to Q$. $\square$

Remark 3.5.2. The proof of *(iii)* of the last Proposition gives a coordinate expression of the 1-form Λ_i. Indeed, from (3.5.25) we have

$$\Lambda_i = (\mathbf{i}_{(\chi_j)_\mathcal{M}}\Theta_\mathcal{M})dg_{ij},$$

where $\zeta_i = g_{ij}\chi_j$ and $\chi_j \in \mathfrak{g}$. We can relate this with the results in [9] where the authors introduced the concept of $\mathcal{M}$-cotangent lift $(\zeta_i)_Q^\mathcal{M}$ of the vector field $(\zeta_i)_Q$ on Q and proved that (see [9, Lemma 3.2])

$$(\zeta_i)_Q^\mathcal{M} = -\pi_{nh}^\#(dJ_i) = (\zeta_i)_\mathcal{M} - (\mathbf{i}_{(\chi_j)_\mathcal{M}}\Theta_\mathcal{M})\pi_{nh}^\#(dg_{ij}).$$

Compare the last equality with (3.5.23). $\diamond$

Following the ideas of [6], in order to obtain a (gauge related) bivector field π_B so that $\pi_B^\sharp(dJ_i) = -Y_i \in \Gamma(\mathcal{V})$ (recall that $Y_i := (\zeta_i)_\mathcal{M}$) we look for a (semi-basic with respect to $\tau_\mathcal{M} : \mathcal{M} \to Q$, see Remark 3.1.3) 2-form B such that

$$\mathbf{i}_{Y_i}B|_\mathcal{C} = \Lambda_i|_\mathcal{C} \qquad \text{for each } i = 1, \cdots, l. \tag{3.5.26}$$

Moreover, since π_B has to describe the nonholonomic dynamics (1.4.31), then we have to impose also the dynamical condition

$$\mathbf{i}_{X_{nh}}B|_\mathcal{C} = 0. \tag{3.5.27}$$

Next we analyze the set of equations (3.5.26) for the case where the infinitesimal generators Y_i are linearly independent, i.e., for a *free* and proper G-action. In this case, the vertical spaces V and $\mathcal{V}$ are of constant rank (and isomophic) as well as $S = D \cap V$ and $\mathcal{S} = \mathcal{C} \cap \mathcal{V}$. Moreover, it is clear that $\text{rank}(S) = \text{rank}(\mathfrak{g}_\mathcal{S})$.

Observe that Lemma 3.5.1 give conditions on the coefficients of the 1-forms Λ_i, $i = 1, \cdots, l$, with respect to suitable bases that we choose in the following Proposition. In particular the properties in Lemma 3.5.1 guarantees that the system on the coefficients of B formed by equations (3.5.26) is compatible.

Proposition 3.5.3. *Consider the nonholonomic system given by the triple $(\mathcal{M}, \pi_{nh}, \mathcal{H}_\mathcal{M})$ with a G-symmetry given by a free and proper action. If $d = \text{rank}(D)$ and $l = \text{rank}(S)$, then the system of equations (3.5.26) is a system with $\binom{d}{2}$ variables and $(d-1)+(d-2)+\cdots+(d-l)$ equations that always has a (not necessarily unique) solution.*

Proof. From Remark 3.1.3 we suppose $\mathbf{i}_X B \equiv 0$, for any $X \in \Gamma(\mathcal{W})$, then we are interested only on the restriction $B|_{\mathcal{C}}$. Moreover since B is semi-basic with respect to $\mathcal{M} \to Q$, it is constructed by pull-backs (w.r.t the bundle $\tau_{\mathcal{M}} : \mathcal{M} \to Q$) of 1-forms dual to a basis of D. Thus, by skew-symmetry, B has $\binom{d}{2} = \frac{d(d-1)}{2}$ unknown coefficients.

The computation of the number of conditions is facilitated if we choose an adequate basis of sections of $\mathcal{C}$. Consider the vector fields $Y_i = (\zeta_i)_{\mathcal{M}}$ and complete a basis of $\mathcal{C}$ with sections X_α, $\alpha = 1, \cdots, \lambda := d - l$. Using indices $\alpha = 1, \cdots, \lambda$; $i = 1, \cdots, l$; $a = 1, \cdots, k$, $M = 1, \cdots, d$, we get

$$\begin{aligned} T\mathcal{M} &= span\{X_\alpha, Y_i, Z_a, \frac{\partial}{\partial p_M}\}, \\ T^*\mathcal{M} &= span\{X^\alpha, Y^i, Z^a, dp_M\}, \\ \mathcal{C} &= span\{X_\alpha, Y_i, \frac{\partial}{\partial p_M}\}. \end{aligned} \tag{3.5.28}$$

The semi-basic 2-form (w.r.t. $\mathcal{M} \to Q$) B in that basis has the form

$$B = a_{\alpha,\beta} X^\alpha \wedge X^\beta + b_{\alpha,i} X^\alpha \wedge Y^i + c_{i,j} Y^i \wedge Y^j, \tag{3.5.29}$$

where we use Einstein convention and the coefficients $a_{\alpha,\beta}$, $b_{\alpha,i}$ and $c_{i,j}$ are skew-symmetric. First we compute

$$\mathbf{i}_{Y_1} B|_{\mathcal{C}} = -b_{\alpha,1} X^\alpha + c_{1,i} Y^i,$$

with $i = 2, \cdots, l$ and $\alpha = 1, \cdots, \lambda$, and imposing that $\mathbf{i}_{Y_1} B|_{\mathcal{C}} = \Lambda_1|_{\mathcal{C}}$ we get $l + \lambda - 1$ conditions.

Continuing the process and using the skew-symmetry of $c_{i,j}$ we observe that $\mathbf{i}_{Y_2} B|_{\mathcal{C}} = \Lambda_2|_{\mathcal{C}}$ gives new $l + \lambda - 2$ conditions, until the case $\mathbf{i}_{Y_l} B|_{\mathcal{C}} = \Lambda_l|_{\mathcal{C}}$ that gives λ conditions on the coefficients of B. Then the total number of conditions is

$$(l + \lambda - 1) + (l + \lambda - 2) + \cdots + \lambda = \frac{(2\lambda + l - 1)l}{2}. \tag{3.5.30}$$

On the other hand the number of unknown coefficients of B is

$$\binom{d}{2} = \binom{l + \lambda}{2} = \frac{l^2 + 2l\lambda + \lambda^2 - l - \lambda}{2}. \tag{3.5.31}$$

Subtracting (3.5.30) from (3.5.31) we get $\frac{\lambda^2 - \lambda}{2}$ which is equal to zero if $\lambda = 0$ or $\lambda = 1$ and it is positive for $\lambda > 1$.

Then, the system of equations (3.5.26) has at least the same number of unknowns than the number of equations. We claim that given 1-forms Λ_i, $i = 1, \cdots, l$, verifying the properties of Lemma 3.5.1, we can find exactly all the $b_{\alpha,i}$ and $c_{i,j}$ in (3.5.29) and there are no conditions on the $a_{\alpha,\beta}$, giving the non-unicity if $\lambda \geq 2$.

Indeed, Λ_1 should have the form

$$\Lambda_1|_{\mathcal{C}} = A_{1,\alpha} X^\alpha + B_{1,i} Y^i,$$

with $B_{1,1} = 0$. Analogously, Λ_2 has the form

$$\Lambda_2|_{\mathcal{C}} = A_{2,\alpha} X^\alpha + B_{2,i} Y^i,$$

with $B_{2,2} = 0$, and $B_{2,1} = -B_{1,2}$. Continuing further we get, for $i = l$,

$$\Lambda_l|_{\mathcal{C}} = A_{l,\alpha} X^\alpha + B_{l,i} Y^i,$$

with $B_{l,l} = 0$, and $B_{l,i} = -B_{i,l}$.

On the other hand, using (3.5.29), we get

$$\mathbf{i}_{Y_i} B|_{\mathcal{C}} = -b_{\alpha,i} X^\alpha + c_{i,j} Y^j,$$

with $c_{i,j} = -c_{j,i}$. Setting $\mathbf{i}_{Y_i} B|_{\mathcal{C}} = \Lambda_i|_{\mathcal{C}}$, for $i = 1, \cdots, l$, we observe that given the $A_{i,\alpha}$ and $B_{i,j}$ we find the coefficients of B as

$$b_{\alpha,i} = -A_{i,\alpha}, \quad c_{i,j} = B_{i,j},$$

and this ensures that the system has at least one solution. Hence

$$B = \Lambda_{i,j} Y^i \wedge Y^j + \Lambda_{i,\alpha} X^\alpha \wedge Y^i + a_{\alpha,\beta} X^\alpha \wedge X^\beta, \tag{3.5.32}$$

where $\Lambda_i = \Lambda_{i,j} Y^i + \Lambda_{i,\alpha} X^\alpha$. The $a_{\alpha,\beta}$ in (3.5.32) are free. $\square$

The special case when rank(D) − rank$(S) = 1$

Here we also assume that the G-action acts freely and properly on Q and thus G acts also freely and properly on the constraint manifold $\mathcal{M}$. Then, for the case where $\mathrm{rank}(D) - \mathrm{rank}(S) = 1$ we will see that we have a unique (semi-basic) 2-form B satisfying (3.5.26) and, moreover, under an extra (technical) assumption, this 2-form B will satisfy the dynamical condition (3.5.27). It is worth noticing that the example treated in this dissertation is of this type away from the singularities, but also the examples of ball in a cylinder [5, 10], the solid of revolution rolling on a plane [6, 47], etc.

Denote by $d = \mathrm{rank}(D)$ and $l = \mathrm{rank}(S)$ so that $d - l = 1$ and suppose that $\{J_1, ..., J_l\}$ are l horizontal gauge momenta (functionally independent) of the nonholonomic system with $\{\zeta_1, ..., \zeta_l\}$ the associated horizontal gauge symmetries (also linearly independent). First we will set a (local) basis of $T(\mathcal{M}/G)$ and $T^*(\mathcal{M}/G)$. Let us consider a basis of sections $\{(\zeta_i)_Q, X\}$ of $\Gamma(D)$. Then we have the basis of TQ given by $\mathfrak{B}_{TQ} = \{(\zeta_i)_Q, X, Z_a\}$ for $\{Z_a\}$ a basis of section of W and its dual basis $\mathfrak{B}_{T^*Q} = \{\alpha_i, \alpha_X, \epsilon^a\}$ with associated coordinates (p_i, p_x, p_a). As we already did in (3.5.28), we obtain that

$$\begin{aligned} T^*\mathcal{M} &= \mathrm{span}\{\tilde{\alpha}_i, \tilde{\alpha}_X, \tilde{\epsilon}^a, dp_i, dp_x\} \quad \text{and} \\ T\mathcal{M} &= \mathrm{span}\{X_i, \bar{X}, \bar{Z}_a, \frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_x}\}, \end{aligned} \tag{3.5.33}$$

where $\tilde{\alpha}_i = \rho^*\alpha_i$, $\tilde{\alpha}_X = \rho^*\alpha_X$ and $\tilde{\epsilon}^a = \rho^*\epsilon^a$ and where the two bases are dual. Now we can observe that the horizontal gauge momenta J_i verify that $J_i = p_i$ and thus it is straightforward to check that the functions p_i and p_x are G-invariant. Then, we obtain that

$$T(\mathcal{M}/G) = \text{span}\{\mathcal{X} := T\tau(\bar{X}), \frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_x}\}, \tag{3.5.34}$$

with its associated dual basis

$$T^*(\mathcal{M}/G) = \text{span}\{\alpha_{\mathcal{X}}, dp_i, dp_x\}. \tag{3.5.35}$$

Observe also that, in this case, $\dim(\mathcal{M}/G) = l + 2$. Then, we observe that

Lemma 3.5.4. *If the bivector field π_{red}^B on $\mathcal{M}/G$ has $\{J_1, ..., J_l\}$ Casimirs, then it has the form*

$$\pi_{red}^B = \mathcal{X} \wedge \frac{\partial}{\partial p_x},$$

and it is Poisson.

Proof. Since $\dim(\mathcal{M}/G) = l + 2$, and π_{red}^B has l Casimirs, the common level sets of the (independent) J_i, $i = 1, \cdots, l$ are 2-dimensional submanifolds. Then, the bivector π_{red}^B has a regular integrable characteristic distribution and then it is twisted Poisson. In fact, π_{red}^B is Poisson because the associated leaves are 2-dimensional.

To get the form of π_{red}^B, we start writing the bivector field π_B of the basis (3.5.33). By the observation we made in the end of Sec. 3.2, we see that π_B has the form

$$\pi_B = X_i \wedge \frac{\partial}{\partial p_i} + \bar{X} \wedge \frac{\partial}{\partial p_x} + \sum f_{ij} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_j} + \sum g_i \frac{\partial}{\partial p_x} \wedge \frac{\partial}{\partial p_i},$$

for f_{ij}, g_i functions on $\mathcal{M}$. Then, computing the reduced bivector π_{red}^B, and using that $p_i = J_i$ are Casimirs of π_{red}^B we get

$$(\pi_{red}^B)^\sharp(dp_i) = 0, \quad (\pi_{red}^B)^\sharp(dp_x) = -\mathcal{X}, \quad (\pi_{red}^B)^\sharp(\alpha_{\mathcal{X}}) = \frac{\partial}{\partial p_x}. \tag{3.5.36}$$

This implies that $\pi_{red}^B = \mathcal{X} \wedge \frac{\partial}{\partial p_x}$. $\square$

Proposition 3.5.5. *Let $(\mathcal{M}, \pi_{nh}, \mathcal{H}_{\mathcal{M}})$ be a nonholonomic system with a G-symmetry given by a free and proper action and assume that $\{J_1, ..., J_l\}$ are l horizontal gauge momenta of the nonholonomic system. If $d = \text{rank}(D)$ and $l = \text{rank}(S)$ so that $d - l = 1$, then*

- (i) *there is a unique (semi-basic) 2-form B satisfying* (3.5.26).
- (ii) *If B satisfies* (3.5.26) *for each $i = 1, ..., l$, and the equilibrium set $\mathcal{E}$ of the reduced dynamics X_{red} has empty interior, then the dynamical condition* (3.5.27) *is automatically satisfied.*

Proof. (i) If $d = l + 1$ then in the proof of Prop. 3.5.3, we have $\lambda = 1$, and we verified than in that case $\binom{d}{2} = (d-1) + (d-2) + ... + (d-l)$.

(ii) If $X_{nh}(m) \notin \mathcal{S}_m$, for all $m \in \mathcal{M}$, then any vector field $X \in \Gamma(\mathcal{C})$ can be written $X = aX_{nh} + b_iY_i + c_M\frac{\partial}{\partial p_M}$. Thus, using Lemma 3.5.1 (i),

$$B(X_{nh}, X) = b_iB(X_{nh}, Y_i) = -b_i\Lambda_i(X_{nh}) = 0.$$

If there is some point m_0 where $X_{nh}(m_0) \in \mathcal{S}_{m_0}$, then using the basis of $\mathcal{C}$ in (3.5.28) we write $X_{nh} = wX_1 + v_iY_i + u_M\frac{\partial}{\partial \tilde{p_M}}$ and we get

$$\mathbf{i}_{X_{nh}}B = w\,\mathbf{i}_{X_1}B + v_i\Lambda_i. \tag{3.5.37}$$

We call $\tilde{\mathcal{E}}$ the inverse image of $\mathcal{E}$ under the orbit map and observe that $\tilde{\mathcal{E}} \subset \mathcal{M}$ has also empty interior, then this closed set coincides with its boundary. Moreover, all points m such that $X_{nh}(m) \in \mathcal{S}_m$ are in $\tilde{\mathcal{E}}$, then we can approximate any point $m \in \tilde{\mathcal{E}}$ by a sequence of points (m_n) where $X_{nh}(m_n) \notin \mathcal{S}_{m_n}$. Thus from (3.5.37) we have

$$w(m_n)\,\mathbf{i}_{X_1}B|_{m_n} = -v_i(m_n)\Lambda_i|_{m_n}.$$

By continuity of the functions w, v_i, taking the limit when $m_n \to m$ and using that $w(m) = 0$, we get from the latter equation that

$$v_i(m)\Lambda_i|_m = 0,$$

which implies from (3.5.37) that for any $m \in \tilde{\mathcal{E}}$ we have

$$\mathbf{i}_{X_{nh}(m)}B|_m = w(m)\mathbf{i}_{X_1}B|_m = 0.$$

□

Hence, we conclude that if $\text{rank}(D) = \text{rank}(S) + 1$ and we have $\{J_1, ..., J_l\}$ (independent) horizontal gauge momenta, there is a unique 2-form B that transforms these functions into Casimirs of the reduced bracket $\{\cdot,\cdot\}^B_{red}$. Moreover, this 2-form B will satisfy the *dynamical condition* (3.5.26) and thus $\{\cdot,\cdot\}^B_{red}$ will describe the reduced dynamics (3.1.5). Finally, we observe that if B satisfies (3.5.26), then the reduced bracket $\{\cdot,\cdot\}^B_{red}$ is a rank 2 Poisson bracket and, in particular, for the choice of coordinates done in (3.5.34), it has the form given in Lemma 3.5.4.

We enforce the fact that in many interesting examples, away from the singularities, the action is free and proper, they satisfy the condition $\text{rank}(D) = \text{rank}(S) + 1$ and they admit l horizontal gauge momenta for $l = \text{rank}(S)$.

Remark 3.5.6. If $d = l$ then there is also a unique 2-form B satisfying (3.5.26) since $\binom{d}{2} = (d-1) + (d-2) + ... + (d-l)$. However, this case is not interesting for us since if $d = l$ then $X_{nh} \in \Gamma(\mathcal{S})$ and thus $X_{red} \equiv 0$ on $\mathcal{M}/G$. ⋄

3.6 Examples

3.6.1 The nonholonomic particle

We continue with the example treated in Sec. 1.5.1 and Sec. 2.4.1. We have seen that the reduced bracket $\{\cdot,\cdot\}_{red}$ is Poisson and then no gauge transformation is needed.

It is known (see e.g. [6, 9]) that

$$J = \frac{1}{\sqrt{1+y^2}}\tilde{p_x}$$

is a first integral of the system. In fact, J is a G-invariant horizontal gauge momenta, and verifies $\pi^\sharp_{nh}(dJ) \in \Gamma(\mathcal{V})$, then it induces a Casimir $\bar{J}$ in the reduced space $\mathcal{M}/G \simeq \mathbb{R}^3$, such that $\rho^*\bar{J} = J$. The symplectic leaves of reduced Poisson bracket are 2-dimensional and are determined by the level sets of the Casimir $\bar{J}$.

3.6.2 The vertical rolling disk

We continue with the Example from Sections 1.5.2 and 2.4.2 where we showed that the reduced bracket $\{\cdot,\cdot\}_{red}$ is Poisson, and then no gauge transformation is needed.

The distribution $S = D \cap V$ is given by $S = span\{Y_\theta = R\cos\phi\frac{\partial}{\partial x} + R\sin\phi\frac{\partial}{\partial y} + \frac{\partial}{\partial\theta}\}$, and it is induced by the bundle $\mathfrak{g}_S \to Q$ generated by $\eta = (R\cos\phi, R\sin\phi, 1)$, i.e. $\eta_Q = Y_\theta$. In fact η is a horizontal gauge symmetry [9] with associated gauge momentum $J_\eta = (\frac{R^2m}{E} + 1)\tilde{p_\theta}$. It is easy to verify that $\pi^\sharp_{nh}(dJ_\eta) \in \Gamma(\mathcal{V})$, then, since J_η is G-invariant, it induces a Casimir of the reduced bracket $\{\cdot,\cdot\}_{red}$. The symplectic leaves are the 2-dimensional level sets of the Casimir and are diffeomorphic to cylinders $S^1 \times \mathbb{R}$ with coordinates $(\phi, \tilde{p_\phi})$.

3.6.3 Snakeboard

We have seen in Section 2.4.3 that the snakeboard has a symmetry such that the reduced G-action induces a twisted Poisson structure in the quotient 5-dimensional manifold $\mathcal{M}/G \simeq \mathbb{T}^2 \times \mathbb{R}^3$.

Using the distribution D given in (1.5.39) and the vertical distribution V given in (2.4.28), we compute $S = D \cap V = span\{\frac{\partial}{\partial\psi}\}$ and hence $\mathfrak{g}_S = span\{(0,0,1)\}$, where we use the canonical basis of $\mathfrak{g} \simeq \mathbb{R}^3$. Then, taking the constant section $\eta = (0,0,1)$ we verify that the associated component of the nonholonomic moment map J_η is the coordinates $\tilde{p_\psi}$ (see Section 1.5.3), which is also a first integral of the dynamics, and hence a horizontal gauge momentum. We verify that $\pi^\sharp_{nh}(dJ_\eta) = -Y_\psi = -\frac{\partial}{\partial\psi}$ is vertical, and then, since $\tilde{p_\psi}$ is G-invariant, we conclude that it induces a Casimir of the twisted Poisson bracket $\{\cdot,\cdot\}_{red}$. The level sets of the Casimir $\tilde{p_\psi}$ in $\mathcal{M}/G$ are the almost symplectic leaves of the reduced bracket $\{\cdot,\cdot\}_{red}$.

3.6.4 Rigid body and Chaplygin ball

We continue with the example of Sections 1.5.4 and 2.4.4 where we showed that the reduced bracket $\{\cdot,\cdot\}_{red}$ has a nonintegrable characteristic distribution.

Consider the 2-form B on $\mathcal{M}$ biven by $B = -r^2m\langle\boldsymbol{\Omega}, \boldsymbol{\lambda}\times\boldsymbol{\lambda}\rangle$, which is the non-basic part (with respect to $\mathcal{M} \to \mathcal{M}/G$) of the 2-form $\langle J, K_\mathcal{W}\rangle$ given in (2.4.35) Following [5, Sec. 7.3], we have that B is basic with respect to $\mathcal{M} \to Q$, G-invariant and then induces a G-invariant bivector field π_B gauge related to π_{nh}. In fact, in [8] it has been verified that $\mathbf{i}_{X_{nh}}B = 0$, so that the bivector π_B also describes de dynamics of the nonholonomic system. Consequently the 2-form $\langle J, K_{\mathcal{W}}\rangle + B$ is basic with respect to $\mathcal{M} \to \mathcal{M}/G$ and then, by [8, Cor. 4.14] (see also the discussion after the Jacobiator formula (3.1.7)), the reduced bracket $\{\cdot,\cdot\}_{red}^B$ is $(d\mathcal{B}_{red})$-twisted Poisson, where $\rho^*\mathcal{B}_{red} = \langle J, K_{\mathcal{W}}\rangle + B$. In fact, in the local coordinates $(\boldsymbol{\gamma}, \tilde{\boldsymbol{M}})$ of $\mathcal{M}/G$, we have

$$\mathcal{B}_{red} = -r^2 m\langle \boldsymbol{\gamma}, \boldsymbol{\Omega}\rangle\langle \boldsymbol{\gamma}, d\boldsymbol{\gamma} \times d\boldsymbol{\gamma}\rangle, \tag{3.6.38}$$

where $\boldsymbol{\Omega}$ is related to $\tilde{\boldsymbol{M}}$ by (1.5.47). Consequently, the reduced bracket $\{\cdot,\cdot\}_{red}^B$ has an integrable characteristic distribution.

Using the basis of sections of $\mathfrak{g}_S \to Q$ given after (2.4.33), it has been verified in [9] that $\eta = (1, -y, x)$ is a horizontal gauge symmetry, such that $\eta_{\mathcal{M}} = \langle \boldsymbol{\gamma}, \boldsymbol{X}^L\rangle$, with associated horizontal gauge momentum $J_\eta = \langle J^{nh}, \eta\rangle = \mathbf{i}_{\langle \boldsymbol{\gamma}, \boldsymbol{X}^L\rangle}\Theta_{\mathcal{M}} = \langle \boldsymbol{\gamma}, \tilde{\boldsymbol{M}}\rangle$. Since $\langle \boldsymbol{\gamma}, \tilde{\boldsymbol{M}}\rangle$ is G-invariant, it has been proved in [8] that it is a Casimir of the reduced bracket $\{\cdot,\cdot\}_{red}^B$. The leaves of the characteristic distribution of $\{\cdot,\cdot\}_{red}^B$ are the 4-dimensional level sets of the Casimir $\langle \boldsymbol{\gamma}, \tilde{\boldsymbol{M}}\rangle$, and are diffeomorphic to TS^2.

In fact, the reduced bracket $\{\cdot,\cdot\}_{red}^B$ is conformally Poisson, see [8]. Then, after a time reparametrization using the conformal factor, there is a Poisson bracket on $\mathcal{M}/G$ describing the reduced dynamics. On the 4-dimensional symplectic leaves there are two first integrals in involution, the reduced Hamiltonian H_{red} and the function $J = \langle \tilde{\boldsymbol{M}}, \tilde{\boldsymbol{M}}\rangle$ such that the joint level sets are compact on the leaf. It follows from the Arnold-Liouville Theorem (see [1, 3]) that these joint level sets are invariant two-tori and the dynamics are quasi-periodic on them, i.e. the system is completely integrable on each symplectic leaf.

3.6.5 Homogeneous ball in a cylinder

We continue with the example of Sections 1.5.5 and 2.4.5. From the expression of the nonholonomic dynamics X_{nh} computed in (1.5.55) we see that $\tilde{p_\theta}$ is a first integral of the the system. In fact, the system has two horizontal gauge momenta (see e.g. [10]),

$$J_1 = \tilde{p_\theta}, \quad J_2 = \frac{r}{I}\tilde{M}_n - \frac{zr^2}{E(R-r)^2}\tilde{p_\theta},$$

which are G-invariant. The functions J_1 and J_2 do not induce Casimirs of $\{\cdot,\cdot\}_{red}$, because one verifies that $\pi_{nh}^\sharp(dJ_i) \notin \Gamma(V)$.

Now we perform a gauge transformation by a 2-form B, semi-basic with respect to $\mathcal{M} \to Q$ and such that $\boldsymbol{i}_X B = 0$, for $X \in \Gamma(W)$. Starting with an ansatz on the form

$$B = a\, dz \wedge d\theta + b\, d\theta \wedge \beta_n + c\, \beta_n \wedge dz,$$

and imposing $\pi_B^\sharp(dJ_i) = V_i$, we get for $i = 1$ that $\pi_B^\sharp(d\tilde{p_\theta}) = -Y_\theta$, or equivalently

$$\boldsymbol{i}_{Y_\theta}B = \Lambda_1 = -\boldsymbol{i}_{Y_\theta}\Omega_{\mathcal{C}} + dp_\theta,$$

which joint with the dynamical condition $\boldsymbol{i}_{X_{nh}}B = 0$ allow us to find the coefficients of B. We get

$$B = \frac{R}{r^2}\tilde{M}_n dz \wedge d\theta + \frac{RI}{E}\tilde{p_z} d\theta \wedge \beta_n + \frac{RI}{E(R-r)^2}\tilde{p_\theta}\beta_n \wedge dz,$$

and one verifies that $\pi_B^\sharp(dJ_2) \in \Gamma(V)$. As a consequence, J_1 and J_2 induces Casimirs $\bar{J}_1$ and $\bar{J}_2$ of the reduced bracket $\{\cdot,\cdot\}_{red}^B$ in the quotient manifold $\mathcal{M}/G$ and, since $dim(\mathcal{M}/G) = 4$, the 2-dimensional level sets of the Casimirs define a foliation which is symplectic by dimensionality. Thus, the reduced bracket $\{\cdot,\cdot\}_{red}^B$ is Poisson. See [5], for another proof using the Jacobiator formula 3.1.7.

3.6.6 Body of revolution on a plane

We complete our study started in Sections 1.5.6 and 2.4.6 performing a gauge transformation leading to the hamiltonization of the nonholonomic system formed by a body of revolution rolling on a plane. Following [6, 36], this mechanical system admits two horizontal gauge momenta J_1 and J_2 which are G-invariant and associated to two horizontal gauge symmetries ζ_1 and ζ_2 described in [6, Theo. 3.7 & Cor. 3.8].

In order have $\pi_B^\sharp(dJ_i) = -(\zeta_i)_\mathcal{M}$, by Prop. 3.3.2 it is enough to look for a 2-form B such that $\mathbf{i}_{(\zeta_i)_\mathcal{M}} B = \Lambda_i$, where $\Lambda_i = -\mathbf{i}_{(\zeta_i)_\mathcal{M}}\Omega_\mathcal{C} + dJ_i$. The computations are simplified when the vertical symmetry condition is verified and in this case Λ_i becomes

$$\Lambda_i = \mathbf{i}_{(\zeta_i)_\mathcal{M}}\langle J, K_\mathcal{W}\rangle + j_k df_{ik},$$

where we write $\zeta_i = f_{ik}\eta_k \in \Gamma(\mathfrak{g}_S)$, with corresponding horizontal gauge momenta $J_i := J_{\zeta_i} = f_{ik}j_k$ and $j_k = \mathbf{i}_{(\eta_k)_\mathcal{M}}\Theta_\mathcal{M}$. For this example we have (see [6]):

$$\begin{aligned} (\eta_1)_\mathcal{M} &= -X_3^L - \left(\tilde{M}_2\frac{\partial}{\partial M_1} - \tilde{M}_1\frac{\partial}{\partial M_2}\right), \qquad (\eta_2)_\mathcal{M} = \langle \boldsymbol{\gamma}, \boldsymbol{Y}\rangle, \\ j_1 &= -\tilde{M}_3, \qquad j_2 = \langle \boldsymbol{\gamma}, \tilde{\boldsymbol{M}}\rangle, \end{aligned} \tag{3.6.39}$$

and the 2-form $\langle J, K_\mathcal{W}\rangle$ was computed and written as

$$\langle J, K_\mathcal{W}\rangle = \langle \boldsymbol{K}, d\boldsymbol{\lambda}\rangle, \tag{3.6.40}$$

where $\boldsymbol{K}$ is given in (1.5.62). The functions f_{ij} depend only on $\tau_1 = \gamma_3$ and verify one explicit ODE (see [6, Teo. 3.7]) implying that J_1 and J_2 are first integrals of X_{nh}. The equality coming from the ODE appearing in the computation of Λ_1 and Λ_2 are

$$\begin{aligned} j_1 df_{11} + j_2 df_{12} &= (f_{11}\mathcal{Q} + f_{12}\mathcal{P})d\gamma_3, \\ j_1 df_{21} + j_2 df_{22} &= (f_{21}\mathcal{Q} + f_{22}\mathcal{P})d\gamma_3, \end{aligned}$$

where $d\gamma_3 = \gamma_1\lambda_2 - \gamma_2\lambda_1$ and $\mathcal{P}$, $\mathcal{Q}$ are defined in (1.5.63).

We look for a 2-form B having the form

$$B = a\,\lambda_2\wedge\lambda_3 + b\,\lambda_3\wedge\lambda_1 + c\,\lambda_1\wedge\lambda_2, \tag{3.6.41}$$

which is semi-basic with respect to $\mathcal{M}\to Q$ and then induces a bivector field π_B. We have also supposed that $\mathbf{i}_X B = 0$ for any $X \in \Gamma(\mathcal{W})$.

Writing $\zeta_1 = f_{11}\eta_1 + f_{12}\eta_2$ and $\zeta_2 = f_{21}\eta_1 + f_{22}\eta_2$, we write the system of equations for the coefficients of B,

$$f_{11}\mathbf{i}_{(\eta_1)_\mathcal{M}}B + f_{12}\mathbf{i}_{(\eta_2)_\mathcal{M}}B = \Lambda_1, \quad f_{21}\mathbf{i}_{(\eta_1)_\mathcal{M}}B + f_{22}\mathbf{i}_{(\eta_2)_\mathcal{M}}B = \Lambda_2. \tag{3.6.42}$$

The system above has a unique solution given by

$$a = m\varrho\langle\boldsymbol{\gamma}, \boldsymbol{s}\rangle\Omega_1, \quad b = m\varrho\langle\boldsymbol{\gamma}, \boldsymbol{s}\rangle\Omega_2, \quad c = m\varrho\langle\boldsymbol{\gamma}, \boldsymbol{s}\rangle\Omega_3. \tag{3.6.43}$$

and then the G-invariant 2-form B is

$$B = m\varrho\langle\boldsymbol{\gamma}, \boldsymbol{s}\rangle\langle\boldsymbol{\Omega}, d\boldsymbol{\lambda}\rangle, \tag{3.6.44}$$

and since the horizontal gauge momenta J_1 and J_2 are G-invariant, they induce Casimirs of the reduced bracket $\{\cdot,\cdot\}^B_{red}$ on the regular stratum $\bar{\mathcal{M}}_{reg}$ of the stratified differential space $\mathcal{M}/G$. Observe that on $\bar{\mathcal{M}}_{reg}$ we have $\mathrm{rank}(D) - \mathrm{rank}(S) = 1$ and that the dynamical condition $\mathbf{i}_{X_{nh}}B = 0$ is verified as was proved in Section 3.5. Moreover, the reduced bracket $\{\cdot,\cdot\}^B_{red}$ is Poisson on $\mathcal{M}/G$ as was verified in [6] using the Jacobiator formula 3.1.7.

Chapter 4

The homogeneous ball on a convex surface of revolution

4.1 Qualitative description of the system

We are interested in studying geometric properties of the equations of motion of a homogeneous ball rolling on a convex surface of revolution [24, 40, 52, 77, 79, 88]. More precisely, consider the motion of a homogeneous ball of mass m and radius R rolling without sliding under the influence of gravity on the interior side of a convex surface of revolution $\Sigma \subset \mathbb{R}^3$. We denote by (x, y) the coordinates of the projection of the center of mass of the ball to the plane $z = 0$. The homogeneity of the ball means that the inertia tensor has the form $\mathbb{I} = I \cdot id$, where I is a positive constant and id the 3×3 identity matrix. See Fig. 4.1.

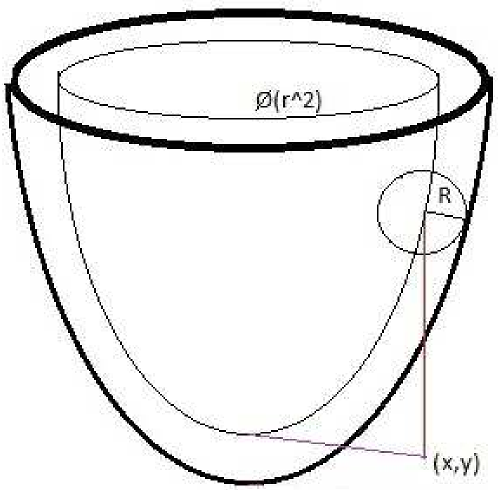


Figure 4.1: The ball rolling on the surface of revolution.

We fix a reference orthonormal frame $\{\hat{e}_x, \hat{e}_y, \hat{e}_z\}$ in space (*space frame*), a moving orthonormal frame $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ attached to the ball (*body frame*) and denote by g the orthogonal matrix relating both frames. Therefore the configuration space Q of the mechanical system is $\mathbb{R}^2 \times SO(3)$ with coordinates (x, y, g).

This mechanical system has two kinds of constraints. The holonomic one imposing the movement of the ball over the surface $\Sigma \subset \mathbb{R}^3$, and the nonholonomic one given by the non-sliding condition described by the following vector equation relating the angular velocity $\vec{\omega}$ and the velocity of the center of mass $\vec{v}$:

$$\vec{\omega} \times \vec{a} = -\vec{v}, \tag{4.1.1}$$

where $\times$ denotes the usual vector product in $\mathbb{R}^3$ and $\vec{a}$ is a vector joining the center of mass of the ball with the contact point with the surface. Denoting by $\vec{n}$ the exterior unit normal vector to the surface, we have $\vec{a} = R\vec{n}$. Following [24], the angular momentum with respect to the contact point is given by

$$\begin{aligned}\vec{M} &= I\vec{\omega} + mR^2\vec{n} \times (\vec{\omega} \times \vec{n}) \\ &= I\vec{\omega} + mR^2\vec{\omega} - mR^2(\vec{\omega} \cdot \vec{n})\vec{n}.\end{aligned}$$

The equations of motion can be found by considering Newton's second law for translations and rotations:

$$\begin{aligned} m\dot{\vec{v}} &= \vec{N} + \vec{F}, \\ I\dot{\vec{\omega}} &= R\vec{n} \times \vec{N}, \end{aligned} \tag{4.1.2}$$

where $\vec{N}$ denotes the reaction force at the contact point and $\vec{F}$ the external force applied to the center of mass [24]. The force $\vec{F}$ is the gradient of the potential energy $ma_g z$, where a_g is the acceleration of gravity, and z the height (vertical position) of the center of mass of the ball which is constrained to lie on the surface Σ. Eliminating $\vec{N}$ from (4.1.2) and using (4.1.1), we get

$$\begin{aligned} \dot{\vec{M}} &= mR^2\dot{\vec{n}} \times (\vec{\omega} \times \vec{n}) + \vec{M}_F \\ \dot{\vec{r}} + R\dot{\vec{n}} &= -R\vec{\omega} \times \vec{n}, \end{aligned} \tag{4.1.3}$$

where $\vec{M}_F$ denotes the external moment of force. See a more complete discussion on these equations in [24]. These equations of motion do not include the orientation of the ball, so they can be considered as the equations after reduction by the symmetry given by $SO(3)$. Following [24] note that the equations of motion (4.1.3) are valid on any smooth surface. In this dissertation we restrict ourselves to the case of a smooth convex surface of revolution with vertical axis of symmetry (parallel to the gravity force) which has the effect of introducing a new S^1-symmetry. The action associated to the S^1-symmetry leaves invariant the configurations where the ball is spinning in the bottom of the surface, thus the action is not *free*. This fact has many implications, e.g. the reduced space now is not a smooth manifold and the rank of the vertical distribution defined in the beginning of Section 2.3.1 can vary.

Borisov, Mamaev and Kilin [24] have shown, in local coordinates, that the reduced system is described by a Poisson bracket in the sense that the reduced equations of motion are given by the vector field (I.2) (after a time reparametrization). The properties of this reduced Poisson bracket have also been studied by Ramos [77] (observing that, by a dimensional argument, reparametrization of time is not necessary), see also Fasso-Giacobbe-Sansonetto [40]. However, it was not known if such reduced Poisson bracket came from a reduction of an almost Poisson bivector. In this Chapter we will show that there exists an almost Poisson bracket $\{\cdot,\cdot\}_B$ describing the dynamics of the nonholonomic system such that, after a reduction by symmetries, induces a reduced bracket $\{\cdot,\cdot\}^B_{red}$ which is Poisson.

4.2 The geometric approach and reduction by symmetries

The (convex) surface Σ is parametrized in Cartesian coordinates by

$$\Sigma = \{(x, y, z) \in \mathbb{R}^3 : z = \phi(x^2 + y^2)\}, \tag{4.2.4}$$

where the smooth function ϕ describes the profile of the curve defined by the center of mass of the ball. The function $\phi : \mathbb{R}^+ \to \mathbb{R}$ is related to the function $\varphi : \mathbb{R} \to \mathbb{R}$ given by $\varphi(s) = \phi(s^2)$, and which verifies the smoothness and convexity conditions, $\varphi'(0^+) = \varphi'(0^-) = 0$, $\varphi''(s) \geq 0$. To ensure that the ball has only one contact point with the surface we ask the curvature of $\varphi(s)$ to be at most $1/R$.

Denoting by $\vec{n} = \vec{n}(x, y) = (n_1, n_2, n_3)$ the exterior unit normal to Σ, the following standard formulas will be used in the sequel to simplify and re-write expressions involving the normal vector and the derivatives of the function ϕ defining Σ,

$$\frac{n_1}{n_3} = 2x\phi', \quad \frac{n_2}{n_3} = 2y\phi', \quad n_3 = -\frac{1}{(1 + 4(x^2 + y^2)(\phi')^2)^{1/2}}, \tag{4.2.5}$$

where ϕ' denotes the derivative of ϕ. Observe that n_3 is never zero.

Recall that the configuration manifold Q is given by the projection to the plane $z = 0$ of the coordinates of the center of mass of the ball and an orthogonal matrix indicating the orientation of the ball, thus $Q = \mathbb{R}^2 \times SO(3)$.

Using the holonomic constraint (4.2.4) and relations (4.2.5), the mechanical Lagrangian $L : TQ \to \mathbb{R}$ becomes

$$\begin{aligned} L =& \frac{m}{2}\left((1 + \phi'^2 4x^2)\dot{x}^2 + \phi'^2 8xy\dot{x}\dot{y} + (1 + \phi'^2 4y^2)\dot{y}^2\right) + \frac{I}{2}(\omega_1^2 + \omega_2^2 + \omega_3^2) \\ &- ma_g\phi(x^2 + y^2) \\ =& \frac{m}{2n_3^2}\left((1 - n_2^2)\dot{x}^2 + 2n_1n_2\dot{x}\dot{y} + (1 - n_1^2)\dot{y}^2\right) + \frac{I}{2}(\omega_1^2 + \omega_2^2 + \omega_3^2) \\ &- ma_g\phi(x^2 + y^2). \end{aligned} \tag{4.2.6}$$

4.2.1 The constraints and the nonholonomic bivector field

Let us denote by $(\omega_1, \omega_2, \omega_3)$ the coordinates associated to the right invariant frame $\{X_1, X_2, X_3\}$ of $TSO(3)$, i.e. the ω_i, $i = 1, 2, 3$, are the components of the angular velocity $\vec{\omega}$ in the spatial frame. Then, the non-sliding constraints (4.1.1) are written

$$\dot{x} = -R(\omega_2 n_3 - \omega_3 n_2),$$
$$\dot{y} = -R(\omega_3 n_1 - \omega_1 n_3),$$

inducing therefore the *constraint* 1*-forms*

$$\begin{aligned} \epsilon^1 &= dx - R(\rho_3 n_2 - \rho_2 n_3), \\ \epsilon^2 &= dy - R(\rho_1 n_3 - \rho_3 n_1), \end{aligned} \tag{4.2.7}$$

where we have denoted by $\{\rho_1, \rho_2, \rho_3\}$ the right Maurer-Cartan forms in $T^*SO(3)$, dual to $\{X_1, X_2, X_3\}$. The constraint distribution D on Q is the annihilator of ϵ^1 and ϵ^2 and is given by

$$D = span\left\{Y_x := \frac{\partial}{\partial x} + \frac{n_2}{Rn_3}X_n - \frac{1}{Rn_3}X_2,\ Y_y := \frac{\partial}{\partial y} - \frac{n_1}{Rn_3}X_n + \frac{1}{Rn_3}X_1,\ X_n\right\}, \tag{4.2.8}$$

where $X_n := \sum_{i=1}^3 n_i X_i$.

Consider now the basis of TQ given by

$$\mathfrak{B}_{TQ} = \{Y_x, Y_y, X_n, Z_1, Z_2\}, \tag{4.2.9}$$

where Z_1 and Z_2 are vector fields defined by

$$Z_1 := \frac{1}{Rn_3}X_2 - \frac{n_2}{Rn_3}X_n, \quad Z_2 := -\frac{1}{Rn_3}X_1 + \frac{n_1}{Rn_3}X_n. \tag{4.2.10}$$

Observe that the dual frame of (4.2.9) is

$$\mathfrak{B}_{T^*Q} = \{dx, dy, \beta_n, \epsilon^1, \epsilon^2\}, \tag{4.2.11}$$

with

$$\beta_n = \sum n_i \rho_i, \tag{4.2.12}$$

with associated coordinates $(\dot{x}, \dot{y}, \omega_n, v_1, v_2)$ for $\omega_n = \vec{\omega} \cdot \vec{n} = \sum n_i \omega_i$ the normal component of the angular velocity $\vec{\omega}$.

Remark 4.2.1. The vector fields Z_1 and Z_2 induce a splitting $TQ = D \oplus W$ where $W = span\{Z_1, Z_2\}$. The fact that W chosen in this way is adapted to the symmetries will be shown in Section 4.2.3. $\diamond$

Remark 4.2.2. The sections Z_1 and Z_2 in (4.2.10) where computed so that they complete a basis of TQ dual to $\{dx, dy, \beta_n, \epsilon^1, \epsilon^2\}$. The choice of $\{dx, dy, \beta_n\}$ as a basis of the annihilator of W was motivated by [40], which uses coordinates $(\dot{x}, \dot{y}, \omega_n)$. $\diamond$

Remark 4.2.3. Using

$$\begin{aligned} \rho_1 &= n_1\beta_n + \frac{n_1 n_2}{Rn_3}(dx - \epsilon^1) + \frac{1 - n_1^2}{Rn_3}(dy - \epsilon^2), \\ \rho_2 &= n_2\beta_n + \frac{n_2^2 - 1}{Rn_3}(dx - \epsilon^1) + \frac{-n_1 n_2}{Rn_3}(dy - \epsilon^2), \\ \rho_3 &= n_3\beta_n + \frac{n_2 n_3}{Rn_3}(dx - \epsilon^1) + \frac{-n_1 n_3}{Rn_3}(dy - \epsilon^2), \end{aligned} \tag{4.2.13}$$

one also verify that $d\beta_n$ is not in the differential ideal generated by $\{dx, dy, \beta_n\}$, then W is not integrable. On the other hand, the following formulas will also appear in the computations:

$$
\begin{aligned}
\rho_1 \wedge \rho_2 &= B\beta_n \wedge (dx - \epsilon^1) + A(dy - \epsilon^2) \wedge \beta_n - \frac{1}{R^2}(dy - \epsilon^2) \wedge (dx - \epsilon^1), \\
\rho_2 \wedge \rho_3 &= -\frac{1}{R}(dx - \epsilon^1) \wedge \beta_n - \frac{B}{R}(dx - \epsilon^1) \wedge (dy - \epsilon^2), \\
\rho_3 \wedge \rho_1 &= -\frac{1}{R}(dy - \epsilon^2) \wedge \beta_n - \frac{A}{R}(dy - \epsilon^2) \wedge (dx - \epsilon^1).
\end{aligned} \tag{4.2.14}
$$

◇

Computation of the constraint manifold $\mathcal{M}$ and the distribution $\mathcal{C}$

In this section we use the notations:

$$a_x = 1 + \phi'^2 4x^2, \quad a_y = 1 + \phi'^2 4y^2, \quad b_{xy} = \phi'^2 4xy.$$

Therefore in the basis $\{dx, dy, \rho_1, \rho_2, \rho_3\}$ the kinetic energy metric is written in matrix form

$$\kappa = \begin{pmatrix} ma_x & mb_{xy} & 0 & 0 & 0 \\ mb_{xy} & ma_y & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix}, \tag{4.2.15}$$

or equivalently,

$$\frac{1}{2}\kappa = \frac{ma_x}{2}(dx \otimes dx) + \frac{ma_y}{2}(dy \otimes dy) + \frac{mb_{xy}}{2}(dx \otimes dy) + \frac{I}{2}(\rho_1 \otimes \rho_1 + \rho_2 \otimes \rho_2 + \rho_3 \otimes \rho_3).$$

We now compute the constraint manifold $\mathcal{M} = \kappa^\flat(D) = Leg(D) \subset T^*Q$ defined in (1.4.22). Using the basis of D given in (4.2.8), we have

$$\kappa^\flat(D) = \dot{x}\kappa^\flat(Y_x) + \dot{y}\kappa^\flat(Y_y) + \tilde{\omega_n}\kappa^\flat(X_n).$$

Writing $\kappa^\flat(D)$ in the basis (4.2.11) of T^*Q,

$$\kappa^\flat(D) = \tilde{p_x}dx + \tilde{p_y}dy + \tilde{M}_n\beta_n + \tilde{M}_x\epsilon^1 + \tilde{M}_y\epsilon^2,$$

we get the relations

$$
\begin{aligned}
\tilde{p_x} &= \dot{x}\kappa(Y_x, Y_x) + \dot{y}\kappa(Y_y, Y_x) + \tilde{\omega_n}\kappa(X_n, Y_x) \\
\tilde{p_y} &= \dot{x}\kappa(Y_x, Y_y) + \dot{y}\kappa(Y_y, Y_y) + \tilde{\omega_n}\kappa(X_n, Y_y) \\
\tilde{M}_n &= \dot{x}\kappa(Y_x, X_n) + \dot{y}\kappa(Y_y, X_n) + \tilde{\omega_n}\kappa(X_n, X_n) \\
\tilde{M}_x &= \dot{x}\kappa(Y_x, Z_2) + \dot{y}\kappa(Y_y, Z_2) + \tilde{\omega_n}\kappa(X_n, Z_2) \\
\tilde{M}_y &= \dot{x}\kappa(Y_x, Z_1) + \dot{y}\kappa(Y_y, Z_1) + \tilde{\omega_n}\kappa(X_n, Z_1).
\end{aligned}
$$

The components of the matrix $\kappa(\cdot,\cdot)$ can be computed from (4.2.15). Denoting $E := I + mR^2$, we get the relations between velocities and momenta:

$$\begin{aligned} \dot{x}E &= -\tilde{p_y}R^2 n_1 n_2 + \tilde{p_x}R^2(1-n_1^2), \\ \dot{y}E &= -\tilde{p_x}R^2 n_1 n_2 + \tilde{p_y}R^2(1-n_2^2), \\ \tilde{\omega_n}I &= \tilde{M_n}. \end{aligned} \tag{4.2.16}$$

On the other hand

$$\begin{aligned} \tilde{M_x} &= -\tilde{p_x} + \dot{x}ma_x + \dot{y}mb_{xy}, \\ \tilde{M_y} &= -\tilde{p_y} + \dot{x}mb_{xy} + \dot{y}ma_y, \end{aligned} \tag{4.2.17}$$

and using (4.2.16) we have

$$\begin{aligned} \dot{x}ma_x + \dot{y}mb_{xy} &= \frac{R^2 m}{E}\tilde{p_x}, \\ \dot{x}mb_{xy} + \dot{y}ma_y &= \frac{R^2 m}{E}\tilde{p_y}. \end{aligned} \tag{4.2.18}$$

Then, from (4.2.17) and (4.2.18), the constraint manifold $\mathcal{M}$ is given by

$$\mathcal{M} = \Big\{(x, y, g, \tilde{p_x}, \tilde{p_y}, \tilde{M_n}, \tilde{M_x}, \tilde{M_y}) \ : \ \tilde{M_x} = -\frac{I}{E}\tilde{p_x}, \ \tilde{M_y} = -\frac{I}{E}\tilde{p_y}\Big\}, \tag{4.2.19}$$

where recall that $E = I + mR^2$.

Next, we compute the constraint distribution $\mathcal{C}$ on $\mathcal{M}$. Consider the basis of $T^*\mathcal{M}$ given by

$$\mathfrak{B}_{T^*\mathcal{M}} = \Big\{\tilde{dx}, \tilde{dy}, \tilde{\beta_n}, \tilde{\epsilon}^1, \tilde{\epsilon}^2, d\tilde{p_x}, d\tilde{p_y}, d\tilde{M_n}\Big\}, \tag{4.2.20}$$

where $\tilde{dx} = \tau_{\mathcal{M}}^* dx$, $\tilde{dy} = \tau_{\mathcal{M}}^* dy$, $\tilde{\beta_n} = \tau_{\mathcal{M}}^*\beta_n$, $\tilde{\epsilon}^1 = \tau_{\mathcal{M}}^*\epsilon^1$, $\tilde{\epsilon}^2 = \tau_{\mathcal{M}}^*\epsilon^2$ and, as usual, $\tau_{\mathcal{M}} : \mathcal{M} \to Q$ is the canonical projection.

Dualizing the basis in (4.2.20) we get the associated vector fields in $\mathcal{M}$, which we denote with a tilde to distinguish them from the corresponding fields on Q,

$$\mathfrak{B}_{T\mathcal{M}} = \Big\{\tilde{Y_x}, \tilde{Y_y}, \tilde{X_n}, \tilde{Z_1}, \tilde{Z_2}, \frac{\partial}{\partial\tilde{p_x}}, \frac{\partial}{\partial\tilde{p_y}}, \frac{\partial}{\partial\tilde{M_n}}\Big\}. \tag{4.2.21}$$

Hence, the constraint subbundle $\mathcal{C}$ defined in (1.4.23) is given by

$$\mathcal{C} = span\Big\{\tilde{Y_x}, \tilde{Y_y}, \tilde{X_n}, \frac{\partial}{\partial\tilde{p_x}}, \frac{\partial}{\partial\tilde{p_y}}, \frac{\partial}{\partial\tilde{M_n}}\Big\}. \tag{4.2.22}$$

Computation of the nonholonomic bivector π_{nh} and the dynamics X_{nh}

Now we are ready to we compute the nonholonomic bivector field π_{nh} (1.4.26), and afterwards the nonholonomic vector field $X_{nh} = -\pi_{nh}^\sharp(dH_{\mathcal{M}})$ describing the dynamics.

Proposition 4.2.4. *The nonholonomic bivector field π_{nh} on $\mathcal{M}$ defining the nonholonomic dynamics is given by*

$$\begin{aligned}\pi_{nh} =&\tilde{Y_x} \wedge \frac{\partial}{\partial \tilde{p_x}} + \tilde{Y_y} \wedge \frac{\partial}{\partial \tilde{p_y}} + \tilde{X_n} \wedge \frac{\partial}{\partial \tilde{M_n}} \\ &+ \tilde{M_n} D^n_{xy} \frac{\partial}{\partial \tilde{p_x}} \wedge \frac{\partial}{\partial \tilde{p_y}} + \frac{I}{E}(\tilde{p_x} D^x_{yn} + \tilde{p_y} D^y_{yn}) \frac{\partial}{\partial \tilde{p_y}} \wedge \frac{\partial}{\partial \tilde{M_n}} \\ &- \frac{I}{E}(\tilde{p_x} D^x_{xn} + \tilde{p_y} D^y_{xn}) \frac{\partial}{\partial \tilde{M_n}} \wedge \frac{\partial}{\partial \tilde{p_x}},\end{aligned} \tag{4.2.23}$$

where D^n_{xy}, D^x_{xn}, D^y_{yn}, D^y_{xn} and D^x_{yn} are basic functions on $\mathcal{M}$ (with respect to the bundle $\tau_{\mathcal{M}} : \mathcal{M} \to Q$) given by

$$\begin{aligned} D^n_{xy} &= \frac{1}{Rn_3}\left(n^x_1 + n^y_2 + \frac{1}{R}\right), \\ D^x_{xn} &= -D^y_{yn} = R\left(-n^y_1 n_3 + n_1 n^y_3 + \frac{n_1 n_2}{Rn_3}\right), \\ D^y_{xn} &= R\left(n^x_1 n_3 - n_1 n^x_3 - \frac{n_1^2 + n_3^2}{Rn_3}\right), \\ D^x_{yn} &= R\left(-n^y_2 n_3 + n_2 n^y_3 + \frac{n_2^2 + n_3^2}{Rn_3}\right), \end{aligned} \tag{4.2.24}$$

with n^x_i and n^y_i the partial derivatives of n_i, $i = 1, 2, 3$, with respect to x and y, respectively.

Proof. First observe that the Liouville 1-form in T^*Q in the basis (4.2.11) is written as $\Theta_Q = \tilde{p_x}\tilde{dx} + \tilde{p_y}\tilde{dy} + \tilde{M_n}\tilde{\beta_n} + \tilde{M_x}\tilde{\epsilon}^1 + \tilde{M_y}\tilde{\epsilon}^2$, and its restriction to $\mathcal{M}$ is

$$\Theta_{\mathcal{M}} = \tilde{p_x}\tilde{dx} + \tilde{p_y}\tilde{dy} + \tilde{M_n}\tilde{\beta_n} - \frac{I}{E}\tilde{p_x}\tilde{\epsilon}^1 + -\frac{I}{E}\tilde{p_y}\tilde{\epsilon}^2. \tag{4.2.25}$$

Using the definition of $\Omega_{\mathcal{M}}$ given in (1.4.24) we obtain that

$$\begin{aligned}\Omega_{\mathcal{M}} =& -d\tilde{p_x} \wedge \tilde{dx} - d\tilde{p_y} \wedge \tilde{dy} - d\tilde{M_n} \wedge \tilde{\beta_n} \\ &+ \frac{I}{E} d\tilde{p_x} \wedge \tilde{\epsilon}^1 + \frac{I}{E} d\tilde{p_y} \wedge \tilde{\epsilon}^2 - \tilde{M_n} d\tilde{\beta_n} + \frac{I}{E}\tilde{p_x} d\tilde{\epsilon}^1 + \frac{I}{E}\tilde{p_y} d\tilde{\epsilon}^2,\end{aligned}$$

and its pointwise restriction to $\mathcal{C}$ gives

$$\Omega_{\mathcal{C}} = \left(-d\tilde{p_x} \wedge \tilde{dx} - d\tilde{p_y} \wedge \tilde{dy} - d\tilde{M_n} \wedge \tilde{\beta_n} - \tilde{M_n} d\tilde{\beta_n} + \frac{I}{E}\tilde{p_x} d\tilde{\epsilon}^1 + \frac{I}{E}\tilde{p_y} d\tilde{\epsilon}^2\right)|_{\mathcal{C}}. \tag{4.2.26}$$

Denoting by $n^x = (n^x_1, n^x_2, n^x_3)$ (respectively $n^y = (n^y_1, n^y_2, n^y_3)$) the component-wise partial derivatives of $\vec{n} = (n_1, n_2, n_3)$ with respect to x (respectively to y), we have the following relation

$$\langle \vec{n}, n^x \rangle = \langle \vec{n}, n^y \rangle = 0,$$

To compute $d\tilde{\beta}_n$, using (4.2.13) we obtain at first

$$dn_1 \wedge \rho_1|_{\mathcal{C}} + dn_2 \wedge \rho_2|_{\mathcal{C}} + dn_3 \wedge \rho_3|_{\mathcal{C}} = \left(n_1^x \frac{1}{Rn_3} + n_2^y \frac{1}{Rn_3} \right) dx \wedge dy|_{\mathcal{C}}.$$

On the other hand, to compute $\sum_i n_i d\rho_i|_{\mathcal{C}}$, we use formulas (4.2.14) restricted to $\mathcal{C}$ and we get

$$(n_1 d\rho_1 + n_2 d\rho_2 + n_3 d\rho_3)|_{\mathcal{C}} = \frac{1}{R^2 n_3} dx \wedge dy|_{\mathcal{C}}.$$

Consequently,

$$d\tilde{\beta}_n|_{\mathcal{C}} = \frac{1}{Rn_3} \left(n_1^x + n_2^y + \frac{1}{R} \right) \tilde{dx} \wedge \tilde{dy}|_{\mathcal{C}}. \tag{4.2.27}$$

Moreover, from (4.2.7) we also get

$$\begin{aligned}
d\tilde{\epsilon}^1|_{\mathcal{C}} = &- R\left(-n_1^y n_3 + n_1 n_3^y + \frac{n_1 n_2}{Rn_3} \right) \tilde{dx} \wedge \tilde{\beta}_n|_{\mathcal{C}} \\
&- R\left(-n_2^y n_3 + n_2 n_3^y + \frac{n_2^2 + n_3^2}{Rn_3} \right) \tilde{dy} \wedge \tilde{\beta}_n|_{\mathcal{C}}, \\
d\tilde{\epsilon}^2|_{\mathcal{C}} = &- R\left(n_1^x n_3 - n_1 n_3^x - \frac{n_1^2 + n_3^2}{Rn_3} \right) \tilde{dx} \wedge \tilde{\beta}_n|_{\mathcal{C}} \\
&- R\left(n_1^y n_3 - n_1 n_3^y - \frac{n_1 n_2}{Rn_3} \right) \tilde{dy} \wedge \tilde{\beta}_n|_{\mathcal{C}}.
\end{aligned} \tag{4.2.28}$$

Therefore, using (4.2.27) and (4.2.28), the expression (4.2.26) becomes

$$\begin{aligned}
\Omega_{\mathcal{C}} = (&- d\tilde{p}_x \wedge \tilde{dx} - d\tilde{p}_y \wedge \tilde{dy} - d\tilde{M}_n \wedge \tilde{\beta}_n \\
&- \tilde{M}_n D^n_{xy} \tilde{dx} \wedge \tilde{dy} - \frac{I}{E}(\tilde{p}_x D^x_{xn} + \tilde{p}_y D^y_{xn}) \tilde{dx} \wedge \tilde{\beta}_n \\
&- \frac{I}{E}(\tilde{p}_x D^x_{yn} + \tilde{p}_y D^y_{yn}) \tilde{dy} \wedge \tilde{\beta}_n)|_{\mathcal{C}},
\end{aligned} \tag{4.2.29}$$

where D^n_{xy}, D^x_{xn}, D^y_{xn}, D^x_{yn} and D^y_{yn} are given in (4.2.24). Finally, the nonholonomic bivector is computed using (1.4.26), and we get the desired expression (4.2.23).

□

Equivalently, we can write the nonholonomic bivector as

$$\begin{aligned}
\pi^{\#}_{nh}(\tilde{dx}) &= \frac{\partial}{\partial \tilde{p}_x}, \qquad \pi^{\#}_{nh}(\tilde{dy}) = \frac{\partial}{\partial \tilde{p}_y}, \qquad \pi^{\#}_{nh}(\tilde{\beta}_n) = \frac{\partial}{\partial \tilde{M}_n}, \\
\pi^{\#}_{nh}(d\tilde{p}_x) &= -\tilde{Y}_x + \tilde{M}_n D^n_{xy} \frac{\partial}{\partial \tilde{p}_y} + \frac{I}{E}(\tilde{p}_x D^x_{xn} + \tilde{p}_y D^y_{xn}) \frac{\partial}{\partial \tilde{M}_n}, \\
\pi^{\#}_{nh}(d\tilde{p}_y) &= -\tilde{Y}_y - \tilde{M}_n D^n_{xy} \frac{\partial}{\partial \tilde{p}_x} + \frac{I}{E}(\tilde{p}_x D^x_{yn} + \tilde{p}_y D^y_{yn}) \frac{\partial}{\partial \tilde{M}_n}, \\
\pi^{\#}_{nh}(d\tilde{M}_n) &= -\tilde{X}_n - \frac{I}{E}(\tilde{p}_x D^x_{xn} + \tilde{p}_y D^y_{xn}) \frac{\partial}{\partial \tilde{p}_x} - \frac{I}{E}(\tilde{p}_x D^x_{yn} + \tilde{p}_y D^y_{yn}) \frac{\partial}{\partial \tilde{p}_y}.
\end{aligned}$$

Moreover, since $\tilde{\epsilon}^1|_{\mathcal{C}} \equiv 0$ and $\tilde{\epsilon}^2|_{\mathcal{C}} \equiv 0$, the definition of the nonholonomic bivector implies that

$$\pi^{\#}_{nh}(\tilde{\epsilon}^1) = \pi^{\#}_{nh}(\tilde{\epsilon}^2) = 0.$$

Now, using Prop. 4.2.4 we compute the nonholonomic dynamics $X_{nh} = -\pi^{\#}_{nh}(dH_{\mathcal{M}})$ associated to our example. From the Lagrangian (4.2.6) the restricted Hamiltonian $H_{\mathcal{M}}$ is given by

$$\begin{aligned} H_{\mathcal{M}} =& \frac{R^2}{2E}((1-n_1^2)\tilde{p_x}^2 + (1-n_2^2)\tilde{p_y}^2 - 2\tilde{p_x}\tilde{p_y}n_1n_2) \\ &+ \frac{\tilde{M_n}^2}{2I} + ma_g\phi(x^2+y^2), \end{aligned} \tag{4.2.30}$$

and therefore the nonholonmic vector field X_{nh} is written

$$\begin{aligned} X_{nh} &= \dot{x}\tilde{Y_x} + \dot{y}\tilde{Y_y} + \tilde{\omega_n}\tilde{X_n} \\ &+ \left(\dot{y}\tilde{M_n}D^n_{xy} + (\tilde{p_x}n_1 + \tilde{p_y}n_2)(\tilde{p_x}n_1^x + \tilde{p_y}n_2^x) + \tilde{\omega_n}\frac{I}{E}(\tilde{p_x}D^x_{xn} + \tilde{p_y}D^y_{xn})\right)\frac{\partial}{\partial\tilde{p_x}} \\ &+ \left(-\dot{x}\tilde{M_n}D^n_{xy} + (\tilde{p_x}n_1 + \tilde{p_y}n_2)(\tilde{p_x}n_1^y + \tilde{p_y}n_2^y) + \tilde{\omega_n}\frac{I}{E}(\tilde{p_x}D^x_{yn} + \tilde{p_y}D^y_{yn})\right)\frac{\partial}{\partial\tilde{p_y}} \\ &+ \frac{I}{E}\left(-\dot{x}(\tilde{p_x}D^x_{xn} + \tilde{p_y}D^y_{xn}) - \dot{y}(\tilde{p_x}D^x_{yn} + \tilde{p_y}D^y_{yn})\right)\frac{\partial}{\partial\tilde{M_n}} \\ &- ma_g\phi'\left(2x\frac{\partial}{\partial\tilde{p_x}} + 2y\frac{\partial}{\partial\tilde{p_y}}\right), \end{aligned} \tag{4.2.31}$$

where, by Legendre transform, we have the following relations between velocities and momenta,

$$\begin{aligned} \dot{x} =& \frac{R^2}{E}\left(\tilde{p_x}(1-n_1^2) - \tilde{p_y}n_1n_2\right), \quad \tilde{\omega_n} = \frac{\tilde{M_n}}{I}, \\ \dot{y} =& \frac{R^2}{E}\left(\tilde{p_y}(1-n_2^2) - \tilde{p_x}n_1n_2\right). \end{aligned} \tag{4.2.32}$$

4.2.2 The G-symmetry and the reduced bracket $\{\cdot,\cdot\}_{red}$

Consider the compact Lie group

$$G = S^1 \times SO(3), \tag{4.2.33}$$

where $SO(3)$ acts by right action and S^1 by left action on $Q = \mathbb{R}^2 \times SO(3)$. More precisely, the action by an element $(\varphi, h) \in S^1 \times SO(3)$ on $(x,y,g) \in Q$ is given by

$$(\varphi, h)\cdot(x,y,g) = (R_\varphi(x,y), \hat{R}_\varphi\, g\, h), \tag{4.2.34}$$

where R_φ denotes the 2×2 rotation matrix of angle φ and $\hat{R}_\varphi$ denotes the 3×3 rotation matrix of angle φ with respect to the z-axis.

We show that the G-action is a symmetry of the nonholonomic system. It is enough to show that the action preserves the Lagrangian L and the distribution D. It is easier start with the Lagrangian

$$L = \frac{1}{2}||\dot{\boldsymbol{x}}||^2 + \frac{1}{2}||\boldsymbol{\omega}||^2 - ma_g\phi(x^2+y^2),$$

with $\boldsymbol{x} = (x, y, z)$ taking in consideration along the computations the holonomic constraint for z given in (4.2.4). First, we observe that the right $SO(3)$-action only acts on $\boldsymbol{\omega}$ by an orthogonal transformation which preserves the norm. Second, the left S^1 action leaves invariant the (spatial) angular velocity $\boldsymbol{\omega}$ (which is left invariant) and preserves $x^2 + y^2$ and $||\dot{\boldsymbol{x}}||$. Then, the Lagrangian L is G-invariant.

On the other hand consider the expression of Y_x, Y_y and X_n which generate the constraint distribution D (see (4.2.8)). Observe that the vector fields Y_x, Y_y and X_n are $SO(3)$-invariant because the vector fields X_i, $i = 1, 2, 3$ are right invariant by definition and the right $SO(3)$ action does not act on the (x, y)-coordinates. Now, for the left S^1-action, consider a rotation matrix R_φ with respect to the vertical axis and angle φ, then

$$\begin{aligned}
((R_\varphi)_*(Y_x))|_{R_\varphi\cdot q} &= \cos\varphi Y_x|_{R_\varphi\cdot q} + \sin Y_y|_{R_\varphi\cdot q},\\
((R_\varphi)_*(Y_y))|_{R_\varphi\cdot q} &= -\sin\varphi Y_x|_{R_\varphi\cdot q} + \cos Y_y|_{R_\varphi\cdot q},\\
((R_\varphi)_*(X_n))|_{R_\varphi\cdot q} &= X_n|_{R_\varphi\cdot q},
\end{aligned}$$

then the distribution D is G-invariant.

The Lie algebra $\mathfrak{g}$ of G is isomorphic to $\mathbb{R}\times\mathbb{R}^3$ and thus we work with the following basis of $\mathfrak{g}$,

$$\{(1, \mathbf{0}), (0, \mathbf{e}_i)\}, \quad i = 1, 2, 3, \tag{4.2.35}$$

where $\mathbf{e}_i$ denotes the i-th canonical basis vector of $\mathbb{R}^3$. The infinitesimal generator with respect to the S^1-action is

$$U_0 := (1, \mathbf{0})_Q = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + X_3.$$

If we denote by $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)$ and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ the rows of the matrix $g \in SO(3)$, then the infinitesimal generators associated to the $SO(3)$-action are given by

$$(0, \mathbf{e_i})_Q = \alpha_i X_1 + \beta_i X_2 + \gamma_i X_3, \quad i = 1, 2, 3. \tag{4.2.36}$$

Denoting $\mathbf{U} = ((0, \mathbf{e_1})_Q, (0, \mathbf{e_2})_Q, (0, \mathbf{e_3})_Q)$, we see that $X_1 = \langle \boldsymbol{\alpha}, \mathbf{U}\rangle$, $X_2 = \langle \boldsymbol{\beta}, \mathbf{U}\rangle$ and $X_3 = \langle \boldsymbol{\gamma}, \mathbf{U}\rangle$, consequently the vertical (generalized) distribution V is given by

$$V = span\{U_0, X_1, X_2, X_3\}, \tag{4.2.37}$$

and the G-symmetry satisfies the dimension assumption (2.3.4). We observe that the rank of V is 3 for $(x, y) = (0, 0)$ and it is 4 elsewhere, showing that the action is not free (not even locally free).

Remark 4.2.5. We express the vector fields appearing in (4.2.37) in the basis associated to the splitting $TQ = D \oplus W$, see Rmk. 4.2.1. Recalling the expressions for X_n, Z_1 and Z_2 in (4.2.10), we get the relations

$$\begin{pmatrix} n_1 & n_2 & n_3 \\ -An_1 & -An_2 + \frac{1}{Rn_3} & -An_3 \\ -Bn_1 - \frac{1}{Rn_3} & -Bn_2 & -Bn_3 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} X_n \\ Z_2 \\ Z_1 \end{pmatrix},$$

with inverse

$$\begin{pmatrix} n_1 & 0 & -Rn_3 \\ n_2 & Rn_3 & 0 \\ n_3 & -Rn_2 & Rn_1 \end{pmatrix} \begin{pmatrix} X_n \\ Z_2 \\ Z_1 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}.$$

On the other hand, from the definitions of Y_x and Y_y in the basis of D given in (4.2.8), we have

$$\frac{\partial}{\partial x} = Y_x + Z_1, \quad \frac{\partial}{\partial y} = Y_y + Z_2.$$

Consequently we can write

$$U_0 = (1, \mathbf{0})_Q = -y(Y_x + Z_1) + x(Y_y + Z_2) + n_3 X_n - Rn_2 Z_2 + Rn_1 Z_1,$$

and then the vertical distribution (4.2.37) is also written

$$V = span\{U_0, X_n, Z_2, Z_1\}.$$

$\diamond$

Now we describe the reduced space $\mathcal{M}/G$ as a stratified differential space and write the reduced dynamics. The reduction of $\mathcal{M}$ by the symmetry group $G = S^1 \times SO(3)$ is performed by stages as in [40, 52]. The reduction by $SO(3)$ gives the smooth manifold $\mathcal{M}/SO(3)$ and results in the elimination of the coordinate g of $\mathcal{M}$. Furthermore, from (4.2.49) we see that S^1 acts on $\mathcal{M}/SO(3)$ by $\varphi \cdot (x, y, \tilde{p_x}, \tilde{p_y}, \tilde{M_n}) = (R_\varphi(x, y), R_\varphi(\tilde{p_x}, \tilde{p_y}), \tilde{M_n})$.

Since $(0, 0, 0, 0, M_n)$ is a fixed point for any rotation R_φ, the S^1-action is not free and the reduction is performed using invariant theory as in [40, 52, 77]. The S^1-invariant polynomials on $\mathcal{M}/SO(3)$ for this action are given by

$$\begin{aligned} p_0 &= \tilde{p_x}^2 + \tilde{p_y}^2, \\ p_1 &= x^2 + y^2, \\ p_2 &= x\tilde{p_x} + y\tilde{p_y}, \\ p_3 &= x\tilde{p_y} - y\tilde{p_x}, \\ p_4 &= \tilde{M_n}. \end{aligned} \tag{4.2.38}$$

Since the S^1-action is not free but it is proper the reduced space $\mathcal{M}/G$ is a stratified differential space and the ring of smooth functions on $\mathcal{M}/G$ is identified with the ring $C^\infty(\mathcal{M})^G$ of G-invariant functions on $\mathcal{M}$, see Section 2.1.3. By invariant theory, the p_i, $i = 0, \cdots, 4$, form a basis of $C^\infty(\mathcal{M})^G$ and hence the polynomials p_i can also be considered as coordinates on $\mathcal{M}/G$ in the sense that $\mathcal{M}/G$ is described by the following semi-algebraic subset of $\mathbb{R}^5$,

$$\{p = (p_0, p_1, p_2, p_3, p_4) \in \mathbb{R}^5 : p_0 \geq 0,\ p_1 \geq 0,\ p_0 p_1 = p_2^2 + p_3^2\}.$$

The stratified orbit space $\mathcal{M}/G$ has two strata corresponding to the orbit types. The 1-dimensional *singular stratum* associated to the S^1 isotropy type is given by

$$M_1 = \{p = (p_0, p_1, p_2, p_3, p_4) \in \mathbb{R}^5 : p_0 = p_1 = p_2 = p_3 = 0\}, \tag{4.2.39}$$

and corresponds to the situation where the ball lies at the bottom of the surface and is spinning about the vertical axis.

The other 4-dimensional stratum, called *regular stratum*, is the complement of M_1 in $\mathcal{M}/G$ and is given by

$$\begin{aligned} M_4 = \{p = (p_0, p_1, p_2, p_3, p_4) \in \mathbb{R}^5 : p_0 \geq 0,\ p_1 \geq 0,& \\ p_0 p_1 = p_2^2 + p_3^2,\ p_0^2 + p_1^2 > 0\}.& \end{aligned} \tag{4.2.40}$$

It can be understood as the manifold which is the orbit space of the submanifold $\mathcal{M}_{reg}$ of $\mathcal{M}$ where the action is free. Hence $M_4 = \mathcal{M}_{reg}/G$ and we use the notation $\bar{\mathcal{M}}_{reg}$ instead of M_4. It has been observed that the regular stratum $\bar{\mathcal{M}}_{reg}$ is diffeomorphic to $S^2 \times \mathbb{R}^2$, see [40, 52]. Observe that possible trajectories in the regular stratum include the case of the ball passing by the bottom of the surface Σ, $(x, y) = (0, 0)$, but with non-zero velocity. On the other hand, from (4.2.32), we obtain the following relations

$$\begin{aligned} \tilde{p_x} &= \dot{x}\frac{1 - n_2^2}{n_3^2 R^2}E + \dot{y}\frac{n_1 n_2}{n_3^2 R^2}E, \\ \tilde{p_y} &= \dot{x}\frac{n_1 n_2}{n_3^2 R^2}E + \dot{y}\frac{1 - n_1^2}{n_3^2 R^2}E, \end{aligned} \tag{4.2.41}$$

and rewriting the polynomials p_i, $i = 0, \cdots, 4$ in the velocity coordinates, we observe that trajectories with $p_3 = 0$ are those where the ball moves only in the radial direction, i.e. in the intersection of the surface with a vertical plane passing by the origin, while when $p_2 = 0$ the ball moves in a circular trajectory at a constant height. The variables p_3 and p_4 will be important when we will consider first integrals in Section 4.3.1.

Remark 4.2.6. Equivalently, we can understand the dynamics by studying the reduced equations of motion obtained by Hermans [52]. We recall the reduction in the form presented in [40] and then relate it to our variables p_i, $i = 0, \ldots, 4$ defined in (4.2.38). Consider the invariant polynomials in positions and velocities:

$$\begin{aligned} &\bar{p_0} = \frac{\dot{x}^2 + \dot{y}^2}{2}, \quad \bar{p_1} = \frac{x^2 + y^2}{2}, \quad \bar{p_2} = x\dot{x} + y\dot{y}, \\ &\bar{p_3} = x\dot{y} - y\dot{x}, \quad \bar{p_4} = R\omega_n. \end{aligned}$$

In these variable de reduced dynamics is given by,

$$X_{nh}^{red} = \bar{p_2}F_0\frac{\partial}{\partial \bar{p_0}} + \bar{p_2}\frac{\partial}{\partial \bar{p_1}} + F_2\frac{\partial}{\partial \bar{p_2}} + \bar{p_2}\bar{p_4}F_3\frac{\partial}{\partial \bar{p_3}} + \bar{p_2}\bar{p_3}F_4\frac{\partial}{\partial \bar{p_4}}. \tag{4.2.42}$$

The functions F_3 and F_4 depend only on $\bar{p_1}$. Adapting the formulas of [40] to our notations we have

$$\begin{aligned}
F_0(\bar{p_0},\bar{p_1},\bar{p_2},\bar{p_3},\bar{p_4}) &= \left[4(\frac{I}{E}\bar{p_3}\bar{p_4} - 2\bar{p_2}^2\phi'(\bar{p_1}))\phi''(\bar{p_1}) - 2a_g\frac{mR^2}{E}\phi'(\bar{p_1}) - 8\bar{p_0}\phi'(\bar{p_1})^2\right] g(\bar{p_1}),\\
F_2(\bar{p_0},\bar{p_1},\bar{p_2},\bar{p_3},\bar{p_4}) &= \left[2\bar{p_0} - 2(\frac{I}{E}\bar{p_3}\bar{p_4} + 2a_g\frac{mR^2}{E}\bar{p_1})\phi'(\bar{p_1}) + 16\bar{p_1}\bar{p_2}^2\phi'(\bar{p_1})\phi''(\bar{p_1})\right] g(\bar{p_1}),\\
F_3(\bar{p_1}) &= \frac{I}{E}\left(2\phi'(\bar{p_1}) + 8\bar{p_1}\phi''(\bar{p_1})\right) g(\bar{p_1}),\\
F_4(\bar{p_1}) &= \left(8\phi'(\bar{p_1})^3 - 4\phi''(\bar{p_1})\right) g(\bar{p_1}),\\
g(\bar{p_1}) &= \frac{1}{1+2\bar{p_1}\Psi'(\bar{p_1})^2}.
\end{aligned} \tag{4.2.43}$$

Observe that $g(\bar{p_1})$ is equal to n_3^2, the square of the z-component of the unit normal vector $\vec{n}$. Using Eqs. (4.2.41) one can find the relations between the $\bar{p_i}$ and p_i. We get:

$$\begin{aligned}
p_0 &= \frac{2E^2}{n_3^4R^4}\bar{p_0} - \frac{E^2}{n_3^4R^4}(1+n_3^2)(n_2\dot{x} - n_1\dot{y})^2\\
p_1 &= 2\bar{p_1}, \quad p_2 = \frac{E}{n_3^2R^2}\bar{p_2}, \quad p_3 = \frac{E}{R^2}\bar{p_3}, \quad p_4 = \frac{I}{R}\bar{p_4}.
\end{aligned} \tag{4.2.44}$$

From the latter relations and the chain rule we compute the relevant terms of the reduced dynamics X_{red} in our variables p_i. ◇

The reduced dynamics is computed projecting the nonholonomic vector field X_{nh} given in (4.2.31) by the quotient map $\rho : \mathcal{M} \to \mathcal{M}/G$ and has been presented in [52] (see also [40]). In our reduced variables p_i, $i = 0, \cdots, 4$, the reduced nonholonomic vector field X_{red} has the form:

$$X_{red} = \bar{F}_0\frac{\partial}{\partial p_0} + \frac{2R^2n_3^2}{E}p_2\frac{\partial}{\partial p_1} + \bar{F}_2\frac{\partial}{\partial p_2} + \frac{Rn_3^2}{E}\bar{F}_3p_2p_4\frac{\partial}{\partial p_3} + \frac{R^3In_3^2}{E^2}p_2p_3\bar{F}_4\frac{\partial}{\partial p_4}, \tag{4.2.45}$$

where $n_3, \bar{F}_3, \bar{F}_4$ are basic functions with respect to the bundle $\mathcal{M}/G \to Q/G$ (thinking of Q/G as a differential space) given by

$$\begin{aligned}
n_3 = n_3(p_1) &= -\frac{1}{(1+4p_1\phi'(p_1)^2)^{1/2}},\\
\bar{F}_3 = \bar{F}_3(p_1) &= (2\phi'(p_1) + 4p_1\phi''(p_1))n_3^2,\\
\bar{F}_4 = \bar{F}_4(p_1) &= (8\phi'(p_1)^3 - 4\phi''(p_1))n_3^2,
\end{aligned} \tag{4.2.46}$$

and $\bar{F}_0$ and $\bar{F}_2$ are functions on $\mathcal{M}/G$ that are computed from (4.2.43) and (4.2.44).

As it has been remarked in [40], the interval of continuity of the $\bar{F}_i$ contains a neighbourhood of zero, so the reduced dynamics is well defined for $p_1 = 0$ (and $p_0 \neq 0$, in the regular stratum), that is for orbits passing at the bottom of the surface with non-zero momentum. Note that X_{red} is a vector field in $\mathbb{R}^5$ which is tangent to the space $\mathcal{M}/G$ as a stratified space, that is X_{red} is tangent to each strata [40]. In the singular stratum M_1 it reduces to the equation $\dot{p_4} = 0$ with trivial solution and on the regular stratum it defines a smooth vector field on the smooth manifold M_4.

The equilibria of the reduced equations of motion (or *relative equilibria*) are of two types: the singular equilibria which are all the points of the singular stratum with $\tilde{M}_n$ constant, and the points of the regular stratum verifying $p_2 = 0$ and $\bar{F}_2(p_0, p_1, p_2, p_3, p_4) = 0$, which describe circular motions at constant height, see [40, 52, 79, 88]. Hermans [52] has shown that away from the equilibrium points all the orbits of the reduced dynamics are periodic. The qualitative study of the relative equilibria was started by Routh [79], where he gave necessary conditions for the stability of the relative equilibria by linearizing the dynamics. Afterwards Zenkov [88] proved that linear stability imply nonlinear (orbital) stability. This implies that orbits that are close to one stable relative equilibrium evolve around the periodic circular motion given by the relative equilibrium. Finally, the case of a ball rolling inside a circular cylinder (see Sections 1.5.5, 2.4.5, 3.6.5), can be considered as a limit case of a surface of revolution, and for that example the Routh/Zenkov stability condition always verified (see Eq. (2.2) of [88]). In fact, the equations of motion in this case can be completely integrated and the oscillation of the variable z (height) has explicit formulas, see e.g. [70, 77].

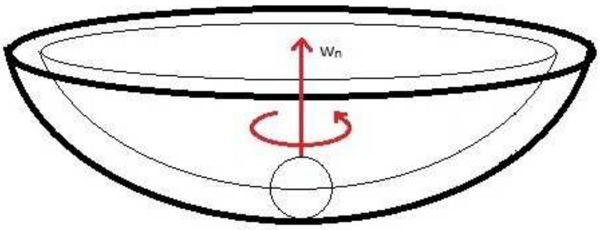


Figure 4.2: Equilibrium in the singular stratum

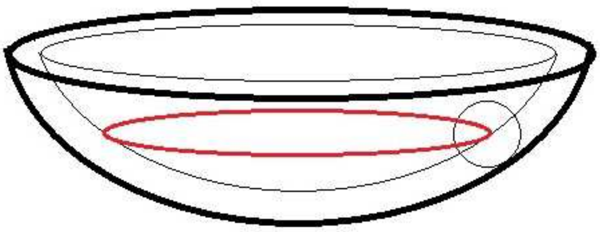


Figure 4.3: Relative equilibrium in the regular stratum

Remark 4.2.7. Since the surface Σ is of revolution we observe that the functions $\bar{F}_3$ and $\bar{F}_4$ can be expressed in terms of the principal curvatures λ_1 and λ_2 associated to Σ. In fact, since

$$\lambda_1 = -\frac{2\phi'}{(1 + 4p_1(\phi')^2)^{1/2}}, \quad \lambda_2 = -\frac{2\phi' + 4p_1\phi''}{(1 + 4p_1(\phi')^2)^{3/2}},$$

the we obtain that
$$\bar{F}_3(p_1) = \frac{\lambda_2}{n_3}, \quad \bar{F}_4(p_1) = \frac{\lambda_1 - \lambda_2}{n_3 p_1}.$$
$\diamond$

In the next section we will compute the 3-form $dJ \wedge K_{\mathcal{W}}$ defined in (2.3.19) and prove that the reduced bracket $\{\cdot,\cdot\}_{red}$ is not Poisson based on formula (2.3.21).

4.2.3 The vertical complement of the constraints and the reduced bracket

We begin by showing that the distribution W generated by the vector fields Z_1 and Z_2 defined in (4.2.10) is G-invariant and is a vertical complement of the constraints: $W \subset V$ and $TQ = D \oplus W$ (see (2.3.6) and Rmk. 4.2.1).

Proposition 4.2.8. *For the mechanical system we are considering, the distribution $W = span\{Z_1, Z_2\}$ where Z_1 and Z_2 are given in (4.2.10) is a G-invariant vertical complement of the constraints induced by an Ad-invariant subbundle $\mathfrak{g}_W \to Q$ of $\mathfrak{g} \times Q \to Q$.*

Proof. From (4.2.9) we see that $TQ = D \oplus W$. To see that $W \subset V$, we observe that $Z_1 = (\xi_1)_Q$ and $Z_2 = (\xi_2)_Q$, where ξ_1 and ξ_2 are sections of the bundle $\mathfrak{g} \times Q \to Q$ given at the point $q = (x, y, g)$ by
$$\xi_1|_q = \langle \mathbf{A}g, (0, \mathbf{e})\rangle, \quad \xi_2|_q = \langle \mathbf{B}g, (0, \mathbf{e})\rangle, \tag{4.2.47}$$
where $(0, \mathbf{e}) = ((0, \mathbf{e}_1), (0, \mathbf{e}_2), (0, \mathbf{e}_3))$ with $(0, \mathbf{e}_i)$ given in (4.2.35) and $\mathbf{A}$, $\mathbf{B}$ are the vectors
$$\mathbf{A} = -\frac{1}{Rn_3}\Big(n_1n_2, \;\; n_2^2 - 1\,, \;\; n_2n_3\Big), \quad \mathbf{B} = \frac{1}{Rn_3}\Big(n_1^2 - 1\,, \;\; n_1n_2\,, \;\; n_1n_3\Big).$$
Next we verify that W is G-invariant. First observe that, from (4.2.10), the vector fields Z_1 and Z_2 are $SO(3)$-invariant(the right action of $SO(3)$ does not involves the (x, y)-coordinates and the fields X_i, $i = 1, 2, 3$ are right invariant). On the other hand, in order to show that Z_1 and Z_2 are S^1-invariant, consider a rotation R_φ of angle φ with respect to the z-axis and observe that
$$\begin{aligned}((R_\varphi)_*(Z_1))\,|_{R_\varphi q} &= \cos\varphi Z_1|_{R_\varphi q} + \sin\varphi Z_2|_{R_\varphi q} \in W_{R_\varphi q},\\ ((R_\varphi)_*(Z_2))\,|_{R_\varphi q} &= -\sin\varphi Z_1|_{R_\varphi q} + \cos\varphi Z_2|_{R_\varphi q} \in W_{R_\varphi q}.\end{aligned}$$
Therefore the subbundle W of TQ is G-invariant. Consider now the vector subbundle $\mathfrak{g}_W \to Q$ of the trivial bundle $\mathfrak{g} \times Q \to Q$ generated by the sections ξ_1 and ξ_2 defined in (4.2.47). It is straightforward to see that W is induced by $\mathfrak{g}_W$ as in (2.3.10) and that $\mathfrak{g}_W \to Q$ is Ad-invariant. $\square$

Remark 4.2.9. The complement W does not satisfy the vertical symmetry condition of [5], in fact W is not even integrable. Moreover, there is no complement satisfying that condition since the Lie group $SO(3)$ is simple and therefore does not have any normal subgroup which could act on the system as a symmetry group, see [6, Remark 2.4]. $\diamond$

The bundle $\mathfrak{g}_S \to Q$ and the distributions $\mathcal{S}$ and $\mathcal{W}$ on the manifold $\mathcal{M}$

Following (2.3.5) we compute $S = span\{S_1 := yY_x - xY_y, S_2 := X_n\}$, and we observe that it also has nonconstant rank, it is equal to 1 when $(x, y) = (0, 0)$ and is equal to 2 elsewhere. The subbundle $\mathfrak{g}_S \to Q$ of $\mathfrak{g} \times Q \to Q$ defined in (2.3.8) is generated by the sections

$$\eta_1|_q = (1, \mathbf{0}) + y\xi_1|_q - x\xi_2|_q - \langle \boldsymbol{\gamma}, (0, \mathbf{e})\rangle|_q, \qquad \eta_2|_q = \langle \vec{n}g, (0, \mathbf{e})\rangle|_q, \tag{4.2.48}$$

where we consider the normal $\vec{n}$ as a row vector, and the section ξ_1 and ξ_2 are given in (4.2.47). Then we have that $(\eta_1)_Q = S_1$ and $(\eta_2)_Q = S_2$. As we observed in Section 2.3.1 (see also [6]), the bundle $\mathfrak{g}_S \to Q$ has constant rank (here it is equal to 2) while the distribution S varies its rank.

On the other hand, using analogous notation as in (4.2.34), the cotangent lift of the G-action to $\mathcal{M}$ is computed to be

$$(\varphi, h) \cdot (x, y, g, \tilde{p_x}, \tilde{p_y}, \tilde{M_n}) = (R_\varphi(x, y), \hat{R}_\varphi\, g\, h, R_\varphi(\tilde{p_x}, \tilde{p_y}), \tilde{M_n}). \tag{4.2.49}$$

The infinitesimal generators of the action on $\mathcal{M}$ in the basis (4.2.35) of the Lie algebra $\mathfrak{g}$ are

$$(1, \mathbf{0})_{\mathcal{M}} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + \tilde{X}_3 + \tilde{p_y}\frac{\partial}{\partial \tilde{p_x}} - \tilde{p_x}\frac{\partial}{\partial \tilde{p_y}}, \tag{4.2.50}$$

$$(0, \mathbf{e_i})_{\mathcal{M}} = \alpha_i \tilde{X}_1 + \beta_i \tilde{X}_2 + \gamma_i \tilde{X}_3, \quad i = 1, 2, 3, \tag{4.2.51}$$

with $\tilde{X}_i = \tau_{\mathcal{M}}^* X_i$ and $\tau_{\mathcal{M}} : \mathcal{M} \to Q$ the canonical projection. Therefore, using the basis (4.2.21) of $T\mathcal{M}$, the vertical distribution $\mathcal{V}$ on $\mathcal{M}$ is given by

$$\mathcal{V} = span\left\{-y\tilde{Y}_x + x\tilde{Y}_y + \tilde{p_y}\frac{\partial}{\partial \tilde{p_x}} - \tilde{p_x}\frac{\partial}{\partial \tilde{p_y}},\ \tilde{X}_n,\ \tilde{Z}_1,\ \tilde{Z}_2\right\}. \tag{4.2.52}$$

The basis of sections $\{\eta_1, \eta_2\}$ in (4.2.48) generating the bundle $\mathfrak{g}_S \to Q$ induces a set of generators $(\eta_1)_{\mathcal{M}}$ and $(\eta_2)_{\mathcal{M}}$ of the distribution $\mathcal{S}$ on $\mathcal{M}$ defined in (2.3.14),

$$\mathcal{S} = span\{(\eta_1)_{\mathcal{M}}, (\eta_2)_{\mathcal{M}}\},$$

where

$$(\eta_1)_{\mathcal{M}} = y\tilde{Y}_x - x\tilde{Y}_y + \tilde{p_y}\frac{\partial}{\partial \tilde{p_x}} - \tilde{p_x}\frac{\partial}{\partial \tilde{p_y}} \quad \text{and} \quad (\eta_2)_{\mathcal{M}} = \tilde{X}_n. \tag{4.2.53}$$

Now, from (2.3.12), the G-invariant vertical complement $\mathcal{W}$ is induced by the basis of sections of the bundle $\mathfrak{g}_W \to Q$ given by $\{\xi_1, \xi_2\}$ where ξ_1 and ξ_2 are defined in (4.2.47). That is

$$\mathcal{W} = span\{(\xi_1)_{\mathcal{M}}, (\xi_2)_{\mathcal{M}}\},$$

and observe that, in this case, $(\xi_1)_{\mathcal{M}} = \tilde{Z}_1$ and $(\xi_2)_{\mathcal{M}} = \tilde{Z}_2$. From Proposition 4.2.8 the bundle $\mathfrak{g}_W \to Q$ is Ad-invariant and then $\mathcal{W}$ is also G-invariant.

The 3-form $dJ \wedge K_{\mathcal{W}}$

Next, using Lemma 2.3.7 we compute the $\mathcal{W}$-curvature $K_{\mathcal{W}}$ in the adapted basis (4.2.20) of $T^*\mathcal{M}$.

Lemma 4.2.10. *For the vertical complement $\mathcal{W} = span\{\tilde{Z}_1, \tilde{Z}_2\}$, the $\mathcal{W}$-curvature $K_{\mathcal{W}}$ written in the basis of constant sections of $\mathfrak{g}$, $\{(1, \mathbf{0}), (0, \mathbf{e_i})\}$, $i = 1, 2, 3$, is given by,*

$$K_{\mathcal{W}} = \tilde{dx} \wedge \tilde{\beta}_n \otimes \langle -D^x_{xn}\mathbf{A}g - D^y_{xn}\mathbf{B}g, (0, \mathbf{e})\rangle + \tilde{dy} \wedge \tilde{\beta}_n \otimes \langle -D^x_{yn}\mathbf{A}g - D^y_{yn}\mathbf{B}g, (0, \mathbf{e})\rangle,$$

where $(0, \mathbf{e}) = ((0, \mathbf{e_1}), (0, \mathbf{e_2}), (0, \mathbf{e_3}))$.

Proof. By Lemma 2.3.7 (using $a = 1, 2$ to match with our notations) the $\mathcal{W}$-curvature $K_{\mathcal{W}}$ in the basis $\{\xi_1, \xi_2\}$ of sections of $\mathfrak{g}_W \to Q$, given in (4.2.47) is written,

$$K_{\mathcal{W}}|_{\mathcal{C}} = d\tilde{\epsilon}^1|_{\mathcal{C}} \otimes \xi_1 + d\tilde{\epsilon}^2|_{\mathcal{C}} \otimes \xi_2 \quad \text{and} \quad K_{\mathcal{W}}|_{\mathcal{W}} = 0,$$

which is equivalent to

$$K_{\mathcal{W}}|_{\mathcal{C}} = d\tilde{\epsilon}^1|_{\mathcal{C}} \otimes \langle \mathbf{A}g, (0, \mathbf{e})\rangle + d\tilde{\epsilon}^2|_{\mathcal{C}} \otimes \langle \mathbf{B}g, (0, \mathbf{e})\rangle, \quad \text{and} \quad K_{\mathcal{W}}|_{\mathcal{W}} = 0.$$

Finally from (4.2.28) we obtain,

$$\begin{aligned} K_{\mathcal{W}} =& (-D^x_{xn}\tilde{dx} \wedge \tilde{\beta}_n - D^x_{yn}\tilde{dy} \wedge \tilde{\beta}_n) \otimes \langle \mathbf{A}g, (0, \mathbf{e})\rangle \\ &+ (-D^y_{xn}\tilde{dx} \wedge \tilde{\beta}_n - D^y_{yn}\tilde{dy} \wedge \tilde{\beta}_n) \otimes \langle \mathbf{B}g, (0, \mathbf{e})\rangle \\ =& \tilde{dx} \wedge \tilde{\beta}_n \otimes \langle -D^x_{xn}\mathbf{A}g - D^y_{xn}\mathbf{B}g, (0, \mathbf{e})\rangle + \tilde{dy} \wedge \tilde{\beta}_n \otimes \langle -D^x_{yn}\mathbf{A}g - D^y_{yn}\mathbf{B}g, (0, \mathbf{e})\rangle. \end{aligned}$$

□

The next step is to compute the 3-form $dJ \wedge K_{\mathcal{W}}$ using Proposition 2.3.8. As we saw in Section 2.3.1, this 3-form is the key tool in order to prove whether the reduced bracket $\{\cdot, \cdot\}_{red}$ is Poisson or not (see (2.3.20) and (2.3.21)).

Following the notation of Prop. 2.3.8, the basis of sections of $\mathfrak{g} \times Q$ adapted to the splitting $\mathfrak{g} \times Q = \mathfrak{g}_S \oplus \mathfrak{g}_W$ is given by

$$\mathfrak{B}_{\mathfrak{g}\times Q} = \{\eta_1, \eta_2, \xi_3, \xi_4\}, \tag{4.2.54}$$

where η_1 and η_2 are sections of $\mathfrak{g}_S \to Q$ given in (4.2.48) and ξ_3 and ξ_4 correspond to the sections ξ_1 and ξ_2 of $\mathfrak{g}_W \to Q$ given in (4.2.47), respectively. The components of the moment map J in the basis (4.2.54) are denoted

$$J_1 = \mathbf{i}_{(\eta_1)_{\mathcal{M}}}\Theta_{\mathcal{M}}, \quad J_2 = \mathbf{i}_{(\eta_2)_{\mathcal{M}}}\Theta_{\mathcal{M}},$$

and

$$J_3 = \mathbf{i}_{(\xi_3)_{\mathcal{M}}}\Theta_{\mathcal{M}}, \quad J_4 = \mathbf{i}_{(\xi_4)_{\mathcal{M}}}\Theta_{\mathcal{M}}. \tag{4.2.55}$$

Proposition 4.2.11. *For the choice of the vertical complement $\mathcal{W} = span\{\tilde{Z}_1, \tilde{Z}_2\}$ to the constraints $\mathcal{C}$, the G-invariant 3-form $dJ \wedge K_{\mathcal{W}}$ verifies*

$$dJ \wedge K_{\mathcal{W}}|_{\mathcal{C}} = \left(-\frac{I}{E} d\tilde{p_x} \wedge d\tilde{\epsilon}^1 - \frac{I}{E} d\tilde{p_y} \wedge d\tilde{\epsilon}^2 + \Psi_K \wedge (h_3^K d\tilde{\epsilon}^1 + h_4^K d\tilde{\epsilon}^2) \right) |_{\mathcal{C}}, \quad (4.2.56)$$

where Ψ_K is a semi-basic 1-form (with respect to the bundle $\tau_{\mathcal{M}} : \mathcal{M} \to Q$) given by $\Psi_K = J_1 d\bar{h}_K^1 + J_2 d\bar{h}_K^2 - \frac{I}{E}\tilde{p}_x d\bar{h}_K^3 - \frac{I}{E}\tilde{p}_y d\bar{h}_K^4$, with $\bar{h}_K^L$ the entries of the matrix $\bar{\mathbf{h}}$, the inverse transpose of the 4×4 matrix $\mathbf{h}$ given by

$$\begin{aligned} \eta_i &= h_i^K \chi_K, \quad i = 1, 2, \\ \xi_a &= h_a^K \chi_K, \quad a = 3, 4, \end{aligned}$$

where we used the basis of $\mathfrak{g}$ given by $\{\chi_1 = (1, \mathbf{0}), \chi_{i+1} = (0, \mathbf{e}_i)\}$, $i = 1, 2, 3$, and $h_L^K \in C^\infty(Q)^G$; $K, L = 1, \cdots, 4$.

Proof. Consider the basis of sections $\{\xi_a\}$, $a = 3, 4$, of $\mathfrak{g}_W \to Q$ using the convention of (4.2.54) and recall that $\tilde{Z}_1 = (\xi_3)_{\mathcal{M}}$ and $\tilde{Z}_2 = (\xi_4)_{\mathcal{M}}$. Then, the components J_a, $a = 3, 4$, of the moment map defined in (4.2.55) are written

$$J_3 = \mathbf{i}_{\tilde{Z}_1} \Theta_{\mathcal{M}} = -\frac{I}{E}\tilde{p_x}, \quad J_4 = \mathbf{i}_{\tilde{Z}_2} \Theta_{\mathcal{M}} = -\frac{I}{E}\tilde{p_y},$$

where we used the expression of $\Theta_{\mathcal{M}}$ given in (4.2.25). Consequently from Proposition 2.3.8, using that there $\tilde{\epsilon}^a$, $a = 3, 4$, corresponds to (the pull-back of) the constraint 1-forms $\tilde{\epsilon}^1$ and $\tilde{\epsilon}^2$, we have

$$\begin{aligned} dJ \wedge K_{\mathcal{W}}|_{\mathcal{C}} &= \left(dJ_3 \wedge d\tilde{\epsilon}^1 + dJ_4 \wedge d\tilde{\epsilon}^2 + h_3^K J_L d\bar{h}_K^L \wedge d\tilde{\epsilon}^1 + h_4^K J_L d\bar{h}_K^L \wedge d\tilde{\epsilon}^2 \right) |_{\mathcal{C}} \\ &= \left(-\frac{I}{E} d\tilde{p_x} \wedge d\tilde{\epsilon}^1 - \frac{I}{E} d\tilde{p_y} \wedge d\tilde{\epsilon}^2 + (J_L d\bar{h}_K^L) \wedge (h_3^K d\tilde{\epsilon}^1 + h_4^K d\tilde{\epsilon}^2) \right) |_{\mathcal{C}}. \end{aligned}$$

Finally we denote by Ψ_K the expression $J_L d\bar{h}_K^L = J_1 d\bar{h}_K^1 + J_2 d\bar{h}_K^2 + J_3 d\bar{h}_K^3 + J_4 d\bar{h}_K^4$ which gives the formula in the statement of the Proposition. The fact that $dJ \wedge K_{\mathcal{W}}$ is G-invariant is a consequence of the G-invariance of the complement W (see Prop. 4.2.8 and [5]). □

In other words, using (4.2.28), the 3-form $dJ \wedge K_{\mathcal{W}}$ can be written as,

$$dJ \wedge K_{\mathcal{W}}|_{\mathcal{C}} = \Psi_1 + \Psi_2|_{\mathcal{C}}, \quad (4.2.57)$$

where

$$\begin{aligned} \Psi_1 =& \frac{I}{E} \left(D_{xn}^x d\tilde{p_x} \wedge \tilde{dx} \wedge \tilde{\beta}_n + D_{xn}^y d\tilde{p_y} \wedge \tilde{dx} \wedge \tilde{\beta}_n \right) \\ &+ \frac{I}{E} \left(D_{yn}^x d\tilde{p_x} \wedge \tilde{dy} \wedge \tilde{\beta}_n + D_{yn}^y d\tilde{p_y} \wedge \tilde{dy} \wedge \tilde{\beta}_n \right), \end{aligned} \quad (4.2.58)$$

and Ψ_2 is a semi-basic 3-form (with respect to $\mathcal{M} \to Q$) that verifies

$$\Psi_2|_{\mathcal{C}} = \Psi_K \wedge (h_3^K d\tilde{\epsilon}^1 + h_4^K d\tilde{\epsilon}^2)|_{\mathcal{C}}. \quad (4.2.59)$$

Remark 4.2.12. We give a second proof of the last proposition. Here we use the explicit formula of the $\mathcal{W}$-curvature $K_{\mathcal{W}}$ in Lemma 4.2.10. Using the canonical Liouville 1-form restricted to $\mathcal{M}$, $\Theta_{\mathcal{M}} = \tilde{p_x}dx + \tilde{p_y}dy + \tilde{M_n}\beta_n - \frac{I}{E}\tilde{p_x}\epsilon^1 - \frac{I}{E}\tilde{p_y}\epsilon^2$, we can compute the moment map $J : \mathcal{M} \to \mathfrak{g}^*$, by $\langle J(m), \xi \rangle = \mathbf{i}_{\xi_{\mathcal{M}}}\Theta_{\mathcal{M}}$ in our chosen basis of $\mathfrak{g}$. We obtain for $i = 1, 2, 3$,

$$J = (J_0, J_i),$$

where $J_0 = \mathbf{i}_{(1,\mathbf{0})_{\mathcal{M}}}\Theta_{\mathcal{M}}$ will not appear in the computations, and $J_i = \mathbf{i}_{(0,\mathbf{e_i})_{\mathcal{M}}}\Theta_{\mathcal{M}}$. We have:

$$\begin{aligned} J_i =& \tilde{p_x}\mathbf{i}_{(0,\mathbf{e_i})_{\mathcal{M}}}dx + \tilde{p_y}\mathbf{i}_{(0,\mathbf{e_i})_{\mathcal{M}}}dy + \tilde{M_n}\mathbf{i}_{(\mathbf{0},\mathbf{e_i})_{\mathcal{M}}}\beta_n \\ & - \frac{I}{E}\tilde{p_x}\mathbf{i}_{(\mathbf{0},\mathbf{e_i})_{\mathcal{M}}}\epsilon^1 - \frac{I}{E}\tilde{p_y}\mathbf{i}_{(\mathbf{0},\mathbf{e_i})_{\mathcal{M}}}\epsilon^2. \end{aligned} \tag{4.2.60}$$

Denoting $K^i = h^i_1 d\epsilon^1 + h^i_2 d\epsilon^2$ the coordinates of the $\mathcal{W}$-curvature in our basis of $\mathfrak{g}$, the 3-form $dJ \wedge K_{\mathcal{W}}$ can be written,

$$\begin{aligned} dJ_i \wedge K^i|_{\mathcal{C}} =& \frac{\partial J_i}{\partial \tilde{p_x}}d\tilde{p_x} \wedge K^i + \frac{\partial J_i}{\partial \tilde{p_y}}d\tilde{p_y} \wedge K^i + \frac{\partial J_i}{\partial \tilde{M_n}}d\tilde{M_n} \wedge K^i \\ & + Y_x(J_i)dx \wedge K^i + Y_y(J_i)dy \wedge K^i + X_n(J_i)\beta_n \wedge K^i|_{\mathcal{C}}, \end{aligned} \tag{4.2.61}$$

where in the last equality we have used that $\mathcal{C}$ annihilates ϵ^1 and ϵ^2. By the form of K^i we observe that the last three terms of (4.2.61) are semi-basic with respect to $\mathcal{M} \to Q$, hence they are of the form $\Psi \, dx \wedge dy \wedge \beta_n$ for some function Ψ as in the statement of the Proposition. The function Ψ is G-invariant because the 3-form $dJ \wedge K_{\mathcal{W}}$ is G-invariant. The other terms can be computed using (4.2.60),

$$\begin{aligned} \frac{\partial J_i}{\partial \tilde{p_x}} &= \frac{I}{E}R(\gamma_i n_2 - n_3\beta_i), \\ \frac{\partial J_i}{\partial \tilde{p_y}} &= \frac{I}{E}R(\alpha_i n_3 - n_1\gamma_i), \\ \frac{\partial J_i}{\partial \tilde{M_n}} &= \alpha_i n_1 + \beta_i n_2 + \gamma_i n_3. \end{aligned} \tag{4.2.62}$$

We use the explicit formula of Lemma 4.2.10 to perform the scalar products in the first three terms of the right hand side of 4.2.61. The computations are straightforward and can the shorten using matrix notation in formulas 4.2.62. By instance:

$$\begin{aligned} \frac{\partial J_i}{\partial \tilde{M_n}}d\tilde{M_n} \wedge K^i =& (-D^x_{xn}\mathbf{A}g - D^y_{xn}\mathbf{B}g) \cdot (g^T\vec{n})d\tilde{M_n} \wedge dx \wedge \beta_n \\ & + \left(-D^x_{yn}\mathbf{A}g - D^y_{yn}\mathbf{B}g\right) \cdot (g^T\vec{n})d\tilde{M_n} \wedge dy \wedge \beta_n = 0, \end{aligned}$$

since $\mathbf{B} \cdot \vec{n} = \mathbf{A} \cdot \vec{n} = 0$. In the last formula we considered $\mathbf{A}, \mathbf{B}$ as row vectors and the unit normal $\vec{n}$ as a column vector. Analogous computations gives:

$$\begin{aligned} \frac{\partial J_i}{\partial \tilde{p_x}}d\tilde{p_x} \wedge K^i &= \frac{I}{E}D^x_{xn}d\tilde{p_x} \wedge dx \wedge \beta_n + \frac{I}{E}D^x_{yn}d\tilde{p_x} \wedge dy \wedge \beta_n, \\ \frac{\partial J_i}{\partial \tilde{p_y}}d\tilde{p_y} \wedge K^i &= \frac{I}{E}D^y_{xn}d\tilde{p_y} \wedge dx \wedge \beta_n + \frac{I}{E}D^y_{yn}d\tilde{p_y} \wedge dy \wedge \beta_n. \end{aligned}$$

Replacing the latter formulas in (4.2.61) proves the proposition. $\diamond$

The reduced bracket $\{\cdot,\cdot\}_{red}$ is not Poisson

Finally we arrive to the conclusion of this section:

Theorem 4.2.13. *The reduced bracket $\{\cdot,\cdot\}_{red}$ on $\mathcal{M}/G$ induced by the nonholonomic bracket $\{\cdot,\cdot\}_{nh}$ on $\mathcal{M}$ and the orbit projection $\mathcal{M}\to\mathcal{M}/G$ is not Poisson.*

Proof. Note that

$$
\begin{aligned}
\pi^{\#}_{nh}(d\rho^* p_2) &= \tilde{p_x}\frac{\partial}{\partial\tilde{p_x}} + \tilde{p_y}\frac{\partial}{\partial\tilde{p_y}} \\
&\quad + x\left(-\tilde{Y_x} + \tilde{M}_n D^n_{xy}\frac{\partial}{\partial\tilde{p_y}} + \frac{I}{E}(\tilde{p_x}D^x_{xn} + \tilde{p_y}D^y_{xn})\frac{\partial}{\partial\tilde{M_n}}\right) \\
&\quad + y\left(-\tilde{Y_y} - \tilde{M}_n D^n_{xy}\frac{\partial}{\partial\tilde{p_x}} + \frac{I}{E}(\tilde{p_x}D^x_{yn} + \tilde{p_y}D^y_{yn})\frac{\partial}{\partial\tilde{M_n}}\right), \\
\pi^{\#}_{nh}(d\rho^* p_3) &= \tilde{p_y}\frac{\partial}{\partial\tilde{p_x}} - \tilde{p_x}\frac{\partial}{\partial\tilde{p_y}} \\
&\quad + x\left(-\tilde{Y_y} - \tilde{M}_n D^n_{xy}\frac{\partial}{\partial\tilde{p_x}} + \frac{I}{E}(\tilde{p_x}D^x_{yn} + \tilde{p_y}D^y_{yn})\frac{\partial}{\partial\tilde{M_n}}\right) \\
&\quad - y\left(-\tilde{Y_x} + \tilde{M}_n D^n_{xy}\frac{\partial}{\partial\tilde{p_y}} + \frac{I}{E}(\tilde{p_x}D^x_{xn} + \tilde{p_y}D^y_{xn})\frac{\partial}{\partial\tilde{M_n}}\right), \\
\pi^{\#}_{nh}(d\rho^* p_4) &= -\tilde{X_n} - \frac{I}{E}(\tilde{p_x}D^x_{yn} + \tilde{p_y}D^y_{yn})\frac{\partial}{\partial\tilde{p_y}} - \frac{I}{E}(\tilde{p_x}D^x_{xn} + \tilde{p_y}D^y_{xn})\frac{\partial}{\partial\tilde{p_x}}.
\end{aligned}
$$

We compute $dJ\wedge K_{\mathcal{W}}(\pi^{\#}_{nh}(d\rho^*p_2), \pi^{\#}_{nh}(d\rho^*p_3), \pi^{\#}_{nh}(d\rho^*p_4))$ evaluating each term of the decomposition (4.2.57). Observe that the semi-basic 3-form Ψ_2 appearing in the expression of $dJ\wedge K_{\mathcal{W}}$, when restricted to $\mathcal{C}$ has the form $F(x,y)\cdot dx\wedge dy\wedge\tilde{\beta}_n$, where $F\in C^\infty(\mathcal{M})$. Then

$$
\begin{aligned}
\Psi_1((d\rho^*p_2), \pi^{\#}_{nh}(d\rho^*p_3), \pi^{\#}_{nh}(d\rho^*p_4)) &= \frac{I}{E}y(\tilde{p_x}D^x_{xn} + \tilde{p_y}D^y_{xn}) - \frac{I}{E}x(\tilde{p_x}D^x_{yn} + \tilde{p_y}D^y_{yn}) \\
&\quad + \frac{I}{E}x(\tilde{p_y}D^x_{xn} - \tilde{p_x}D^y_{xn}) + \frac{I}{E}y(\tilde{p_y}D^x_{xn} - \tilde{p_x}D^y_{xn}),
\end{aligned}
$$

and

$$
\Psi_2|_{\mathcal{C}}((d\rho^*p_2), \pi^{\#}_{nh}(d\rho^*p_3), \pi^{\#}_{nh}(d\rho^*p_4))) = F(x,y)\cdot(x^2+y^2).
$$

By the Jacobiator formula (2.3.21) of the reduced bracket $\{\cdot,\cdot\}_{red}$ we conclude that $\{\cdot,\cdot\}_{red}$ is not Poisson.

□

Remark 4.2.14. The reduced bracket $\{\cdot,\cdot\}_{red}$ does not admit leaves. In fact, the existence of 2-dimensional leaves should imply, by dimensionaly, that the bracket $\{\cdot,\cdot\}_{red}$ is Poisson, in contradiction with Thm. 4.2.13. ◇

For completeness we compute the reduced bracket $\{\cdot,\cdot\}_{red}$. It is given in our reduced variables p_i in the following table:

$\{\cdot,\cdot\}_{red}$	p_0	p_1	p_2	p_3	p_4
p_0		$4p_2$	$2p_0 + 2p_4p_3D^n_{xy}$	$-2p_2p_4D^n_{xy}$	$-2\tilde{p_x}C_{xn} - 2\tilde{p_y}C_{yn}$
p_1	$-4p_2$		$-2p_1$	0	0
p_2	$-2p_0 - 2p_4p_3D^n_{xy}$	$2p_1$		$2p_3 - p_4p_0D^n_{xy}$	$-xC_{xn} - yC_{yn}$
p_3	$2p_2p_4D^n_{xy}$	0	$-2p_3 + p_4p_0D^n_{xy}$		$yC_{xn} - xC_{yn}$
p_4	$2\tilde{p_x}C_{xn} + 2\tilde{p_y}C_{yn}$	0	$xC_{xn} + yC_{yn}$	$yC_{xn} - xC_{yn}$	

We used the shorthand notations

$$C_{xn} = \frac{I}{E}(\tilde{p_x}D^x_{xn} + \tilde{p_y}D^y_{xn}), \qquad C_{yn} = \frac{I}{E}(\tilde{p_x}D^x_{yn} + \tilde{p_y}D^y_{yn}), \tag{4.2.63}$$

and verified that the entries of the table are G-invariant. In fact, the three expressions

$$\tilde{p_x}C_{xn} + \tilde{p_y}C_{yn}, \quad xC_{xn} + yC_{yn}, \quad yC_{xn} - xC_{yn}$$

are rewritten in function of the G-invariant variables p_i, $i = 0, \cdots, 4$, after a long computation using the complete expressions (4.2.24).

4.3 Horizontal gauge momenta and hamiltonization

Continuing with the example in Section 4.2, now we compute a dynamical gauge transformation of the bivector π_{nh} given in (4.2.23) such that the reduction of the gauge related bivector π_B gives a Poisson bracket $\{\cdot,\cdot\}^B_{red}$ on $\mathcal{M}/G$.

In this section, we find a 2-form B based in the techniques of Proposition 3.3.2. That is, first we study the existence of two horizontal gauge momenta and then we compute a 2-form B so that (3.3.22) is satisfied. Since these horizontal gauge momenta are G-invariant they induce two Casimirs of the reduced bracket $\{\cdot,\cdot\}^B_{red}$ and then we show that this reduced bracket is Poisson with symplectic leaves given by the common level surfaces of the two Casimirs.

4.3.1 Horizontal gauge momenta associated to $\mathfrak{g}_S \to Q$

Following the ideas of Section 3.3, we look for G-invariant horizontal gauge momenta of the system. Recall from Section 4.2 that the bundle $\mathfrak{g}_S \to Q$ is generated by two sections η_1 and η_2 defined in (4.2.48) and consequently an arbitrary section of $\mathfrak{g}_S \to Q$ is written as

$$\zeta = f_1\eta_1 + f_2\eta_2, \tag{4.3.64}$$

for functions $f_1, f_2 \in C^\infty(Q)$. The nonholonomic moment map associated to the section ζ is

$$J_\zeta = \langle J^{nh}, \zeta\rangle = f_1J_1 + f_2J_2,$$

where, following (3.3.20), the functions J_1 and J_2 are the components of the nonholonomic moment map in the basis $\{\eta_1, \eta_2\}$ of $\mathfrak{g}_S \to Q$, that is,

$$J_1 = \mathbf{i}_{(\eta_1)_\mathcal{M}}\Theta_\mathcal{M} = -y\tilde{p_x} + x\tilde{p_y} = p_3,$$
$$J_2 = \mathbf{i}_{(\eta_2)_\mathcal{M}}\Theta_\mathcal{M} = \tilde{M_n} = p_4,$$

(recall that $S_1 = (\eta_1)_\mathcal{M}$ and $S_2 = (\eta_2)_\mathcal{M}$ were computed in (4.2.53).

Since, in this case, the functions J_1 and J_2 are G-invariant, in order to have a G-invariant horizontal gauge momentum of the form $J_\zeta = f_1 p_3 + f_2 p_4$ we impose that f_1 and f_2 are G-invariant as well. In this case, the function J_ζ defines a reduced function $\bar{J}_\zeta$ on $\mathcal{M}/G$ such that $\rho^*\bar{J}_\zeta = J_\zeta$. Therefore J_ζ is a first integral of X_{nh} if $\bar{J}_\zeta$ is a first integral of X_{red}, i.e. $X_{red}(\bar{J}_\zeta) = 0$. More precisely, the reduced function $\bar{J}_\zeta$ has the form $\bar{J}_\zeta = \bar{f}_1 p_3 + \bar{f}_2 p_4$, where $\bar{f}_1 = \bar{f}_1(p_1)$ and $\bar{f}_2 = \bar{f}_2(p_1)$ are functions on Q/G (seen as a differential space) satisfying

$$X_{red}(\bar{J}) = p_3\bar{f_1}' dp_1(X_{red}) + p_4\bar{f_2}' dp_1(X_{red}) + \bar{f}_1 dp_3(X_{red}) + \bar{f}_2 dp_4(X_{red}) = 0. \quad (4.3.65)$$

Using the reduced dynamics X_{red} given in (4.2.45), the functions $\bar{f}_1$ and $\bar{f}_2$ satisfy the following ODE:

$$\bar{f_1}' + \bar{f}_2\frac{RI}{2E}\bar{F}_4(p_1) = 0, \qquad \bar{f_2}' + \bar{f}_1\frac{1}{2R}\bar{F}_3(p_1) = 0, \quad (4.3.66)$$

where $\bar{F}_3, \bar{F}_4 \in C^\infty(Q/G)$ are given in (4.2.46). The solutions of the linear differential system (4.3.66) exist and are unique in the domain of continuity of the functions $\bar{F}_3$ and $\bar{F}_4$. Moreover starting with two independent initial conditions, the solutions remain independent as long as they exist.

Using the following notation for the two independent solutions,

$$p_1 \mapsto (\bar{f}_1(p_1), \bar{f}_2(p_1)), \qquad p_1 \mapsto (\bar{g}_1(p_1), \bar{g}_2(p_1)),$$

which verify $f_1(p_1)g_2(p_1) - g_1(p_1)f_2(p_1) \neq 0$ on an interval containing the origin, we have two first integrals in $\mathcal{M}/G$ of the form

$$\begin{aligned} \bar{J^{(1)}}(p_1, p_3, p_4) &= \bar{f}_1(p_1)p_3 + \bar{f}_2(p_1)p_4, \\ \bar{J^{(2)}}(p_1, p_3, p_4) &= \bar{g}_1(p_1)p_3 + \bar{g}_2(p_1)p_4. \end{aligned} \quad (4.3.67)$$

It is straightforward to see that the first integrals (4.3.67) are functionally independent in $\mathcal{M}_{reg}/G$ (as it was observed in [40]).

Hence the corresponding G-invariant horizontal gauge momenta on $\mathcal{M}$ are:

$$\begin{aligned} J^{(1)}(x, y, g, \tilde{p_x}, \tilde{p_y}, \tilde{M_n}) &= f_1 \cdot (x\tilde{p_y} - y\tilde{p_x}) + f_2 \cdot \tilde{M_n}, \\ J^{(2)}(x, y, g, \tilde{p_x}, \tilde{p_y}, \tilde{M_n}) &= g_1 \cdot (x\tilde{p_y} - y\tilde{p_x}) + g_2 \cdot \tilde{M_n}, \end{aligned} \quad (4.3.68)$$

where $f_i = f_i(x^2 + y^2) = \rho^*\bar{f}_i(p_1)$, and $g_i = g_i(x^2 + y^2) = \rho^*\bar{g}_i(p_1)$, $i = 1, 2$, and thus we have recovered in a constructive way the first integrals of this mechanical system which where known since the work of Routh [79], see also [40, 52]. In fact by construction, $J^{(1)}$ and $J^{(2)}$ are not only first integrals but are horizontal gauge momenta. We summarize the results of this section by the following

Proposition 4.3.1. *The functions $J^{(1)}$ and $J^{(2)}$ are G-invariant horizontal gauge momenta associated to the horizontal gauge symmetries*

$$\zeta_1 = f_1\eta_1 + f_2\eta_2 \quad \textit{and} \quad \zeta_2 = g_1\eta_1 + g_2\eta_2, \tag{4.3.69}$$

where η_1 and η_2 are given in (4.2.48).

In general these first integrals are not explicitly known except for some particular cases such as the circular paraboloid [24].

4.3.2 The dynamical gauge transformation and the reduced Poisson structure

In this section we use Proposition 3.3.2 and Lemma 3.3.1 to seek for a 2-form B having the property that it induces a dynamical gauge transformation of the bivector π_{nh} such that each of the G-invariant horizontal gauge momenta $J^{(1)}$ and $J^{(2)}$ in (4.3.68) define Casimirs $\bar{J^{(1)}}$ and $\bar{J^{(2)}}$ of the reduced bracket $\{\cdot,\cdot\}^B_{red}$. Studying the properties of the bracket $\{\cdot,\cdot\}^B_{red}$ we show that it is a Poisson bracket on the differential space $\mathcal{M}/G$.

Let us consider a 2-form B on $\mathcal{M}$

$$B = a\,\tilde{dx}\wedge\tilde{dy} + b\,\tilde{dy}\wedge\tilde{\beta}_n + c\,\tilde{\beta}_n\wedge\tilde{dx}, \tag{4.3.70}$$

for a, b, c arbitrary smooth functions on $\mathcal{M}$. Observe that B is semi-basic with respect to the bundle $\mathcal{M}\to Q$ and thus it induces a new bivector field π_B, see Remark 3.1.3 (i). On the other hand, following Remark 3.1.3 (ii), we observe that $\mathbf{i}_Z B \equiv 0$ for any $Z\in\Gamma(\mathcal{W})$.

Recall that $J^{(1)}$ and $J^{(2)}$ given in (4.3.68) are the horizontal gauge momenta with associated gauge symmetries ζ_1 and ζ_2 given in (4.3.69) where f_i, g_i are the pull-back to $\mathcal{M}$ of the functions $\bar{f}_i$, $\bar{g}_i$ which are solutions of (4.3.66). Using Proposition 3.3.2 we will find the corresponding functions a, b, c so that $\pi_B^\#(dJ^{(i)}) = -(\zeta_i)_\mathcal{M} \in \Gamma(\mathcal{V})$, $i = 1, 2$. The infinitesimal generators associated to the horizontal gauge symmetries ζ_1 and ζ_2 in (4.3.69) are given by

$$(\zeta_1)_\mathcal{M} = f_1(\eta_1)_\mathcal{M} + f_2(\eta_2)_\mathcal{M} \quad \text{and} \quad (\zeta_2)_\mathcal{M} = g_1(\eta_1)_\mathcal{M} + g_2(\eta_2)_\mathcal{M},$$

where the infinitesimal generators $(\eta_1)_\mathcal{M}$ and $(\eta_2)_\mathcal{M}$ where computed in (4.2.53).

Following Proposition 3.3.2 (i), each section ζ_1 and ζ_2 induces 1-forms Λ_1 and Λ_2, respectively, defined by

$$\begin{aligned}\Lambda_1 &= -f_1\mathbf{i}_{(\eta_1)_\mathcal{M}}\Omega_\mathcal{C} - f_2\mathbf{i}_{(\eta_2)_\mathcal{M}}\Omega_\mathcal{C} + J_1 df_1 + J_2 df_2 + f_1 dJ_1 + f_2 dJ_2,\\ \Lambda_2 &= -g_1\mathbf{i}_{(\eta_1)_\mathcal{M}}\Omega_\mathcal{C} - g_2\mathbf{i}_{(\eta_2)_\mathcal{M}}\Omega_\mathcal{C} + J_1 dg_1 + J_2 dg_2 + g_1 dJ_1 + g_2 dJ_2,\end{aligned} \tag{4.3.71}$$

and then imposing the condition $\mathbf{i}_{(\zeta_i)_\mathcal{M}}B = \Lambda_i$, $i = 1, 2$, we get

$$\begin{aligned}a &= \tilde{M}_n\left(D^n_{xy} + \bar{F}_3\frac{1}{R}\right),\\ b &= \frac{I}{E}\left(\tilde{p_x}(D^x_{yn} - Ry^2\bar{F}_4) + \tilde{p_y}(D^y_{yn} + Rxy\bar{F}_4)\right),\\ c &= -\frac{I}{E}\left(\tilde{p_x}(D^x_{xn} - Rxy\bar{F}_4) + \tilde{p_y}(D^y_{xn} + Rx^2\bar{F}_4)\right),\end{aligned} \tag{4.3.72}$$

where we used that the functions $\bar{f}_1$ and $\bar{f}_2$ associated to f_1 and f_2 satisfy the differential equations (4.3.66).

Therefore the 2-form B can be written as

$$B = \Phi(x,y)\left(\tilde{\omega_n}\,\tilde{dx}\wedge\tilde{dy} + \dot{x}\,\tilde{dy}\wedge\tilde{\beta}_n + \dot{y}\,\tilde{\beta}_n\wedge\tilde{dx}\right), \tag{4.3.73}$$

where the function Φ is given by

$$\Phi(x,y) = (1-\phi' 2Rn_3)\frac{I}{R^2 n_3}, \tag{4.3.74}$$

and is S^1-invariant because the surface is of revolution. Recall that $\dot{x}$, $\dot{y}$ and $\tilde{\omega_n}$ are related to the coordinates in $\mathcal{M}$ using the formulas in (4.2.32). Then we arrive to the following result:

Proposition 4.3.2. *The 2-form B given in (4.3.73) satisfies that*

(*i*) $\mathbf{i}_{(\zeta_1)_\mathcal{M}}B = \Lambda_1$ *and* $\mathbf{i}_{(\zeta_2)_\mathcal{M}}B = \Lambda_2$, *where* Λ_i *are* 1*-forms on* $\mathcal{M}$ *such that*

$$\pi^\#_{nh}(dJ^{(i)}) = -(\zeta_i)_\mathcal{M} + \pi^\#_{nh}(\Lambda_i) \quad for \quad i = 1,2.$$

(*ii*) $\mathbf{i}_{X_{nh}}B = 0$, *i.e. B defines a dynamical gauge transformation.*

Proof. It only remains to show (*ii*). This is verified by direct computation using the the expression (4.2.31) for X_{nh}. The computations are simplified using the fact that B is semi-basic with respect to $\mathcal{M}\to Q$. Indeed, $\mathbf{i}_{X_{nh}}B = \mathbf{i}_{\dot{x}\tilde{Y_x}+\dot{y}\tilde{Y_y}+\omega_n\tilde{X_n}}B = 0$, where we have used (4.3.73) and the relation $\tilde{M_n} = I\omega_n$ from (4.2.32). $\square$

Remark 4.3.3. We observe that our mechanical system is an example where the difference between the ranks of D and S is 1 (in the regular stratum). Using Proposition 3.5.5 we have that the semi-basic 2-form B is uniquely determined by the conditions (3.5.26) and that it induces a dynamical gauge transformation. $\diamond$

Using (4.2.29) and (4.3.72) the gauge transformed bivector is given by

$$\begin{aligned}\pi_B =& \tilde{Y_x}\wedge\frac{\partial}{\partial\tilde{p_x}} + \tilde{Y_y}\wedge\frac{\partial}{\partial\tilde{p_y}} + \tilde{X_n}\wedge\frac{\partial}{\partial\tilde{M_n}}\\ &-\tilde{M_n}F_3\frac{1}{R}\frac{\partial}{\partial\tilde{p_x}}\wedge\frac{\partial}{\partial\tilde{p_y}} - \frac{I}{E}R(x\tilde{p_y}-y\tilde{p_x})F_4 y\frac{\partial}{\partial\tilde{p_y}}\wedge\frac{\partial}{\partial\tilde{M_n}}\\ &+\frac{I}{E}R(x\tilde{p_y}-y\tilde{p_x})F_4 x\frac{\partial}{\partial\tilde{M_n}}\wedge\frac{\partial}{\partial\tilde{p_x}}.\end{aligned} \tag{4.3.75}$$

As a consequence of Proposition 4.3.2 (*ii*) the nonholonomic system is also described by the triple $(\mathcal{M},\pi_B,H_\mathcal{M})$. Moreover since B is G-invariant, we obtain a reduced almost Poisson structure $(\mathcal{M}/G,\{\cdot,\cdot\}^B_{red})$ on the differencial space $\mathcal{M}/G$ as in (3.1.4).

Recall that the G-invariant horizontal gauge momenta $J^{(1)}$ and $J^{(2)}$ on $\mathcal{M}$ induce the functions $\bar{J}^{(1)}$ and $\bar{J}^{(2)}$ on $\mathcal{M}/G$ given in (4.3.67). Then, from Prop. 4.3.2 (*i*) we have the following

Corollary 4.3.4. *The functions $\bar{J^{(1)}}$ and $\bar{J^{(2)}}$ on $\mathcal{M}/G$ given in (4.3.67) are Casimirs of the reduced bracket $\{\cdot,\cdot\}^B_{red}$ on $\mathcal{M}/G$ induced by the gauge related bivector field π_B.*

Proof. Since B is defined so that $\mathbf{i}_{(\zeta_i)_\mathcal{M}}B = \Lambda_i$ where Λ_i is given in (4.3.71), then by Proposition 3.3.2 (ii) we have that $\pi^\#_B(dJ^{(i)}) = -(\zeta_i)_\mathcal{M}$, $i = 1, 2$. By the G-invariance of $J^{(i)}$, Lemma 3.3.1 implies that $\bar{J^{(i)}}$ are Casimirs of the reduced bracket $\{\cdot,\cdot\}^B_{red}$. $\square$

Next we show that the bracket $\{\cdot,\cdot\}^B_{red}$ is Poisson using two different arguments. The first one, Theorem 4.3.5, works away from the singular stratum and has an easier and more conceptual proof. On the other hand, Theorem 4.3.6 shows that the Jacobi identity of $\{\cdot,\cdot\}^B_{red}$ is zero over the differential space $\mathcal{M}/G$ based on the jacobiator formula (3.1.7).

Theorem 4.3.5. *On the manifold $\bar{\mathcal{M}}_{reg} = \mathcal{M}_{reg}/G$, $\{\cdot,\cdot\}^B_{red}$ is a regular (rank 2) Poisson bracket.*

Proof. On the manifold $\bar{\mathcal{M}}_{reg}$, the G-action is free and proper and thus the basis of sections $\{\zeta_1, \zeta_2\}$ of the bundle $\mathfrak{g}_S \to Q$ defines a basis of $\mathcal{S}$ given by $\{(\zeta_1)_\mathcal{M}, (\zeta_2)_\mathcal{M}\}$. Since $\pi^\#_B(dJ^{(1)}) = -(\zeta_1)_\mathcal{M}$ and $\pi^\#_B(dJ^{(2)}) = -(\zeta_2)_\mathcal{M}$ (see (3.3.22)), then by Proposition 3.4.1 the characteristic distribution of $\{\cdot,\cdot\}^B_{red}$ is involutive. Moreover this distribution is also regular, since the leaves are the level sets of the functions $\bar{J^{(1)}}$ and $\bar{J^{(2)}}$ (where as usual, $\bar{J}^{(i)} \in C^\infty(\mathcal{M}/G)$ such that $J^{(i)} = \rho^*\bar{J}^{(i)}$), and thus it is integrable. Finally, since the reduced bracket $\{\cdot,\cdot\}^B_{red}$ on $\bar{\mathcal{M}}_{reg}$ is regular, we conclude that $\{\cdot,\cdot\}^B_{red}$ is twisted Poisson on $\bar{\mathcal{M}}_{reg}$ (see [80] and [8, Cor. 3.7]) and hence it has an associated almost symplectic foliation. On the other hand, since $dim(\bar{\mathcal{M}}_{reg}) = 4$, we observe that this foliation has 2-dimensional leaves and thus they are symplectic. This implies that $\{\cdot,\cdot\}^B_{red}$ is a (rank 2) regular Poisson bracket on $\bar{\mathcal{M}}_{reg}$. $\square$

Note that the argument used in [40] first constructed a Poisson bivector outside of the equilibria of X_{red} and then, using a specific transversality condition (which is verified by the example), then they extended the result to $\bar{\mathcal{M}}_{reg}$.

Theorem 4.3.6. *On the differential space $\mathcal{M}/G$ the reduced bracket $\{\cdot,\cdot\}^B_{red}$ is Poisson.*

Proof. Now we use formula (3.1.7) that encodes the failure of the Jacobi identity of $\{\cdot,\cdot\}^B_{red}$ on a differential space. We begin computing the relevant terms of the 3-form dB, where B is the 2-form given in (4.3.73). The coefficients of B have the form $a = a_n\tilde{M}_n$, $b = b_x\tilde{p_x} + b_y\tilde{p_y}$ and $c = c_x\tilde{p_x} + c_y\tilde{p_y}$, where a_n, b_x, b_y, c_x, c_y are functions in $C^\infty(Q)$ which can be read directly from (4.3.72). Therefore the 3-form dB has the form

$$\begin{aligned} dB = {} & a_n d\tilde{M}_n \wedge \tilde{dx} \wedge \tilde{dy} + b_x d\tilde{p_x} \wedge \tilde{dy} \wedge \tilde{\beta}_n + b_y d\tilde{p_y} \wedge \tilde{dy} \wedge \tilde{\beta}_n \\ & + c_x d\tilde{p_x} \wedge \tilde{\beta}_n \wedge \tilde{dx} + c_y d\tilde{p_y} \wedge \tilde{\beta}_n \wedge \tilde{dx} + \Xi, \end{aligned}$$

where Ξ is a semi-basic 3-form (with respect to $\mathcal{M} \to Q$). Consequently, using the expression of $dJ \wedge K_W$ in Proposition 4.2.11 and formulas (4.3.72), we have,

$$\begin{aligned}(dJ \wedge K_{\mathcal{W}} - dB)|_{\mathcal{C}} = & -\frac{I}{E} R\bar{F}_4(xy\, d\tilde{p_x} \wedge \tilde{\beta}_n \wedge \tilde{dx} - x^2\, d\tilde{p_y} \wedge \tilde{\beta}_n \wedge \tilde{dx} \\ & - y^2\, d\tilde{p_x} \wedge \tilde{dy} \wedge \tilde{\beta}_n + xy\, d\tilde{p_y} \wedge \tilde{dy} \wedge \tilde{\beta}_n)|_{\mathcal{C}} \\ & - \frac{1}{I}\Phi(x,y) d\tilde{M}_n \wedge \tilde{dx} \wedge \tilde{dy}|_{\mathcal{C}} + (\Psi_2 - \Xi)|_{\mathcal{C}},\end{aligned}$$

where Φ is given in (4.3.74) and Ψ_2 in (4.2.59).

Recall that the invariant functions p_i, $i = 0, \cdots, 4$, given in (4.2.38) are seen as coordinates on $\mathcal{M}/G$ and thus we have to compute the expression

$$(dJ \wedge K_{\mathcal{W}} - dB)(\pi_B^\#(d\rho^* p_i), \pi_B^\#(d\rho^* p_j), \pi_B^\#(d\rho^* p_k)) \tag{4.3.76}$$

for all the combinations of distinct p_i, p_j and p_k. Since $\bar{J^{(1)}}$ and $\bar{J^{(2)}}$ depend linearly on p_3 and p_4, see (4.3.67), and using that they are Casimirs of the reduced bracket $\{\cdot,\cdot\}^B_{red}$, it is enough to check the formula of the Jacobiator in (4.3.76) on p_0, p_1 and p_2. We have

$$\begin{aligned}\pi_B^\#(d\rho^* p_0) = & -2\tilde{p_x}\tilde{Y}_x - 2\tilde{p_y}\tilde{Y}_y - 2\tilde{p_x}\tilde{M}_n\bar{F}_3\frac{1}{R}\frac{\partial}{\partial\tilde{p_y}} + 2\tilde{p_y}\tilde{M}_n\bar{F}_3\frac{1}{R}\frac{\partial}{\partial\tilde{p_x}} \\ & - \frac{I}{E}2R(x\tilde{p_y} - y\tilde{p_x})(x\tilde{p_x} + y\tilde{p_y})\bar{F}_4\frac{\partial}{\partial\tilde{M}_n}, \\ \pi_B^\#(d\rho^* p_1) = & 2x\frac{\partial}{\partial\tilde{p_x}} + 2y\frac{\partial}{\partial\tilde{p_y}}, \\ \pi_B^\#(d\rho^* p_2) = & \tilde{p_x}\frac{\partial}{\partial\tilde{p_x}} + \tilde{p_y}\frac{\partial}{\partial\tilde{p_y}} + x\left(-\tilde{Y}_x - \tilde{M}_n\bar{F}_3\frac{1}{R}\frac{\partial}{\partial\tilde{p_y}}\right) \\ & + y\left(-\tilde{Y}_y + \tilde{M}_n\bar{F}_3\frac{1}{R}\frac{\partial}{\partial\tilde{p_x}}\right) - \frac{I}{E}R(x\tilde{p_y} - y\tilde{p_x})(x^2+y^2)\bar{F}_4\frac{\partial}{\partial\tilde{M}_n}.\end{aligned} \tag{4.3.77}$$

First observe that the vector field $\pi_B^\#(d\rho^* p_1)$ is in the kernel of $\Phi(x,y)d\tilde{M}_n \wedge \tilde{dx} \wedge \tilde{dy}$ and of the semi-basic form $\Psi - \Xi$. Then, we observe that the vector fields in (4.3.77) do not depend on $\tilde{X}_n$ and thus we conclude that

$$(dJ \wedge K_{\mathcal{W}} - dB)(\pi_B^\#(d\rho^* p_0), \pi_B^\#(d\rho^* p_1), \pi_B^\#(d\rho^* p_2)) = 0.$$

Hence the reduced bracket $\{\cdot,\cdot\}^B_{nh}$ is Poisson on $\mathcal{M}/G$.

□

For completeness let us compute the reduced bracket $\{\cdot,\cdot\}^B_{red}$ on $\mathcal{M}/G$. Using (4.3.77) and

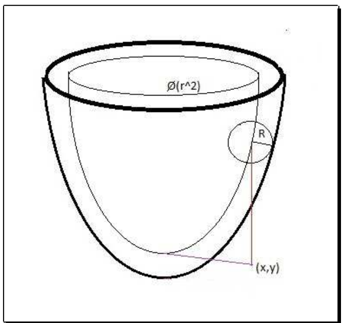
$$\begin{aligned}\pi_B^\#(d\rho^* p_3) &= y\tilde{Y}_x - x\tilde{Y}_y + \left(\tilde{p_y} + x\tilde{M}_n\bar{F}_3\frac{1}{R}\right)\frac{\partial}{\partial\tilde{p_x}} - \left(\tilde{p_x} - y\tilde{M}_n\bar{F}_3\frac{1}{R}\right)\frac{\partial}{\partial\tilde{p_y}}, \\ \pi_B^\#(d\rho^* p_4) &= -\tilde{X}_n + x\frac{I}{E}R(x\tilde{p_y} - y\tilde{p_x})\bar{F}_4\frac{\partial}{\partial\tilde{p_x}} + y\frac{I}{E}R(x\tilde{p_y} - y\tilde{p_x})\bar{F}_4\frac{\partial}{\partial\tilde{p_y}},\end{aligned}$$

we get

$$
\begin{aligned}
&\{p_0, p_1\}^B_{red} = 4p_2, && \{p_0, p_2\}^B_{red} = 2p_0 - 2\bar{F}_3 \tfrac{1}{R} p_3 p_4 \ , \\
&\{p_0, p_3\}^B_{red} = 2\bar{F}_3 \tfrac{1}{R} p_2 p_4, && \{p_0, p_4\}^B_{red} = \tfrac{I}{E} 2R\bar{F}_4 p_3 p_2, \\
&\{p_1, p_2\}^B_{red} = -2p_1, && \{p_1, p_3\}^B_{red} = 0 \ , \\
&\{p_1, p_4\}^B_{red} = 0, && \{p_2, p_3\}^B_{red} = \bar{F}_3 \tfrac{1}{R} p_1 p_4 \ , \\
&\{p_2, p_4\}^B_{red} = \tfrac{I}{E} R\bar{F}_4 p_1 p_3, && \{p_3, p_4\}^B_{red} = 0.
\end{aligned}
$$

4.3.3 Conclusion

In conclusion, we have shown that the mechanical system describing the dynamics of a homogeneous ball rolling without sliding on a convex surface of revolution modelled as the nonholonomic system $(\mathcal{M}, \pi_{nh}, H_{\mathcal{M}})$ having the symmetry group $S^1 \times SO(3)$, admits a gauge transformation by a G-invariant 2-form B, semi-basic with respect to the bundle $\mathcal{M} \to Q$, such that the reduced bracket $\{\cdot,\cdot\}^B_{red}$ on the differential quotient space $\mathcal{M}/G$ is Poisson. The reduced Poisson bracket $\{\cdot,\cdot\}^B_{red}$ has two Casimirs which come from two G-invariant horizontal gauge momenta that are computed using the solutions of the linear differential equation (4.3.66). The common level sets of both Casimirs of $\{\cdot,\cdot\}^B_{red}$ are the symplectic leaves of the reduced Poisson structure $(\mathcal{M}/G, \{\cdot,\cdot\}^B_{red})$ which is rank-two in the regular stratum of $\mathcal{M}/G$. From the bracket $\{\cdot,\cdot\}^B_{red}$ computed above we see that the Poisson structure on the singular stratum (4.2.39) is trivial.



$$
\begin{tikzcd}
\boxed{\begin{array}{c}\text{Nonholonomic bracket}\\ \{\cdot,\cdot\}_{nh} \text{ not Poisson}\end{array}} \arrow[r, "\text{dyn. gauge trans.}"] \arrow[d, "\text{red. by symm.}"] & \boxed{\begin{array}{c}\text{Almost Poisson str.}\\ \pi_B^\sharp(dJ^{(i)}) = -(\zeta_i)_{\mathcal{M}}\end{array}} \arrow[d, "\text{red. by symm.}"] \\
\boxed{\begin{array}{c}\text{Reduced bracket } \{\cdot,\cdot\}_{red}\\ \text{not Poisson}\end{array}} & \boxed{\begin{array}{c}\text{Poisson bracket } \{\cdot,\cdot\}^B_{red}\\ \bar{J^{(1)}},\ \bar{J^{(2)}} \text{ Casimirs}\end{array}}
\end{tikzcd}
$$

Remark 4.3.7. Following [47], which was worked independently, we see that B can be written as a contraction of the 3-form

$$\Pi = \Phi(x,y)\, \tilde{dx} \wedge \tilde{dy} \wedge \tilde{\beta}_n,$$

with the nonholonomic vector field X_{nh}, where $\Phi(x,y)$ is given in (4.3.74). That is, $B = \mathbf{i}_{X_{nh}}\Pi$, which implies immediately that B defines a dynamical gauge transformation. $\diamond$

4.4 Integrability

As mentioned in the introduction, the nonholonomic system treated in this thesis has being studied by several authors without reducing a bivector in $\mathcal{M}$ but instead working directly with the properties of the reduced dynamics in $\mathcal{M}/G$. Fassò, Giacobbe and Sansonetto [40] utilize a specific dynamical property of the reduced system in order to find a Poisson bracket in $\mathcal{M}/G$. Outside the equilibria, the orbits of the reduced dynamics are the fibers of a (locally trivial) fibration with fiber S^1, and the period of the flow is a continuous (and smooth) function of the initial data.

Let $\mathcal{M}_{neq}$ denote the complement of the equilibrium points of the dynamics X_{red} in the regular stratum $\bar{\mathcal{M}}_{reg}$ of $\mathcal{M}/G$. Calling $\bar{E}$ the reduced energy H_{red}, in [40] it is shown that the map $F = (\bar{E}, \bar{J}^{(1)}, \bar{J}^{(2)}) : \mathcal{M}_{neq} \to \mathbb{R}^3$ is a submersion. Given the foliation $\mathcal{F}$ by level sets of $\bar{J}^{(1)}$ and $\bar{J}^{(2)}$, that is $\mathcal{F} = \{\mathcal{F}_{c_1,c_2}\}$, $\mathcal{F}_{c_1,c_2} = \{\bar{J}^{(1)} = c_1,\ \bar{J}^{(2)} = c_2\}$, which is transversal to the level sets of $\bar{E}$, and observing that $\bar{E}$ does not have critical points on $\mathcal{M}_{neq}$, Theorem 2 of [40] allows to conclude that there exist a rank-two Poisson bivector π, with symplectic foliation $\mathcal{F}$, such that the dynamics is Hamiltonian with Hamiltonian $\bar{E}$. Moreover, by construction, $\bar{J}^{(1)}$ and $\bar{J}^{(2)}$ are Casimirs of the Poisson structure π. Under these conditions there exists a semi-local (i.e. in a neighbourhood of a periodic orbit) system of coordinates $(\bar{E}, \bar{J}^{(1)}, \bar{J}^{(2)}, \phi)$ such that the reduced dynamics takes the form $X_{red} = \omega\frac{\partial}{\partial\phi}$ and the bivector has the form $\pi = \omega\frac{\partial}{\partial E} \wedge \frac{\partial}{\partial\phi}$, with ω a non-vanishing function. Some work is required to get a global bivector in $\mathcal{M}_{neq}$. Under more specific hypothesis on the equilibrium set of X_{red}, which are verified in our mechanical example, it is also shown in Theorem 3 of [40] that there exists a rank-two bivector $\tilde{\pi}$ in all $\bar{\mathcal{M}}_{reg}$ which is Poisson, verifies $\tilde{\pi}^{\#}(d\bar{E}) = -X_{red}$ and has $\mathcal{F}$ as symplectic foliation.

4.5 Simulation

We performed a simulation of the unreduced dynamics X_{nh}. We used the expression in polar coordinates obtained in (B.1.2). The parameters used where:

```
% Parameters
m = 1; % mass
R = 0.8; % ball radius
I = 0.4*m*R^2; % inertia homogenous ball
tmax = 12; % time of simulation
```

We use standard methods on numerical integration includes in the routines of the software (GNU Octave v4.2.1). Following the online reference, the routine `ode45` integrates a system of non–stiff ordinary differential equations using second order Dormand-Prince method. This is a fourth–order accurate integrator therefore the local error normally expected is $O(h^5)$. By default, `ode45` uses an adaptive timestep. The tolerance for the timestep computation may be changed by using the options "`RelTol`" and "`AbsTol`".

The parameters used in the simulation of Fig. 4.5 are presented afterwards. The vector of initial conditions `X0` uses the variables $(\theta, r, \phi_n, \tilde{p_\theta}, \tilde{p_T}, \tilde{M}_n)$ used in Appendix B, Eq. (B.1.2) and ϕ_n is the angle associated to $\tilde{M}_n$.

```
% parameters for numeric integration
odeopt = odeset ('RelTol', 0.00001, 'AbsTol', 0.00001, 'InitialStep',
0.08,'MaxStep',0.08);
% integrate, for t=[0,tmax], with given initial conditions and 'parameters'
X0 = [0.0 1.0 1.0 10.0 0 1];
[t,y] = ode45(dydt,[0 tmax], X0, odeopt);
```

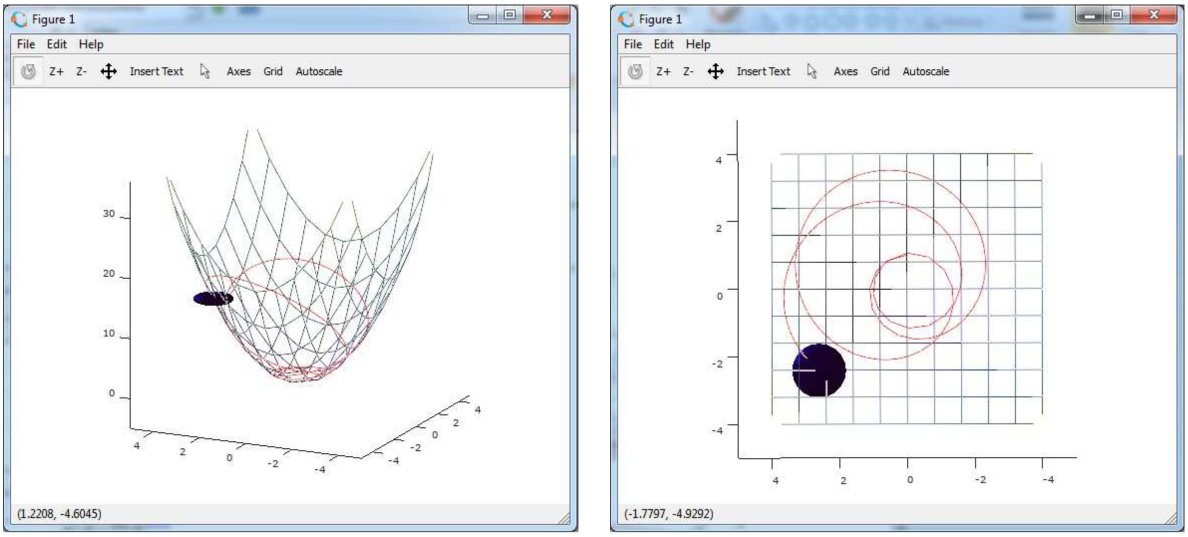


Figure 4.4: 'Homogeneous ball rolling on the paraboloid $z = 1.2 * (x^2 + y^2)$. Line in red indicates the trajectory of the center of mass (left). Integrated trajectory seen from "above" (right).

Appendix A

Twisted Dirac Structures and Gauge Transformations

A basic property of a Poisson bivector is that the characteristic distribution is integrable and defines a foliation by even dimensional leaves which are endowed with a symplectic 2-form. On the other hand almost Poisson and almost Dirac structures do not have any associated foliation in general, however there exist 'intermediate' structures having a foliation carrying leaf-wise 2-forms which are not symplectic. This is the case of twisted Poisson and twisted Dirac structures defined in Section A.3. We start by describing the main properties of the distributions and foliations associated to those twisted structures. One way to generate examples of twisted structures is by means of gauge transformations, see Section A.4. Finally in table A.1 we give an overview of the effect of gauging on the involved distributions and foliations. The main references for this appendix are [8] and [48].

A.1 Almost Dirac and H-twisted Dirac structures

As usual let Q denote a smooth manifold. We start presenting the properties of (almost) Dirac and twisted Dirac structures because they provide a natural framework for defining gauge transformation. Afterwards we will be more interested in the case of almost Poisson and twisted bivector fields.

Definition A.1.1. An *almost Dirac structure* is a subbundle L of $TQ \oplus T^*Q$ that is maximal isotropic (i.e. $L = L^\perp$) with respect to the pairing

$$\langle (X,\alpha),(Y,\beta)\rangle = \alpha(Y) + \beta(X). \tag{A.1.1}$$

Let $H \in \Gamma(\bigwedge^3 T^*Q)$ be a closed 3-form. We say that L defines a *H-twisted Dirac structure* if, in addition, L is involutive with respect to the following bracket on $TQ \oplus T^*Q$,

$$[(X,\alpha),(Y,\beta)]_H = ([X,Y], \pounds_X\beta - \mathbf{i}_Y d\alpha + \mathbf{i}_{X\wedge Y}H), \tag{A.1.2}$$

where $X, Y \in \Gamma(TQ)$ and $\xi, \eta \in \Gamma(T^*Q)$. In this case Q is called an *H-twisted Dirac manifold.* If $H = 0$ then L is simply called a *Dirac structure* and Q a *Dirac manifold.*

Remark A.1.2. Observe that (A.1.2) defines a bracket which not skew-symmetric and it is called a Dorfman bracket. It is known that it coincides with its skew-symmetrization (called the Courant bracket) on a maximal isotropic subspace. The Courant bracket was introduced by Courant in [34]. Dirac structures were studied by Courant and Weinstein to unify Poisson and presymplectic geometry. For more details see for instance [26] and [27]. $\diamond$

The Courant bracket itself already gives some constraints on the type of involutive subbundles L.

Proposition A.1.3. *[48] If $L \subset TQ \oplus T^*Q$ is involutive then L must either be an isotropic subbundle, or a bundle of type $\Delta \oplus T^*Q$ for a nontrivial involutive subbundle Δ of TQ.*

In the maximal isotropic case there exist an equivalence between the involutivity of L and the vanishing of the following operators: for $A, B, C \in \Gamma(TQ \oplus T^*Q)$ define the *Nijenhuis operator*

$$Nij(A, B, C) = \frac{1}{3} cyclic\{\langle [A, B], C \rangle\},$$

and the *Jacobiator*:

$$Jac(A, B, C) = cyclic\{[[A, B], C]\},$$

which verifies (see [48, Prop. 3.16])

$$Jac(A, B, C) = d(Nij(A, B, C)).$$

Proposition A.1.4. *[48] Let L be a maximal isotropic subbundle of $TQ \oplus T^*Q$, then the following are equivalent:*

1. *L is involutive.*
2. *$Nij|_L = 0$.*
3. *$Jac|_L = 0$.*

Let L be an H-twisted Dirac structure. Denote by π_T the projection of $L \subset TQ \oplus T^*Q$ onto the first component, $\pi_T : L \to TQ$ and by $[\cdot, \cdot]_H|_L$ the restriction of the bracket to the subbundle L. It can be shown that taking π_T as anchor map, the triple $(L, \pi_T, [\cdot, \cdot]_H|_L)$ verifies all the axioms of a Lie algebroid and it is a general result in the theory on Lie algebroids that the *characteristic distribution*, $\pi_T(L)$, is an integrable distribution [68]. If the distribution $\pi_T(L)$ has constant rank, we say that L is a *regular* Dirac structure and the induced foliation is regular. In the same way as a Poisson manifold has a foliation by symplectic leaves, each leaf of a Dirac manifold carries a presymplectic 2-form (closed but in general degenerate), and each H-twisted Dirac manifold has a foliation where the induced 2-form on each leaf could be degenerate and non-closed, where the non-closedness is controlled by the 3-form H.

A.2 Examples

1. Let Ω be a closed 2-form and consider the map $\Omega^\flat : TQ \to T^*Q$ given by $\Omega^\flat(X) = \mathbf{i}_X\Omega$. Then $L_\Omega = graph(\Omega^\flat) = \{(X,\alpha) \in TQ \oplus T^*Q : \mathbf{i}_X\Omega = -\alpha\}$ is a Dirac structure.

2. Let π be a Poisson bivector field and define the map $\pi^\sharp : T^*Q \to TQ$ by $\beta(\pi^\sharp(\alpha)) = \pi(\alpha,\beta)$. In this case $L_\pi = \{(X,\alpha) \in TQ \oplus T^*Q : \pi^\sharp(\alpha) = X\}$ is also a Dirac structure. Note that the projection $\pi_{T^*Q} : L_\pi \to T^*Q$ identifies T^*Q with L_π. Thus L_π induces a Lie algebroid structure on T^*Q, with anchor map $\pi^\sharp : T^*Q \to TQ$, and bracket

$$[\alpha,\beta]_\pi = \pounds_{\pi^\sharp(\alpha)}\beta - \mathbf{i}_{\pi^\sharp(\beta)}d\alpha. \tag{A.2.3}$$

   This bracket is $\mathbb{R}$-bilinear, skew-symmetric and satisfies the Leibniz property for Lie algebroids. Note that it is uniquely characterized by $[df,dg]_\pi = d\{f,g\}$ and the Leibniz property for Lie algebroids.

   Now let π be a bivector field (not necessarily Poisson). The failure of the involutivity of the Courant bracket on L_π is measured by $\frac{1}{2}[\pi,\pi]$ and the distribution $\pi^\sharp(T^*Q) = \pi_T(L_\pi)$ is in general non-integrable. Indeed the map $\pi^\sharp$ does not necessarily preserve the bracket but the following formula holds [26]:

$$\pi^\sharp([\alpha,\beta]_\pi) = [\pi^\sharp(\alpha),\pi^\sharp(\beta)] - \frac{1}{2}\mathbf{i}_{\alpha\wedge\beta}[\pi,\pi], \tag{A.2.4}$$

   for $\alpha,\beta \in T^*Q$.

   It is known from the original work of Courant [34] that the subbundle $L \subset TQ \oplus T^*Q$ is the graph of a bivector π if and only if

$$TQ \cap L = \{0\}. \tag{A.2.5}$$

3. Another very important example for us is the following. Let $F \subset TQ$ be regular distribution, Ω a 2-form on Q and define the subbundle L depending of F and Ω_F, $L = L(F,\Omega_F)$, by

$$L := \{(X,\alpha) \in TQ \oplus T^*Q : X \in F, \mathbf{i}_X\Omega|_F = -\alpha|_F\}, \tag{A.2.6}$$

   where $\cdot|_F$ means the pointwise restriction to F. If F is integrable and Ω is closed then L is a Dirac structure.

In the case of regular almost Dirac structures, the last example characterizes uniquely all regular almost Dirac structures in Q.

Proposition A.2.1. *[8] There is a one-to-one correspondence between regular almost Dirac structures $L \subset TQ \oplus T^*Q$ and pairs (F,Ω_F), where F is a regular distribution on Q and $\Omega_F \in \Gamma(\bigwedge^2 F^*)$.*

Proof. Let $L \subset TQ \oplus T^*Q$ be a regular almost Dirac structure. Define $F := pr_1(L)$, which is also a regular (perhaps non integrable) distribution. On the other hand define the section $\Omega_F \in \Gamma(\bigwedge^2 F^*)$, at each $x \in Q$, by

$$\Omega_F|_x(X_x, Y_x) = -\alpha_x(Y_x), \tag{A.2.7}$$

for $X, Y \in \Gamma(F)$ such that $(X_x, \alpha_x) \in L_x$. Using the fact that L is maximal isotropic we can show that Ω_F is well defined, i.e. it is independent of the choice of α.

Conversely, given a regular distribution F and the section Ω_F in $\Gamma(\bigwedge^2 F^*)$, define the subbunble L as the pairs (X, α) whenever $X \in F$ and $\mathbf{i}_X \Omega_F = -\alpha|_F$. □

Using a complement to the distribution F in TQ it is possible to show that there exists a (non-unique) 2-form Ω on Q such that $\Omega|_F = \Omega_F$. From the definition of Ω_F it follows that $Ker(\Omega_F) = L \cap TQ$. As a consequence, L is the graph of a bivector field π if and only if Ω_F is non-degenerate.

A.3 Twisted Poisson and Twisted Dirac structures

Poisson structures have two kinds of integrability. On the one hand, their characteristic foliation $\pi^\sharp(T^*Q)$ is integrable and on the other hand each leaf $\mathcal{O}$ carries a non-degenerate 2-form $\Omega_\mathcal{O}$ which is closed. Twisted Poisson structures retain the integrability of $\pi^\sharp(T^*Q)$ but allow the leafwise 2-form $\Omega_\mathcal{O}$ to be non-closed. For twisted Dirac structures even the nondegeneracy of $\Omega_\mathcal{O}$ is lost.

Theorem A.3.1. *[8] Let $L \subset TQ \oplus T^*Q$ be a regular almost Dirac structure such that $\pi_T(L)$ is an integrable distribution on Q. Then there exists an exact 3-form H such that L is an H-twisted Dirac structure.*

Proof. Using Proposition A.2.1, take the pair (F, Ω_F) corresponding to L. Since F is integrable, it defines a foliation $\mathcal{F}$ tangent to F. For each leaf $\mathcal{O}$ of the foliation $\mathcal{F}$, Ω_F defines a 2-form $\Omega_\mathcal{O}$ which, by the remark after Proposition A.2.1, is the restriction of 2-form Ω on Q. We claim that L is a $d\Omega$-twisted Dirac structure.

Indeed, if (X, α) and (Y, β) are sections of L, the involutivity condition of the twisted bracket is equivalent to

$$\mathbf{i}_{[X,Y]}\Omega|_F = -(\pounds_X\beta - \mathbf{i}_Y d\alpha + \mathbf{i}_{X\wedge Y} d\Omega)|_F. \tag{A.3.8}$$

Indeed, by the integrability of F we have that

$$-(\pounds_X\beta - \mathbf{i}_Y d\alpha + \mathbf{i}_{X\wedge Y} d\Omega)|_F = -\pounds_X(\beta|_F) + \mathbf{i}_Y d(\alpha|_F) - \mathbf{i}_{X\wedge Y} d(\Omega|_F), \tag{A.3.9}$$

but using that $\mathbf{i}_Y \Omega_F = -\beta|_F$, $\mathbf{i}_X \Omega_F = -\alpha|_F$, it is an straightforward calculation to show that the right hand side of (A.3.9) is equal to $\mathbf{i}_{[X,Y]}\Omega_F$. □

A bivector field π such that $graph(\pi^\sharp)$ is an H-twisted Dirac structure is called H-twisted Poisson bivector. The associated bracket $\{\cdot, \cdot\}$ verify the relation:

$$cyclic[\{f, \{g, h\}\}] = -H(X_f, X_g, X_h), \tag{A.3.10}$$

for $f, g, h \in C^\infty(Q)$ and $X_f = \{\cdot, f\}$. That means that the failure of the Jacobi identity is controlled by the closed 3-form H.

Let π be an H-twisted Poisson structure. Note that $\pi^\sharp$ does not preserve the bracket $[\cdot,\cdot]_\pi$ but if we define a new bracket $[\cdot,\cdot]_H$ by

$$[\alpha, \beta]_H := \pounds_{\pi^\sharp(\alpha)}\beta - \mathbf{i}_{\pi^\sharp(\beta)}d\alpha + \mathbf{i}_{\pi^\sharp(\alpha)\wedge\pi^\sharp(\beta)}H, \tag{A.3.11}$$

then it can be shown that π verifies $[\pi^\sharp(\alpha), \pi^\sharp(\beta)] = \pi^\sharp([\alpha, \beta]_H)$. This implies that $(T^*Q, [\cdot,\cdot]_H, \pi^\sharp)$ is a Lie algebroid. The characteristic distribution $\pi^\sharp(T^*Q)$ is then integrable and each leaf $\mathcal{O}$ carries a non-degenerate 2-form $\Omega_{\mathcal{O}}$ which is not closed, in fact we have $d\Omega_{\mathcal{O}} = H|_{\mathcal{O}}$.

Therefore Theorem A.3.1 specializes to:

Corollary A.3.2. *Let π be a bivector field on Q with an integrable regular characteristic distribution. Then there exists an exact 3-form H with respect to which π is H-twisted.*

A.4 Gauge transformations

Let B be a 2-form in Q and let $L \subset TQ \oplus T^*Q$ be an almost Dirac structure. Consider the map $\tau_B : L \to TQ \oplus T^*Q$ given by $(X, \alpha) \mapsto (X, \alpha + \mathbf{i}_X B)$, i.e.

$$\tau_B(L) = \{(X, \alpha + \mathbf{i}_X B) : (X, \alpha) \in L\}. \tag{A.4.12}$$

It can be show that $\tau_B(L)$ is also an almost Dirac structure. If B is closed and L is a Dirac structure then $\tau_B(L)$ is also Dirac. Moreover if L is a H-twisted Dirac structure, then the gauge transformation of L by the 2-form B is $(H - dB)$-twisted Dirac, see [80]. Observe that L and $\tau_B(L)$ have the same characteristic distribution, $\pi_T(L)$. Consequently, when both L and $\tau_B(L)$ are involutive, they have the same associated foliation.

The presymplectic form on each leaf is modified by the restriction of B to the leaf. In fact if L is the almost Dirac structure which corresponds to the pair (F, Ω_F), then $\tau_B(L)$ corresponds to the pair $(F, \Omega_F - B|_F)$. It is also shown in [8] (Proposition 3.11) that two regular almost Dirac structures L_1 and L_2 are gauge related if an only if $\pi_T(L_1) = \pi_T(L_2)$.

Consider now an almost Poisson manifold (P, π) with associated Dirac structure $L_\pi = graph(\pi^\sharp)$ and a 2-form B on Q. The gauge transformation of L_π by B is given by $\tau_B(L_\pi) = \{(X, \alpha + \mathbf{i}_X B) \in TQ \oplus T^*Q : X = \pi^\sharp(\alpha)\}$, which does not necessarily correspond to the graph of a new bivector π^B. We recall that this last condition is true if and only if $\tau_B(L_\pi) \cap TQ = \{0\}$ which is equivalent to the fact that $Id + B^\flat \circ \pi^\sharp$ be invertible [34].

Indeed if π^B is a bivector then we must have

$$(\pi^\sharp(\alpha), \alpha + \mathbf{i}_{\pi^\sharp(\alpha)}B) = ((\pi^B)^\sharp(\alpha + \mathbf{i}_{\pi^\sharp(\alpha)}B), \alpha + \mathbf{i}_{\pi^\sharp(\alpha)}B). \tag{A.4.13}$$

In that case the relation between π and π^B is given by

$$(\pi^B)^\sharp = \pi^\sharp \circ (Id - B^\flat \circ \pi^\sharp)^{-1}. \tag{A.4.14}$$

Table A.1: Foliations associates to the different structures.

Poisson	Dirac
Symplectic foliation	Presymplectic foliation
$d\Omega_{\mathcal{O}} = 0$	$d\Omega_{\mathcal{O}} = 0$
$\Omega_{\mathcal{O}}$ non-degenerate	$\Omega_{\mathcal{O}}$ non-degenerate
H-twisted Poisson	H-twisted Dirac
Almost Symplectic foliation	Foliation
$d\Omega_{\mathcal{O}} = H\|_{\mathcal{O}}$	$d\Omega_{\mathcal{O}} = H\|_{\mathcal{O}}$
$\Omega_{\mathcal{O}}$ non-degenerate	$\Omega_{\mathcal{O}}$ non-degenerate
Regular almost Poisson	Regular almost Dirac
(with integrable characteristic distribution)	(with integrable characteristic distribution)
$\exists$ closed 3-form H s.t.	$\exists$ closed 3-form H s.t.
$d\Omega_{\mathcal{O}} = H\|_{\mathcal{O}}$	$d\Omega_{\mathcal{O}} = H\|_{\mathcal{O}}$
$\Omega_{\mathcal{O}}$ non-degenerate	$\Omega_{\mathcal{O}}$ non-degenerate

It is shown in [80] that if an H-twisted Poisson bivector π is gauge related with another bivector π^B via the 2-form B, then π^B is $(H - dB)$-twisted. Recall that twisted structures have integrable characteristic distributions, then the induced 2-form $\Omega_{\mathcal{O}}$ on a leaf $\mathcal{O}$ is non-degenerate and $d\Omega_{\mathcal{O}} = (H - dB)|_{\mathcal{O}}$. Note that in particular, if π is Poisson, then π^B is $(-dB)$-twisted, so that if B is closed π^B is also Poisson. We present the characteristics of the different structures in Table A.1.

All the work done so far in mechanics involves real Dirac structures. Analogously, a maximal isotropic and involutive complex subbundle $L \subset (TQ \oplus T^*Q) \otimes \mathbb{C}$ is called a complex Dirac structure. Note that the definition still works if $TQ \oplus T^*Q$ is replaced with any real or complex Courant algebroid thus one may speak of Dirac structures in an arbitrary Courant algebroid.

Appendix B

Example in Polar Coordinates

This appendix can be read almost independently of the main text. We use references to the main text only for the theoretical sections, but some definitions are recalled along the presentation. The style is less formal than the main text and some details of computations are presented. Since polar coordinates (r, θ) are not a diffeomorphism in the origin, the result are valid away from the origin, that is when the radial coordinate verifies $r > 0$.

B.1 Example: the homogeneous ball in a convex surface of revolution

Consider the mechanical system presented in Chapter 4. Recall that the configuration manifold Q is given by the position of the center of mass of the ball and an orthogonal matrix relating a fixed frame in space and a frame attached to the ball. In consequence we have $Q = \Sigma \times SO(3)$, where $\Sigma \subset \mathbb{R}^3$ is the surface of revolution where the ball rolls. The surface Σ is parametrized in cylindrical coordinates by

$$\Sigma = \{(r, \theta, z) : z = \phi(r^2)\}.$$

An orthonormal frame $\{\hat{\theta}, \hat{T}\}$ on $T\Sigma$ is obtained by normalizing the vectors of the frame $\{\frac{\partial}{\partial \theta}, \frac{\partial}{\partial r}\}$ associated to the polar coordinates (r, θ) on the plane $\{z = 0\}$, and then induced to $T\Sigma$ by the map $T\phi$.

Using the symmetry of revolution it is also possible to construct the frame $\{\hat{\theta}, \hat{T}\}$ from the canonical basis using 2 rotations. Indeed, let us consider the cylindrical orthonormal moving frame $\{\hat{r}, \hat{\theta}, \hat{z}\}$ with coordinates (v_r, v_θ, v_z). It is related to the canonical frame (denoted $(\hat{x}, \hat{y}, \hat{z})$) by

$$\begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{z} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix},$$

with dual basis $\{\Psi^r, \Psi^\theta, \Psi^z\}$ verifying

$$\begin{pmatrix} \Psi^r \\ \Psi^\theta \\ \Psi^z \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}.$$

Remark B.1.1. We remark that if A is a matrix relating two ordered basis (v_i) and (w_i), with dual bases (v^i) and (w^i), the coordinates of vectors with respect to those basis are related by $(A^T)^{-1}$, the dual basis (v^i) and (w^i) are also related by $(A^T)^{-1}$, and the coordinates with respect to the dual basis are related by A. In particular if A is orthogonal, the same matrix A is used in all cases. $\diamond$

Let ν denote the angle between the exterior normal to the surface and the horizontal. By means of an extra rotation we construct the new frame $(\hat{N}, \hat{\theta}, \hat{T})$ such that $\hat{\theta}$ and $\hat{T}$ are tangent to the surface Σ in the "horizontal" and "vertical" directions, respectively, and $\hat{N}$ is normal to the surface, see Fig. B.1.

The relation between ν and the parametrization ϕ of the surface is given by

$$\sin\nu = \frac{1}{\sqrt{1+(2r\phi')^2}}, \quad \cos\nu = \frac{2r\phi'}{\sqrt{1+(2r\phi')^2}},$$

and the change of frame by

$$\begin{pmatrix} \hat{N} \\ \hat{\theta} \\ \hat{T} \end{pmatrix} = \begin{pmatrix} \cos\nu & 0 & -\sin\nu \\ 0 & 1 & 0 \\ \sin\nu & 0 & \cos\nu \end{pmatrix} \begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{z} \end{pmatrix}.$$

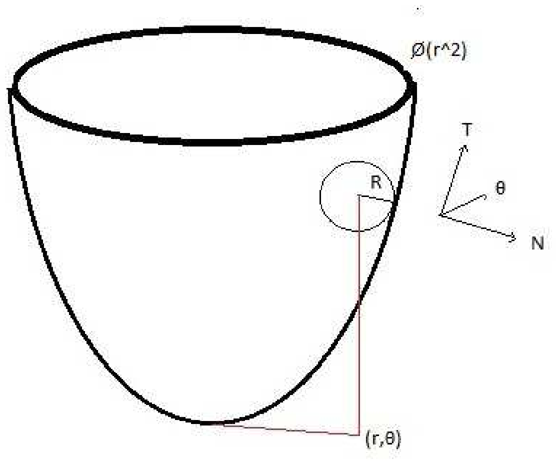


Figure B.1: The ball rolling on the surface of revolution in polar coordinates.

By right trivialization we have a frame of right invariant fields (X_1, X_2, X_3) of $TSO(3)$, and then get a frame for TQ,

$$TQ = span\{\hat{\theta}, \hat{T}, X_1, X_2, X_3\}.$$

Analogously, we perform an orthogonal transformation on the right invariant fields (X_1, X_2, X_3) in order to get a new frame adapted to the surface,

$$\begin{pmatrix} X_n \\ X_\theta \\ X_T \end{pmatrix} = \begin{pmatrix} \cos\nu & 0 & -\sin\nu \\ 0 & 1 & 0 \\ \sin\nu & 0 & \cos\nu \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$
$$= \begin{pmatrix} \cos\nu\cos\theta & \cos\nu\sin\theta & -\sin\nu \\ -\sin\theta & \cos\theta & 0 \\ \sin\nu\cos\theta & \sin\nu\sin\theta & \cos\nu \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}.$$

Since we performed the same rotations this matrix is equal to the matrix relating the frame $(\hat{N}, \hat{\theta}, \hat{T})$ with the canonical frame $(\hat{x}, \hat{y}, \hat{z})$.

Therefore, the new frame for TQ is

$$TQ = span\{\hat{\theta}, \hat{T}, X_n, X_\theta, X_T\}$$

with coordinates $(v_\theta, v_T, \omega_n, \omega_\theta, \omega_T)$, with corresponding dual frame

$$T^*Q = span\{\Psi^\theta, \Psi^T, \beta_n, \beta_\theta, \beta_T\}$$

with coordinates $(p_\theta, p_T, M_n, M_\theta, M_T)$, where the (co)frame $(\beta_n, \beta_\theta, \beta_T)$ is obtained from the components of the right Maurer-Cartan form, (ρ_1, ρ_2, ρ_3), dual to (X_1, X_2, X_3). We have

$$\begin{pmatrix} \beta_n \\ \beta_\theta \\ \beta_T \end{pmatrix} = \begin{pmatrix} \cos\nu\cos\theta & \cos\nu\sin\theta & -\sin\nu \\ -\sin\theta & \cos\theta & 0 \\ \sin\nu\cos\theta & \sin\nu\sin\theta & \cos\nu \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix}.$$

Non sliding constraints

In our rotated frame the non-sliding condition (4.1.1) takes the form

$$\begin{pmatrix} \omega_n \\ \omega_\theta \\ \omega_T \end{pmatrix} \times \begin{pmatrix} R \\ 0 \\ 0 \end{pmatrix} = - \begin{pmatrix} v_n \\ v_\theta \\ v_T \end{pmatrix},$$

which gives the relations

$$v_\theta = -R\omega_T, \quad v_T = R\omega_\theta.$$

Therefore, the constraint 1-forms

$$\epsilon_T := \beta_T + \frac{\psi^\theta}{R}, \quad \epsilon_\theta := \beta_\theta - \frac{\psi^T}{R},$$

define the constraint subbundle D which is the kernel of ϵ_T and ϵ_θ, and is given by

$$D = span\{Y_\theta := -X_T + R\hat{\theta}, \quad Y_T := X_\theta + R\hat{T}, X_n\}.$$

We choose a complement W to the constraint distribution D such that $TQ = D \oplus W$,

$$W = \{X_\theta, X_T\}.$$

In Section B.1, where we introduce the G-symmetry of the system we will see that the complement is *vertical*, $W \subset V$, and verify that it is G-invariant.

Therefore we work with the following frame of TQ adapted to the constraints and symmetries, $TQ = D \oplus W$ with D and W G-invariant distributions, i.e.

$$TQ = span\{Y_\theta, Y_T, X_n, X_\theta, X_T\},$$

with coordinates $(\tilde{v_\theta}, \tilde{v_T}, \tilde{\omega_n}, \tilde{\omega_\theta}, \tilde{\omega_T})$, with dual frame

$$T^*Q = span\{Y^\theta, Y^T, \beta_n, \epsilon_\theta, \epsilon_T\}, \tag{B.1.1}$$

and coordinates $(\tilde{p_\theta}, \tilde{p_T}, \tilde{M_n}, \tilde{M_\theta}, \tilde{M_T})$.

Constraint manifold and nonholonomic bivector

Lagrangian of the system $L : TQ \to \mathbb{R}$ is

$$L = T - ma_g z$$

where a_g is the acceleration of gravity, $z = \phi(r^2)$ the height (vertical position) of the center of mass and $T = \frac{1}{2}\kappa$ is the kinetic energy. In our adapted coordinates we have

$$2T = E\tilde{v_T}^2 + E\tilde{v_\theta}^2 + I(\tilde{\omega_n}^2 + \tilde{\omega_\theta}^2 + \tilde{\omega_T}^2) + 2I(\tilde{v_T}\tilde{\omega_\theta} - \tilde{v_\theta}\tilde{\omega_T}),$$

where $E := I + mR^2$. The kinetic energy metric κ is written in the adapted frame (B.1.1) as

$$\frac{1}{2}\kappa = \frac{E}{2}Y^\theta \otimes Y^\theta + \frac{I}{2}\beta_n \otimes \beta_n + \frac{I}{2}(Y^T \otimes \epsilon_\theta + \epsilon_\theta \otimes Y^T - Y^\theta \otimes \epsilon_T - \epsilon_T \otimes Y^\theta).$$

The constraint manifold $\mathcal{M}$ defined in (1.4.22) is given in the local coordinates associated to (B.1.1) by:

$$\mathcal{M} = \Big\{(r, \theta, g, \tilde{p_\theta}, \tilde{p_T}, \tilde{M_n}, \tilde{M_\theta}, \tilde{M_T}) \;:\; \tilde{M_T} = -\tilde{p_\theta}\frac{I}{E}, \tilde{M_\theta} = \tilde{p_T}\frac{I}{E}\Big\}.$$

The constraint subbundle $\mathcal{C}$ defined in 1.4.23 is given by

$$\mathcal{C} = span\Big\{Y_\theta, Y_T, X_n, \frac{\partial}{\partial\tilde{p_\theta}}, \frac{\partial}{\partial\tilde{p_T}}, \frac{\partial}{\partial\tilde{M_n}}\Big\}$$

Now we compute the nonholonomic bivector field π_{nh}. First observe that the canonical 1-form in T^*Q in the adapted coordinates (B.1.1) is given by

$$\Theta_Q = \tilde{p_\theta}Y^\theta + \tilde{p_T}Y^T + \tilde{M_n}\beta_n + \tilde{M_\theta}\epsilon_\theta + \tilde{M_T}\epsilon_T.$$

Using that $\Omega_\mathcal{M}$ verifies $\Omega_\mathcal{M} = -d\iota_\mathcal{M}^*\Theta_Q$, we obtain the following:

Proposition B.1.2. *The restriction of* $\Omega_{\mathcal{M}}$ *to* $\mathcal{C}$ *for the homogeneous ball rolling on a convex surface of revolution is given by:*

$$\begin{aligned}\Omega_{\mathcal{C}} = &-d\tilde{p_\theta}\wedge Y^\theta - d\tilde{p_T}\wedge Y^T - d\tilde{M}_n\wedge\beta_n\\ &- A(m)Y^T\wedge Y^\theta + B(m)Y^\theta\wedge\beta_n + C(m)\beta_n\wedge Y^T,\end{aligned}$$

where, for $m\in\mathcal{M}$*,*

$$\begin{aligned}A(m) &= \left(\tilde{p_\theta} sin\,\nu\frac{R}{r} - \tilde{M}_n\left(1+cos\,\nu\frac{R}{r} + cos\,\nu\ sin^2\,\nu\frac{\phi'+\phi''2r^2}{\phi'}\frac{R}{r}\right)\right),\\ B(m) &= \tilde{p_T}\frac{I}{E}\left(1+cos\,\nu\frac{R}{r}\right),\\ C(m) &= \tilde{p_\theta}\frac{I}{E}\left(1+cos\,\nu\ \sin^2\,\nu\frac{\phi'+\phi''2r^2}{\phi'}\frac{R}{r}\right).\end{aligned}$$

Proof. This a long but straightforward computation (following the steps in Prop. 4.2.4) based in formulas:

$$\begin{aligned}dY^\Theta|_{\mathcal{C}} &= -sin\,\nu\frac{R}{r}Y^\theta\wedge Y^T,\\ dY^T|_{\mathcal{C}} &= 0,\\ d\beta_n|_{\mathcal{C}} &= -\left(1+cos\,\nu\frac{R}{r}+cos\,\nu\ sin^2\,\nu\frac{\phi'+\phi''2r^2}{\phi'}\frac{R}{r}\right)Y^T\wedge Y^\theta,\\ d\epsilon_\theta|_{\mathcal{C}} &= -\left(1+cos\,\nu\frac{R}{r}\right)Y^\theta\wedge\beta_n,\\ d\epsilon_T|_{\mathcal{C}} &= \left(1+cos\,\nu\ sin^2\,\nu\frac{\phi'+\phi''2r^2}{\phi'}\frac{R}{r}\right)\beta_n\wedge Y^T.\end{aligned}$$

□

Corollary B.1.3. *The nonholonomic bivector* π_{nh} *is given by:*

$$\begin{aligned}\pi_{nh} = &Y_\theta\wedge\frac{\partial}{\partial\tilde{p_\theta}} + Y_T\wedge\frac{\partial}{\partial\tilde{p_T}} + X_n\wedge\frac{\partial}{\partial\tilde{M}_n}\\ &- A(m)\frac{\partial}{\partial\tilde{p_\theta}}\wedge\frac{\partial}{\partial\tilde{p_T}} - B(m)\frac{\partial}{\partial\tilde{p_\theta}}\wedge\frac{\partial}{\partial\tilde{M}_n} + C(m)\frac{\partial}{\partial\tilde{p_T}}\wedge\frac{\partial}{\partial\tilde{M}_n},\end{aligned}$$

where $A(m), B(m)$ *and* $C(m)$ *are given in Proposition B.1.2.*

Or equivalently:

$$\begin{aligned}&\pi_{nh}^\#(Y^\theta) = \frac{\partial}{\partial\tilde{p_\theta}},\qquad \pi_{nh}^\#(Y^T) = \frac{\partial}{\partial\tilde{p_T}},\qquad \pi_{nh}^\#(\beta_n) = \frac{\partial}{\partial\tilde{M}_n},\\ &\pi_{nh}^\#(d\tilde{p_\theta}) = -Y_\theta - A(m)\frac{\partial}{\partial\tilde{p_T}} - B(m)\frac{\partial}{\partial\tilde{M}_n},\\ &\pi_{nh}^\#(d\tilde{p_T}) = -Y_T + A(m)\frac{\partial}{\partial\tilde{p_\theta}} + C(m)\frac{\partial}{\partial\tilde{M}_n},\\ &\pi_{nh}^\#(d\tilde{M}_n) = -X_n + B(m)\frac{\partial}{\partial\tilde{p_\theta}} - C(m)\frac{\partial}{\partial\tilde{p_T}}.\end{aligned}$$

Since $\epsilon_\theta|_{\mathcal{C}} \equiv 0$ and $\epsilon_T|_{\mathcal{C}} \equiv 0$, the definition of the nonholonomic bivector implies that:

$$\pi_{nh}^{\#}(\epsilon_T) = \pi_{nh}^{\#}(\epsilon_\theta) = 0.$$

From Legendre transformation we get the following Hamiltonian restricted to $\mathcal{M}$:

$$H_{\mathcal{M}} = \frac{\tilde{p_T}^2}{2E} + \frac{\tilde{p_\theta}^2}{2E} + \frac{\tilde{M_n}^2}{2I} + ma_g\phi(r^2).$$

The nonholonomic vector field X_{nh} describing the dynamics is given by $X_{nh} = -\pi_{nh}^{\#}(dH_{\mathcal{M}})$. Using that $dr = \Psi^r = R\sin\nu Y^T$ we get

$$\begin{aligned} X_{nh} &= \frac{\tilde{p_\theta}}{E}Y_\theta + \frac{\tilde{p_T}}{E}Y_T - \frac{\tilde{M_n}}{I}X_n \\ &- \left(-\frac{\tilde{p_\theta}^2}{E}\frac{R}{r}\sin\nu + \frac{\tilde{p_\theta}\tilde{M_n}}{E}\frac{R}{r}\cos\nu\right)\frac{\partial}{\partial\tilde{p_T}} - ma_g\cos\nu R\frac{\partial}{\partial\tilde{p_T}} \\ &+ \sin\nu\frac{\tilde{p_T}}{E}\frac{R}{r}\left(\tilde{p_\theta} - \tilde{M_n}\cos\nu\ \sin\nu\frac{\phi' + \phi''2r^2}{\phi'}\right)\frac{\partial}{\partial\tilde{p_\theta}} \\ &- \tilde{p_T}\tilde{p_\theta}\frac{I}{E^2}\frac{R}{r}\cos\nu\left(\sin^2\nu\frac{\phi'+\phi''2r^2}{\phi'} - 1\right)\frac{\partial}{\partial\tilde{M_n}}. \end{aligned} \tag{B.1.2}$$

Symmetries and the 3-form $dJ \wedge K_{\mathcal{W}}$

The system has the symmetry Lie group $G = S^1 \times SO(3)$. In the coordinates (r, θ, g) of Q, the action of G is given by

$$(\varphi, h)\cdot(r,\theta,g) = (r, \theta+\varphi, R_\varphi\, g\, h),$$

where R_φ denotes a rotation of angle φ with respect to the z-axis. The canonical lift of the G-action to T^*Q is given by

$$(\varphi, h)\cdot(r,\theta,g,p_r,p_\theta,M) = (r,\theta+\varphi, R_\varphi\, g\, h, p_r, p_\theta, R_\varphi M).$$

Note that M is right invariant to it is unchanged by the right action by $SO(3)$. The restricted action to $\mathcal{M}$ is given by

$$(\varphi, h)\cdot(r,\theta,g,p_r,p_\theta,\tilde{M_n}) = (r,\theta+\varphi, R_\varphi\, g\, h, p_r, p_\theta, \tilde{M_n}).$$

Since the Lie algebra $\mathfrak{g}$ of G is isomorphic to $\mathbb{R}\times\mathbb{R}^3$ we choose the basis

$$\{(1,\mathbf{0}), (0,\mathbf{e_i})\}, \quad i = 1,2,3, \tag{B.1.3}$$

where $\{\mathbf{e_i}\}$ denotes the canonical basis of $\mathbb{R}^3$. Then, the infinitesimal generator with respect to the S^1 action is

$$\begin{aligned} (1,0)_Q &= r\hat{\theta} + X_3^r \\ &= \frac{r}{R}Y_\theta - sin\,\nu X_n + \left(\frac{r}{R} + cos\,\nu\right)X_T. \end{aligned}$$

We use the following notation for an element g of $SO(3)$,

$$g = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}.$$

Then, the infinitesimal generator with respect to the $SO(3)$ action is given by:

$$(0, \mathbf{e_i})_Q = \alpha_i X_1^r + \beta_i X_2^r + \gamma_i X_3^r, \quad i = 1, 2, 3.$$

Recall the matrix

$$C = \begin{pmatrix} \cos\nu\cos\theta & \cos\nu\sin\theta & -\sin\nu \\ -\sin\theta & \cos\theta & 0 \\ \sin\nu\cos\theta & \sin\nu\sin\theta & \cos\nu \end{pmatrix}$$

which relates the frames (X_n, X_θ, X_T) and (X_1, X_2, X_3) as:

$$\begin{pmatrix} X_n \\ X_\theta \\ X_T \end{pmatrix} = C \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}. \tag{B.1.4}$$

In matrix notation the $SO(3)$-infinitesimal generator can be written

$$\begin{pmatrix} (0, \mathbf{e_1})_Q \\ (0, \mathbf{e_2})_Q \\ (0, \mathbf{e_3})_Q \end{pmatrix} = g^T C^T \begin{pmatrix} X_n \\ X_\theta \\ X_T \end{pmatrix} \tag{B.1.5}$$

The matrices g and C being orthogonal, the vertical space V is also generated by

$$V = span\Big\{ \frac{r}{R} Y_\theta + \Big(\frac{r}{R} + \cos\nu \Big) X_T - \sin\nu X_n, X_n, X_\theta, X_T \Big\},$$

and we see that the symmetry verifies the dimension assumption (2.3.4). Recall that D has constant rank, $rank(D) = 3$. We will first consider the case $r > 0$, where $rank(V) = 4$ and a vertical complement $W \subset V$ has rank 2. In this case the subdistribution $S = V \cap D$ has rank 2, and is generated by

$$S = span\{ \frac{r}{R} Y_\theta, X_n \} = span\{ Y_\theta, X_n \}, \quad r > 0.$$

We choose the vertical complement $W \subset V$ as following and verify that it is G-invariant,

$$W = \{ X_\theta, X_T \}.$$

Indeed, W is invariant by the right action $SO(3)$ because the fields $\{X_1, X_2, X_3\}$ related to $\{X_\theta, X_T\}$ by (B.1.4) are G-invariant. On the other hand, we can verify that both X_n and X_θ are S^1-invariant (left action). We observe that the complement W does not satisfy the vertical-symmetry condition, see Rmk. 2.3.1.

The projection $P_W : TQ \to W$, associated to the splitting $TQ = D \oplus W$ is given by $P_W = \epsilon_\theta \otimes X_\theta + \epsilon_T \otimes X_T$. The associated $\mathfrak{g}$-valued map A_W can be computed from $A_W(v_m) = \xi \Leftrightarrow P_W(v_m) = \xi_M(m)$.

From (B.1.5) we can write the rotated frame (X_n, X_θ, X_T) in function of the $SO(3)$-infinitesimal generators:

$$\begin{pmatrix} X_n \\ X_\theta \\ X_T \end{pmatrix} = Cg \begin{pmatrix} (0, \mathbf{e_1})_Q \\ (0, \mathbf{e_2})_Q \\ (0, \mathbf{e_3})_Q \end{pmatrix},$$

we get

$$\begin{aligned} X_\theta &= (-\alpha_1 \sin\theta + \beta_1 cos\,\theta)(0, \mathbf{e_1})_Q + (-\alpha_2 sin\,\theta + \beta_2 cos\,\theta)(0, \mathbf{e_2})_Q \\ &\quad + (-\alpha_3 sin\,\theta + \beta_3 cos\,\theta)(0, \mathbf{e_3})_Q, \\ X_T &= (\alpha_1 sin\,\nu\, cos\,\theta + \beta_1 sin\,\nu\, \sin\theta + \gamma_1 cos\,\nu)(0, \mathbf{e_1})_Q \\ &\quad + (\alpha_2 sin\,\nu\, cos\,\theta + \beta_2 sin\,\nu\, \sin\theta + \gamma_2 cos\,\nu)(0, \mathbf{e_2})_Q \\ &\quad + (\alpha_3 sin\,\nu\, cos\,\theta + \beta_3 sin\,\nu\, \sin\theta + \gamma_3 cos\,\nu)(0, \mathbf{e_3})_Q. \end{aligned}$$

We introduce some notations to write vector expressions in short form. Using the notations $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)$ and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ for the rows of the matrix g, we define the following vectors $\mathbf{A} = \boldsymbol{\alpha}\cos\nu\cos\theta + \boldsymbol{\beta}\cos\nu\sin\theta - \boldsymbol{\gamma}\sin\nu$, $\mathbf{B} = -\boldsymbol{\alpha}\sin\theta + \boldsymbol{\beta}\cos\theta$ and $\mathbf{C} = \boldsymbol{\alpha}\sin\nu\cos\theta + \boldsymbol{\beta}\sin\nu\sin\theta + \boldsymbol{\gamma}\cos\nu$. Note that $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ are orthonormal.

In that notation X_θ and X_T can be written as

$$\begin{aligned} X_\theta &= \langle \mathbf{B}, (0, \mathbf{e})_Q \rangle, \\ X_T &= \langle \mathbf{C}, (0, \mathbf{e})_Q \rangle, \end{aligned} \tag{B.1.6}$$

where $(0, \mathbf{e}) = ((0, \mathbf{e}_1), (0, \mathbf{e}_2), (0, \mathbf{e}_3))$. Define the following sections of the bundle $\mathfrak{g}_W \to Q$ written in the chosen basis:

$$\begin{aligned} \xi_\theta &= \langle \mathbf{B}, (0, \mathbf{e}) \rangle, \\ \xi_T &= \langle \mathbf{C}, (0, \mathbf{e}) \rangle, \end{aligned}$$

then the expression of the $\mathfrak{g}$-valued 1-form A_W is

$$A_W = \epsilon_\theta \otimes \xi_\theta + \epsilon_T \otimes \xi_T.$$

By Lemma 2.3.7 we compute the $\mathcal{W}$-curvature,

$$K_{\mathcal{W}} = d\epsilon_\theta|_{\mathcal{C}} \otimes \xi_\theta + d\epsilon_T|_{\mathcal{C}} \otimes \xi_T.$$

Using the formulas for the differentials, $d\epsilon_\theta|_\mathcal{C}$ and $d\epsilon_T|_\mathcal{C}$, we obtain

$$
\begin{aligned}
K_\mathcal{W} = & -\left(1+\frac{R}{r}cos\nu\right) Y^\theta \wedge \beta_n \otimes \langle \mathbf{B}, (0,\mathbf{e})\rangle \\
& + \left(1 + cos\,\nu\, sin^2\,\nu \frac{\phi' + \phi'' 2r^2}{\phi'}\frac{R}{r}\right)\beta_n \wedge Y^T \otimes \langle \mathbf{C}, (0,\mathbf{e})\rangle.
\end{aligned}
$$

Our next step is to compute the 3-form, $dJ \wedge K_\mathcal{W}$.

Theorem B.1.4. *The choice of the vertical complement $W = \{X_n, X_\theta\}$ induces the expression for G-invariant 3-form $dJ \wedge K_\mathcal{W}$,*

$$
\begin{aligned}
dJ \wedge K_\mathcal{W} = & -\frac{I}{E}\left(1 + cos\,\nu \frac{R}{r}\right) d\tilde{p_T} \wedge Y^\theta \wedge \beta_n \\
& + \tilde{p_T}\frac{I}{E}\left(1 + cos\,\nu sin^2\,\nu F(r)\frac{R}{r}\right)\left(-\sin\,\nu\frac{R}{r}\right)\beta_n \wedge Y^T \wedge Y^\theta \\
& - \frac{I}{E}\left(1 + cos\,\nu\, sin^2\,\nu\;\; F(r)\frac{R}{r}\right) d\tilde{p_\theta} \wedge \beta_n \wedge Y^T,
\end{aligned}
$$

where $F(r) = \frac{\phi' + \phi'' 2r^2}{\phi'}$.

Proof. Using the canonical 1-form $\Omega_Q = \tilde{p_\theta} Y^\theta + \tilde{p_T} Y^T + \tilde{M_n}\beta_n + \tilde{M_\theta}\epsilon_\theta + \tilde{M_T}\epsilon_T$, and canonical basis of $\mathfrak{g}$, we compute the moment map $J : \mathcal{M} \to \mathfrak{g}^*$, by $\langle J(m), \xi\rangle = \mathbf{i}_{\xi_\mathcal{M}}\Theta_Q$. We obtain

$$
J = (J_0, \tilde{M_n}\mathbf{A} + \tilde{M_\theta}\mathbf{B} + \tilde{M_T}\mathbf{C}),
$$

where $J_0 = \mathbf{i}_{(1,\mathbf{0})_\mathcal{M}}\Theta_Q$ will not appear in the computations of $dJ \wedge K_\mathcal{W}$. In order to compute $dJ|_\mathcal{C}$, we use the following formulas:

$$
\begin{aligned}
d\boldsymbol{\alpha} &= \boldsymbol{\gamma}\rho_2 - \boldsymbol{\beta}\rho_3 \\
d\boldsymbol{\beta} &= -\boldsymbol{\gamma}\rho_1 + \boldsymbol{\alpha}\rho_3 \\
d\boldsymbol{\gamma} &= \boldsymbol{\beta}\rho_1 - \boldsymbol{\alpha}\rho_2.
\end{aligned}
$$

Using $d\nu = -cos\,\nu\, sin^2\,\nu \frac{\phi' + \phi'' 2r^2}{\phi'}\frac{R}{r}Y^T$, and computing $d\mathbf{A}$, $d\mathbf{B}$ and $d\mathbf{C}$ we get

$$
dJ|_\mathcal{C} = (0, \mathbf{A}d\tilde{M_n} + \tilde{M_n}d\mathbf{A} + \mathbf{B}\frac{I}{E}d\tilde{p_T} + \tilde{p_T}\frac{I}{E}d\mathbf{B} + \mathbf{C}(-\frac{I}{E})d\tilde{p_\theta} - p_\theta\frac{I}{E}d\mathbf{C}.
$$

The final expression is complicated but some of the terms will disappear when computing $dJ \wedge K_\mathcal{W}$, giving at the end the formula in the statement of the theorem. $\square$

Reduction by symmetries

Let us consider the reduction of $\mathcal{M}$ by the symmetry group $G = S^1 \times SO(3)$. First, observe that the reduction by the free action of $SO(3)$ eliminates the coordinate g. In order to perform the S^1 reduction in $\mathcal{M}/SO(3)$ we use invariant theory as performed in the literature, see [40, 52]. The standard reduction uses the following coordinates for

$\mathcal{M}/SO(3)$: $(a, \dot{a}, \omega_n)$ where $a = (x, y)$ denotes the Cartesian coordinates of the center of mass projected to the horizontal plane, the derivatives $\dot{a}$ denotes the corresponding velocities and ω_n is the component to the angular velocity normal to the surface. The Lie group S^1 acts in the above coordinates by

$$\theta \cdot (a, \dot{a}, \omega_n) = (R_\theta a, R_\theta \dot{a}, \omega_n),$$

where R_θ is the 2×2 rotation matrix of angle θ in the (x, y)-plane. The action is not free, indeed the point $(0, 0, 0, 0, \omega_n)$ is fixed by the S^1-action. Consider the following invariant polynomials as in [40]:

$$\bar{p_0} = \frac{|\dot{a}|}{2}, \qquad \bar{p_1} = \frac{|a|}{2}, \qquad \bar{p_2} = a \cdot \dot{a}$$
$$\bar{p_3} = x\dot{y} - y\dot{x}, \qquad \bar{p_4} = \omega_n.$$

The invariant polynomials are written in our coordinates in $\mathcal{M}/SO(3)$ as

$$\bar{p_0} = \frac{2R^2}{E^2}(\tilde{p_T}^2 \sin^2 \nu + \tilde{p_\theta}^2), \qquad \bar{p_1} = \frac{1}{2}r^2, \qquad \bar{p_2} = \frac{2Rr}{E}\tilde{p_T} \sin \nu \tag{B.1.7}$$
$$\bar{p_3} = \frac{2Rr}{E}\tilde{p_\theta}, \qquad \bar{p_4} = \frac{R}{I}\tilde{M_n} \tag{B.1.8}$$

The S^1-reduced space is describes by the following semi-algebraic set of $\mathbb{R}^5$,

$$\mathcal{M}/G = \{p = (\bar{p_0}, \bar{p_1}, \bar{p_2}, \bar{p_3}, \bar{p_4}) \in \mathbb{R}^5 : \bar{p_0} \geq 0, \bar{p_1} \geq 0, \quad 4\bar{p_0}\bar{p_1} = \bar{p_2}^2 + \bar{p_3}^2\}. \tag{B.1.9}$$

The singular 1-dimensional stratum, is given by

$$S_1 = \{p \in \mathbb{R}^5 : \bar{p_0} = \bar{p_1} = \bar{p_2} = \bar{p_3} = 0\},$$

and it corresponds to the situation where the ball is at the bottom of the surface and spinning about the vertical axis. The 4-dimensional regular stratum M_4 is given by

$$M_4 = \{p = (\bar{p_0}, \bar{p_1}, \bar{p_2}, \bar{p_3}, \bar{p_4}) \in \mathbb{R}^5 : \bar{p_0} \geq 0, \bar{p_1} \geq 0,\ 4\bar{p_0}\bar{p_1} = \bar{p_2}^2 + \bar{p_3}^2,\ \bar{p_0}^2 + \bar{p_1}^2 > 0\}.$$

The reduced bracket is not Poisson

In Theorem B.1.4 we have computed the 3-form $dJ \wedge K_{\mathcal{W}}$ which is used to verify if the reduced brackets $\{\cdot, \cdot\}_{red}$ is Poisson using the Jacobiator formula (2.3.21).

Proposition B.1.5. *The reduced bracket $\{,\}_{red}$ is not Poisson.*

Proof. We compute $dJ \wedge K_{\mathcal{W}}(\pi^{\#}_{nh}(d\rho^*\bar{p}2), \pi^{\#}_{nh}(d\rho^*\bar{p}3), \pi^{\#}_{nh}(d\rho^*\bar{p}4)) \neq 0$. Then, by the Jacobiator formula (2.3.21) we conclude that the reduced bivector π^{red}_{nh} is not Poisson. $\square$

B.2 First integrals and dynamical gauge transformation

It is known since Routh [79] that the mechanical system we are studying admits two first integrals J_1 and J_2 (besides the energy). In this section we recall the construction of those first integrals using the *ad-hoc* methods of the literature, see [40, 52]. We remark that in Chapter 4 we found the mentioned first integrals using the theory of the nonholonomic moment map and horizontal gauge momenta.

In $\mathcal{M}/G$ the reduced energy is given by

$$\bar{E}(\bar{p_0},\bar{p_1},\bar{p_2},\bar{p_4}) = \bar{p_0} + \frac{A}{2}\bar{p_4}^2 + \frac{1}{2}\bar{p_2}^2\Psi'(\bar{p_1})^2 + B\Psi(\bar{p_1}),$$

where $A = \frac{I}{E}$, $B = \frac{ma_g r^2}{E}$, and $\Psi(\frac{r^2}{2}) = \phi(r^2)$. The other integrals of motion J_1 and J_2 are constructed from solutions of the system of equations (B.2.10), which comes from the reduced dynamics on the regular stratum M_4 given in [52] and [40]:

$$\frac{d\bar{p_3}}{d\bar{p_1}} = F_3(\bar{p_1})\bar{p_4}, \quad \frac{d\bar{p_4}}{d\bar{p_1}} = F_4(\bar{p_1})\bar{p_3}, \tag{B.2.10}$$

where

$$F_3(\bar{p_1}) = A\left(\Psi'(\bar{p_1}) + 2\bar{p_1}\Psi''(\bar{p_1})\right) g(\bar{p_1}), \tag{B.2.11}$$

$$F_4(\bar{p_1}) = \left(\Psi'(\bar{p_1})^3 - \Psi''(\bar{p_1})\right) g(\bar{p_1}), \tag{B.2.12}$$

$$g(\bar{p_1}) = \frac{1}{1 + 2\bar{p_1}\Psi'(\bar{p_1})^2}. \tag{B.2.13}$$

Using the following notation for two independent solutions of (B.2.10)

$$p_1 \mapsto (\sigma_3(\bar{p_1}), \sigma_4(\bar{p_1})), \quad p_1 \mapsto (\tau_3(\bar{p_1}), \tau_4(\bar{p_1})),$$

which verify $\sigma_3(\bar{p_1})\tau_4(\bar{p_1}) - \sigma_3(\bar{p_1})\tau_4(\bar{p_1}) \neq 0$ in some interval containing the origin, one can verify that the two functions

$$\bar{J_1}(\bar{p_1},\bar{p_3},\bar{p_4}) := \bar{p_3}\sigma_4(\bar{p_1}) - \bar{p_4}\sigma_3(\bar{p_1}), \tag{B.2.14}$$

$$\bar{J_2}(\bar{p_1},\bar{p_3},\bar{p_4}) := \bar{p_3}\tau_4(\bar{p_1}) - \bar{p_4}\tau_3(\bar{p_1}), \tag{B.2.15}$$

are constant of motion of the reduced system on $\mathcal{M}/G$. In general these integral are not explicitly known except for some particular cases. In Section B.3 we consider the case of the circular paraboloid where $\bar{J_1}$ and $\bar{J_2}$ can be explicitly computed.

Hermans [52] has proved that the orbits of the reduced dynamics are only equilibrium points or periodic orbits. In fact, the periodic orbits are the connected components of the fibers of the submersion $(\bar{E},\bar{J_1},\bar{J_2}) : M_4\backslash\mathcal{E} \to \mathbb{R}^3$, where $\mathcal{E}$ is the equilibrium set of the reduced dynamics, see [40].

The J_i are gauge momenta

For each conserved quantity J_i, $i = 1, 2$, in (B.2.14) we look for a vector $X_i \in \Gamma(\mathcal{S})$ such that $\mathbf{i}_{X_i}\Theta_{\mathcal{M}} = J_i$. Using the formulas for J_i and the fact that the respective X_i belongs both to $\Gamma(\mathcal{C})$ and $\Gamma(\mathcal{V})$ we get

$$X_1 = \frac{2Rr}{E}\sigma_4(\bar{p_1})Y_\theta - \frac{2R}{I}\sigma_3(\bar{p_1})X_n, \tag{B.2.16}$$

$$X_2 = \frac{2Rr}{E}\tau_4(\bar{p_1})Y_\theta - \frac{2R}{I}\tau_3(\bar{p_1})X_n. \tag{B.2.17}$$

Using the formulas of the infinitesimal generators of the action on $\mathcal{M}$,

$$\begin{aligned}(1, \mathbf{0})_{\mathcal{M}} =& \frac{r}{R}Y_\theta - sin\,\nu X_n + \left(\frac{r}{R} + cos\,\nu\right) X_T,\\ (0, \boldsymbol{e_i})_{\mathcal{M}} =& \alpha_i X_1^r + \beta_i X_2^r + \gamma_i X_3^r, \quad i = 1, 2, 3,\end{aligned}$$

we can write X_1 and X_2 in function of the infinitesimal generators. Indeed, since $S = span\{Y_\theta, X_n\}$, it is generated by sections η_1, η_2 of the bundle $\mathfrak{g}_S \to Q$, i.e

$$(\eta_1)_{\mathcal{M}} = Y_\theta, \quad (\eta_2)_{\mathcal{M}} = X_n,$$

then defining

$$\zeta_1 = \frac{2Rr}{E}\sigma_4(\bar{p_1})\eta_1 - \frac{2R}{I}\sigma_3(\bar{p_1})\eta_2, \quad \zeta_2 = \frac{2Rr}{E}\tau_4(\bar{p_1})\eta_1 - \frac{2R}{I}\tau_3(\bar{p_1})\eta_2,$$

we have that $(\zeta_1)_{\mathcal{M}} = X_1$ and $(\zeta_2)_{\mathcal{M}} = X_2$. This shows that the J_1 and J_2 are horizontal gauge momenta in the sense of [41] (see algo [9]). Indeed we have found sections $\xi_i \in \Gamma(\mathfrak{g}_{\mathcal{S}})$ such that $X_i = (\xi_i)_{\mathcal{M}}$.

The dynamical gauge transformation and the reduced Poisson structure

Define the G-invariant 2-form:

$$B = \tilde{M_n}(1+\cos(\nu)\frac{R}{r})Y^\theta \wedge Y^T + \tilde{p_\theta}\frac{I}{E}(1+\cos(\nu)\frac{R}{r})Y^T \wedge \beta_n + \tilde{p_T}\frac{I}{E}(1+\cos(\nu)\frac{R}{r})\beta_n \wedge Y^\theta$$

This form B is semi-basic with respect to the fibre bundle $\mathcal{M} \to Q$ and verifies that $\mathbf{i}_{X_{nh}}B = 0$, i.e. it a dynamical gauge transformation.

The 2-section $\Omega_{\mathcal{C}} + B$ is given by

$$\begin{aligned}\Omega_{\mathcal{C}} + B =& -d\tilde{p_\theta} \wedge Y^\theta - d\tilde{p_T} \wedge Y^T - d\tilde{M_n} \wedge \beta_n\\ &- \left(\tilde{p_\theta} sin\,\nu\frac{R}{r} - \tilde{M_n} cos\,\nu sin^2\,\nu \frac{\phi' + \phi'' 2r^2}{\phi'}\frac{R}{r}\right) Y^T \wedge Y^\theta\\ &+ \tilde{p_\theta}\frac{I}{E}\frac{R}{r}cos\,\nu\left(cos\,\nu\, sin\,\nu\frac{\phi' + \phi'' 2r^2}{\phi'} - 1\right)\beta_n \wedge Y^T.\end{aligned}$$

and the gauge transformed bivector is biven by

$$\begin{aligned}
\pi_B^\#(Y^\theta) &= \frac{\partial}{\partial p_\theta}, \quad \pi_B^\#(Y^T) = \frac{\partial}{\partial p_T}, \quad \pi_B^\#(\beta_n) = \frac{\partial}{\partial \tilde{M}_n},\\
\pi_B^\#(dp_\theta) &= -Y_\theta - \left(\tilde{p_\theta} sin\,\nu \frac{R}{r} - \tilde{M}_n cos\,\nu\, sin^2\,\nu \frac{\phi' + \phi"2r^2}{\phi'}\frac{R}{r}\right)\frac{\partial}{\partial \tilde{p_T}},\\
\pi_B^\#(dp_T) &= -Y_T + \left(\tilde{p_\theta} sin\,\nu \frac{R}{r} - \tilde{M}_n cos\,\nu\, sin^2\,\nu \frac{\phi' + \phi"2r^2}{\phi'}\frac{R}{r}\right)\frac{\partial}{\partial \tilde{p_\theta}}\\
&\quad + \tilde{p_\theta}\frac{I}{E}\frac{R}{r} cos\,\nu \left(sin^2\,\nu \frac{\phi' + \phi''2r^2}{\phi'} - 1\right)\frac{\partial}{\partial \tilde{M}_n},\\
\pi_B^\#(d\tilde{M}_n) &= -X_n - \tilde{p_\theta}\frac{I}{E}\left(cos\,\nu \sin^2\nu \frac{\phi' + \phi''2r^2}{\phi'}\frac{R}{r} - \frac{R}{r} cos\,\nu\right)\frac{\partial}{\partial \tilde{p_T}}.
\end{aligned}$$

We verify that $\pi_B^\#(dH_\mathcal{M}) = \pi_{nh}^\#(dH_\mathcal{M}) = -X_{nh}$, that is the dynamics is preserved.

First integrals and how to find the gauge transformation

In this section we will study how to compute the 2-form B giving the gauge transformation from the knowledge of first integrals.

We compute $\pi_{nh}^\#(dJ_i) = V_i + W_i$, for $i = 1, 2$, with $V_i \in \Gamma(\mathcal{V})$ being the vertical part.

$$\begin{aligned}
\pi_{nh}^\#(dJ_1) &= V_1 + W_1 = -\frac{2Rr}{E}\sigma_4(p_1)Y_\theta + \frac{2R}{I}\sigma_3(p_1)X_n + W_1\\
\pi_{nh}^\#(dJ_2) &= V_2 + W_2 = -\frac{2Rr}{E}\tau_4(p_1)Y_\theta + \frac{2R}{I}\tau_3(p_1)X_n + W_2.
\end{aligned}$$

The gauge transformation B is constructed so that the new bivector π_B keeps only the vertical part. More precisely for a B of the form $B = aY^\theta \wedge Y^T + bY^T \wedge \beta_n + c\beta_n \wedge Y^\theta$, imposing the condition $\mathbf{i}_{V_i}(\Omega_\mathcal{C} + B) = -dJ_i$, for $i = 1, 2$, determines a unique B, such that

$$\begin{aligned}
a &= \tilde{M}_n\left(1 + cos(\nu)\frac{R}{r}\right),\\
b &= \tilde{p_\theta}\frac{I}{E}\left(1 + cos(\nu)\frac{R}{r}\right),\\
c &= \tilde{p_T}\frac{I}{E}\left(1 + cos(\nu)\frac{R}{r}\right).
\end{aligned}$$

By construction π_B verifies

$$\begin{aligned}
\pi_B^\#(dJ_1) &= V_1 = -\frac{2Rr}{E}\sigma_4(p_1)Y_\theta + \frac{2R}{I}\sigma_3(p_1)X_n\\
\pi_B^\#(dJ_2) &= V_2 = -\frac{2Rr}{E}\tau_4(p_1)Y_\theta + \frac{2R}{I}\tau_3(p_1)X_n.
\end{aligned}$$

Theorem B.2.1. *The reduced bracket $\{\cdot,\cdot\}_{red}^B$ in the differential space $\mathcal{M}/G$ is Poisson.*

Proof. We get:

$$(dJ\wedge K_{\mathcal{W}}-dB)|_{\mathcal{C}}=\frac{IR}{Er}\cos^3\nu d\tilde{p_\theta}\wedge\beta_n\wedge Y^T-\left(1+\cos\nu\frac{R}{r}\right)d\tilde{M_n}\wedge Y^\theta\wedge Y^T. \quad \text{(B.2.18)}$$

Recall that the invariant functions $\bar{p_i}$, $i=0,\cdots,4$ are generators of $C^\infty(\mathcal{M})^G$, so we can verify the following equality:

$$(dJ\wedge K_{\mathcal{W}}+dB)(\pi_{nh}^\#(d\rho^*\bar{p_i}),\pi_{nh}^\#(d\rho^*\bar{p_j}),\pi_{nh}^\#(d\rho^*\bar{p_k}))=0,$$

on all the combinations of $\bar{p_i}$ using their expression in our adapted coordinates (B.1.7). By the formula of the Jacobiator of the reduced bracket we conclude that the reduced bracket $\{\cdot,\cdot\}_{red}^B$ is Poisson. □

Theorem B.2.2. *The first integrals $\bar{J_i}$ are casimirs of the bracket $\{\cdot,\cdot\}_{red}^B$*

Proof. As the vector fields associated to the conserved quantities J_i are vertical w.r.t $\rho:\mathcal{M}\to\mathcal{M}/G$, i.e. $\pi_B^\#(dJ_1)=V_1$, both J_1 and J_2 are casimirs of the reduced bracket $\{\cdot,\cdot\}_{red}^B$. □

In Section B.2 we observed that the J_i are gauge momenta, i.e. here exists sections $\xi_i\in\Gamma(\mathfrak{g}_{\mathcal{S}})$ such that $\mathbf{i}_{(\xi_i)_{\mathcal{M}}}\Theta_{\mathcal{M}}=J_i$.

From (B.2.16) we observe that the $(\xi_i)_{\mathcal{M}}=-V_i$, $i=1,2$, which shows:

$$\pi_B^\#(dJ_1)=-(\xi_i)_{\mathcal{M}}.$$

B.3 Particular case: the homogeneous ball on a paraboloid of revolution

In the particular case where the surface of revolution is a circular paraboloid the conserves quantities J_i, $i=1,2$ can be explicitly integrated. Indeed, in that case take $\Psi(\bar{p_1})=2\bar{p_1}=r^2$. In that case the functions F_3 and F_4 in (B.2.11) are given by

$$F_3(\bar{p_1})=\frac{2I}{E(1+8\bar{p_1})},\quad F_4(\bar{p_1})=\frac{8}{1+8\bar{p_1}}.$$

From the system of differential equations (B.2.10) we get the following equation,

$$\bar{p_3}''+\frac{8}{1+8\bar{p_1}}\bar{p_3}'-\frac{2I}{E}\frac{8}{(1+8\bar{p_1})^2}\bar{p_3}=0,$$

where $'$ denotes the derivative with respect to $\bar{p_1}$. This Euler equation can be solved using the ansatz: $p_3=(1+8p_1)^\mu$. Replacing this in () we get $\mu^2=\frac{1}{2}\sqrt{\frac{I}{E}}$, and the general solution of the system () is:

$$\begin{aligned}\bar{p_3}=&A_1(1+8\bar{p_1})^\mu+B_1(1+8\bar{p_1})^{-\mu}\\ \bar{p_4}=&\frac{4E\mu}{I}\left(A_1(1+8\bar{p_1})^\mu-B_1(1+8\bar{p_1})^{-\mu}\right),\end{aligned}$$

where A_1 and B_1 are integration constants. Plugging the initial values $(1,0)$ and $(0,1)$ we get two solutions (σ_3, σ_4) and (τ_3, τ_4), respectively. Computing the Wronskian we see that they are independent. Recall the formula for the first integrals.

$$\begin{aligned}\bar{J}_1 &= \bar{p_3}\sigma_4(\bar{p_1}) - \bar{p_4}\sigma_3(\bar{p_1})\\ \bar{J}_2 &= \bar{p_4}\sigma_3(\bar{p_1}) - \bar{p_3}\sigma_4(\bar{p_1}).\end{aligned}$$

Two explicit solutions are:

$$\begin{aligned}\bar{J}_1 =& \frac{4Rr\mu}{I}\tilde{p_\theta}\left((1+8\bar{p_1})^{\mu} - (1+8\bar{p_1})^{-\mu}\right) - \frac{R}{I}\tilde{M_n}\left((1+8\bar{p_1})^{\mu} + (1+8\bar{p_1})^{-\mu}\right)\\ \bar{J}_2 =& \frac{Rr}{E}\tilde{p_\theta}\left((1+8\bar{p_1})^{\mu} + (1+8\bar{p_1})^{-\mu}\right) - \frac{R}{4E\mu}\tilde{M_n}\left((1+8\bar{p_1})^{\mu} - (1+8\bar{p_1})^{-\mu}\right).\end{aligned}$$

Now we can perform the same computations to find the gauge transformation B at the end of the last section. The general computation performed gives a B which does not depend on the form of the surface. Actually, this work started analysing this particular case before considering the the general case.

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